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A MEAN FIELD LIMIT FOR A LATTICE CARICATURE OF DYNAMIC ROUTING IN CIRCUIT SWITCHED NETWORKS[1]

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Simulation studies of circuit switched networks with dynamic alternate routing reveal the existence of hysteresis phenomena, which suggest that such networks can admit more than one regime of operation for the same offered traffic. Such behavior is also suggested by a detailed analytical model due to Marbukh and a simpler model due to Gibbens, Hunt and Kelly. In these models, a limit is taken as the network size becomes large, and one finds a limiting ODE describing the proportions of network links in different states. The possibility of multiple regimes of operation shows up through the fact that the ODE has multiple equilibrium points for certain ranges of parameters.

The kinds of limits considered by Marbukh and Gibbens, Hunt and Kelly do not take into account the spatial extent of the network. In an attempt to preserve the spatial characteristics, we consider a lattice model similar to that of Gibbens, Hunt and Kelly. We derive a mean field limit for this lattice model. This is an integrodifferential equation which describes how the spatial distribution of the network evolves in time. The mean field equation also admits multiple spatially homogeneous equilibrium solutions for certain ranges of the parameters, which may be loosely thought of as the different operating regimes. This equation may be particularly useful in understanding the exchange between the operating regimes, that is, questions like "for what parameter values is a hot spot of heavy loading in the system likely to take over the whole network by knock-on effects?"

1. Introduction. This paper discusses circuit switched networks with dynamic alternate routing. The purpose of dynamic routing schemes is to adaptively adjust traffic patterns in the network in response to demand, so as to make better use of spare capacity and to provide robustness to failures or overloads. Such schemes have been the topic of considerable recent interest [1–5, 7, 11–16, 18–20, 22, 23], primarily because it has only recently been possible to implement them in practice, and because they offer improved performance over the traditional hierarchical routing schemes.

A difficulty associated with dynamic routing schemes is the potential for metastable states. Several simulation based studies of such routing schemes [1, 2, 4, 7, 12, 16, 22] have revealed the existence of hysteresis phenomena, which suggest that the network may have several qualitatively different regimes of

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operation for the same offered traffic, spending long periods of time in one or the other regime and sometimes moving from one to the other in response to fluctuations in the demand. Intuitively, a situation where most calls are using alternate routes is likely to persist for a while because arriving calls will then find the network close to saturation and will be unable to make their direct connections. On the other hand, for the same offered traffic, it might also be the case that if most of the calls in progress are using their direct route, arriving calls will be able to make their direct connection. Important performance characteristics of the network such as blocking probabilities typically differ considerably between regimes. All the same, the improvement in performance over hierarchical routing schemes is such that dynamic routing schemes are being implemented in real-world networks [12, 23], with control schemes, such as trunk reservation for directly routed traffic. These, if suitably chosen, mitigate the effects of the potential multiplicity of operating regimes [1, 2, 4, 11, 12, 16, 23].

The possibility of metastable regimes of operation is also predicted in analytical models for dynamic routing such as the ones studied by Kelly [15], Krupp [16], Marbukh [18, 19] and Gibbens, Hunt and Kelly [12]. In [15] and [16] simple fixed-point approximations for the blocking probability are written, and it is found they have multiple solutions for certain ranges of the parameters. The models in [12] and [18] and [19] are more detailed. ODE limits are found for the fraction of network links that are in a given state as the network size becomes large. This work is briefly discussed in Section 2.

The focus of this paper is on understanding the interaction between the operating regimes using particle system techniques [10, 17]. To describe the dynamic exchange between different operating regimes, we need simple equations that describe how the spatially distributed network state evolves over time. Motivated by this, we consider a lattice model in Section 3, which is analogous to the model of Gibbens, Hunt and Kelly [12]. We find a mean field limit for this lattice model [8, 9, 24]. This is an integrodifferential equation describing the time evolution of the spatially distributed network state. This equation also admits multiple spatially homogeneous time-invariant solutions for certain ranges of the parameters, which may be loosely thought of as the different operating regimes. The model described in Section 3 is at best a crude caricature of the situation in a real network; nevertheless, the resulting equation may be of some use in understanding the behavior of real networks with dynamic routing. The main results are stated as Theorems 1 and 2 in Section 3.

Section 4 is devoted to the proof of Theorem 1, which depends on a dual particle process. The proof of Theorem 2 is somewhat more technical and is carried out in Section 5. Some concluding remarks are made in Section 6.

2. ODE limits. In this section we first review the ODE limit of Marbukh [18, 19], and then we review the ODE limit of Gibbens, Hunt and Kelly [12] in order to motivate the investigation in the following sections.

Marbukh [18, 19] analyzes several dynamic routing strategies in large completely connected networks. In the simplest version of the model of Marbukh, we are given a completely connected network on n nodes, with a two-way communication link between each pair of nodes, consisting of C circuits. Call requests between any pair of nodes a and b arrive according to independent Poisson processes of rate ν. If link (a, b) is not saturated, the call occupies one circuit in the link. If link (a, b) is saturated, the call randomly chooses a third node c such that each of the channels (a, c) and (c, b) has at least one circuit free and simultaneously occupies one circuit on each of these links. If there is no such c, the call is lost. Each call holds the circuits it occupies for an exponential time of mean 1, after which it simultaneously releases them.

Let N denote $\binom{n}{2}$. Let $X_{ab}^n(t)$, $1 \le a, b \le n$, denote the number of circuits occupied on link (a, b) at time t in the network with n nodes. [Note that $(X_{ab}^n, 1 \le a, b \le n)$ are not enough to specify the state of the system, because two-link calls simultaneously release the circuits they occupy.] In Marbukh [18] the following (as yet unproved) assumptions are made:

1. For any $a_1, b_1, \ldots, a_l, b_l$,

$$\lim_{n \to \infty} P\big(X_{a_1 b_1}^n(t) = k_1, \ldots, X_{a_l b_l}^n(t) = k_l\big) = \prod_{j=1}^{l} \lim_{n \to \infty} P\big(X_{a_j b_j}^n(t) = k_j\big),$$

   the convergence to the limits being uniform over t in any finite interval $[0, T]$. (In particular, the initial conditions are assumed to be of product type.)
2. $\lim_{n \to \infty} P(X_{ab}^n(t) = k) = \gamma_k(t)$ is independent of a, b.

These assumptions are related to the hypothesis of propagation of chaos in statistical mechanics [25, 26] and are quite likely true. Under these assumptions one finds that $(\gamma_0, \gamma_1, \ldots, \gamma_C)$ satisfy a differential equation on the C-dimensional simplex. This differential equation takes the form

$$\begin{aligned}
\dot\gamma_0 &= \gamma_1 - \big(\nu + 2\nu\gamma_C(1-\gamma_C)^{-1}\big)\gamma_0, \\
\dot\gamma_k &= (k+1)\gamma_{k+1} + \big(\nu + 2\nu\gamma_C(1-\gamma_C)^{-1}\big)\gamma_{k-1} \\
&\quad - \big(k + \nu + 2\nu\gamma_C(1-\gamma_C)^{-1}\big)\gamma_k, \qquad 0<k<C, \\
\dot\gamma_C &= -C\gamma_C + \big(\nu + 2\nu\gamma_C(1-\gamma_C)^{-1}\big)\gamma_{C-1},
\end{aligned} \tag{2.1}$$

where the left-hand sides are time derivatives.

When one looks for equilibrium points of (2.1), one finds the following: If $\nu > C/2$, then $(\gamma_0, \ldots, \gamma_C) = (0, 0, \ldots, 1)$ is a stable equilibrium. Further, for

$C \geq 3$, there is a value $\nu^* > C/2$ such that for all $\nu < \nu^*$ there is another stable equilibrium $(\gamma_0^*, \dots, \gamma_C^*)$ given by

$$\gamma_k^* = \frac{\left(\nu + 2\nu\gamma_C^*(1-\gamma_C^*)^{-1}\right)^k C!}{\left(\nu + 2\nu\gamma_C^*(1-\gamma_C^*)^{-1}\right)^C k!}\gamma_C^*, \tag{2.2}$$

$$\gamma_C^* = E\left(\nu + 2\nu\gamma_C^*(1-\gamma_C^*)^{-1}, C\right),$$

where $E(\nu, C) = \nu^C/C!/\sum_{l=0}^C \nu^l/l!$. In particular, for $C/2 < \nu < \nu^*$, (2.1) have more than one stable equilibrium.

