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AN EXPONENTIAL PROBABILITY BOUND FOR THE ENERGY OF A TYPE OF GAUSSIAN PROCESS

BY L. H. KOOPMANS[1] AND C. QUALLS[2]

The University of New Mexico

Real-valued stochastic processes of the form $x(t) = \int A(t, \lambda)Z(d\lambda)$ are considered, where $Z(\lambda)$ is a zero mean Gaussian process with independent increments and $\int\int |A(t, \lambda)|^2 F(d\lambda)\, dt < \infty$, where $F(d\lambda) = E|Z(d\lambda)|^2$. It is shown that the energy of $x(t)$, $\int x^2(t)\, dt$, is a well-defined random variable and an exponential bound for $P(\int x^2(t)\, dt - E \int x^2(t)\, dt \geqq \varepsilon)$ is derived. This bound is used to obtain an exponential bound for crossing probabilities $P(|y(t)| > a$ for some $t)$ where $y(t) = \int h(t-\tau)x(\tau)\, d\tau$, $\int h^2(t)\, dt < \infty$.

1. Introduction and summary. Let $\{Z(\lambda) : -\infty < \lambda < \infty\}$ be a measurable, zero mean, complex-valued Gaussian process with independent increments on a probability space $(\Omega, \mathscr{A}, P)$. This process will be thought of as the spectral process of a real-valued stationary time series in continuous time. As such, it generates a complex-valued spectral measure $Z(A)$ on the F-measurable subsets of the real line with the property that $Z(-A) = \overline{Z(A)}$, where F is the spectral distribution function of the time series. The Lebesgue–Stieltjes measure induced by F will also be denoted by F and it is related to Z by the expression $F(A \cap B) = EZ(A)\overline{Z(B)}$. Thus, in particular $F(A) = E|Z(A)|^2$. Let $A(t, \lambda)$ be a complex-valued $L \times F$ measurable function (where L denotes Lebesgue measure on $(-\infty, \infty)$) such that

$$A(t, -\lambda) = \overline{A(t, \lambda)} \tag{1.1}$$

and

$$\int\int |A(t, \lambda)|^2 F(d\lambda)\, dt < \infty\,. \tag{1.2}$$

(All integrals are over the range $(-\infty, \infty)$ unless specified otherwise.) The basic result of this paper is the following theorem.

THEOREM A. *Let $A(t, \lambda)$ be an $L \times F$ measurable function satisfying conditions* (1.1) *aud* (1.2) *and consider the stochastic process defined by the stochastic integral*

$$x(t) = \int A(t, \lambda)Z(d\lambda)\,. \tag{1.3}$$

Then the energy of this process, $\int x^2(t)\, dt$, is a well-defined random variable for which

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the following simple exponential probability bound is valid:

$$P(\int x^2(t)\,dt - E\int x^2(t)\,dt \geqq \varepsilon) \leqq \left(1+\frac{\varepsilon}{\Lambda}\right)^{\frac{1}{2}} \exp\left(-\frac{\varepsilon}{2\Lambda}\right), \tag{1.4}$$

for all $\varepsilon \geqq 0$, *where* $\Lambda^2 = \frac{1}{2}\operatorname{Var}\int x^2(t)\,dt$.

This inequality is based on an elegant inequality for quadratic forms in symmetric random variables recently published by Hanson and Wright [2]. We have modified their inequality slightly and have applied it to the simpler case of normal random variables yielding an exponential bound which we feel is of interest in its own right. The details of this modification will be given in Section 2. In Section 3 we establish Theorem A.

