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TRAVEL AND SOJOURN TIMES IN STOCHASTIC NETWORKS[1]

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This study concerns travel times in a stochastic network in which units move among the nodes that process the units. The network may be closed or open, there may be several types of units and the processing at each node may depend on the numbers of units at the other nodes. This is a Jackson network when the nodes operate independently. The travel time on a "route" in this network is the time it takes for an arbitrary unit to traverse one of a series of nodes that constitute the route, when the network is in equilibrium. An example is the time for a unit to move from one set of nodes to another. We present an expression for the expectation of a general travel time. We also characterize the distribution of the travel time, and the sojourn times at the nodes, on an overtake-free path. This includes the known results on the product-form distribution of sojourn times at the nodes on overtake-free paths in Jackson networks.

1. Introduction. For a stochastic network, such as a Jackson network, some typical travel issues are as follows:

1. How long does it take for a unit to travel from one sector (i.e., set of nodes) to another when the network is in equilibrium?
2. How much time does a unit spend in a sector during its stay in an open network?
3. How much time does a unit spend as a certain type of unit during its stay in a sector?

Such travel and sojourn times are the subject of this study.

We consider travel times for a Markovian network process, like the processes in [14], [12], [33] and [27], in which the processing rate of units at a node may depend on the locations of units throughout the network, and its equilibrium distribution is not of product form. A Jackson network is a special case. We define a general route $\mathscr{R}$ as a collection of feasible paths, called simple routes. The travel time on a route $\mathscr{R}$ is the time it takes for a unit to traverse one of the simple routes in $\mathscr{R}$ chosen under the dynamics of the process, in equilibrium. There is a one-to-one correspondence between the family of travel times and the family of essentially all stopping times for a certain Markov

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chain representing the movement of a single unit among the nodes in discrete time.

Even a simply-stated travel time, such as the time to travel from one sector to another, is rather intricate. Some of the complicating factors are the following:

1. A travel time is not a stopping time of the network process.
2. Units may overtake one another as they traverse the route.
3. The processing at the nodes on a route may involve processor sharing.
4. A unit may begin a traverse of a route and not complete it (e.g., a unit may start to travel from sector J to sector K but may return to J or exit the network before reaching K).
5. The status of whether a unit is traversing a route completely is typically not known until the route is completed.
6. The distribution of a travel time is not with respect to the underlying probability law of the network process, but it is with respect to the Palm probability of the process conditioned that a unit begins a complete traverse of the route.

Our main results are an expression for the expectation of a travel time for a general route, and a characterization of the distribution of sojourn and travel times on an overtake-free route. To prove these, we represent the network by a Markov process that is more encompassing than the usual process depicting the numbers of units at the nodes. This larger process monitors the route-traversing status of each unit by looking into the future—it records whether or not the unit will eventually complete the route. Functionals of this process that determine travel times are (a) the point process of the number of units that begin the route in a time period and eventually complete it, and (b) the process representing the numbers of units at the nodes at any time that are undergoing a complete traverse of the route.

We use laws of large numbers for these functionals and properties of Palm probabilities to obtain the expected travel time on a route. This expectation is directly proportional to the expected number of units traversing the route completely; the expectation of the travel time is under the Palm probability of the process and the expectation of the number of traversing units is under the usual probability law of the process. We also establish a strong law of large numbers for travel times. A corollary of our result is a Little law for the mean sojourn time in a node or sector of a network. This also follows by standard Little laws for queues as surveyed in [30] or [32], although it apparently has not appeared in the literature.

Our characterization of the distributions of sojourn and travel times on an overtake-free route is an extension of several earlier results. Suppose $W_{r_1}, \dots, W_{r_l}$ are the sojourn times of a unit that traverses the nodes $r = (r_1, \dots, r_l)$ in that order, and r is an overtake-free simple route (Definition 5.1). Walrand and Varaiya [31] showed that, for an open Jackson network with unlimited capacity, the $W_{r_1}, \dots, W_{r_l}$ are independent exponential random variables; Reich [22] proved this initially for queues in tandem. For a closed

Jackson network, Kelly and Pollett [15] proved an analogous result that the joint distribution of $W_{r_1}, \ldots, W_{r_l}$ is a certain product form. Related results are in [2], [5]–[8], [11], [15]–[25], [30], [31] and [4] is a survey.

We found that it is natural to treat these two results for open and closed networks as one. For our more general network process, we prove that $W_{r_1}, \ldots, W_{r_l}$ has a similar product-form distribution regardless of whether the network is closed, or open with limited capacity, or open with unlimited capacity. In the latter case, the distribution of $W_{r_1}, \ldots, W_{r_l}$ is such that they are independent exponential random variables. This result extends our knowledge of sojourn times on overtake-free paths in networks with dependent nodes including open Jackson networks with limited capacity. We also show that the distribution of the travel time on an overtake-free route $\mathscr{R}$ is the mixture of distributions of random variables $W_{r_1} + \cdots + W_{r_l}$ over all simple routes r in $\mathscr{R}$. The distributions of these sojourn times are with respect to certain Palm probabilities; this fact was only implicit in earlier analyses. Our use of Palm probabilities provides short proofs and exposes some unnoticed features of sojourn times.

The rest of this study is organized as follows. Section 2 describes the network and travel times on general routes in the network. Our results on expected travel times are presented in Section 3, and proved in Section 4. Distributions of sojourn and travel times on overtake-free routes are in Section 5. We end in Section 6 by discussing how our results also apply to networks with several types of units.

2. Preliminaries: Description of the network process and travel times. We shall consider a basic stochastic network process, or queueing network process, that is a generalization of the classical Jackson [13], Gordon–Newell [10], BCMP [3], Kelly [14] and Whittle [33] processes. Specifically, we consider a network in which discrete units (or customers) move among m nodes, labeled $1, \ldots, m$, that process the units. We take m to be finite; our results, with slight modifications, extend to infinite-node networks. The state of the network is represented by the stochastic process

$$X(t) \equiv \big(X_1(t), \ldots, X_m(t)\big), \qquad t \geq 0,$$

that records the numbers of units at the respective nodes at time t. We assume the units are indistinguishable. We discuss later how the results apply to multiple types of units. A typical state of the process $\{X(t)\colon t \geq 0\}$ is a vector $n \equiv (n_1, \ldots, n_m)$ with nonnegative integer-valued entries.

