The Annals of Applied Probability
1992, Vol. 2, No. 4, 1009–1018

THE HEIGHT OF A RANDOM PARTIAL ORDER: CONCENTRATION OF MEASURE

By Béla Bollobás[1] and Graham Brightwell

University of Cambridge and London School of Economics

The problem of determining the length L_n of the longest increasing subsequence in a random permutation of $\{1, \dots, n\}$ is equivalent to that of finding the height of a random two-dimensional partial order (obtained by intersecting two random linear orders). The expectation of L_n is known to be about $2\sqrt{n}$. Frieze investigated the concentration of L_n about this mean, showing that, for $\varepsilon > 0$, there is some constant $\beta > 0$ such that

$$\Pr\big(|L_n - \mathbf{E}L_n| \geq n^{1/3+\varepsilon}\big) \leq \exp(-n^{\beta}).$$

In this paper we obtain similar concentration results for the heights of random k-dimensional orders, for all $k \geq 2$. In the case $k = 2$, our method replaces the $n^{1/3+\varepsilon}$ above with $n^{1/4+\varepsilon}$, which we believe to be essentially best possible.

The study of random d-dimensional orders was begun by Winkler [15, 16]. His model was to construct an order $<$ on the set $[n] \equiv \{1, \dots, n\}$ as follows. From the $n!$ possible linear orders on $[n]$, d orders $\prec_1, \dots, \prec_d$ are chosen independently and uniformly at random, with replacement. Then the order $<$ is set equal to the intersection of $\prec_1, \dots, \prec_d$, so that $x < y$ iff $x \prec_i y$ for each i.

One advantage of this model of random orders is that there is a natural equivalent formulation. Consider the d-dimensional unit cube $[0,1]^d$ with the standard product measure, and take n points at random in this cube. The usual coordinatewise order on the cube induces a partial order on the set of n points, and it is easy to see that the probability of any particular partial order arising here is the same as in the model described above. The two natural alternative formulations make the model relatively easy to work with.

One of the problems considered by Winkler in [15] was that of determining the height $L_{n,d}$ of a random d-dimensional order (the length of the longest chain). Some results were obtained in [15], and then Bollobás and Winkler [5] proved the following result.

Theorem 1. *For every $d \geq 2$, there is a constant c_d such that $L_{n,d} n^{-1/d}$ tends to c_d in probability.*

Received November 1991; revised January 1992.
[1]Research supported in part by NSF Grant DMS-88-06097.
AMS 1980 *subject classifications*. 06A10, 60C05, 05A99.
Key words and phrases. Partial order, height, random orders, Ulam's problem, increasing subsequences.

Bollobás and Brightwell [4] extended Theorem 1 to a somewhat more general class of models for random partial orders. (Everything we prove in this paper can be extended with a little effort into that setting.)

For $d = 2$, the problem of finding the height of a random two-dimensional order is equivalent to the well-known combinatorial problem of finding the length of a longest increasing subsequence of a random permutation of the set $[n]$. This is often referred to as Ulam's problem, since it was apparently first raised in [13]. The case $d = 2$ of Theorem 1 was in fact first proved in this formulation by Hammersley [7]. The constant c_2 is known to be equal to 2: Logan and Shepp [9] proved $c_2 \geq 2$ and Veršik and Kerov [14] showed $c_2 \leq 2$. A combinatorial proof of this last result was supplied by Pilpel [11].

The constants c_d for $d \geq 3$ are unknown, although Bollobás and Winkler give some reasonable bounds in [5].

Theorem 1 gives no indication of either the speed of convergence of $\mathbf{E}L_{n,d}n^{-1/d}$ to c_d, or of the concentration of $L_{n,d}$ about its expectation. The main aim of this paper is to give some results in these directions.

Concerning the latter problem, Frieze [6] proved that $L_{n,d}$ is sharply concentrated about its mean, for $d = 2$. We state his result in a slightly stronger form than that given in [6], since this assertion follows immediately from the proof.

THEOREM 2. *Suppose that $\alpha > 1/3$ and $\beta < \min\{\alpha, 3\alpha - 1\}$. Then, for n sufficiently large,*

$$\Pr(|L_{n,2} - \mathbf{E}L_{n,2}| \geq n^{\alpha}) < \exp(-n^{\beta}).$$

As we shall show later, it is possible to combine a result of this kind with the methods of Bollobás and Winkler to get information about the rate of convergence of $\mathbf{E}L_{n,2}$ to $2\sqrt{n}$.