Stochastic processes of the form (1.3) have been of interest for some time. (See Priestley (1965).) With the condition (1.1) they constitute a large and important class of real-valued (possibly) non-stationary processes. With the addition of condition (1.2) the processes become decidedly non-stationary and, in fact, have finite energy, with probability 1. In this form they are useful as models for transient phenomena or for non-transient phenomena over finite time intervals. In a study of the failure probability for linear structures given in [3], such a process (an earthquake) was used as the input to a linear system (an idealized physical structure) with impulse response function $h(t)$, $\int h^2(t)\,dt < \infty$. If $y(t) = \int h(t-u)x(u)\,du$ is the output of the system, then an investigation of the probability of failure of the structure leads naturally to evaluation of the crossing probability $P(|y(t)| > a$ for some t, $-\infty < t < \infty)$. In Section 4 an elementary application of inequality (1.4) will be shown to yield the bound

$$P(|y(t)| > a \text{ for some } t) \leqq \left(1+\frac{\varepsilon(a)}{\Lambda}\right)^{\frac{1}{2}} \exp\left(-\frac{\varepsilon(a)}{2\Lambda}\right), \tag{1.5}$$

where $\varepsilon(a) = (a^2 - \int h^2(t)\,dt\,E\int x^2(t)\,dt)/\int h^2(t)\,dt$ and Λ is as in (1.4). This bound is of exponential form as are the asymptotic values of crossing probabilities (for large a) given in the literature (e.g., [5] Lemma 2.9) and has the practical advantage of being valid for all values of a for which $\varepsilon(a) \geqq 0$, and thus, readily usable in applications.

2. Modification of the Hanson–Wright inequality. Since we deal with the special case of normal random variables, the derivation of our version of the Hanson–Wright inequality is considerably shorter than the one in [2] and is given below for completeness.

THEOREM 1. *Let* $A = [a_{ij}]$ *be a real, symmetric* $N \times N$ *matrix and* $\mathbf{u} = (u_1, \cdots, u_N)'$ *a vector of independent, standard normal* ($\mathscr{N}(0,1)$) *random variables. Then, if* $S = \sum_{i=1}^N \sum_{j=1}^N a_{ij}(u_iu_j - E(u_iu_j))$, *for every* $\varepsilon \geqq 0$

$$P(S \geqq \varepsilon) \leqq \left(1+\frac{\varepsilon}{\Lambda}\right)^{\frac{1}{2}} \exp\left(-\frac{\varepsilon}{2\Lambda}\right), \tag{2.1}$$

where $\Lambda^2 = \sum_{i=1}^N \sum_{j=1}^N a_{ij}^2 = \frac{1}{2}\operatorname{Var} S$.

PROOF. We use a basic probability inequality (see, e.g. Loève, page 255); namely

$$P(S \geqq \varepsilon) \leqq Ee^{\theta(S-\varepsilon)}\,, \tag{2.2}$$

which is valid for all $\theta \geqq 0$. Since $\mathbf{u}$ is orthogonally invariant, we may transform A to diagonal form $\operatorname{diag}(b_1, \cdots, b_N)$ and obtain the quadratic form

$$S = \sum_{i=1}^N b_i(w_i^2 - 1)\,, \qquad \text{where } \mathbf{w} = D\mathbf{u} \text{ is } N(\mathbf{0}, I)\,.$$

The moment generating function of $w_i^2 - 1$ is easily calculated and is given by

$$E \exp \lambda(w_i^2 - 1) = \frac{e^{-\lambda}}{(1 - 2\lambda)^{\frac{1}{2}}} \qquad \text{for} \quad \lambda < \tfrac{1}{2}\,. \tag{2.3}$$

Now, define a function $C(\lambda)$ by the expression

$$\exp(C(\lambda)\lambda^2) = \frac{e^{-\lambda}}{(1 - 2\lambda)^{\frac{1}{2}}} \qquad \text{for} \quad 0 < |\lambda| < \tfrac{1}{2}\,, \tag{2.4}$$

and

$$C(0) = 1\,.$$

By expanding $\log(1 - 2\lambda)$ in a power series about $\lambda = 0$ it can be verified that

$$C(\lambda) = 1 + 2\sum_{k=1}^\infty \frac{(2\lambda)^k}{k + 2} \qquad \text{for} \quad |\lambda| < \tfrac{1}{2}\,.$$