Gibbens, Hunt and Kelly [12] consider a simplified model for dynamic alternate routing which bypasses the spatial features of the network. Consider a collection of N links, each of which consists of C circuits. At each link, calls arrive according to a Poisson process of rate ν. If its link is not saturated, the call occupies one circuit on the link. If its link is saturated, the call chooses two distinct links at random from the remaining $N-1$ links, and if neither is saturated, the call occupies one circuit from each of these two links. Otherwise the call is blocked and rejected from the system. Each occupied circuit is held for an independent exponential time of mean 1. (Note that when a call occupies two circuits after making a successful choice of alternate route, it is assumed that these circuits are released independently.)

Let $\gamma_k^N(t)$, $0 \leq k \leq C$, denote the fraction of the N links that have k occupied circuits at time t. Then $(\gamma_0^N, \gamma_1^N, \dots, \gamma_C^N)$ evolves as a Markov process on the C-dimensional simplex. In [12], an ODE limit is found for the evolution as $N \to \infty$. Namely, if the initial conditions $(\gamma_0^N(0), \gamma_1^N(0), \dots, \gamma_C^N(0))$ converge weakly to a limit $(\gamma_0(0), \gamma_1(0), \dots, \gamma_C(0))$, then the process converges to the deterministic process given by

$$\begin{aligned} \dot\gamma_0 &= \gamma_1 - (\nu + 2\nu\gamma_C(1-\gamma_C))\gamma_0, \\ \dot\gamma_k &= (k+1)\gamma_{k+1} + (\nu + 2\nu\gamma_C(1-\gamma_C))\gamma_{k-1} \\ &\quad -(k + \nu + 2\nu\gamma_C(1-\gamma_C))\gamma_k, \qquad 0<k<C, \\ \dot\gamma_C &= -C\gamma_C + (\nu + 2\nu\gamma_C(1-\gamma_C))\gamma_{C-1}, \end{aligned} \tag{2.3}$$

with the appropriate initial conditions.

When one looks for equilibrium points of (2.3), one finds the following: The equilibrium points are given by the solutions of

$$\gamma_k^* = \frac{(\nu + 2\nu\gamma_C^*(1-\gamma_C^*))^k C!}{(\nu + 2\nu\gamma_C^*(1-\gamma_C^*))^C k!}\gamma_C^*, \tag{2.4}$$

$$\gamma_C^* = E(\nu + 2\nu\gamma_C^*(1-\gamma_C^*), C).$$

When C is large enough, one finds that there is a range of ν in which (2.4) admit three solutions, two of which are stable. To the left and right of this range, there is a unique stable solution. For a graph of the solution set of (2.4) see Figure 1(i) of [12].

3. Lattice caricature. In this section we analyze a lattice caricature for dynamic alternate routing which has the virtue of preserving spatial features of the system. We find a mean field limit for this lattice model. This is an integrodifferential equation which describes how the spatially distributed network state evolves over time; see (3.1). The main results are Theorems 1 and 2.

Let $\mathbb{Z}^d/M$ denote the lattice in $\mathbb{R}^d$ consisting of points all of whose coordinates are rational with denominator dividing M. The points of $\mathbb{Z}^d/M$ are called *sites*. Let W denote $\{0, 1, \dots, C\}$. We consider a Markov process $(\eta_t^M, t \geq 0)$ on $W^{\mathbb{Z}^d/M}$ which caricatures a circuit switched network with dynamic routing (the statements below are true for any d, but the situations $d = 1$ and $d = 2$ are likely to be of most interest). We use η to denote a generic element of $W^{\mathbb{Z}^d/M}$ and call $\eta(x)$ the *value* at site x. Let M^* denote $\binom{(2M+1)^d - 1}{2}$. The Markov process is described by the transitions

$$\eta(x) \to \eta(x) - 1 \qquad \text{at rate } \eta(x),$$

$$\eta(x) \to \eta(x) + 1 \qquad \text{at rate } \nu \text{ if } \eta(x) \neq C,$$

$$(\eta(x), \eta(y), \eta(z)) \to (\eta(x), \eta(y) + 1, \eta(z) + 1) \quad \text{at rate } \nu/M^*$$

$$\text{if } x, y, z \text{ are distinct sites with}$$

$$\eta(x) = C,\ \eta(y) < C,\ \eta(z) < C \text{ and } y, z \in x + [-1, 1]^d.$$

There is no difficulty constructing such a Markov process even though the number of sites is infinite. See Liggett [17], Chapter 1, Section 3, for details; Theorem 3.9 of that section applies directly.

We think of each site in the lattice as representing a link in our network, which consists of C circuits. We think of the value at a site as giving the number of occupied circuits in the corresponding link. Occupied circuits become free at rate 1. At each link there is a Poisson process of calls with rate ν. Each call occupies one circuit on its link if available; if the link is saturated the call randomly picks two other links which are in its $[-1, 1]^d$ neighborhood, and uses one circuit from each of these links if possible. Otherwise the call is blocked and rejected from the system. Note that because we have a compressed lattice, the interaction actually has range M on the scale of links.

For $x \in \mathbb{Z}^d/M$, let $u_M(t, x, k)$ denote $P(\eta_t^M(x) = k)$, $0 \leq k \leq C$. We extend the definition of $u_M(t, \cdot, k)$ to $\mathbb{R}^d$ by setting $u_M(t, x, k) = u_M(t, [x]_M, k)$ for $x \in \mathbb{R}^d$, where $[x]_M$ denotes the minimum element in $\mathbb{Z}^d/M$ which dominates x in the usual partial order on $\mathbb{R}^d$. Let $u(0, x, k)$, $0 \leq k \leq C$, be continuous functions with bounded derivative and with $\sum_{k=0}^{C} u(0, x, k) = 1$. Let $u(t, x, k)$,

$0 \le k \le C$, denote the solution of the integrodifferential equations

$$
\begin{aligned}
\frac{\partial u(t,x,0)}{\partial t} &= u(t,x,1) \\
&\quad - \nu\Big[1 + 2^{1-2d}\iint_{q,r\in[-1,1]^d} u(t,x+q,C) \\
&\qquad \times (1 - u(t,x+q+r,C))\,dq\,dr\Big]u(t,x,0),
\end{aligned}
$$

$$
(3.1)\quad
\begin{aligned}
\frac{\partial u(t,x,k)}{\partial t} &= (k+1)u(t,x,k+1) \\
&\quad + \nu\Big[1 + 2^{1-2d}\iint_{q,r\in[-1,1]^d} u(t,x+q,C) \\
&\qquad \times (1 - u(t,x+q+r,C))\,dq\,dr\Big]u(t,x,k-1) \\
&\quad - \Big(k + \nu\Big[1 + 2^{1-2d}\iint_{q,r\in[-1,1]^d} u(t,x+q,C) \\
&\qquad \times (1 - u(t,x+q+r,C))\,dq\,dr\Big]\Big)u(t,x,k) \\
&\qquad\qquad \text{for } 0 < k < C,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial u(t,x,C)}{\partial t} &= -Cu(t,x,C) \\
&\quad + \nu\Big(1 + 2^{1-2d}\iint_{q,r\in[-1,1]^d} u(t,x+q,C) \\
&\qquad \times (1 - u(t,x+q+r,C))\,dq\,dr\Big)u(t,x,C-1).
\end{aligned}
$$

Then we have the following theorem.

THEOREM 1. *Fix $T < \infty$. Suppose that we start $(\eta_t^M)_t$ with initial configuration the product measure having $P(\eta_0^M(x) = k) = u_M(0,x,k)$, $0 \le k \le C$. If $u_M(0,x,k) \to u(0,x,k)$ as $M \to \infty$ uniformly on compact sets, $0 \le k \le C$, then $u_M(t,x,k) \to u(t,x,k)$ for all $t \in [0,T]$, $x \in \mathbb{R}^d$ and $0 \le k \le C$.*