The basic tool used by Frieze in [6] is a martingale inequality (similar to Lemma 4 below). He dealt with the Ulam's problem formulation, looking directly for a long increasing subsequence of a random permutation. Our approach in this paper will again be to use a martingale inequality, but we consider the formulation of the problem as that of finding a long chain in a random d-dimensional partial order. This enables us not only to extend Theorem 2 to the case $d > 2$, but also to improve the exponents in the case $d = 2$.

Thus we show that we have sharp concentration about the mean not only for $L_{n,2}$, but also for each $L_{n,d}$ with $d \geq 2$. Also, we prove that, in the case $d = 2$, the conclusion of Theorem 2 holds with n^{α} replaced by $n^{1/4}\log n$: We suspect that the exponent $1/4$ here is best possible. One consequence of our results is that the variance of $L_{n,d}$ is at most $n^{1/d}\log^2 n$; another is that $|c_d n^{1/d} - \mathbf{E}L_{n,d}|$ is bounded above by $n^{1/2d}\log^{3/2} n$.

It turns out to be convenient to follow Bollobás and Winkler, and to consider a slightly different model of d-dimensional random orders. We set $X = [0,1]^d$, and consider a Poisson process with density n in X. This process will thus give us a set S of about n points in X, which comes equipped with the coordinatewise order $<$. We shall prove our strong concentration results for this model, and then indicate how to recover the results for the model where exactly n points are chosen at random in X.

For S a set of points in $[0,1]^d$, we let $H(S)$ be the height of the partial order induced on S. If S is generated by a Poisson process of density n in the cube $[0,1]^d$, we denote by $H \equiv H_{n,d}$ the random variable $H(S)$. We shall prove the following result.

THEOREM 3. *For each integer $d \geq 2$, there is a constant K_d such that, for n sufficiently large,*

$$\Pr\left(|H_{n,d} - \mathbf{E}H_{n,d}| > \frac{\lambda K_d n^{1/2d} \log n}{\log\log n} \right) \leq 4\lambda^2 \exp(-\lambda^2)$$

for every λ with $2 < \lambda < n^{1/2d}/\log\log n$.

Note that the case $d = 2$ extends Theorem 2 for every λ in the stated range. Our methods can also be used to obtain somewhat weaker results for values of λ larger than those given above.

We shall use the "method of bounded differences," involving the application of martingale inequalities. The basic principle of this method is to perform some process of subdivision so that the random variable $H_{n,d}$ does not depend too crucially on how the random process behaves on each individual part. In this case, what we shall do is divide the cube $X = [0,1]^d$ into a moderate number of "slices," so that a long chain in $(S, <)$ is very unlikely to contain many points from any one slice.

We set $m = \lceil dn^{1/d} \rceil$, and partition the cube into m slices: For $j = 1, \dots, m$, we set

$$X_j = \left\{ (x_1, \dots, x_d) \in X : j - 1 \leq \frac{m}{d} \sum_1^d x_i < j \right\}.$$

Let $S_j = S \cap X_j$ for each j. Thus the X_j form a partition of X (except for the top point), and the random sets S_j are mutually independent.

Our intention is to apply the following lemma. This can essentially be found in the articles by Bollobás [2, 3] or McDiarmid [10], and follows simply from Azuma's inequality [1] for martingale convergence, or from a slight variant due to Hoeffding [8].

LEMMA 4. *Suppose that $Z = Z(U)$ is a random variable, where $U = (U_1, \dots, U_m)$, and the U_i are chosen independently from probability spaces Ω_i. Suppose also that, whenever U and V differ in only one coordinate (i.e., $U_i = V_i$ for all but one index i), we have $|Z(U) - Z(V)| \leq k$. Then, for any real a, we have*

$$\Pr(|Z - \mathbf{E}Z| > a) \leq 2\exp(-a^2/2mk^2).$$

Ideally, we would like to apply Lemma 4 with $U = (S_1, \dots, S_m)$ and $Z = H$. This will not quite work since H *could* increase substantially if, for instance, one of the S_j contained a huge number of points. However, we do know that this is a very unlikely occurrence.