By a term by term comparison of this expression it is evident that $C(\lambda) \leqq C(|\lambda|)$ and that $C(\lambda)$ is an increasing function of λ when $\lambda > 0$. Thus, since $\max_i |b_i| \leqq (\sum_{i=1}^\infty b_i^2)^{\frac{1}{2}} = (\operatorname{tr} A^2)^{\frac{1}{2}} = \Lambda$, we will have

$$C(\theta b_i) \leqq C(\theta|b_i|) \leqq C(\theta\Lambda) \qquad \text{for} \quad 0 < \theta < \frac{1}{2\Lambda}\,.$$

Now, combining this fact and expressions (2.2)—(2.4), we obtain

$$\begin{aligned} P(S \geqq \varepsilon) &\leqq e^{-\varepsilon\theta} \prod_{i=1}^N E \exp \theta b_i(w_i^2 - 1) \\ &\leqq e^{-\varepsilon\theta} \exp[\theta^2 \textstyle\sum_{i=1}^N C(\theta|b_i|) b_i^2] \\ &\leqq e^{-\varepsilon\theta} \exp[\theta^2 C(\theta\Lambda)\Lambda^2] \\ &= e^{-\varepsilon\theta} \frac{e^{-\theta\Lambda}}{(1 - 2\theta\Lambda)^{\frac{1}{2}}} \qquad \text{for } 0 < \theta < \frac{1}{2\Lambda}\,. \end{aligned}$$

Finally, the minimization of the right-hand side of this inequality as a function of θ yields inequality (2.1).

The expression for Λ^2 is an immediate consequence of the representation of S in terms of the w_i's and the fact that $\operatorname{Var}(w_i^2 - 1) = 2$.

3. The exponential bound for a class of complex quadratic forms. Let $\{\Delta_k : k = 0, \pm 1, \cdots, \pm n\}$ be a finite F-measurable partition of the real line with the property that $\Delta_{-k} = -\Delta_k$ for all k. (Thus, $\Delta_0 = \{0\}$.) Let $I_k(\lambda)$ denote the set characteristic function of Δ_k and define the product simple function

$$\mathscr{F}(\lambda, \mu) = \sum_{j=-n}^n \sum_{k=-n}^n C_{j,k} I_j(\lambda) I_k(\mu)\,,$$

where the complex numbers $C_{j,k}$ are chosen in such a way that $\mathscr{F}(\lambda, \mu)$ satisfies the symmetry conditions

$$\mathscr{F}(\mu, \lambda) = \overline{\mathscr{F}(\lambda, \mu)} \quad \text{and} \quad \mathscr{F}(-\lambda, -\mu) = \overline{\mathscr{F}(\lambda, \mu)} . \tag{3.1}$$

That is, $C_{k,j} = \bar{C}_{j,k}$ and $C_{-j,-k} = \bar{C}_{k,j}$. Then,

$$\begin{aligned} T &= \textstyle\int\int \mathscr{F}(\lambda, \mu) Z(d\lambda)\overline{Z(d\mu)} \\ &= \textstyle\sum_{j=-n}^{n} \sum_{k=-n}^{n} C_{j,k} Z(\Delta_j)\overline{Z(\Delta_k)} \end{aligned}$$

is a real-valued quadratic form in complex normal random variables.

LEMMA. *Inequality* (2.1) *is valid with* $S = T - ET$ *and*

$$\Lambda^2 = \tfrac{1}{2} \operatorname{Var} T = \tfrac{1}{2} \textstyle\int\int |\mathscr{F}(\lambda, \mu)|^2 F(d\lambda)F(d\mu) .$$

Moreover

$$ET^2 = [\textstyle\int \mathscr{F}(\lambda, \lambda)F(d\lambda)]^2 + \int\int |\mathscr{F}(\lambda, \mu)|^2 F(d\lambda)F(d\mu) . \tag{3.2}$$