In order to discuss both open and closed networks together, we assume that the process X may represent either type of network and distinguish this by the form of its state space as follows:

Closed network. A fixed number of units $\bar{\nu}$ move among the m nodes and the state space of X is

$$E \equiv \{n\colon |n| = \bar{\nu}\}, \quad \text{where } |n| \equiv n_1 + \cdots + n_m.$$

Open network. Units enter the network from the outside, hereafter called node 0, and move among the nodes for a while and eventually exit to node 0. The state space of X is

$$E \equiv \begin{cases} \{n: |n| \le \bar{\nu}\}, & \text{if the network has finite capacity } \bar{\nu}, \\ \{n: |n| < \infty\}, & \text{if the network has unlimited capacity } (\bar{\nu} = \infty). \end{cases}$$

The units move one at a time between nodes as follows. Suppose the process X is in state n. A transition of X is triggered by the movement of one unit from some node j to another node k in the node set

$$M \equiv \begin{cases} \{1, \dots, m\}, & \text{if the network is closed,} \\ \{0, 1, \dots, m\}, & \text{if the network is open.} \end{cases}$$

Then the new state of X is denoted by $T_{jk}n$, which is the vector n with one less unit at node j and one more unit at node k. For example, $T_{03}n$ is the vector n with n_3 replaced by $n_3 + 1$. The movement of the units and their processing at the nodes are such that X is a Markov jump process. We assume that its transition rates

$$q(n, n') \equiv \lim_{t \downarrow 0} t^{-1} P\{X(t) = n' | X(0) = n\}$$

are of the form

$$(1) \qquad q(n, n') = \begin{cases} \lambda_{jk}\phi_j(n), & \text{if } n' = T_{jk}n \in E \text{ for some } j, k \in M, \\ 0, & \text{otherwise.} \end{cases}$$

Here $\lambda_{jk} \ge 0$ and ϕ_j is a positive function on E. The value $\phi_j(n)\Sigma_k \lambda_{jk}$ is the rate at which units depart from node j when X is in state n. The $\phi_j(n)$ can be viewed as a service intensity at j that may depend on the entire vector n. This generality, which is not present in the classical models, allows for a variety of congestion-dependent processing schemes [27]. λ_{jk} is the intensity at which units departing from j move to k. We define the *routing probability*

$$(2) \qquad p_{jk} = \lambda_{jk} / \sum_l \lambda_{jl}, \qquad j, k \in M,$$

which is the probability that a unit departing from node j moves to k. We assume, with no loss in generality, that the Markov matrix $\{p_{jk}\}$ is irreducible. Then there are positive numbers w_j that satisfy the *routing equations*

$$(3) \qquad w_j \sum_{k \in M} \lambda_{jk} = \sum_{k \in M} w_k \lambda_{kj}, \qquad j \in M,$$

and $w_0 = 1$ when the network is open. Then $w_j / \Sigma_k w_k$ is the equilibrium distribution for $\{p_{jk}\}$. Under these assumptions, the process X is irreducible. We will also assume that X is positive recurrent. This is automatically true when the state space E is finite. And it is true for the open network with infinite E, under condition (4), if and only if the sum of the terms on the right side of (5) below over all n in E is finite.

The following result from [27] characterizes the equilibrium distribution of the process X.

THEOREM 2.1. *Suppose there is an $l \in M$ such that, for each $n \in E$ and $j, k \in M$,*

$$\phi_j(n)\phi_k(T_{jl}n)\phi_l(T_{kl}n) = \phi_k(n)\phi_j(T_{kl}n)\phi_l(T_{jl}n). \tag{4}$$

Then the equilibrium distribution of X is

$$\pi(n) = c\Phi(n)\prod_{j=1}^{m} w_j^{n_j}, \qquad n \in E, \tag{5}$$

where c is a normalizing constant and Φ is a positive function on E defined recursively as follows. For $\nu = 0, 1, \ldots, \bar{\nu}$, define

$$E_\nu = \begin{cases} \{n \in E: |n| = \nu\}, & \textit{if the network is open}, \\ \{n \in E: n_l = \bar{\nu} - \nu\}, & \textit{if the network is closed}, \end{cases}$$

where l is any fixed element of M. Then set $\Phi(n) = 1$, $n \in E_0$, and, for $\nu = 1, 2, \ldots, \bar{\nu}$, define

$$\Phi(n) = \phi_l(T_{jl}n)\phi_j(n)^{-1}\Phi(T_{jl}n), \qquad n \in E_\nu. \tag{6}$$

This definition is equivalent to

$$\Phi(n) = \prod_{k=1}^{\nu} \phi_l(s_{k-1})\phi_{j_k}(s_k)^{-1}, \qquad n \in E_\nu,$$

where s_k is any sequence such that $s_k \in E_k$, and $s_k = T_{lj_k}s_{k-1}$, $k = 1, \ldots, \bar{\nu}$.

One can prove Theorem 2.1 by showing that the π given by (5) and (6) satisfies the *partial balance equations*

$$\pi(n)\sum_{k \in M} q(n, T_{jk}n) = \sum_{k \in M} \pi(T_{jk}n)q(T_{jk}n, n), \qquad j \in M, n \in E.$$

This says that when X is in state n the rate of movement of units out of j equals the rate of movement into j. Summing these equations over j yields the total balance equations that π must satisfy. The equilibrium distribution π is not of product form. In fact any positive probability on E can be the equilibrium distribution of such a network. The normalizing constant and other performance parameters can be computed by Monte Carlo estimators. These and other properties are discussed in [27] and [29].