Our approach is to define a variant H' of H which does satisfy the hypotheses of Lemma 4, and then to estimate separately the (very small) probability that H' is much different from H. To this end, we set $k = 2(d+1)\log n/\log\log n$, $U = (S_1, \dots, S_m)$ as above, and let $H' \equiv H'_{n,d}(U)$ be a length of the longest chain in $(S, <)$ including at most k points from each S_j.

LEMMA 5. *For each integer $d \geq 2$, there is a constant K'_d such that*

$$\Pr\left(|H'_{n,d} - \mathbf{E}H'_{n,d}| > \frac{\lambda K'_d n^{1/2d} \log n}{\log\log n}\right) \leq 2\exp(-\lambda^2)$$

for every positive real n and λ.

PROOF. We apply Lemma 4 with $m = \lceil dn^{1/d} \rceil$, $U = (S_1, \dots, S_m)$, $Z(U) = H'(U)$ and $k = 2(d+1)\log n/\log\log n$. Evidently the hypotheses of Lemma 4 are satisfied. For every λ, we set

$$a = \lambda k\sqrt{2m} \leq 3(d+1)\sqrt{2d}\,\frac{\lambda n^{1/2d}\log n}{\log\log n}.$$

Then Lemma 4 gives the required result, with $K'_d = 3(d+1)\sqrt{2d}$. □

Lemma 5 contains the heart of the proof of Theorem 3. It remains to estimate the error term $H - H'$. One can easily check that this is in fact almost always 0, but we need to prove more, namely that the probability that $H - H'$ is large is extremely small. The proof of this is fairly simple, although a little technical.

It is convenient to consider a different subdivision of the cube, into subcubes rather than slices. Set l to be the greatest integer so that $l^d \leq n$. Now, for $J = (j_1, \dots, j_d) \in [l]^d$, we define

$$Y_J = \{(x_1, \dots, x_d) : j_i - 1 \leq lx_i < j_i \text{ for each } i\}$$

and $S_J = S \cap Y_J$. Again, the Y_J form a partition of X, except for the top point, and the S_J are mutually independent. Note that each cube Y_J has volume l^{-d}, which is at most $2/n$ if n is sufficiently large.

For $J \in [l]^d$, set $T_J = \max\{0, |S_J| - k/(d+1)\}$, and $T = \sum_{J \in [l]^d} T_J$. So T counts the total "surplus" of points in the various subcubes.

LEMMA 6. *For any set S of points in $[0,1]^d$, we have $H - H' \leq T$.*

PROOF. Given any set S of points in $[0,1]^d$, we delete T points from S to form a set S' such that each subcube Y_J contains at most $k/(d+1)$ points from S'. The set S' inherits the order $<$ from S. We fix some longest chain C in $(S', <)$, and suppose that C contains more than k points from some slice X_j. Let $x = (x_1, \dots, x_d)$ and $y = (y_1, \dots, y_d)$ be the lowest and highest points of C in X_j. Then $\sum_1^d (y_i - x_i) < d/m \leq n^{-1/d} \leq 1/l$, since x and y are in the same slice. Therefore there is at most one integer between lx_i and l_{yi}, for each i, and so C passes through at most $d + 1$ of our subcubes Y_J between x and y. Thus C contains at most k points in X_j, a contradiction. Therefore the height of $(S', <)$ is at most H' and so, a little crudely, the height H of $(S, <)$ is at most $H' + T$, as required. □

Of course, Lemma 6 implies that $\mathbf{E}(H - H') \leq \mathbf{E}T$. The remaining ingredients of our proof of Theorem 3 are the bounds for T given in the next lemma.

LEMMA 7.

(i) *For every* $d \geq 2$ *and sufficiently large* n,

$$\Pr\left(T > \frac{\lambda n^{1/2d} \log n}{\log\log n}\right) \leq 2\lambda^2 e^{-\lambda^2}$$

for every λ *with* $2 \leq \lambda < n^{1/2d} \log\log n$.

(ii) *For* n *sufficiently large*, $\mathbf{E}T \leq 1$.

PROOF. We assume throughout that n is large enough for various inequalities to hold: In particular, we assume that the volume of each cube Y_J is at most $2/n$.