PROOF. The proof is based on a simple reduction of the complex quadratic form to a form in real random variables and coefficients. Associated with the complex-valued spectral measure $Z(A)$ are two zero mean real-valued spectral measures $X(A)$ and $Y(A)$ such that $Z(A) = X(A) - iY(A)$ for every F-measurable set A. It is seen from the properties of $Z(A)$ that $X(A)$ and $Y(B)$ are independent, zero mean, normal random variables with $X(-A) = X(A)$, $Y(-A) = -Y(A)$ and $EX(A)X(B) = \frac{1}{2}[F(A \cap B) + F(A \cap (-B))]$, $EY(A)Y(B) = \frac{1}{2}[F(A \cap B) - F(A \cap (-B))]$. It follows that if $A \subset (0, \infty)$, $\operatorname{Var} X(A) = \operatorname{Var} Y(A) = \frac{1}{2}F(A)$.

To avoid difficulties we will assume that $F(\{0\}) = 0$ for the time being and will show how to remove this restriction later. Then, $P(Z(\Delta_0) = 0) = 1$. Then it is easy to establish that

$$\begin{aligned} &\textstyle\int\int \mathscr{F}(\lambda, \mu) Z(d\lambda)\overline{Z(d\mu)} \\ &\quad = 2 \textstyle\int_0^\infty \int_0^\infty \{[\operatorname{Re} \mathscr{F}(\lambda, \mu) + \operatorname{Re} \mathscr{F}(\lambda, -\mu)]X(d\lambda)X(d\mu) \\ &\qquad + [-\operatorname{Im} \mathscr{F}(\lambda, \mu) + \operatorname{Im} \mathscr{F}(\lambda, -\mu)]X(d\lambda)Y(d\mu) \\ &\qquad + [\operatorname{Im} \mathscr{F}(\lambda, \mu) + \operatorname{Im} \mathscr{F}(\lambda, -\mu)]Y(d\lambda)X(d\mu) \\ &\qquad + [\operatorname{Re} \mathscr{F}(\lambda, \mu) - \operatorname{Re} \mathscr{F}(\lambda, -\mu)]Y(d\lambda)Y(d\mu)\} . \end{aligned} \tag{3.5}$$

Thus, if $X_k = X(\Delta_k)/(\frac{1}{2}F(\Delta_k))^{\frac{1}{2}}$ and $Y_k = Y(\Delta_k)/(\frac{1}{2}F(\Delta_k))^{\frac{1}{2}}$ when $F(\Delta_k) \neq 0$ and $X_k = Y_k = 0$ if $F(\Delta_k) = 0$, then T can be put in the form required by Theorem 1 by the representation

$$T = \textstyle\sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_{j,k} X_j X_k + \beta_{j,k} X_j Y_k + \gamma_{j,k} Y_j X_k + \delta_{j,k} Y_j Y_k \tag{3.4}$$

where (from (3.3)),

$$\begin{aligned} \alpha_{j,k} &= (\operatorname{Re} C_{j,k} + \operatorname{Re} C_{j,-k})(F(\Delta_j)F(\Delta_k))^{\frac{1}{2}} \\ \beta_{j,k} &= (-\operatorname{Im} C_{j,k} + \operatorname{Im} C_{j,-k})(F(\Delta_j)F(\Delta_k))^{\frac{1}{2}} \\ \gamma_{j,k} &= (\operatorname{Im} C_{j,k} + \operatorname{Im} C_{j,-k})(F(\Delta_j)F(\Delta_k))^{\frac{1}{2}} \\ \delta_{j,k} &= (\operatorname{Re} C_{j,k} - \operatorname{Re} C_{j,-k})(F(\Delta_j)F(\Delta_k))^{\frac{1}{2}} . \end{aligned} \tag{3.5}$$

It is easily checked that the appropriate symmetry properties for these coefficients follow from (3.1).