The following examples illustrate the difference between classified networks with independently operating nodes, and a network with dependently operating nodes.

EXAMPLE 2.2 (Jackson networks). Suppose the nodes operate independently such that the departure intensity $\phi_j(n)$ for $j \neq 0$ is a function $\phi_j(n_j)$ of only n_j. If the network is closed, then (4) is automatically satisfied and π

reduces to the product form

$$\pi(n) = c \prod_{j=1}^{m} w_j^{n_j} \prod_{\nu=1}^{n_j} \phi_j(\nu)^{-1}.$$

If the network is open, then one can show that (4) is equivalent to $\phi_0(n) = \alpha(|n|)$ for some positive function α. In this case, π is

$$\pi(n) = c \prod_{k=1}^{|n|} \alpha(k-1) \prod_{j=1}^{m} w_j^{n_j} \prod_{\nu=1}^{n_j} \phi_j(\nu)^{-1}.$$

EXAMPLE 2.3 (Tree-like network with load balancing). Suppose the network is open and the routing intensities λ_{jk} are such that the network is tree-like: There is one root node and each unit moves from the root to a leaf node. Suppose $\phi_j(n) = \gamma_j(d_j(n))$, where $d_j(n)$ is the total number of units at all the nodes up the tree from j (those that can eventually be reached from j) and γ_j is a positive function. Then $\gamma_j(d_j(n))$ is the intensity at which the $d_j(n)$ units higher up in the tree "pull" units from j to balance the congestion [$\gamma_j(\nu)$ decreases as ν increases]. Also, assume as above that $\phi_0(n) = \alpha(|n|)$. Then (4) is clearly satisfied and

$$\pi(n) = c \prod_{k=1}^{|n|} \alpha(k-1) \prod_{j=1}^{m} w_j^{n_j} \prod_{\nu=1}^{d_j(n)} \gamma_j(\nu)^{-1}.$$

We shall study travel times for the basic network process described in Theorem 2.1. Our definition of a travel time is as follows. A *simple route* of the network is a vector $r = (r_1, \dots, r_l)$ of nodes in M such that $\lambda_{r_1 r_2} \cdots \lambda_{r_{l-1} r_l} > 0$ and only r_l may be node 0. A unit upon entering r_1 *traverses the route* r if it proceeds to nodes $r_2, \dots, r_l$ in that order in the next $l-1$ moves.

DEFINITION 2.4. A *route* $\mathscr{R} \equiv (\mathscr{R}, \mathscr{S}, \{S_r\colon r \in \mathscr{R}\})$ consists of a collection $\mathscr{R}$ of simple routes, a start set $\mathscr{S}$ and a collection of time-recording sets S_r with the following properties:

(a) The start set is any subset

$$\mathscr{S} \subset \{(j,k)\colon j \in M, \text{and } k = r_1 \text{ for some } r \in \mathscr{R}\}.$$

A unit triggers the start of the route $\mathscr{R}$ when and only when it moves from node j to node k for some $(j,k) \in \mathscr{S}$.

(b) Each $r = (r_1, \dots, r_l) \in \mathscr{R}$ is such that $(r_s, r_{s+1}) \notin \mathscr{S}$, for $1 \le s \le l-2$. This ensures that a unit's move on r from r_s to r_{s+1} cannot trigger a new start on the route $\mathscr{R}$. It may be possible that $(r_{l-1}, r_l) \in \mathscr{S}$, in which case the last move on r from r_{l-1} to r_l triggers a new start.

(c) No route in $\mathscr{R}$ is the initial segment of another: There are no two routes r, r' in $\mathscr{R}$ with $(r'_1, \dots, r'_l) = r$ and $l < l'$. This ensures that a unit ends the route $\mathscr{R}$ when and only when it ends one of the simple routes in $\mathscr{R}$.

(d) For each route $r \in \mathscr{R}$, the $S_r \subset \{1, \dots, l\}$ is a subset of stages at which travel time is recorded.

A unit *traverses the route* $\mathscr{R}$ if it moves from j to k for some $(j, k) \in \mathscr{S}$ and, therefore, it traverses one of the simple routes in $\mathscr{R}$ completely beginning with node k. The unit's *travel time on the route* $\mathscr{R}$ is the time spent in nodes $\{r_s: s \in S_r\}$ on the simple route r in $\mathscr{R}$ it traverses.

Note that the travel time on a route pertains to a "complete traverse" of the route. If a unit begins a route and exits before completing it, then we do not regard this as a travel time (e.g., a unit beginning a traverse from sector J to sector K may exit the network or return to J before hitting K). In other words, we consider travel times that are manifest. This convention ensures that the travel time is finite; it may take any value in $(0, \infty)$ since the sojourn time of X in any state is exponentially distributed.

EXAMPLE 2.6 (Travel times). A basic example is the time it takes for a unit to travel from J to K. Its defining sets are

$$\mathscr{S} = J \times I, \qquad \mathscr{R} = \bigcup_{l=1}^{\infty} I^l \times K, \qquad S_r = \{1, \dots, l-1\},$$

where $I = M \setminus (J \cup K)$. The sets $\{S_r\}$ are useful for representing various scenarios for recording travel times on a route. For instance, if one were interested in the time a unit spends in the network sector I in the first five movements while traveling between J and K, then one would use $S_r = \{s: r_s \in I,\ s \leq 5\}$. A few more travel times are:

1. The total time a unit spends in J as it passes through an open network.
2. The time between a unit's one-step move from J to K and its next one-step move from J' to K'.
3. The time it takes for a unit to visit node j five times.

We show later that the family of all routes (and hence travel times) is equivalent to the family of all stopping times of the Markov transition probabilities p_{jk} for a single unit moving in M.

3. Expected travel times. Throughout the rest of this study, we let $\{X(t): t \geq 0\}$ denote the network process described in Theorem 2.1 and let $\mathscr{R}$ denote a route for the network. We shall consider a generic travel time $T \equiv T(\mathscr{R})$ for a unit traversing $\mathscr{R}$ when the system is in equilibrium. This requires an explanation.