We set $M_r = |\{J \in [l]^d\colon |S_J| > r\}|$, for each integer r, and note that

$$T = \sum_{r=k/(d+1)}^{\infty} M_r.$$

Each subcube has volume at most $2/n$, so the probability that a subcube contains more than r points from a Poisson process of density n is at most

$$\sum_{i=r+1}^{\infty} \frac{e^{-2}2^i}{i!} < \frac{e^{-2}2^r}{r!} < \exp(-r \log r + 2r),$$

provided $r \geq 3$. Therefore the probability that there are m subcubes each containing more than r points of S is at most

$$\binom{n}{m}(\exp(-r\log r + 2r))^m < [n \exp(-r(\log r - 2))]^m.$$

If $r \geq k/(d+1) = 2\log n/\log\log n$, and n is suitably large, then $\log r \geq 3 + \log n/r$, and so the expression inside the square brackets is at most e^{-r}, and the probability that $M_r \geq m$ is at most e^{-rm}.

To prove (i), we fix any λ with $2 \leq \lambda < n^{1/2d} \log\log n$. Setting $m = 1$ and $r = \lambda^2$ gives that the probability of any cube containing more than λ^2 points is at most $e^{-\lambda}$. Also, the probability that $M_r \geq \lambda^2/r$ for any r with $k/(d+1) \leq r \leq \lambda^2$ is at most $\lambda^2 e^{-\lambda^2}$.

Thus, with probability at least $1 - 2\lambda^2 e^{-\lambda^2}$, we have

$$T = \sum_{r=k/(d+1)}^{\infty} M_r < \sum_{r=1}^{\lambda^2} \lambda^2/r < 4\lambda^2 \log\lambda < 4\lambda \frac{n^{1/2d}\log n}{2d\log\log n} < \frac{\lambda n^{1/2d}\log n}{\log\log n},$$

as required.

To prove (ii), we note that

$$\begin{aligned}\mathbf{E}T &= \sum_{r=k/(d+1)}^{\infty} \mathbf{E}M_r \\ &\leq \sum_{r=k/(d+1)}^{\infty} l^d\left(\frac{e^{-2}2^i}{i!}\right) \\ &\leq n\exp(-r\log r + 2r) \leq e^{-r} < 1,\end{aligned}$$

using the estimates above. □

Theorem 3 now follows easily from the preceding lemmas.

Proof of Theorem 3. Set $K_d = K'_d + 2$. Now we have

$$\begin{aligned}&\Pr\left(|H - \mathbf{E}H| > \frac{\lambda K_d n^{1/2d}\log n}{\log\log n}\right) \\ &\quad\leq \Pr\left(|H - \mathbf{E}H'| > \frac{\lambda(K_d - 1)n^{1/2d}\log n}{\log\log n}\right) \\ &\quad\leq \Pr\left(|H' - \mathbf{E}H'| > \frac{\lambda K'_d n^{1/2d}\log n}{\log\log n}\right) + \Pr\left(T > \frac{\lambda n^{1/2d}\log n}{\log\log n}\right) \\ &\quad\leq 2e^{-\lambda^2} + 2\lambda^2 e^{-\lambda^2} \leq 4\lambda^2 e^{-\lambda^2}.\end{aligned}$$
□

Let us next adapt Theorem 2 to the model where exactly n points are chosen from $[0,1]^d$. Recall that the height of a random partial order in this model is denoted $L_{n,d}$.

Theorem 8. *For every integer $d \geq 2$, there is a constant C_d such that, for n sufficiently large,*

$$\Pr\left(|L_{n,d} - \mathbf{E}L_{n,d}| > \frac{\lambda C_d n^{1/2d}\log n}{\log\log n}\right) \leq 80\lambda^2\exp(-\lambda^2)$$

for every λ with $2 < \lambda < n^{1/2d}/\log\log n$.

Proof. We shall prove that, for sufficiently large n,

$$\Pr\left(|L_{n,d} - \mathbf{E}H_{n,d}| > \frac{\lambda(K_d + 4)n^{1/2d}\log n}{\log\log n}\right) \leq 80\lambda^2\exp(-\lambda^2) \tag{1}$$

for every λ in the range, where K_d is as in Theorem 3. One can check that (1) implies that the expectation of $L_{n,d}$ is bounded within $\mathbf{E}H_{n,d} \pm C'_d n^{1/2d} \log n/\log\log n$, which implies the full result with $C_d = K_d + 4 + C'_d$. The details of this routine calculation are omitted, and we confine ourselves to the proof of (1).