Theorem 1 can be applied directly to (3.4) to obtain the first part of the lemma. Since $T = S + ET$, Var S = Var T and $ET^2 = (ET)^2 + \text{Var}(S) = (ET)^2 + 2\Lambda^2$. By recalling that Λ^2 is the sum of squares of the elements of A and by evaluating ET from (3.4) we obtain the expression

$$ET^2 = [\sum_{j=1}^n (\alpha_{jj} + \beta_{jj} + \gamma_{jj} + \delta_{jj})]^2 + 2 \sum_{j=1}^n \sum_{k=1}^n (\alpha_{j,k}^2 + \beta_{j,k}^2 + \gamma_{j,k}^2 + \delta_{j,k}^2) . \tag{3.6}$$

Thus, by virtue of (3.5) and the symmetry properties of the $C_{j,k}$'s and F,

$$\begin{aligned} ET^2 &= [2 \sum_{j=1}^n C_{j,j} F(\Delta_j)]^2 + 2 \sum_{j=1}^n \sum_{k=1}^n (|C_{j,k}|^2 + |C_{j,-k}|^2) F(\Delta_j) F(\Delta_k) \\ &= [\int \mathscr{F}(\lambda, \lambda) F(d\lambda)]^2 + \int \int |\mathscr{F}(\lambda, \mu)|^2 F(d\lambda) F(d\mu) . \end{aligned}$$

This readily yields Var $T = \int \int |\mathscr{F}(\lambda, \mu)|^2 F(d\lambda) F(d\mu)$.

To remove the restriction $F(\{0\}) = 0$, the representation (3.4) can be extended to a double sum from 0 to n in j and k by taking $\alpha_{0,0} = C_{0,0} F(\{0\})$, $\beta_{0,0} = \gamma_{0,0} = \delta_{0,0} = 0$, $\alpha_{0,k} = \alpha_{k,0} = \text{Re}\, C_{k,0} (\frac{1}{2} F(\{0\}) F(\Delta_k))^{\frac{1}{2}}$, $\beta_{0,k} = \gamma_{k,0} = \text{Im}\, C_{k,0} (\frac{1}{2} F(\{0\}) F(\Delta_k))^{\frac{1}{2}}$, and $\beta_{k,0} = \gamma_{k,0} = \delta_{0,k} = \delta_{k,0} = 0$ for $k \geqq 1$.

The inequality (2.1) is certainly still valid and the additional terms acquired in (3.6) by extending the sums to the lower index zero are seen to account for the spectral mass at the origin and along the coordinate axes. Thus, (3.2) is correct in general. This proves the lemma.

We proceed to the proof of Theorem A. The class of simple functions with rectangular (product) sets of constancy is dense in $\mathscr{L}_2(L \times F)$. Thus, because of (1.2), there exists a sequence of such simple functions, $\{A_n(t, \lambda)\}$, such that

$$\int \int |A_n(t, \lambda) - A(t, \lambda)|^2 \, dt F(d\lambda) \to 0 \qquad \text{as } n \to \infty . \tag{3.7}$$

These functions can be modified (if necessary) to satisfy (1.1) without changing their convergence properties by replacing $A_n(t, \lambda)$ by $\overline{A_n(t, -\lambda)}$ for $\lambda \leqq 0$. Consequently, we assume (1.1) to be satisfied by $A_n(t, \lambda)$ for all n. Now, let

$$x_n(t) = \int A_n(t, \lambda) Z(d\lambda) .$$

Then

$$T_n = \int x_n^2(t) \, dt = \int \int \mathscr{F}_n(\lambda, \mu) Z(d\lambda) \overline{Z(d\mu)} ,$$

where

$$\mathscr{F}_n(\lambda, \mu) = \int A_n(t, \lambda) \overline{A_n(t, \mu)} \, dt .$$

It is easy to verify that $\mathscr{F}_n(\lambda, \mu)$ satisfies (3.1) and the lemma can be applied to yield

$$P(T_n - ET_n \geqq \varepsilon) \leqq \left(1 + \frac{\varepsilon}{\Lambda_n}\right)^{\frac{1}{2}} \exp\left(\frac{\varepsilon}{2\Lambda_n}\right) , \qquad \varepsilon \geqq 0 , \tag{3.8}$$

where $\Lambda_n^2 = \frac{1}{2} \text{Var}\, T_n$.