To define T precisely, we must specify an assumption on how the nodes process the units. Typically, the total processing intensity $\phi_j(n)$ at node j is allocated to the n_j units there according to some *processor-sharing rule*. For instance, a *first-come-first-serve, single-server rule* allocates all the $\phi_j(n)$ to the first of the n_j units to enter node j, and an *egalitarian processor-sharing rule* allocates an equal amount of the $\phi_j(n)$, namely $\phi_j(n)/n_j$, to each of the n_j units. For our network, we shall assume that the processing rule at node j allocates the proportion $\alpha_{j\nu}(n_j)$ to the νth of the n_j units to enter j, where

$\sum_{\nu=1}^{n_j}\alpha_{j\nu}(n_j) = 1$. This assumption is consistent with X being a Markov process; the numbers of units at the nodes do not depend on the processing rule, and so we did not specify a rule when defining X.

We shall now represent the network by a Markov process that "looks" into the future of the process X to determine the route-traversing status of each unit. To this end, we will label the units as follows. If a unit is traversing $r \in \mathscr{R}$ completely and it is at node r_s, for some $s = 1, \dots, l$, then the unit is said to be of type sr (the unit is at stage s on the router). Otherwise, the unit is of type 0: It is not traversing any r, or it is making only a partial traverse of some r. We represent the evolution of the network by the process

$$Y(t) \equiv \big(Y_{j\nu}(t): \nu = 1, \dots, X_j(t),\ j = 1, \dots, m\big), \qquad t \geq 0,$$

where $Y_{j\nu}(t)$ denotes the type of the νth of the $X_j(t)$ units to enter node j. Under the assumptions on X and the processor-sharing rule, the process Y is Markovian. Its transition rates are described below in the proof of Theorem 3.1, where it is also established that Y is irreducible and positive recurrent. Note that Y is not simply a network process with multiple types of units as in section 6; the equilibrium distribution of Y is more intricate (in fact we do not know it).

With the broader depiction of the network by Y, we can now lay out the rest of our notation. The number of units that begin a traverse of $\mathscr{R}$ up to time t and complete it is

$$N_{\mathscr{R}}(t) \equiv \sum_{u \leq t} 1((Y(u-), Y(u)) \in B), \qquad t \geq 0. \tag{7}$$

Here B denotes the set of pairs (y, y') representing the movement of a unit from j to k for some $(j, k) \in \mathscr{S}$ and the unit entering k is of type $1r$, for some $r \in \mathscr{R}$, signaling that it will complete the route r. Let $\{Y^*(t)\colon t \geq 0\}$ denote a stationary version of Y (we use an asterisk to represent "stationary version"). Similarly, let $N_{\mathscr{R}}^*$ denote a stationary version of $N_{\mathscr{R}}$; this is defined as in (7) with Y^* in place of Y.

We shall consider the Palm probability distribution P^0 of the process Y^* "conditioned" on the event that a unit begins the route $\mathscr{R}$ at time 0 and completes it. The P^0 is defined (see for instance [1] or [9]) by

$$P^0(A) \equiv E\left[\int_{(0,1]} 1(Y^*(\cdot + t) \in A) N_{\mathscr{R}}^*(dt)\right] \Big/ E[N_{\mathscr{R}}^*(0,1]],$$

where A is a measurable set of sample paths of the Y process in which at time 0 a unit begins a traverse of $\mathscr{R}$ and completes it.

The travel time on $\mathscr{R}$ of a unit entering it at time 0 is what we are denoting by T. The distribution of T is therefore with respect to P^0. We will also consider the sequence of times $T_1, T_2, \dots$, where $T_k \equiv T_k(\mathscr{R})$ is the travel time of the kth unit to begin and complete a traverse of $\mathscr{R}$. These times may be overlapping when units may overtake one another and do not finish $\mathscr{R}$ in the order they begin it. By the standard ergodic limit theorem,

$$\lim_{\nu \to \infty} \nu^{-1} \sum_{k=1}^{\nu} T_k = E^0(T) \qquad \text{a.s.}$$

Note that the sequence T_k is generally not stationary under the original underlying probability distribution of the process Y (or even under the distribution of Y^*). However, T_k is stationary under the Palm probability P^0 of Y^*, and the distribution of each T_k under P^0 is the same as that for T. This follows because P^0 is invariant under a shift in time by an amount τ_k, where τ_k is the time at which the kth unit begins a complete traverse of $\mathscr{R}$.

Our analysis will involve two more stochastic processes. The number of units that move from node j to node k in $[0, t]$ is

$$N_{jk}(t) \equiv \sum_{u \le t} 1\big(X(u) = T_{jk}X(u-)\big), \qquad t \ge 0. \tag{8}$$

The number of units that are traversing $\mathscr{R}$ at time t and complete it is

$$X_{\mathscr{R}}(t) \equiv \sum_{j=1}^{m} \sum_{\nu=1}^{X_j(t)} 1(Y_{j\nu} = sr \text{ for some } r \in \mathscr{R},\, s \in S_r), \qquad t \ge 0. \tag{9}$$

This counts units that are in a stage $s \in S_r$ at which travel time is recorded. Let $X^*, N^*_{jk}, X^*_{\mathscr{R}}$, denote stationary versions of the processes $X, N_{jk}, X_{\mathscr{R}}$, respectively. The processes $N^*_{jk}, X^*_{\mathscr{R}}$ are expressible as in (8) and (9), respectively, with X^*, Y^* used in place of X, Y. With a slight abuse of notation, we let $EX^*_{\mathscr{R}} \equiv EX^*_{\mathscr{R}}(t)$, $EN^*_{\mathscr{R}} \equiv EN^*_{\mathscr{R}}(1)$ and we define EX^*_j, EN^*_{jk} similarly.