Taking the limit as $M \to \infty$ can be visualized as replacing the network by a continuous family of links, so that it makes sense to talk about the probability that the state of the link at spatial location x at time t is k. Theorem 1 is a statement about pointwise convergence of such probabilities. There is a corresponding functional limit theorem. This functional limit theorem allows us to describe the limit behavior of an arbitrary choice of spatial integrals $\phi^{(1)}, \dots, \phi^{(n)}$ at times $t_1, \dots, t_n \in [0, T]$ as long as the $\phi^{(i)}$ decrease sufficiently rapidly. This allows us, for example, to describe the evolution in time of spatial averages of the state over compact regions of the lattice (which caricatures compact regions of our network with dynamic routing). Let $C^\infty(\mathbb{R}^d)$ denote the space of infinitely differentiable functions on $\mathbb{R}^d$. For $f \in C^\infty(\mathbb{R}^d)$, let

$$\| f \|_{\alpha, \beta} = \sup_{x \in \mathbb{R}^d} |x^\alpha D^\beta f(x)|,$$

where $\alpha = (\alpha_1, \dots, \alpha_d)$, $\beta = (\beta_1, \dots, \beta_d)$, $x^\alpha = \prod_{j=1}^d x_j^{\alpha_j}$ and $D^\beta f = [(\partial^{\beta_1 + \cdots + \beta_d}) / (\partial^{\beta_1} x_1 \cdots \partial^{\beta_d} x_d)] f$. Let $\mathscr{S}(\mathbb{R}^d)$ denote the Schwarz space of rapidly decreasing functions on $\mathbb{R}^d$ and $\mathscr{S}^*(\mathbb{R}^d)$ the space of Schwarz distributions, which is its topological dual. We recall that $\mathscr{S}(\mathbb{R}^d)$ consists of precisely those $f \in C^\infty(\mathbb{R}^d)$ such that $\| f \|_{\alpha, \beta} < \infty$ for all α, β. Further, $\mathscr{S}(\mathbb{R}^d)$ is a locally convex topological linear space with topology given by the family of seminorms $\| f \|_{\alpha, \beta}$. The underlying functional analysis is clearly discussed in the book of Yosida [27]; see especially page 146 ff. Then we have the following theorem.

THEOREM 2. *Given $\phi_k \in \mathscr{S}(\mathbb{R}^d)$, $0 \le k \le C$, let*

$$X_t^M(\phi) = \frac{1}{M^d} \sum_{x \in \mathbb{Z}^d / M} \sum_{k=0}^{C} \phi_k(x) 1\big(\eta_t^M(x) = k\big).$$

View $X_\cdot^M$ as an element of $D([0, T], (\mathscr{S}^*(\mathbb{R}^d))^{C+1})$. Then $X_\cdot^M \Rightarrow X_\cdot$, where $\Rightarrow$ denotes weak convergence in $D([0, T], (\mathscr{S}^*(\mathbb{R}^d))^{C+1})$ and

$$X_\cdot(\phi) = \sum_{k=0}^{C} \int_{x \in \mathbb{R}^d} \phi_k(x) u(\cdot, x, k)\, dx.$$

When we look for spatially homogeneous solutions of (3.1) which are time invariant, we are led to the same equations (2.4) found by Gibbens, Hunt and Kelly. Thus we see that for large enough C, there is a range of ν over which (3.1) admit three spatially homogeneous solutions. These may be loosely thought of as different phases associated to the network. The exchange between the phases can be studied by numerically integrating the equations from the appropriate initial conditions. Such work is currently in progress [5]. Similar equations can also be written for more complicated dynamic routing

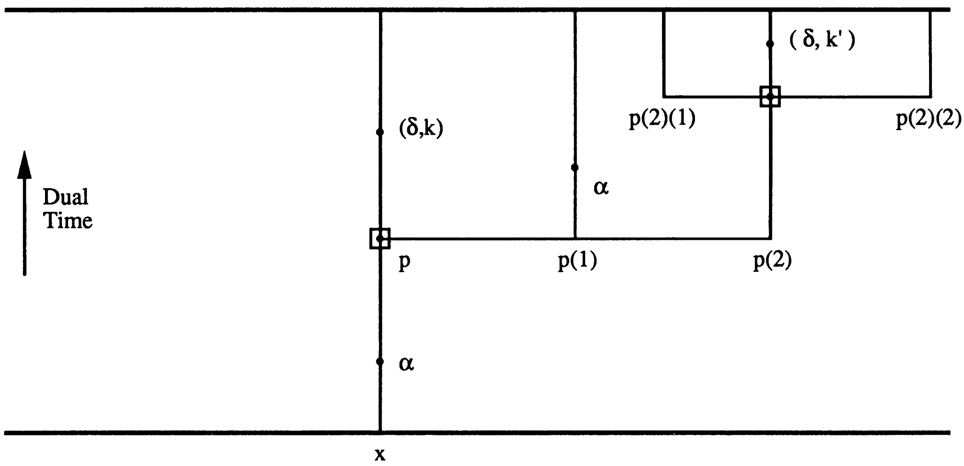

FIG. 1.

schemes, including trunk reservation, using the techniques of the next section. The study of these equations may also be useful in making comparisons between schemes.

4. Proof of Theorem 1. In this section we prove Theorem 1. The proof of Theorem 2 will be completed in Section 5.

The key idea is to consider an evolving system of particles, from which we can construct processes which are in some sense dual to the processes $(\eta_t^M, t \geq 0)$ on $[0, T]$. Given $x \in \mathbb{R}^d$, we construct a process $(Z_s^x, 0 \leq s \leq T)$. Z_s^x is a triple $(\mathscr{P}_s, \mathscr{X}_s, F_s)$. $\mathscr{P}_s$ is a set of particles which are alive at time s. A particle alive at s stays alive on $[s, T]$. $\mathscr{X}_s$: $\mathscr{P}_s \to \mathbb{R}^d$ gives the location of the live particle p. Once a particle becomes alive, its location does not change, so for simplicity we write $x(p) \in \mathbb{R}^d$ for the location of particle p.

We think of a space–time picture $\mathbb{R}^d \times [0, T]$, with time increasing as we go toward the top of the page. See Figure 1. Each live particle p has associated with it independent Poisson processes, $D(p, k)$, $1 \leq k \leq C$, of rate 1, $A(p)$ of rate ν and $Q(p)$ of rate 2ν. If a particle p becomes alive at time s at location $x(p)$, we draw the vertical line $\{x(p)\} \times [s, T]$ and place marks on the line as follows: At the times of $D(p, k)$ we write a (δ, k) on the line and at the times of $A(p)$ we write an α.

Particles become alive by the following mechanism: Suppose p is already alive. At the times of $Q(p)$ the particle p chooses $z(1), z(2) \in [-1, 1]^d$ independently and uniformly and places one new particle at $x(p) + z(1)$ and one new particle at $x(p) + z(1) + z(2)$. We refer to these as $p(1)$ and $p(2)$ respectively and also as respectively the first and second particle generated by p, at the point of $Q(p)$ in question. (Note that each live particle may actually generate several new particles over its life.) We draw a horizontal line between p and $p(1)$ and between $p(1)$ and $p(2)$ at the time p generated $p(1)$ and $p(2)$. Thus, in our space–time picture, there is a branching tree growing from a root at $\{x\} \times 0$ and consisting of vertical and horizontal lines.

We have now specified $(Z_s^x,\ 0 \le s \le T)$, except to say what F_s is. We now think of each particle as capable of supporting any of the $C + 1$ values in W, and of the value of a particle as changing with time. We have

$$F_s \colon W \to 2^{W^{\mathscr{P}_s}}.$$

$\mathbf{k} \in W^{\mathscr{P}_s}$ is such that $\mathbf{k} \in F_s(k)$ iff when we assign value $\mathbf{k}(p)$ to each $p \in \mathscr{P}_s$ and run backwards from s to 0, the resulting value at the root is k. Running backwards consists of the following rules: A (δ, k) is interpreted as a potential death and decreases the value of the corresponding particle by 1 if its current value is less than or equal to k and not already 0. An α is interpreted as a birth and increases the value of the corresponding particle by 1 (unless it is already C). When we meet a horizontal line p–$p(1)$–$p(2)$, we think of $p(1)$ as a link checking p and $p(2)$ for alternate routing. Namely, if $p(1)$ has value C and p and $p(2)$ both have values less than C, the value of p increases by 1, whereas if either $p(1)$ has value less than C or if any one of p and $p(2)$ has value C, the value of p remains unchanged. Note that we need only be concerned about the value of p, because running backwards in time, $p(1)$ and $p(2)$ are no longer present immediately after we cross the p–$p(1)$–$p(2)$ line. At the end of running backwards, we have a single value at the root particle.

Given $x \in \mathbb{Z}^d/M$, from $(Z_s^x,\ 0 \le s \le T)$ we construct a process $(Z_s^{x,M},\ 0 \le s \le T)$ by moving particles to lattice sites. $Z_s^{x,M}$ is a triple $(\mathscr{P}_s, \mathscr{X}_s^M, F_s)$ with $\mathscr{P}_s$ and F_s exactly the same as in Z_s^x. Then $\mathscr{X}_s^M \colon \mathscr{P}_s \to \mathbb{Z}^d/M$ gives the location of the live particles. Since this does not change with time, we let $x^M(p)$ denote the location of particle p in $Z_s^{x,M}$. The determination of $x^M(p)$ is somewhat ungainly; the key features necessary are that particles should not be moved by too much, and they should be placed on lattice sites with the correct distribution. $x^M(p)$ is determined at the time of birth of p by looking at the particle q which gave birth to p. Let $z(1) = x(q(1)) - x(q)$ and $z(2) = x(q(2)) - x(q(1))$ at the time that q gave birth to p.