Also, for every m and n $\mathscr{F}_n(\lambda, \mu) - \mathscr{F}_m(\lambda, \mu)$ is a simple function with the same symmetry properties and (3.2) can be applied to obtain

$$E(T_n - T_m)^2 = [\int (\mathscr{F}_n(\lambda, \lambda) - \mathscr{F}_m(\lambda, \lambda))F(d\lambda)]^2 + \int \int |\mathscr{F}_n(\lambda, \mu) - \mathscr{F}_m(\lambda, \mu)|^2 F(d\lambda)F(d\mu) . \tag{3.9}$$

By a standard argument based on (3.7), the two terms on the right-hand side of (3.9) can be shown to go to zero. Thus, $\{T_n\}$ is a Cauchy sequence in $\mathscr{L}_2(P)$. It follows that there exists a random variable T with $ET^2 < \infty$ and $E(T_n - T)^2 \to 0$; thus $E|T_n - T| \to 0$ as $n \to \infty$. Moreover, $ET_n \to ET$, Var $T_n \to$ Var T and $P(T_n \in A) \to P(T \in A)$ uniformly for all Borel sets A. Consequently, we can pass to the limit on both sides of (3.8) to obtain

$$P(T - ET \geqq \varepsilon) \leqq \left(1 + \frac{\varepsilon}{\Lambda}\right)^{\frac{1}{2}} \exp\left(-\frac{\varepsilon}{2\Lambda}\right), \qquad \varepsilon \geqq 0 , \tag{3.10}$$

where $\Lambda^2 = \frac{1}{2}$ Var T.

Now, (Doob, page 430), it is possible to select an $L \times P$ measurable version of $x(t) = \int A(t, \lambda)Z(d\lambda)$. Thus, $\int x^2(t)\, dt$ is a well-defined random variable in $\mathscr{L}_1(P)$ since (Fubini) $E \int x^2(t)\, dt = \int Ex^2(t)\, dt = \int \int |A(t, \lambda)|^2 F(d\lambda)\, dt < \infty$. Now, a straightforward application of the Cauchy-Schwarz inequality and the equation

$$\int E|x_n(t) - x(t)|^2\, dt = \int \int |A_n(t, \lambda) - A(t, \lambda)|^2 F(d\lambda)\, dt$$

establishes that $E|T_n - \int x^2(t)\, dt| \to 0$ as $n \to \infty$. The uniqueness of $\mathscr{L}_1(P)$ limits yields $T = \int x^2(t)\, dt$ a.s. With this substitution in (3.10), Theorem A is proved.

COROLLARY. *Let $A(t, \lambda)$ be the function given in Theorem* A. *Then,*

$$\Lambda^2 = \frac{1}{2} \int \int |\mathscr{F}(\lambda, \mu)|^2 F(d\lambda)F(d\mu) , \tag{3.11}$$

where

$$\mathscr{F}(\lambda, \mu) = \int A(t, \lambda)\overline{A(t, \mu)}\, dt .$$

The proof of the corollary is easily obtained by a convergence argument based on (3.7). This result, along with the previously derived expression

$$E \int x^2(t)\, dt = \int \int |A(t, \lambda)|^2 F(d\lambda)\, dt \tag{3.12}$$

provides a means for computing the parameters of inequality (1.4) of Theorem A.

EXAMPLE. An important subclass of the processes defined in Section 1 is of the following type: Let $\{u(t): -\infty < t < \infty\}$ be a stationary Gaussian process with $Eu(t) = 0$, and covariance function $R(t)$ continuous at $t = 0$. Then $R(t) = \int e^{it\lambda}F(d\lambda)$ where F is the spectral distribution function of the process and $u(t)$ has the representation $u(t) = \int e^{it\lambda}Z(d\lambda)$ where $Z(A)$ is a Gaussian spectral measure of the type defined in Section 1.