Our results will also involve the following probabilities for the movement of a single unit. Let $\xi_1, \xi_2, \dots$ be a Markov chain with transition probabilities $p_{jk} \equiv \lambda_{jk}/\sum_l \lambda_{jl}$ representing the discrete-time movement in M of a single unit. The probability that the unit starting at r_1 completes the simple route $r = (r_1, \dots, r_l)$ is

$$p(r) \equiv p_{r_1 r_2} \cdots p_{r_{l-1} r_l}$$

and the probability that the unit starting at k finishes the route $\mathscr{R}$ is

$$f_k(\mathscr{R}) \equiv \sum_{r \in \mathscr{R}} p(r) 1(r_1 = k).$$

The probability that the unit in equilibrium is traversing $\mathscr{R}$ completely and is at a stage where travel time is recorded, conditioned that the unit is at node j, is

$$\begin{aligned} p_j(\mathscr{R}) &\equiv \sum_{r \in \mathscr{R}} \sum_{s \in S_r} P\{(\xi_0, \xi_1) \in \mathscr{S}, (\xi_1, \dots, \xi_l) = r \mid \xi_s = j\} \\ &= \sum_{(i,k) \in \mathscr{S}} \sum_{r \in \mathscr{R}} \sum_{s \in S_r} P\{\xi_0 = i\} p_{ik} p(r) 1(r_1 = k, r_s = j)/P\{\xi_s = j\}, \end{aligned}$$

where ξ_ν is stationary, so that $P\{\xi_\nu = j\} = w_j/\sum_{k\in M} w_k$. Then

$$p_j(\mathscr{R}) = w_j^{-1} \sum_{(i,k)\in\mathscr{S}} w_i p_{ik} \sum_{r\in\mathscr{R}} p(r) \sum_{s\in\mathscr{S}_r} 1(r_1 = k, r_s = j).$$

The following result gives an expression for the expectation of a travel time and some related laws of large numbers.

THEOREM 3.1. *For a route $\mathscr{R}$ of the network process X,*

$$E^0[T(\mathscr{R})] = EX_{\mathscr{R}}^*/EN_{\mathscr{R}}^*, \tag{10}$$

where

$$EX_{\mathscr{R}}^* = \sum_{j=1}^{m} p_j(\mathscr{R}) EX_j^*, \tag{11}$$

$$EN_{\mathscr{R}}^* = \sum_{(j,k)\in\mathscr{S}} f_k(\mathscr{R}) EN_{jk}^*. \tag{12}$$

Furthermore, with probability 1,

$$\lim_{\nu\to\infty} \nu^{-1} \sum_{k=1}^{\nu} T_k(\mathscr{R}) = E^0[T(\mathscr{R})], \tag{13}$$

$$\lim_{t\to\infty} t^{-1} \int_0^t X_{\mathscr{R}}(u)\, du = EX_{\mathscr{R}}^*, \tag{14}$$

$$\lim_{t\to\infty} t^{-1} N_{\mathscr{R}}(t) = EN_{\mathscr{R}}^*. \tag{15}$$

Our proof of Theorem 3.1 is in the next section. Although the expression (10) is rather simple, the computation of its terms may be difficult for complicated routes. Tractable examples are given shortly. Note that the probabilities $p_j(\mathscr{R})$, $f_k(\mathscr{R})$ are functions of $\mathscr{R}$, w_j, p_{jk} that do not depend on the distribution π for X^*, while EN_{jk}^* (which we describe shortly) and EX_j^* depend on π but not on $\mathscr{R}$. The simplest travel time is the following sojourn time.

EXAMPLE 3.2 (A Little law for sojourn times). Suppose W_J is the sojourn time of a unit in the set of nodes J. Then (10) yields the Little law

$$E^0(W_J) = L_J/\lambda_J,$$

where

$$L_J = \sum_{j\in J} EX_j^*, \qquad \lambda_J = a^* \sum_{i\notin J} w_i \sum_{j\in J} \lambda_{ij}.$$

These quantities are also limiting averages as in (13)–(15). This result can also be obtained directly, without the Y process, by Little laws as in [30] or [33]; it apparently has not appeared in the literature.

In light of this example, one might wonder whether there is a significant class of travel times that can be analyzed as easily without the cumbersome look-ahead process Y. Unfortunately, essentially any travel time aside from sojourn times, even the time to go from J to K, requires the information from Y. The sojourn time in J is transparent because once a unit enters J there is no possibility it will not complete the sojourn, and the route-traversing status of the unit is determined solely by its current location (look-ahead information is not needed). Due to the subtleties of travel times, their properties have not been uncovered as quickly as other network properties have over the last 30 years. Bear in mind that the dependent nodes and nonproduct-form equilibrium of the network process are not a complicating factor of our analysis. We chose the basic network process as a vehicle for studying travel times because, for our purposes, it is essentially as simple as the Jackson process, but it encompasses so many more networks like Example 2.3; see [27] and [29] for further illustrations.

In Theorem 3.1, the expected number EN_{jk}^* of units that move from j to k per unit time is called the *throughput from j to k*. An expression for this quantity is as follows. Here and later we refer to the following set (i.e., E with its capacity reduced by 1 unit):

$$E' = \begin{cases} E, & \text{if the network is open with unlimited capacity,} \\ \{n: |n| \leq \bar{\nu} - 1\}, & \text{if the network is open with capacity } \bar{\nu}, \\ \{n: |n| = \bar{\nu} - 1\}, & \text{if the network is closed with } \bar{\nu} \text{ units.} \end{cases}$$

PROPOSITION 3.3. *For each $j, k \in M$,*

$$EN_{jk}^* = \lim_{t \to \infty} t^{-1} N_{jk}(t) = a^* w_j \lambda_{jk} \quad a.s., \tag{16}$$

where

$$a^* = w_l^{-1} \sum_{n \in E'} \pi(T_{0l} n) \phi_l(T_{0l} n) \tag{17}$$

for any fixed $l \in M$, and $l = 0$ is a convenient choice when the network is open. If there is a node $l \in M$ such that $\phi_l(n)$ is independent of n_j and $\phi_j(n)$ is independent of n_l for each $j \neq l$, then

$$a^* = \begin{cases} 1, & \textit{if the network is open with } \bar{\nu} = \infty, \\ c/c', & \textit{otherwise}, \end{cases} \tag{18}$$

where c' is the normalizing constant of the distribution π in (5) *on E'.*

PROOF. By a law of large numbers for Markov processes,

$$EN_{jk}^* = \lim_{t \to \infty} t^{-1} \sum_{u \leq t} 1(X(u) = T_{jk} X(u-)) = \sum_{n \in E} \pi(n) q(n, T_{jk} n) \quad \text{a.s.}$$