Suppose $p = q(1)$. To determine $x^M(p)$, we split the $[-1,1]^d$ cube around $x^M(q)$ into $(2M + 1)^d - 1$ regions and associate to each region a unique lattice site in $\{x^M(q) + [-1,1]^d \cap \mathbb{Z}^d/M\} - \{x^M(q)\}$ in such a way that no point is more than $\sqrt{d}/M$ away from its assigned lattice site. We let $x^M(p)$ be the assigned lattice site of $x^M(q) + z(1)$. The required splitting of $[-1,1]^d$ can be done in several ways. The specific technique adopted is not important; it is enough to observe that it can be done, as the reader can easily check.

Suppose $p = q(2)$. To determine $x^M(p)$, we first determine $x^M(q(1))$ as above. Then we split the $[-1,1]^d$ cube around $x^M(q(1))$ into $(2M + 1)^d - 2$ regions and associate to each region a unique lattice site in $\{x^M(q(1)) + [-1,1]^d \cap \mathbb{Z}^d/M\} - \{x^M(q), x^M(q(1))\}$. The splitting of $[-1,1]^d$ should be done in such a way that no point is more than $\sqrt{d}/M$ away from its assigned lattice site. We let $x^M(p)$ be the assigned lattice site of $x^M(q(1)) + z(1)$. Once again the required splitting can be done in several different ways. Now each particle lives on a lattice site. Note that more than one particle may be alive at a site.

Finally, given $x \in \mathbb{Z}^d/M$, we construct a process $(\tilde{Z}_s^{x,M}, 0 \le s \le T)$ from $(Z_s^{x,M}, 0 \le s \le T)$ by prohibiting more than one particle to live at a lattice site. $\tilde{Z}_s^{x,M}$ is a triple $(\tilde{\mathscr{P}}_s, \tilde{\mathscr{X}}_s^M, \tilde{F}_s)$ with $\tilde{\mathscr{P}}_s \subseteq \mathscr{P}_s$ and $\tilde{\mathscr{X}}_s^M\colon \tilde{\mathscr{P}}_s \to \mathbb{Z}^d/M$ the restriction of $\mathscr{X}_s^M$ to $\tilde{\mathscr{P}}_s$. Thus we continue to let $x^M(p)$ denote the location of particle $p \in \tilde{\mathscr{P}}_s$. $(\tilde{Z}_s^{x,M}, 0 \le s \le T)$ differs from $(Z_s^{x,M}, 0 \le s \le T)$ in that if a particle attempts to be born at a site that is already occupied, the birth is aborted. Note that when p gives birth, $p(1)$ and $p(2)$ are treated individually, that is, it is possible that one of them gets aborted while the other does not. $\tilde{F}_s\colon W \to 2^{W^{\tilde{\mathscr{P}}_s}}$ is once again defined so that $\mathbf{k} \in W^{\tilde{\mathscr{P}}_s}$ is such that $\mathbf{k} \in \tilde{F}_s(k)$ iff when we assign the value $\mathbf{k}(p)$ to each $p \in \tilde{\mathscr{P}}_s$ and run backwards from s to 0, the resulting value at the root is k. Running backwards proceeds according to the same rules as for $(Z_s^{x,M}, 0 \le s \le T)$ when we encounter an α or a (δ, k) or when we encounter a horizontal line of the type p–$p(1)$–$p(2)$. If we encounter a horizontal line of the type p–q–$p(2)$ or p–$p(1)$–q or p–q–r, we once again interpret the central element as a link making a virtual attempt to carry out alternate routing at the end links, but we now have to maintain the values of all the particles that continue to be alive when we cross this horizontal link, working our way into the past.

The main use of the dual process is that it yields the following duality relation:

$$P\big(\eta_s^M(x) = k\big) = P\Big(\big[\eta_0^M(x^M(p)), p \in \tilde{\mathscr{P}}_s\big] \in \tilde{F}_s(k)\Big). \tag{4.1}$$

This is seen because, by definition, the particles $\tilde{\mathscr{P}}_s$ are sitting at precisely those sites whose values at time 0 potentially influence the value of site x at time s, and because $\tilde{F}_s(k)$ consists of precisely those configurations of values at these sites which result in the value k at site x at time s. See Figure 1, and think of the process $(\eta_t^M, t \ge 0)$ as starting at the dual time plane s and running toward the bottom of the page, the aim being to determine the value at site x at the dual time plane 0.

Let A denote the event that $\tilde{\mathscr{P}}_s = \mathscr{P}_s$, $0 \le s \le T$. Note that $\tilde{F}_s = F_s$ on A. Then we have the following lemma.

LEMMA 1. $\lim_{M \to \infty} P(A) = 1$.

PROOF. Let us condition on $\{|\mathscr{P}_T| = m\}$. Then, at the time that any particle gives birth, the probability in $(Z_s^{x,M}, 0 \le s \le T)$ that it places any one of its children at a site already occupied is bounded above by $m/[(2M+1)^d - 1]$. Clearly, on the event $\{|\mathscr{P}_T| = m\}$, each particle can give birth to a total of at most m particles. Thus the probability that some particle places some one of its children on a site that is already occupied is bounded above by $m^3/(2M)^d$. This gives

$$P(A) \le \sum_{m=1}^{\infty} \frac{m^3}{(2M)^d} P(|\mathscr{P}_T| = m).$$

Now $|\mathscr{P}_T|$ is the same in distribution as the total number of particles alive at time T in a continuous-time branching process in which particles have birth rate 2ν and the number of offspring is 3 and starting with one particle alive at time 0. We observe that this has bounded third moment. Letting $M \to \infty$ yields the lemma. □

We also need the following lemma.

LEMMA 2. *If $z_1, z_2, \dots, z_m$ and $u_1, u_2, \dots, u_m$ are nonnegative numbers less than or equal to* 1, *then*

$$|z_1 z_2 \cdots z_m - u_1 u_2 \cdots u_m| \le \sum_{i=1}^{m} |z_i - u_i|.$$

PROOF (Lemma 2.3 of [24]). The proof is by induction on m using the identity

$$z_1 z_2 \cdots z_m - u_1 u_2 \cdots u_m = (z_1 - u_1) z_2 \cdots z_m + u_1 (z_2 \cdots z_m - u_2 \cdots u_m). \qquad \square$$

We are now in a position to show that $\lim_{M \to \infty} u_M(s, x, k)$ exists. In fact, we will show that for each $x \in \mathbb{R}^d$ the limit exists uniformly over $s \in [0, T]$ and $0 \le k \le C$. We write

$$\begin{aligned}
u_M(s, x, k) &= P(\eta_s^M(x) = k) \\
&= P\big(\big[\eta_0^M(x^M(p)), p \in \tilde{\mathscr{P}}_s\big] \in \tilde{F}_s(k)\big) \\
&\le (1 - P(A)) + P\big(\big[\eta_0^M(x^M(p)), p \in \tilde{\mathscr{P}}_s\big] \in \tilde{F}_s(k); A\big) \\
&= (1 - P(A)) + P\big(\big[\eta_0^M(x^M(p)), p \in \mathscr{P}_s\big] \in F_s(k); A\big) \\
&= (1 - P(A)) + E\Bigg(\Bigg(\sum_{\mathbf{k} \in F_s(k)} \prod_{p \in \mathscr{P}_s} 1\big(\eta_0^M(x^M(p)) = \mathbf{k}(p)\big)\Bigg) 1(A)\Bigg) \\
&= (1 - P(A)) + E\Bigg(\Bigg(\sum_{\mathbf{k} \in F_s(k)} \prod_{p \in \mathscr{P}_s} u_M(0, x^M(p), \mathbf{k}(p))\Bigg) 1(A)\Bigg) \\
&\le (1 - P(A)) + E\Bigg(\sum_{\mathbf{k} \in F_s(k)} \prod_{p \in \mathscr{P}_s} u_M(0, x^M(p), \mathbf{k}(p))\Bigg).
\end{aligned}$$

Here the sequence of steps is as follows: The first equality is by definition. The second equality comes from the duality relation (4.1). The third inequality is elementary. The fourth equality is because $\tilde{\mathscr{P}}_s = \mathscr{P}_s$ on A. The fifth equality comes from spelling out the configurations that result in the value k at site x at time s. The sixth equality comes from independence of the initial conditions from site to site and because the event A is independent of the initial conditions. The last inequality is obvious.