Suppose then that (1) fails for some λ. Consider two independent Poisson processes on $[0,1]^d$, one, S, with density $n - \sqrt{n}$, and the second, S', with density $2\sqrt{n}$. With probability at least $1/10$, S has at most n points, and $S \cup S'$ has at least n. If this is the case, we select a set S_0 by taking all points of S and choosing at random $n - |S|$ points of S'. This procedure, if successful, generates n points uniformly, independently from $[0,1]^d$. Therefore, since (1) fails, we have

$$\Pr\left(|S| \le n \le |S \cup S'| \text{ and } |H(S_0) - \mathbf{E}H_{n,d}| > \frac{\lambda(K_d + 4)n^{1/2d}\log n}{\log\log n}\right)$$
$$> 8\lambda^2\exp(-\lambda^2),$$

and thus, since $H(S) \le H(S_0) \le H(S \cup S')$,

$$\begin{aligned}\Pr\Bigg(H(S) \le \mathbf{E}H_{n,d} - \frac{\lambda(K_d + 4)n^{1/2d}\log n}{\log\log n} \\ \text{or } H(S \cup S') \ge \mathbf{E}H_{n,d} + \frac{\lambda(K_d + 4)n^{1/2d}\log n}{\log\log n}\Bigg) \\ > 8\lambda^2\exp(-\lambda^2).\end{aligned} \tag{2}$$

Now observe that

$$\mathbf{E}(H_{n,d} - H(S)) \le \mathbf{E}(H_{\sqrt{n},d}) \le c_d n^{1/2d} \le 3n^{1/2d},$$

where the last two inequalities are known from [5].

But, from Theorem 3, if n is sufficiently large then, with probability at least $1 - 4\lambda^2 e^{-\lambda^2}$,

$$\begin{aligned}H(S) &\ge \mathbf{E}H_{n-\sqrt{n},d} - \frac{\lambda K_d(n-\sqrt{n})^{1/2d}\log(n-\sqrt{n})}{\log\log(n-\sqrt{n})} \\ &\ge \mathbf{E}H_{n,d} - \frac{\lambda(K_d + 4)n^{1/2d}\log n}{\log\log n}\end{aligned}$$

and similarly, with probability at least $1 - 4\lambda^2 e^{-\lambda^2}$,

$$H(S \cup S') \le \mathbf{E}H_{n,d} + \frac{\lambda(K_d + 4)n^{1/2d}\log n}{\log\log n}.$$

These two inequalities contradict (2), so (1) indeed holds. □

For any natural numbers n and d, define $c_{n,d}$ by

$$\mathbf{E}H_{n,d} = c_{n,d} n^{1/d}.$$

Theorem 1 tells us that $c_{n,d}$ tends to some constant c_d, but gives us no information as to the speed of convergence. Using Theorem 3, we shall produce what could turn out to be a reasonably good estimate for this rate of convergence. We work with $H_{n,d}$ for convenience: A similar statement can be proved for $L_{n,d}$.

THEOREM 9. *For every integer $d \geq 2$, and all sufficiently large n,*

$$c_d \geq c_{n,d} \geq c_d - \frac{12K_d \log^{3/2} n}{n^{1/2d} \log\log n},$$

where K_d is as in Theorem 3.

PROOF. We shall proceed by comparing $c_{n,d}$ with $c_{2^d n,d}$.

The upper bound was shown in [5], but we repeat the argument here for ease of reference. Consider the cube $Z = [0,2]^d$, and a Poisson process S of density n in Z. The expectation of $H(S)$ is just $c_{2^d n,d} \cdot 2n^{1/d}$. But $H(S)$ is at least the length of the longest chain passing through the point $(1,1,\dots,1)$ of Z, which is the sum of $H(S \cap [0,1]^d)$ and $H(S \cap [1,2]^d)$. These heights are independent random variables distributed as $H_{n,d}$, and therefore the expected length of the longest chain passing through the midpoint of Z is exactly $2c_{n,d}n^{1/d}$. This proves that $c_{2^d n,d} \geq c_{n,d}$ for every n and d, and therefore that $c_d \geq c_{n,d}$.

For the lower bound, we shall prove that, for sufficiently large n,

$$\Pr\left(H_{2^d n,d} - 2\mathbf{E}H_{n,d} > \frac{2\lambda K_d n^{1/2d} \log n}{\log\log n} \right) \leq 2^{d+3} n \lambda^2 e^{-\lambda^2} \tag{3}$$

for every λ with $2 \leq \lambda n^{1/2d}/\log\log n$. (In fact, we shall only use this for the particular value $\lambda = 2\sqrt{\log n}$.)

