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UNIMODALITY OF THE DISTRIBUTION OF AN ORDER STATISTIC[1]

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A distribution function $G(x)$, on the real line, is called unimodal if there exists a value $x = a$, such that $G(x)$ is convex for $x < a$ and concave for $x > a$. Given that $G(x)$ is unimodal, a condition is given for the unimodality of $G^r(x)$, where r denotes a positive integer. $G^r(x)$ represents the distribution function of the largest observed value in a sample of r observations from the distribution $G(x)$. Some of the standard distributions, such as, the normal, gamma, Poisson and binomial distributions satisfy the given condition. An application of the given result to a problem of estimating the largest parameter is given.

1. Main results. A distribution function $G(x)$, on the real line, is called unimodal if there exists a value $x = a$, such that, $G(x)$ is convex for $x < a$ and concave for $x > a$. Let $r \geqq 1$ be a positive number. We consider necessary conditions for the unimodality of $G^r(x)$, given that $G(x)$ is unimodal. When r is a positive integer, $G^r(x)$ represents the distribution function of the largest observed value in a sample of r observations from the distribution $G(x)$.

THEOREM 1.1. *Let the distribution G have density g, and let $1/g$ be convex. Then the distribution of each order statistic in a sample from G is unimodal.*

PROOF. The density function of the sth order statistic in a sample of size r from G is given by

$$f(x) = r\binom{r-1}{s-1}g(x)G^{s-1}(x)(1 - G(x))^{r-s}\,. \tag{1.1}$$

The condition that $1/g$ is convex implies that g is right differentiable and that $g'(x)/g^2(x)$ is non-increasing in x, where g' denotes the right derivative of g. Therefore

$$f'(x) = r\binom{r-1}{s-1}G^{s-1}(x)(1 - G(x))^{r-s}g^2(x)\left\{\frac{g'(x)}{g^2(x)} + \frac{s-1}{G(x)} - \frac{r-s}{1-G(x)}\right\} \tag{1.2}$$

where f' denotes the right derivative of f. As $g'(x)/g^2(x)$ is non-increasing in x, the quantity inside the braces on the right-hand side of (1.2) is non-increasing in x. Therefore, $f'(x)$ has at most one change of sign as x varies from $-\infty$ to ∞, and if it changes sign, the sign changes from positive to negative. Thus f is unimodal. □

Note that $1/g$ is convex $\Leftrightarrow g'/g^2 \downarrow$, where g' is the right derivative of g. If $\log g$ is concave (g is a PF_2 density), then $1/g$ is convex. But the Cauchy dis-

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tribution is a counter example to the converse implication. The conditions of Theorem 1.1 are satisfied by the Cauchy, Normal, Gamma, etc.

THEOREM 1.2. *Let the distribution G be unimodal with mode at a, and density g. Let g be continuous at the mode, and $1/g$ be convex in $x > a$. Then $G^r(x)$ is unimodal for $r \geqq 1$.*

PROOF. Let $h(x) = rG^{r-1}(x)g(x)$ denote the density function of $G^r(x)$. Clearly, $h(x)$ is non-decreasing for $x \leqq a$. In $x > a$, the condition that $1/g$ is convex implies (as it follows from the proof of Theorem 1.1) that $h(x)$ is either non-increasing or it is first increasing then decreasing. Therefore, $G^r(x)$ is unimodal. □

Let $H_r(x) = 1 - (1 - G(x))^r$. When r is a positive integer, H_r represents the distribution of the smallest observed value in a sample of r observations from the distribution G.

Analogous to the result of Theorem 1.2, we have the following result. We omit the proof.

THEOREM 1.1*. *Let the distribution G be unimodal with mode at a, and density g. Let g be continuous at the mode, and $1/g$ be convex in $x < a$. Then $H_r(x)$ is unimodal for $r \geqq 1$.*

Let G be a discrete distribution on the real line, given by

$$P\{X = a_n\} = g_n\,, \qquad n = 0, \pm 1, \pm 2, \cdots \tag{1.3}$$

where $a_n < a_{n+1}$ for each n, and $\sum_{n=-\infty}^{\infty} g_n = 1$. The distribution is called unimodal if there exists a value of n equal to m, say, such that g_n is non-decreasing (non-increasing) in n for $n < (\geqq) m$.

The following theorem gives a necessary condition for the unimodality of $G^r(x)$ where r is a positive integer.