By Lemma 1, it suffices to show that

$$\lim_{M\to\infty} E\Bigl(\sum_{\mathbf{k}\in F_s(k)} \prod_{p\in\mathscr{P}_s} u_M(0, x^M(p), \mathbf{k}(p))\Bigr)$$

exists uniformly over $s \in [0, T]$. This is the content of the following lemma.

LEMMA 3.

$$\lim_{M\to\infty} E\Biggl(\Biggl|\sum_{\mathbf{k}\in F_s(k)} \prod_{p\in\mathscr{P}_s} u_M(0, x^M(p), \mathbf{k}(p)) - \sum_{\mathbf{k}\in F_s(k)} \prod_{p\in\mathscr{P}_s} u(0, x(p), \mathbf{k}(p))\Biggr|\Biggr) = 0$$

uniformly over $s \in [0, T]$.

PROOF. For any $N > 0$, we have

$$P(|\mathscr{P}_T| > N) \le \frac{E|\mathscr{P}_T|^2}{N^2},$$

where $|\mathscr{P}_T|$ denotes the number of particles alive at T. Let K denote $E|\mathscr{P}_T|^2$. Using Lemma 2, we write

$$E\Biggl(\Biggl|\sum_{\mathbf{k}\in F_s(k)} \prod_{p\in\mathscr{P}_s} u_M(0, x^M(p), \mathbf{k}(p)) - \sum_{\mathbf{k}\in F_s(k)} \prod_{p\in\mathscr{P}_s} u(0, x(p), \mathbf{k}(p))\Biggr|\Biggr)$$

$$\le P(|\mathscr{P}_T| > N) + E\Bigl(\sum_{\mathbf{k}\in F_s(k)} \sum_{p\in\mathscr{P}_s} |u_M(0, x^M(p), \mathbf{k}(p)) - u(0, x(p), \mathbf{k}(p))| 1(|\mathscr{P}_T| \le N)\Bigr) \tag{4.2}$$

$$\leq \frac{K}{N^2} + (C+1)^N E\Bigg(\sup_{\mathbf{k} \in F_s(k)} \sum_{p \in \mathscr{P}_s} |u_M(0, x^M(p), \mathbf{k}(p)) - u(0, x^M(p), \mathbf{k}(p))| 1(|\mathscr{P}_T| \leq N) \Bigg)$$

$$+ (C+1)^N E\Bigg(\sup_{\mathbf{k} \in F_s(k)} \sum_{p \in \mathscr{P}_s} |u(0, x^M(p), \mathbf{k}(p)) - u(0, x(p), \mathbf{k}(p))| 1(|\mathscr{P}_T| \leq N) \Bigg).$$

Given $\varepsilon > 0$, we first select N so that $(K/N^2) < (\varepsilon/3)$, thus bounding the first term in (4.2). Next, we select M_1 so that for all $M \geq M_1$ we have

$$|u_M(0, y, k) - u(0, y, k)| < \frac{\varepsilon}{3N(C+1)^N}$$

for all $y \in x + [-N, N]^d$ and all $0 \leq k \leq C$. We can do this by the assumption of uniform convergence of initial conditions on compact sets. Now we observe that on $\{|\mathscr{P}_T| \leq N\}$ the maximum number of jumps that can occur between the initial position x and the location of any particle at time s is N. Consequently, the above inequality can be used to bound the second term in (4.2) by $(\varepsilon/3)$ for $M \geq M_1$. Further, we also learn that the difference between $x(p)$ and $x^M(p)$ for any particle p alive at s is at most $((N\sqrt{d})/M)$. We have

$$|u(0, x^M(p), \mathbf{k}(p)) - u(0, x(p), \mathbf{k}(p))| \leq \frac{N\sqrt{d}}{M} \sup_{k} \sup_{y \in x + [-N, N]^d} |\nabla u(0, y, k)|,$$

where ∇u denotes the spatial gradient of u. Hence we can find M_2 so that for all $M \geq M_2$,

$$|u(0, x^M(p), \mathbf{k}(p)) - u(0, x(p), \mathbf{k}(p))| < \frac{\varepsilon}{3N(C+1)^N}$$

for all $p \in \mathscr{P}_s$. This bounds the third term in (4.2) by $\varepsilon/3$. Letting $\varepsilon \to 0$ gives the lemma. This completes the proof of the existence of $\lim_{M \to \infty} u_M(s, x, k)$ uniformly over $s \in [0, T]$ and $0 \leq k \leq C$. $\square$

Let $\lim_{M\to\infty} u_M(t,x,k)$ be denoted $v(t,x,k)$. To complete the proof of Theorem 1, we need to show that $v(t,x,k)$ satisfies (3.1). First we need the following lemma.

LEMMA 4. *For any $x, y, z \in \mathbb{Z}^d/M$ and $0 \le k_1, k_2, k_3 \le C$, we have*

$$(4.3)\qquad \begin{aligned} &P\big(\eta_s^M(x) = k_1, \eta_s^M(y) = k_2, \eta_s^M(z) = k_3\big) \\ &\quad = P\big(\eta_s^M(x) = k_1\big)P\big(\eta_s^M(y) = k_2\big)P\big(\eta_s^M(z) = k_3\big) + o(1) \end{aligned}$$

uniformly for $s \in [0,T]$ and uniformly over x, y, z and k_1, k_2, k_3.

PROOF. Starting with a single particle at x, y and z respectively, we construct independent processes of particles $(Z_s^x, 0 \le s \le T)$, $(Z_s^y, 0 \le s \le T)$ and $(Z_s^z, 0 \le s \le T)$ as described earlier. We write $(\mathscr{P}_s^x, \mathscr{X}_s^x, F_s^x)$ for Z_s^x and similarly for Z_s^y and Z_s^z. We also think of the three processes together as a single process $(Z_s^{xyz}, 0 \le s \le T)$, with Z_s^{xyz} a triple $(\mathscr{P}_s^{xyz}, \mathscr{X}_s^{xyz}, F_s^{x,y,z})$, where $\mathscr{P}_s^{xyz} = \mathscr{P}_s^x \cup \mathscr{P}_s^y \cup \mathscr{P}_s^z$, and $F_s^{xyz}\colon W \times W \times W \to 2^{W^{\mathscr{P}_s^{xyz}}}$ is given by

$$\mathbf{k} \in F_s^{xyz}(k_1, k_2, k_3) \quad \text{iff} \begin{cases} \big[\mathbf{k}(p), p \in \mathscr{P}_s^x\big] \in F_s^x(k_1), \\ \big[\mathbf{k}(p), p \in \mathscr{P}_s^y\big] \in F_s^y(k_2) \text{ and} \\ \big[\mathbf{k}(p), p \in \mathscr{P}_s^z\big] \in F_s^z(k_3). \end{cases}$$

Since the position of a particle does not change once it is born, we let $x(p)$ denote the position of particle p. From the above processes we construct $(Z_s^{x,M}, 0 \le s \le T)$, $(Z_s^{y,M}, 0 \le s \le T)$ and $(Z_s^{z,M}, 0 \le s \le T)$ by moving particles to lattice sites. These three processes can be thought of together as a single process $(Z_s^{xyz,M}, 0 \le s \le T)$. We let $x^M(p)$ denote the position of particle p in this process. Finally, we construct the process $(\tilde{Z}_s^{xyz,M}, 0 \le s \le T)$ from $(Z_s^{xyz,M}, 0 \le s \le T)$ by prohibiting more than one particle to live at a lattice site. Namely, when a particle is born in $Z_s^{xyz,M}$ and attempts to occupy a site that is already occupied, the birth is aborted. $\tilde{Z}_s^{xyz,M}$ is a triple $(\tilde{\mathscr{P}}_s^{xyz}, \tilde{\mathscr{X}}_s^{xyz,M}, \tilde{F}_s^{xyz})$ with $\tilde{\mathscr{P}}_s^{xyz} \subseteq \mathscr{P}_s^{xyz}$ and $\tilde{F}_s^{xyz}\colon W \times W \times W \to 2^{W^{\tilde{\mathscr{P}}_s^{xyz}}}$ defined so that $\mathbf{k} \in W^{\tilde{\mathscr{P}}_s^{xyz}}$ is such that $\mathbf{k} \in \tilde{F}_s^{xyz}(k_1, k_2, k_3)$ iff when we assign the value $\mathbf{k}(p)$ to each $p \in \tilde{\mathscr{P}}_s^{xyz}$ and run backwards from s to 0, the resulting value at the roots x, y and z are k_1, k_2 and k_3 respectively. Further, positions in $(\tilde{Z}_s^{xyz,M}, 0 \le s \le T)$ are the same as in $(Z_s^{xyz,M}, 0 \le s \le T)$, so that we continue to let $x^M(p)$ denote the position of particle p.