Again, we generate a set S by taking a Poisson process of density n on $Z = [0,2]^n$. The probability that S has more than $2 \cdot 2^d n$ points is negligible, so we randomly choose a labeling $(x_1,\dots,x_m)$ of S with $m \leq 2^{d+1}n$. We consider the probability that the point labeled x_i is the midpoint of a chain of length at least $2\mathbf{E}H_{n,d} + 2\lambda K_d n^{1/2d} \log n/\log\log n$ in S. Wherever x_i lies in Z, either the volume between $(0,\dots,0)$ and x_i or the volume between x_i and $(2,\dots,2)$ is at most 1. Therefore the probability that there are chains both above and below x of length $\mathbf{E}H_{n,d} + \lambda K_d n^{1/2d} \log n/\log\log n$ is at most $4\lambda^2 e^{-\lambda^2}$, by Theorem 3. So the probability that some x_i is the midpoint of such a long chain is at most $2^{d+1}n$ times this, establishing (3).

Inequality (3) enables us to estimate the expected value of $H_{2^d n,d}$ as follows. Set $\lambda = 2\sqrt{\log n}$, and $r = 2\lambda K_d n^{1/2d} \log n/\log\log n$, so that $\Pr(H_{2^d n,d} \geq 2\mathbf{E}H_{n,d} + r_0) < 2^{-d}n^{-1}$ for sufficiently large n. Now, just using the crude bond $H_{2^d n,d} \le |S|$ for the large deviations, we have

$$\mathbf{E}H_{2^d n,d} \le 2\mathbf{E}H_{n,d} + r_0 + 2 \cdot 2^d n \Pr(H_{2^d n,d} \ge 2\mathbf{E}H_{n,d} + r_0) + \sum_{m > 2^{d+1} n} m \Pr(|S| = m)$$

$$\le 2\mathbf{E}H_{n,d} + \frac{4K_d n^{1/2d} \log^{3/2} n}{\log\log n} + 2 + 1.$$

This shows that

$$c_{2^d n,d} \le c_{n,d} + \frac{3K_d \log^{3/2} n}{n^{1/2d} \log\log n},$$

and so

$$c_{n,d} \ge c_d - \sum_{j=0}^{\infty} \frac{3K_d \log^{3/2}(2^{jd} n)}{(2^{jd} n)^{1/2d} \log\log(2^{jd} n)}$$

$$\ge c_d - \frac{3K_d \log^{3/2} n}{n^{1/2d} \log\log n} \sum_{j=0}^{\infty} 2^{-j/2} \frac{\log^{3/2}(2^{jd} n)}{\log^{3/2} n}$$

$$\ge c_d - \frac{12K_d \log^{3/2} n}{n^{1/2d} \log\log n},$$

as required. □

In this paper we have been concerned with proving that $H_{n,d}$ is reasonably close to $c_d n^{1/d}$. Theorems 3 and 9 combine to show that $H_{n,d}$ is very unlikely to differ by more than $n^{1/2d} \log n$ from this amount, and in particular that the variance of $H_{n,d}$ is at most $O(n^{1/d} \log^2 n)$. (Steele [12] has shown that the variance of $L_{n,2}$ is at most $n^{1/2}$.) It seems to us that the exponent of $1/2d$ is likely to be correct, and we venture the following conjecture.

CONJECTURE 10. *There is a constant c such that, for every fixed d and sufficiently large n, the variance of* $L_{n,d}$ *is at least* $n^{1/d}/\log^c n$.

Probably Conjecture 10 is true with $c = 0$. Note that we make this conjecture even for $L_{n,d}$, where one might initially expect the variance to be lower than for $H_{n,d}$. However, the two variances are not actually likely to be too different, since the height of the random partial order will almost never be affected by points a long way off the diagonal, and in either model the number of points in a region around the diagonal is asymptotically a Poisson random variable with mean n times the volume of the region.

On the other hand, we think it quite possible that Theorem 9 is not tight, and that $c_{n,d}$ in fact converges to c_d faster than is given by that result.

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Department of Pure Mathematics and
Mathematical Statistics
University of Cambridge
Cambridge CB2 1SB
England

Department of Statistical and
Mathematical Sciences
London School of Economics
London WC2A 2AE
England