By (5) and (6), we have $\pi(n)\phi_j(n) = w_j w_l^{-1}\pi(T_{jl}n)\phi_l(T_{jl}n)$, and so

$$EN_{jk}^* = w_j\lambda_{jk}w_l^{-1}\sum_{n\in E}\pi(T_{jl}n)\phi_l(T_{jl}n).$$

Then changing the variable n to $T_{0j}n$ yields (16). Expression (18) is a special case of (17). □

We now present some examples of Theorem 3.1. We will use the Markov chain ξ_ν representing the movement of one unit by p_{jk}. We also use its reversed-time version, which is a Markov chain with transition probabilities

$$p_{jk}^{\leftarrow} \equiv w_k w_j^{-1} p_{kj}, \qquad j, k \in M.$$

COROLLARY 3.4. *Let J, K be disjoint subsets of M, let $I \subset L$ be subsets of $M\setminus(J \cup K \cup \{0\})$, and let T be the time a unit spends in I while traveling in L between J and K. Then*

$$E^0(T) = \sum_{i\in I}\left[\alpha_i^{\leftarrow}\alpha_i EX_i^*\right]\Big/\left[a^*\sum_{j\in J}w_j\sum_{l\in L}\lambda_{jl}\alpha_l\right],$$

where α_i is the absorption probability given by

$$\alpha_i = \sum_{k\in K}p_{ik} + \sum_{l\in L}p_{il}\alpha_l, \qquad i \in L$$

and $\alpha_i = 1$ or 0 according as $i \in K$ or $\in M\setminus(L \cup K)$, and $\alpha_i^{\leftarrow}$ is defined similarly with $p_{jk}^{\leftarrow}$ and J in place of p_{jk} and K. The α_i is the probability that the chain ξ_ν starting at i eventually enters K before entering $M\setminus(L \cup K)$.

Before establishing this result, we record some special cases.

1. The expected total time spent in I while traveling through an open network is

$$\sum_{i\in I}EX_i^*\Big/\left(a^*\sum_{l=1}^{m}\lambda_{0l}\right).$$

2. The expected time to travel from J to K is

$$\sum_{i\in L}\left[\alpha_i^{\leftarrow}\alpha_i EX_i^*\right]\Big/\left[a^*\sum_{j\in J}w_j\sum_{l\in L}\lambda_{jl}\alpha_l\right],$$

where $L \equiv M\setminus(J \cup K \cup \{0\})$.

3. The expected time to enter K in an open network is

$$\sum_{i\in L}\left[\alpha_i^{\leftarrow}\alpha_i EX_i^*\right]\Big/\left[a^*\sum_{l\in L}\lambda_{0l}\alpha_l\right],$$

where $L \equiv M\setminus(K \cup \{0\})$.

4. The expected time between visits to J in a closed network is

$$\sum_{i \notin J} EX_i^* \Big/ \Big(a^* \sum_{j \in J} w_j \sum_{l \notin J} \lambda_{jl} \Big).$$

5. The expected time between visits to J in an open network is as in Corollary 3.4 with $K = J$ and $I = L \equiv M \setminus (J \cup \{0\})$.
6. The expected time spent in I for a unit that enters I from J and exits I into K is given by Corollary 3.4 with $L = I$.

It is natural to view some travel times, including those above, as a duration of time between certain one-step movements or transitions of a single unit. To this end, we say that a unit makes a *$\mathscr{K}$-transition* at step v if $(\xi_{\nu-1}, \xi_\nu) \in \mathscr{K}$, where $\mathscr{K} \subset M \times M$.

COROLLARY 3.5. *Suppose T is the amount of time a unit spends in I between an $\mathscr{J}$-transition and a $\mathscr{K}$-transition, and the unit makes only $\mathscr{L}$-transitions between the $\mathscr{J}$-transition and $\mathscr{K}$-transition. Assume $\mathscr{L} \subset \{1, \dots, m\}^2$ and $\mathscr{J} \cap \mathscr{L}$ and $\mathscr{K} \cap \mathscr{L}$ are empty. Then*

$$E^0(T) = \sum_{i \in I} \frac{EX_i^* \left[\sum_k \overleftarrow{p}_{ik} \, \alpha^{\leftarrow}(i,k)\right]\left[\sum_l p_{il} \alpha(i,l)\right]}{\left[a^* \sum_{(j,k) \in \mathscr{J}} w_j \lambda_{jk} \sum_l p_{kl} \alpha(k,l)\right]}, \tag{19}$$

where

$$\alpha(i,j) = \sum_{(j,k) \in \mathscr{K}} p_{jk} + \sum_{(j,l) \in \mathscr{L}} p_{jl} \alpha(j,l), \qquad (i,j) \in \mathscr{L} \tag{20}$$

and $\alpha(i,j) = 1$ or 0 according as $(i,j) \in \mathscr{K}$ or $\notin \mathscr{K} \cup \mathscr{L}$; and

$$\alpha^{\leftarrow}(i,j) = \sum_{(k,i) \in \mathscr{J}} \overleftarrow{p}_{ik} + \sum_{(l,i) \in \mathscr{L}} \overleftarrow{p}_{il} \, \alpha^{\leftarrow}(i,l), \qquad (j,i) \in \mathscr{L}$$

and $\alpha^{\leftarrow}(i,j) = 1$ or 0 according as $(j,i) \in \mathscr{J}$ or $\notin \mathscr{J} \cup \mathscr{L}$.