Let B denote the event that there is no site which supports more than one particle in $(Z_s^{xyz,M}, 0 \le s \le T)$, that is, B is the event $\{\tilde{\mathscr{P}}_s^{xyz} = \mathscr{P}_s^{xyz}\}$. Suppose we can show that

$$(4.4)\qquad \lim_{M\to\infty} P(B) = 1$$

uniformly over the choice of x, y, z. Then we can write the left-hand side of

(4.3) as

$$
\begin{aligned}
&P\big(\eta_s^M(x) = k_1, \eta_s^M(y) = k_2, \eta_s^M(z) = k_3\big) \\
&\quad = P\big(\eta_s^M(x) = k_1, \eta_s^M(y) = k_2, \eta_s^M(z) = k_3; B\big) + o(1) \\
&\quad = P\Big(\big[\eta_0^M(x^M(p)), p \in \tilde{\mathscr{P}}_s^{xyz}\big] \in \tilde{F}_s^{xyz}(k_1, k_2, k_3); B\Big) + o(1) \\
&\quad = P\big(\big[\eta_0^M(x^M(p)), p \in \mathscr{P}_s^{xyz}\big] \in F_s^{xyz}(k_1, k_2, k_3); B\big) + o(1) \\
&\quad = E\Bigg(\Bigg(\sum_{\mathbf{k} \in F_s^{xyz}(k_1, k_2, k_3)} \prod_{p \in \mathscr{P}_s^{xyz}} 1\big(\eta_0^M(x^M(p)) = \mathbf{k}(p)\big)\Bigg)1(B)\Bigg) + o(1) \\
&\quad = E\Bigg(\Bigg(\sum_{\mathbf{k} \in F_s^{xyz}(k_1, k_2, k_3)} \prod_{p \in \mathscr{P}_s^{xyz}} u_M(0, x^M(p), \mathbf{k}(p))\Bigg)1(B)\Bigg) + o(1) \\
&\quad = E\Bigg(\sum_{\mathbf{k} \in F_s^{xyz}(k_1, k_2, k_3)} \prod_{p \in \mathscr{P}_s^{xyz}} u_M(0, x^M(p), \mathbf{k}(p))\Bigg) + o(1) \\
&\quad = E\Bigg(\sum_{\mathbf{k}^x \in F_s^x(k_1)} \prod_{p \in \mathscr{P}_s^x} u_M(0, x^M(p), \mathbf{k}^x(p))\Bigg) \\
&\qquad \times E\Bigg(\sum_{\mathbf{k}^y \in F_s^y(k_2)} \prod_{p \in \mathscr{P}_s^y} u_M(0, x^M(p), \mathbf{k}^y(p))\Bigg) \\
&\qquad \times E\Bigg(\sum_{\mathbf{k}^z \in F_s^z(k_3)} \prod_{p \in \mathscr{P}_s^z} u_M(0, x^M(p), \mathbf{k}^z(p))\Bigg) + o(1).
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
P\big(\eta_s^M(x) = k_1\big) &= P\Big(\big[\eta_0^M(x^M(p)), p \in \tilde{\mathscr{P}}_s^x\big] \in \tilde{F}_s^x(k_1); B\Big) + o(1) \\
&= P\big(\big[\eta_0^M(x^M(p)), p \in \mathscr{P}_s^x\big] \in F_s^x(k_1); B\big) + o(1) \\
&= E\Bigg(\Bigg(\sum_{\mathbf{k}^x \in F_s^x(k_1)} \prod_{p \in \mathscr{P}_s^x} 1\big(\eta_0^M(x^M(p)) = \mathbf{k}^x(p)\big)\Bigg)1(B)\Bigg) + o(1) \\
&= E\Bigg(\Bigg(\sum_{\mathbf{k}^x \in F_s^x(k_1)} \prod_{p \in \mathscr{P}_s^x} u_M(0, x^M(p), \mathbf{k}^x(p))\Bigg)1(B)\Bigg) + o(1) \\
&= E\Bigg(\sum_{\mathbf{k}^x \in F_s^x(k_1)} \prod_{p \in \mathscr{P}_s^x} u_M(0, x^M(p), \mathbf{k}^x(p))\Bigg) + o(1),
\end{aligned}
$$

and similarly for $P(\eta_s^M(y) = k_2)$ and $P(\eta_s^M(x) = k_3)$. From this, the lemma follows directly because the $o(1)$ terms are all uniform in $s \in [0, T]$ and uniform over the choice of x, y, z, k_1, k_2, k_3.

It remains to show (4.4). The argument for this is similar to that for Lemma 1. We condition on $\{|\mathscr{P}_T^{xyz}| = m\}$. Then, at the time that any particle gives birth, the probability in $(Z_s^{xyz, M},\ 0 \le s \le T)$ that it places any one of its children at a site already occupied is bounded above by $m/[(2M+1)^d - 1]$. On the event $\{|\mathscr{P}_T^{xyz}| = m\}$, each particle can give birth to a total of at most m particles. Thus the probability that some particle places one of its children on a site that is already occupied is bounded above by $m^3/(2M)^d$. This gives

$$P(A) \le \sum_{m=1}^{\infty} \frac{m^3}{(2M)^d} P\big(|\mathscr{P}_T^{xyz}| = m\big).$$

Finally, observe that $|\mathscr{P}_T^{xyz}|$ is the same in distribution as the total number of particles alive at time T in a continuous-time branching process in which particles give birth at rate 2ν and the number of offspring is 3, and starting with three particles alive at time 0. This has bounded third moment, so we may let $M \to \infty$ to prove (4.4). The estimate is clearly uniform over the choice of x, y, z. □

Let $\mathscr{E}$ denote $\{e_1, e_2 \in ([-1,1]^d \cap \mathbb{Z}^d/M)\colon e_1 \ne 0,\ e_2 \ne 0,\ e_1 + e_2 \ne 0\}$. A simple generator calculation gives

$$\begin{aligned}
\frac{\partial u_M(t,x,0)}{\partial t} &= u_M(t,x,1) - \nu u_M(t,x,0) \\
&\quad - \frac{\nu}{M^*} \sum_{\mathscr{E}} P\big(\eta_t^M(x) = 0, \eta_t^M(x+e_1) = C, \\
&\qquad\qquad \eta_t^M(x+e_1+e_2) < C\big), \\
\frac{\partial u_M(t,x,k)}{\partial t} &= (k+1) u_M(t,x,k+1) + \nu u_M(t,x,k-1) \\
&\quad + \frac{\nu}{M^*} \sum_{\mathscr{E}} P\big(\eta_t^M(x) = k-1\ \eta_t^M(x+e_1) = C, \\
&\qquad\qquad \eta_t^M(x+e_1+e_2) < C\big) \\
&\quad - (k+\nu) u_M(t,x,k) \\
&\quad - \frac{\nu}{M^*} \sum_{\mathscr{E}} P\big(\eta_t^M(x) = k, \eta_t^M(x+e_1) = C, \\
&\qquad\qquad \eta_t^M(x+e_1+e_2) < C\big) \\
&\qquad\qquad\qquad \text{for } 0 < k < C, \\
\frac{\partial u_M(t,x,C)}{\partial t} &= -C u_M(t,x,C) + \nu u_M(t,x,C-1) \\
&\quad + \frac{\nu}{M^*} \sum_{\mathscr{E}} P\big(\eta_t^M(x) = C-1, \eta_t^M(x+e_1) = C, \\
&\qquad\qquad \eta_t^M(x+e_1+e_2) < C\big).
\end{aligned} \tag{4.5}$$