This process is filtered by a time invariant linear transformation with frequency response function $B(\lambda)$ with $\int |B(\lambda)|^2 F(d\lambda) < \infty$ (so that an output with finite power is obtained). Then the result is subjected to a real-valued tapering function $G(t)$ with $\int G^2(t)\,dt < \infty$. The output of these two operations is of the form $x(t) = \int A(t, \lambda)Z(d\lambda)$ with $A(t, \lambda) = G(t)B(\lambda)e^{it\lambda}$. It follows from (3.11) and (3.12) that

$$E \int x^2(t)\,dt = \int G^2(t)\,dt \int |B(\lambda)|^2 F(d\lambda) \tag{3.13}$$

and

$$\Lambda^2 = \tfrac{1}{2}\operatorname{Var} \int x^2(t)\,dt = \tfrac{1}{2}\int\int |B(\lambda)|^2|B(\mu)|^2|\mathscr{G}(\lambda - \mu)|^2 F(d\lambda)F(d\mu) \tag{3.14}$$

where

$$\mathscr{G}(\lambda) = \int G^2(t)e^{it\lambda}\,dt .$$

As a special case, the tail probabilities of $\int_0^T u^2(t)\,dt$ can be bounded by taking $Eu^2(t) = \int F(d\lambda) = K < \infty$, $B(\lambda) \equiv 1$ and $G(t) = I_T(t)$, where I_T is the set characteristic function of the interval $[0, T]$. The evaluation of the parameters of inequality (1.4) by (3.13) and (3.14) yields

$$E \int_0^T u^2(t)\,dt = KT , \qquad \Lambda^2 = \tfrac{1}{2}\int_0^T \int_0^T R^2(t - s)\,dt\,ds .$$

Thus, with $\delta = \varepsilon/\Lambda$, inequality (1.4) can be put in the form

$$P\left(\int_0^T u^2(t)\,dt \geqq KT + \delta\Lambda\right) \leqq (1 + \delta)^{\frac{1}{2}}e^{-\delta/2} .$$

4. An exponential bound for crossing probabilities. Let $\{x(t) : -\infty < t < \infty\}$ be as given in Section 1 and let $h(t)$ be a real-valued, L-measurable function on $(-\infty, \infty)$ for which $\int h^2(t)\,dt < \infty$. Then an $L \times P$ measurable version of

$$y(t) = \int h(t - \tau)x(\tau)\,d\tau$$

exists, and

$$|y(t)|^2 \leqq \int h^2(t - \tau)\,d\tau \cdot \int x^2(\tau)\,d\tau = \int h^2(\tau)\,d\tau \int x^2(\tau)\,d\tau \quad \text{a.e.}$$

by the Cauchy-Schwarz inequality. Thus, the event $[\sup_t |y(t)| \leqq a]$ contains the event $[MN < a]$ where $M^2 = \int x^2(t)\,dt$, $N^2 = \int h^2(t)\,dt$ from which it follows that

$$P(\sup_t |y(t)| > a) \leqq P(M^2 \geqq a^2/N^2) .$$

Inequality (1.5) is an immediate consequence of this result and inequality (1.4).

This bound has numerous applications, since many processes of interest can be obtained by linearly filtering Gaussian processes of the prescribed variety. In fact, the class of processes $\{y(t)\}$ for which inequality (1.5) is valid is a rather large subclass of the original one defined in Section 1, since

$$y(t) = \int B(t, \lambda)Z(d\lambda)$$

where $B(t, \lambda) = \int h(t - \tau)A(\tau, \lambda)\,d\tau$ is easily seen to satisfy conditions (1.1) and (1.2).

This paper was motivated by a problem in earthquake engineering and a numerical example of the use of inequality (1.5) in this context is given in [3].

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Dept. of Mathematics and Statistics
University of New Mexico
Albuquerque, New Mexico 87106