REMARKS. The $\alpha(i,j)$ is the probability that the two-dimensional chain $(\xi_{\nu-1}, \xi_\nu)$ starting at (i,j) moves in $\mathscr{L}$ and eventually enters K before entering $M^2 \setminus (\mathscr{K} \cup \mathscr{L} \cup \{(0,0)\})$. These absorption probabilities are the unique solution to (20). The $\alpha^{\leftarrow}(i,j)$ are analogous absorption probabilities for the reversed-time chain $(\xi_\nu, \xi_{\nu-1})$, and the entry set is $\mathscr{J}$ (instead of $\mathscr{K}$). The travel time T in Corollary 3.4 is the special case of T in Corollary 3.5 with $\mathscr{J} = J \times L$, $\mathscr{L} = L \times L$, $\mathscr{K} = L \times K$ and $L = M \setminus (J \cup K \cup \{0\})$.

PROOF. The T is the travel time on the route with start set $\mathscr{J}$ and

$$\mathscr{R} = \{(r_1, \dots, r_l) : (r_{l-1}, r_l) \in \mathscr{K}, (r_{s-1}, r_s) \in \mathscr{L} \setminus (\mathscr{J} \cup \mathscr{K}), 2 \le s < l\},$$

$$S_r = \{s : r_s \in I\}, \qquad r \in \mathscr{R}.$$

Then by elementary reasoning for the Markov chain $\{\xi_\nu\}$, it follows that

$$f_k(\mathscr{R}) = \sum_{(k,l)\in\mathscr{L}} p_{kl}\alpha(k,l),$$

$$p_i(\mathscr{R}) = \left[\sum_k \overleftarrow{p}_{ik}\, \alpha^{\leftarrow}(i,k)\right]\left[\sum_l p_{il}\alpha(i,l)\right].$$

Using these expressions in (10) along with Proposition 3.3 yields the expression for $E^0(T)$. □

The following are travel times whose expected values are given by (19).

1. The time between which a unit makes a one-step transition from J to K and its next one-step transition from J' to K'. Here $I = M$, $\mathscr{S} = J \times K$, $\mathscr{K} = J' \times K'$ and $\mathscr{L} = M^2 \setminus \mathscr{K}$.
2. The time a unit spends in I in an open network as it moves through nodes with increasing indices. Here $\mathscr{S} = \{0\} \times M$, $\mathscr{K} = M \times \{0\}$ and $\mathscr{L} = \{(j,k)$: $j < k\}$.
3. The time between two successive one-step transitions of a unit from J to K and each transition of the unit takes it to a lower indexed node or to the next highest indexed node. Here $I = M$, $\mathscr{S} = \mathscr{K} = J \times K$ and $\mathscr{L} = \{(j,k)$: $k < j$ or $k = j + 1$ and $j \in M\}$.

We end this section by describing how routes and hence travel times are associated with stopping times of the movement a single unit via the Markov chain ξ_ν with transition probabilities p_{jk}. For the start set $\mathscr{S}$, consider the stopping time

$$\tau_0 = \min\{\nu \geq 2 \colon \xi_\nu = 0 \text{ or } (\xi_{\nu-1}, \xi_\nu) \in \mathscr{S}\}.$$

This minimum is $+\infty$ when the set is empty. The τ_0 is the time, measured in node movements, at which the single unit moving under p_{jk} either exits the network or enters the start set $\mathscr{S}$.

FACT 3.6. There is a one-to-one correspondence between the family of routes $\mathscr{R}$ and the family of finite stopping times τ of ξ_ν that satisfy $2 \leq \tau \leq \tau_0$.

PROOF. If τ is a finite stopping time of ξ_ν with $2 \leq \tau < \tau_0$, then the set

$$\mathscr{R}_\tau \equiv \{(r_1, \dots, r_2) \colon \tau = l \text{ when } \xi_1 = r_1, \dots, \xi_l = r_l,\ l \geq 2\}$$

is such that $(\mathscr{R}_\tau, \mathscr{S}, \{S_r\})$ is a route for any S_r. Conversely, if $(\mathscr{R}, \mathscr{S}, \{S_r\})$ is a route, then $\tau \equiv \min\{\nu \geq 2\colon\ \xi_1, \dots, \xi_\nu \in \mathscr{R}\}$ is a stopping time of ξ_ν with $2 \leq \tau \leq \tau_0$. □

4. Proof of Theorem 3.1. We begin by describing the structure of the process $Y(t) = (Y_{j\nu}(t))$, where $Y_{j\nu}(t)$ denotes the type of the νth of the $X_j(t)$ units to enter node j. Under the assumptions on X and the processor-sharing rule described above, it follows that the process Y is Markovian. Suppose Y is in state $y = (y_{j\nu}\colon\ \nu = 1, \dots, n_j(y),\ j = 1, \dots, m)$ and the νth of the $n_j(y)$ units to enter j moves to node k ($j, k \neq 0$). Let y' denote the resulting state of

Y. The transition rate for the movement from y to y' is

$$\lambda_{jk}\phi_j(n_1(y),\dots,n_m(y))\alpha_{j\nu}(n_j(y))\gamma_k(y_{j\nu},y'_{kn_k(y')}), \tag{21}$$

where

$$\begin{aligned} &\gamma_k(sr,(s+1)r)=1, && \text{for } r\in\mathscr{R},\ 1\le s<l,\\ &\gamma_k(\alpha,0)=1-f_k(\mathscr{R}), && \text{for } \alpha=0 \text{ or } lr \text{ for some } r\in\mathscr{R},\\ &\gamma_k(\alpha,1r')=p(r'), && \text{for } r'\in\mathscr{R}, \text{ where } \alpha=0 \text{ or } lr \text{ for some } r\in\mathscr{R},\\ &\gamma_k(\alpha,\beta)=0, && \text{otherwise.}\end{aligned}$$

One can easily specify transition rates similar to (21) for units moving from 0 to k or from j to 0. Recall that X is irreducible and $\mathscr{R}$ consists of feasible paths. Consequently, the process Y is irreducible on its (relevant) state space. Furthermore, it is positive recurrent as we will show in the next part of the proof. Our argument does not require knowledge of the equilibrium distribution of Y (we do not know its form and suspect that it is as intractable as some of the infamous networks involving blocking or routing to the shortest queue). We are only using Y as a vehicle for analyzing the other processes.