Using Lemma 4, we observe that

$$
\begin{aligned}
&u_M(t,x,k)\iint_{q,r\in[-1,1]^d} u_M(t,x+q,C)(1-u_M(t,x+q+r,C))\,dq\,dr\\
&\quad = M^{-2d}\sum_{\mathscr{E}} P\big(\eta_t^M(x)=k\big)P\big(\eta_t^M(x+e_1)=C\big)\\
&\qquad \times P\big(\eta_t^M(x+e_1+e_2)<C\big)+o(1) \qquad\qquad (4.6)\\
&\quad = M^{-2d}\sum_{\mathscr{E}} P\big(\eta_t^M(x)=k,\,\eta_t^M(x+e_1)=C,\,\eta_t^M(x+e_1+e_2)<C\big)\\
&\qquad + o(1)
\end{aligned}
$$

uniformly over $t\in[0,T]$, $0\le k\le C$, and $x\in\mathbb{Z}^d/M$. Note that the integrand in (4.6) is uniformly bounded and that we have pointwise convergence of u_M to v. From Lemma 3, for any $x\in\mathbb{R}^d$, the right-hand sides of each equation in (4.5) converge as $M\to\infty$ to the corresponding right-hand sides of (3.1) (with v written instead of u) uniformly over $t\in[0,T]$ and $0\le k\le C$. To complete the proof, it therefore suffices to show that for every $0\le k\le C$,

$$\frac{\partial u_M(t,x,k)}{\partial t}\to\frac{\partial v(t,x,k)}{\partial t}$$

as $M\to\infty$. Writing

$$u_M(t,x,k)=u_M(0,x,k)+\int_0^t\frac{\partial u_M(s,x,k)}{\partial s}\,ds,$$

we see that it suffices to prove that $\partial u_M(s,x,k)/\partial s$ converges uniformly over $s\in[0,T]$ and $0\le k\le C$ for any fixed x. Namely, it suffices to prove that for any $\varepsilon>0$, we can find $M(\varepsilon,x)$ so large that

$$\left|\frac{\partial u_M(s,x,k)}{\partial s}-\frac{\partial u_N(s,x,k)}{\partial s}\right|<\varepsilon$$

for all $s\in[0,T]$, $0\le k\le C$ and all $M,N>M(\varepsilon,x)$. Defining

$$
\begin{aligned}
\Sigma_M(t,x,k) = (2M)^{-2d}\sum_{\mathscr{E}} P\big(\eta_t^M(x)=k,\,&\eta_t^M(x+e_1)=C,\\
&\eta_t^M(x+e_1+e_2)<C\big),
\end{aligned}
$$

straightforward algebra on (4.5) shows that it is enough for us to show that for any $\varepsilon>0$, we can find $M(\varepsilon,x)$ so large that

$$\text{(i)}\qquad |u_M(s,x,k)-u_N(s,x,k)|<\varepsilon$$

and

$$\text{(ii)}\qquad |\Sigma_M(s,x,k)-\Sigma_N(s,x,k)|<\varepsilon$$

for all $s\in[0,T]$, $0\le k\le C$ and $M,N>M(\varepsilon,x)$. Part (i) follows from the triangle inequality and Lemma 3. Part (ii) follows directly from writing

$$
\begin{aligned}
&\Sigma_M(t,x,k) - \Sigma_N(t,x,k) \\
&\quad = \Bigg[2^{-2d} u_N(t,x,k) \\
&\qquad \times \iint_{q,r\in[-1,1]^d} u_N(t,x+q,C)(1-u_N(t,x+q+r,C))\,dq\,dr \\
&\qquad\qquad - \Sigma_N(t,x,k)\Bigg] \\
&\qquad + \Bigg[\Sigma_M(t,x,k) - 2^{-2d} u_M(t,x,k) \\
&\qquad\quad \times \iint_{q,r\in[-1,1]^d} u_M(t,x+q,C)(1-u_M(t,x+q+r,C))\,dq\,dr\Bigg] \\
&\qquad + \Bigg[2^{-2d} u_M(t,x,k) \\
&\qquad\quad \times \iint_{q,r\in[-1,1]^d} u_M(t,x+q,C)(1-u_M(t,x+q+r,C)) \\
&\qquad\quad - 2^{-2d} u(t,x,k) \\
&\qquad\quad \times \iint_{q,r\in[-1,1]^d} u(t,x+q,C)(1-u(t,x+q+r,C))\,dq\,dr\Bigg] \\
&\qquad + \Bigg[2^{-2d} u(t,x,k) \iint_{q,r\in[-1,1]^d} u(t,x+q,C)(1-u(t,x+q+r,C)) \\
&\qquad\quad - 2^{-2d} u_N(t,x,k) \\
&\qquad\quad \times \iint_{q,r\in[-1,1]^d} u_N(t,x+q,C)(1-u_N(t,x+q+r,C))\,dq\,dr\Bigg].
\end{aligned}
$$

The first and second terms on the right converge to 0 uniformly over $t \in [0,T]$ and $x \in \mathbb{R}^d$, as we argued earlier using Lemma 4. The third and fourth terms converge to 0 uniformly over $t \in [0,T]$ and $0 \le k \le C$ for fixed $x \in \mathbb{R}^d$. It follows that (ii) holds, and we have completed the proof of Theorem 1.

5. Proof of Theorem 2. In this section we prove Theorem 2. The proofs are somewhat technical.

To begin, the tightness of the laws of $X^M_\cdot$ in the space $D([0,T], (\mathscr{S}^*(\mathbb{R}^d)^{C+1}))$ follows from the tightness of the laws of $X^M_\cdot(\phi)$ in $D([0,T], \mathbb{R}^d)$ for each $\phi = (\phi_0, \dots, \phi_C) \in \mathscr{S}(\mathbb{R}^d)^{C+1}$. This follows from a general result of

Mitoma [21]. The result we are appealing to is Theorem 4.1 of Mitoma [21]. This theorem is couched in the language of probability distributions P_n on $D([0,T], E^*)$, where E is a nuclear Frechet space and E^* is its dual space. To apply this result in our context, it suffices to observe that $\mathscr{S}(\mathbb{R}^d)^{C+1}$ is a nuclear Frechet space and $(\mathscr{S}^*(\mathbb{R}^d))^{C+1}$ is its topological dual. This follows from the facts that $\mathscr{S}(\mathbb{R}^d)$ is a nuclear Frechet space ([27], page 293), the topological vector product of nuclear Frechet spaces is a nuclear Frechet space ([27], page 293), and the topological dual of a product of topological vector spaces is the product of their topological duals. It will therefore suffice to fix $\phi \in \mathscr{S}(\mathbb{R}^d)^{C+1}$ and prove that the laws of $X^M_\cdot(\phi)$ are tight in $D([0,T], \mathbb{R}^d)$ and to identify the limit as being the one given in Theorem 2. From Theorem 5.3, (2) of Mitoma [21], the latter will be accomplished if we can show that for any $t_i \in [0,T]$, $1 \le i \le n$, and for any $\phi^{(1)}, \dots, \phi^{(n)} \in \mathscr{S}(\mathbb{R}^d)^{C+1}$,

$$(5.1)\qquad \lim_{M\to\infty} P\left(\left| X^M_{t_i}(\phi^{(i)}) - \int_{\mathbb{R}^d} \sum_{k=0}^{C} \phi^{(i)}_k(x) u(t_i, x, k)\, dx \right| < \varepsilon \ \forall\, 1 \le i \le n\right) = 1.$$

(5.1) follows easily once we verify

$$(5.2)\qquad \lim_{M\to\infty} E\big(X^M_t(\phi)\big) = \int_{\mathbb{R}^d} \sum_{k=0}^{C} \phi_k(x) u(t,x,k)\, dx$$

and

$$(5.3)\qquad \lim_{M\to\infty} E\big(X^M_t(\phi) - E\big(X^M_t(\phi)\big)\big)^2 = 0$$

for all $\phi \in \mathscr{S}(\mathbb{R}^d)^{C+1}$. To verify (5.2), we write

$$\begin{aligned}
&\lim_{M\to\infty} E\big(X^M_t(\phi)\big)\\
&\quad = \lim_{M\to\infty} \frac{1}{M^d} \sum_{x\in\mathbb{Z}^d/M} \sum_{k=0}^{C} \phi_k(x) u_M(t,x,k)\\
&\quad = \lim_{M\to\infty} \Bigg[\sum_{x\in\mathbb{Z}^d/M} \sum_{k=0}^{C} \int_{y\in\mathbb{R}^d:\, [y]_M = x} [\phi_k(x) - \phi_k(y)] u_M(t,y,k)\, dy\\
&\qquad\qquad + \sum_{k=0}^{C} \int_{y\in\mathbb{R}^d} \phi_k(y) u_M(t,y,k)\, dy \Bigg].
\end{aligned}$$

The first term on the right clearly goes to 0 for any $\phi \in \mathscr{S}(\mathbb{R}^d)^{C+1}$, whereas the second term on the right converges to $\sum_{k=0}^{C} \int_{\mathbb{R}^d} \phi_k(x) u(t,x,k)\, dx$ by the pointwise convergence of $u_M(t,x,k)$ to $u(t,x,k)$ and the dominated convergence theorem.