THEOREM 1.3. *Let the discrete distribution G be given by (1.3). Let G be unimodal with mode at a_m, and let r be a positive integer. If g_n/g_{n+1} is non-decreasing in n for $n \geqq m$ then $G^r(x)$ is unimodal.*

PROOF. Let $g_n^{(r)}$ represent the frequency function $G^r(x)$, given by

$$\begin{aligned} g_n^{(r)} &= G^r(a_n) - G^r(a_{n-1}) \\ &= g_n A_n \end{aligned} \tag{1.4}$$

where

$$A_n = \sum_{s=0}^{r-1} G^{r-s-1}(a_n) G^s(a_{n-1})\,.$$

Let

$$p_n(s) = G^{r-s-1}(a_{n+1}) G^s(a_n)/A_{n+1}$$

denote the frequency function of a probability distribution on the set of integers $0, 1, \cdots, r-1$. Then

$$A_n/A_{n+1} = \sum_{s=0}^{r-1} \left(\frac{G(a_{n-1})}{G(a_{n+1})}\right)^{r-s-1} p_n(s)\,. \tag{1.5}$$

Now, $G(a_{n+1})/G(a_n) = (1 + g_{n+1}/G(a_n))$ is non-increasing in n for $n \geqq m$, by the unimodality of G. Therefore, $p_n(s)$ has monotone likelihood ratio property in the parameter $n(\geqq m)$. As $G(a_{n-1})/G(a_{n+1}) = 1 - (g_n + g_{n+1})/G(a_{n+1})$ is non-decreasing in n for $n \geqq m$, from (1.5), and the application of Lemma 1.1, given at the end of this section, we have that A_n/A_{n+1} is non-decreasing in n for $n \geqq m$.

From (1.4) we have that

$$g_{n+1}^{(r)} - g_n^{(r)} = A_{n+1}(g_{n+1} - (g_n A_n)/A_{n+1}) \,. \tag{1.6}$$

From the monotonicity property of g_n/g_{n+1} under the condition of the theorem, and the monotonicity property of A_n/A_{n+1}, given above, we have that as n varies from m to ∞, the quantity on the right-hand side of (1.6) changes sign at most once, and if it changes sign, it changes from positive to negative. Also, from (1.4) it is seen that $g_n^{(r)}$ is non-decreasing in n for $n < m$. Therefore, there exists an integer $m' \geqq m$, such that $g_n^{(r)}$ is non-decreasing in n for $n < m'$ and non-increasing in n for $n \geqq m'$. Hence $G^r(x)$ is unimodal. □

For the distribution $H_r(x)$ as defined preceding Theorem 1.1*, we have the following result. We omit the proof.

THEOREM 1.3*. *Let the distribution G be given by* (1.3). *Let G be unimodal with mode at a_m, and let r be a positive integer. If g_n/g_{n+1} is non-decreasing in n for $n \leqq m$ then $H_r(x)$ is unimodal.*

The lemma required for Theorem 1.3 is stated below. The lemma follows from Theorem 3.4 (a), page 285 of Karlin [2]. A distribution on the real line with density $f(x, \theta)$, depending on a real-valued parameter θ, is said to have monotone likelihood ratio (mlr) property in θ if

$$f(x, \theta)f(x', \theta') - f(x, \theta')f(x', \theta) \geqq 0 \tag{1.7}$$

for $x \leqq x'$ and $\theta \leqq \theta'$. Let E_θ denote expectation with respect to the given distribution.

LEMMA 1.1. *Let the distribution of X have monotone likelihood ratio property in θ, and let $\psi(x, \theta)$ be non-decreasing in x and θ. Then $E_\theta \psi(X, \theta)$ is non-decreasing in θ.*

2. Application. The distribution of the largest order statistic arises in several statistical problems. In the following problem, for example, it is of particular interest to ascertain whether the distribution is unimodal. The conditions of the theorems given in the preceding section are convenient to check for the unimodality of $G^r(x)$, from the parent distribution G.

Let $F(x, \theta)$ denote a family of distributions on the real line, indexed by a real-valued parameter θ. It is assumed that $F(x, \theta)$ is non-increasing in θ for each x, and that $g(x, \theta) = -\partial F(x, \theta)/\partial\theta$ satisfies the relation (1.7) with g substituted for f. Let $\theta_1, \cdots, \theta_k$ denote k unknown values of the parameter θ, and $X_1, \cdots, X_k$ represent k independent observations from the distributions $F(x, \theta_1), \cdots, F(x, \theta_k)$, respectively. Let $\theta^* = \max(\theta_1, \cdots, \theta_k)$ and $t = \max(X_1, \cdots, X_k)$.

Given θ^* and $\alpha (0 < \alpha < 1)$, two numbers $c = c(\theta^*)$ and $d = d(\theta^*)$ can be determined such that $c + d > 0$, and

$$P\{t - c \leqq \theta^* \leqq t + d\} \geqq \alpha . \tag{2.1}$$

For estimating θ^* it is shown (see Alam [1], Theorem 2.1) that when $F^k(x, \theta)$ is continuous and unimodal, the values of c and d are uniquely determined for which the length of the interval $(t - c, t + d)$ is minimized. The relation (2.1) provides an interval estimate of θ^*, the largest parameter.

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