To prove (5.3), we write

$$E\big(X_t^M(\phi) - E(X_t^M(\phi))\big)^2$$
$$= E\left(\frac{1}{M^d}\sum_{x\in\mathbb{Z}^d/M}\sum_{k=0}^{C}\phi_k(x)\big[1(\eta_t^M(x)=k)-u_M(t,x,k)\big]\right)^2$$
$$= E\Bigg(\frac{1}{M^{2d}}\sum_{x\in\mathbb{Z}^d/M}\sum_{y\in\mathbb{Z}^d/M}\sum_{k=0}^{C}\sum_{l=0}^{C}\phi_k(x)\phi_l(y)$$
$$\times\big[1(\eta_t^M(x)=k)-u_M(t,x,k)\big]\big[1(\eta_t^M(y)=l)-u_M(t,y,l)\big]\Bigg),$$

and we note that

$$E\big(\big[1(\eta_t^M(x)=k)-u_M(t,x,k)\big]\big[1(\eta_t^M(y)=l)-u_M(t,y,l)\big]\big)$$
$$= P\big(\eta_t^M(x)=k,\eta_t^M(y)=l\big) - P\big(\eta_t^M(x)=k\big)P\big(\eta_t^M(y)=l\big)$$
$$= o(1)$$

uniformly in x, y, k, and l. This last can be proved in a manner directly analogous to the proof of Lemma 4. Then (5.3) follows easily.

It remains to verify the tightness of the laws of $X_\cdot^M(\phi)$ in $D([0,T],\mathbb{R}^d)$ for each $\phi = (\phi_0,\phi_1,\dots,\phi_C)\in\mathscr{S}(\mathbb{R}^d)^{C+1}$. To do this, it suffices to verify that

$$\sup_M E\left(\sup_{0\le t\le T}\big(X_t^M(\phi)\big)^2\right) < \infty \tag{5.4}$$

and that for each $\varepsilon > 0$, we can find $\delta > 0$ and M_1 so that

$$\sup_{M>M_1} P\left(\sup_{0\le s,t\le T;\,|t-s|<\delta}|X_t^M(\phi)-X_s^M(\phi)|>\varepsilon\right)<\varepsilon; \tag{5.5}$$

see Billingsley [6], Theorem 15.6. Condition (5.4) can be verified by writing

$$\big(X_t^M(\phi)\big)^2 = \frac{1}{M^{2d}}\sum_{x\in\mathbb{Z}^d/M}\sum_{y\in\mathbb{Z}^d/M}\sum_{k=0}^{C}\sum_{l=0}^{C}\phi_k(x)\phi_l(y)$$
$$\times 1\big(\eta_t^M(x)=k,\eta_t^M(y)=l\big)$$
$$\le\left(\frac{1}{M^d}\sum_{x\in\mathbb{Z}^d/M}\sum_{k=0}^{C}|\phi_k(x)|\right)^2$$
$$=\left(\frac{1}{M^d}\sum_{|x|\le 1}\sum_{k=0}^{C}|\phi_k(x)| + \frac{1}{M^d}\sum_{|x|>1}\sum_{k=0}^{C}|\phi_k(x)|\right)^2,$$

where $|x|$ denotes $\sup_{1\le i\le d}|x_i|$. Now

$$\frac{1}{M^d}\sum_{|x|\le 1}\sum_{k=0}^{C}|\phi_k(x)| \le \sum_{k=0}^{C}\|\phi_k\|_{0,0}<\infty,$$

where $\|\phi_k\|_{0,0} = \sup_{x\in\mathbb{R}^d}|\phi_k(x)|$ and

$$\begin{aligned}\frac{1}{M^d}\sum_{|x|>1}\sum_{k=0}^{C}|\phi_k(x)| &= \frac{1}{M^d}\sum_{|x|>1}\sum_{k=0}^{C}\frac{|x|^{d+1}|\phi_k(x)|}{|x|^{d+1}}\\ &\le \frac{1}{M^d}\sum_{|x|>1}\sum_{k=0}^{C}\frac{\|\phi_k\|^*_{d+1,0}}{|x|^{d+1}}\\ &\le K\sum_{k=0}^{C}\|\phi_k\|^*_{d+1,0},\end{aligned}$$

where

$$\|\phi_k\|^*_{d+1,0} = \sup_{1\le i\le d}\ \sup_{x\in\mathbb{R}^d}|x_i^{d+1}\phi_k(x)|$$

and $K < \infty$ is a constant.

To verify (5.5), we first choose N so large that

$$\frac{1}{M^d}\sum_{|x|>N}\sum_{k=0}^{C}\frac{\|\phi_k\|^*_{d+1,0}}{|x|^{d+1}} < \frac{\varepsilon}{4}.$$

This ensures that even if all the sites outside $[-N, N]^d$ were to change state, the change in $X_t^M(\phi)$ would be bounded above by $\varepsilon/4$. Next, we select $r > 0$ so that $r\sum_{k=0}^{C}\|\phi_k\|_{0,0} < \varepsilon/4$. Note that

$$\begin{aligned}&|X_t^M(\phi) - X_s^M(\phi)|\\ &\quad= \frac{1}{M^d}\left|\sum_{x\in\mathbb{Z}^d/M}\sum_{k=0}^{C}\phi_k(x)\big(1(\eta_t^M(x) = k) - 1(\eta_s^M(x) = k)\big)\right|\\ &\quad\le \frac{1}{M^d}\sum_{x\in\mathbb{Z}^d/M}\sum_{k=0}^{C}|\phi_k(x)|\big(1(\eta_t^M(x) \ne \eta_s^M(x))\big)\\ &\quad\le \frac{1}{M^d}\sum_{k=0}^{C}\|\phi_k\|_{0,0}\sum_{x\in\mathbb{Z}^d/M}\big(1(\eta_t^M(x) \ne \eta_s^M(x))\big).\end{aligned}$$

Hence, in order for the event in (5.5) to occur, there must be some interval $I_n = [n(\delta/2), (n+1)\delta/2]$ such that at least rM^d of the sites in $[-N, N]^d$ change value during this time interval. Since at any point of any of the Poisson process associated to the sites in $[-N-1, N+1]^d$ at most two sites in $[-N, N]^d$ can change value, there is an upper bound to the overall rate of such changes. Let $\mu = (3\nu + C)M^d(2N+3)^d\delta/2$. Then we have

$$P(J_n > rM^d) \le P(N_\mu > rM^d),$$

where J_n denotes the number of sites that change value during I_n, and N_μ denotes a Poisson random variable of mean μ. From this, one easily gets

$$P(J_n > rM^d) \le \exp\left((3\nu + C)M^d(2N+3)^d\left(\frac{\delta}{2}\right) - rM^d\right).$$

Selecting δ so small that

$$(3\nu + C)(2N + 3)^d \left(\frac{\delta}{2}\right) < \frac{r}{2}$$

gives

$$P\left(J_n > rM^d \text{ for some } n = 1, \dots, \left[\frac{T}{\delta}\right] + 1\right) \le \left(\left[\frac{T}{\delta}\right] + 1\right) \exp\left(-\frac{rM^d}{2}\right),$$

which goes to 0 as $M \to \infty$. Thus we can choose M_1 so that for $M > M_1$, the event in (5.5) has probability less than ε. This completes the proof of Theorem 2.

6. Concluding remarks. We have studied a lattice caricature of circuit switched networks with dynamic alternate routing using particle system techniques. The main result of the investigation is to prove the existence of an integrodifferential equation which describes in a law of large numbers sense the behavior of large-scale networks with long-range routing. The model is highly simplified and structured; nevertheless it captures an essential feature of such schemes observed in simulations, namely the existence of multiple operating regimes for certain ranges of parameters. In a spatially distributed network, there will typically be regions where one or another of the regimes is in effect. We are currently investigating the equations both analytically and numerically to see if they provide insights into the nature of the exchange between the different regimes of operation.

The technique of proof is quite general and can be directly carried over to write mean field limits for other models, including models with trunk reservation, which are of considerable practical interest. A problem of particular interest seems to be to introduce distributed control and revenue ideas into the model. This should lead to a set of controlled integrodifferential equations, using which problems of distributed steering of the network to maximize revenue can be formulated and studied using classical deterministic techniques. We are at present investigating distributed trunk reservation control with such applications in view.

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