To prove $E^0(T)=EX^*_{\mathscr{R}}/EN^*_{\mathscr{R}}$, we will use the regenerative structure of Y. To this end, consider a fixed node k and $r\in\mathscr{R}$ such that $(j,k)\in\mathscr{S}$ for some j and $r_1=k$. Let y^0 denote a state of Y with $n_k(y^0)=\nu_k$, $n_j(y^0)=0$, for $j\ne k$, and $y_{k\nu}=1r$, $1\le\nu\le\nu_k$, for fixed ν_k [i.e., each unit in k came from some node j such that $(j,k)\in\mathscr{S}$ and each of the ν_k units will make a complete traverse of r]. Such a state of Y is always possible for a feasible ν_k ($\nu_k=\bar\nu$ when the network is closed, and ν_k can be any integer less than $\bar\nu$ for an open network). Let $0<\tau_1<\tau_2<\cdots$ denote the successive times at which the process Y enters the state y^0. We first show that $E(\tau_2-\tau_1)$ is finite. Let n^0 be the state defined by $n^0_k=\nu_k$ and $n^0_j=0$, $j\ne k$, and let $Z_1,Z_2,\dots$ denote the times between successive returns of X to n^0. Then

$$\tau_2-\tau_1=_d\sum_{i=1}^{\gamma}Z_i,$$

where γ denotes the number of returns of X to n^0 between two successive returns of Y to the state y^0. Then γ is a geometric random variable independent of the Z_i's and its parameter is the probability

$$\left[\sum_{j\in M}w_jp_{jr_1}p(r)1((j,r_1)\in\mathscr{S})\right]^{\nu_k}$$

that each of the ν_k units at node k starts the route and traverses r completely. Therefore $E(\tau_2-\tau_1)=EZ_1E\gamma<\infty$. This also proves that Y is positive recurrent.

Since Y is Markovian, it is regenerative at the times $\tau_1,\tau_2,\dots$, and so we have the following consequences. The process $X_{\mathscr{R}}$ is regenerative at the times τ_i with respect to the increasing σ-fields of events of Y, which includes the events of $X_{\mathscr{R}}$; see Serfozo [28] for the subtleties of choosing various σ-fields for regenerations. Then by the strong laws of large numbers for stationary and regenerative processes,

$$
\begin{aligned}
EX^*_{\mathscr{R}} &= \lim_{t\to\infty} t^{-1}\int_0^t X^*_{\mathscr{R}}(u)\,du \\
&= \lim_{t\to\infty} t^{-1}\int_0^t X_{\mathscr{R}}(u)\,du \qquad (22)\\
&= E\left[\int_{\tau_1}^{\tau_2} X_{\mathscr{R}}(u)\,du\right]\Big/ E(\tau_2-\tau_1) \quad \text{a.s.}
\end{aligned}
$$

Similarly, the increments of the process $N_{\mathscr{R}}$ are regenerative at the times τ_i, and so

$$
(23)\quad EN^*_{\mathscr{R}} = \lim_{t\to\infty} t^{-1}N_{\mathscr{R}}(t) = E[N_{\mathscr{R}}(\tau_2) - N_{\mathscr{R}}(\tau_1)]/E(\tau_2-\tau_1) \quad \text{a.s.}
$$

Also, the sequence T_i is regenerative over the discrete times $N(\tau_i)$, and so

$$
(24)\qquad E^0(T) = \lim_{\nu\to\infty} \nu^{-1}\sum_{i=1}^{\nu} T_i = E\left[\sum_{i=N_{\mathscr{R}}(\tau_1)+1}^{N_{\mathscr{R}}(\tau_2)} T_i\right]\Big/ E[N_{\mathscr{R}}(\tau_2) - N_{\mathscr{R}}(\tau_1)].
$$

Next, observe that, by the definitions of T_i, $X_{\mathscr{R}}$ and τ_i, it follows that

$$
(25)\qquad \sum_{i=N_{\mathscr{R}}(\tau_1)+1}^{N_{\mathscr{R}}(\tau_2)} T_i = \int_{\tau_1}^{\tau_2} X_{\mathscr{R}}(u)\,du - W_1 + W_2.
$$

Here W_i denotes the remaining time spent on the route $\mathscr{R}$ after time τ_i for those units that entered it in the time period $(\tau_{i-1}, \tau_i]$ (here $\tau_0 = 0$) (there are ν_k such units by the definition of y^0). Note that the integral in (25) is just another way of recording the total time that units spend traversing $\mathscr{R}$ during the period $(\tau_1, \tau_2]$. Since Y regenerates at each τ_i, the W_1, W_2 are independent and identically distributed. Thus, the expectation of the left side of (25) equals the expectation of the integral in (25). Using this result in (24) and then applying (22) and (23) we obtain $E^0(T) = EX^*_{\mathscr{R}}/EN^*_{\mathscr{R}}$, which is assertion (10).

To prove (11), first observe that we can write

$$
(26)\qquad X^*_{\mathscr{R}}(t) = \sum_{j=1}^{m}\sum_{\nu=1}^{X^*_j(t)} U_{j\nu},
$$

where $U_{j\nu} = 1(Y^*_{j\nu} \in \{sr\colon r\in\mathscr{R},\ s\in S_r\})$. Let $\xi^*_1, \xi^*_2, \ldots$ denote a stationary Markov chain on M with transition probabilities p_{jk}. Under the assumptions on X, if a unit is at node j, then its history of nodes visited is independent of the number of units at j. Using this and elementary sample path probabilities of Markov chains, we have

$$
E[U_{j\nu}\,|\,X^*_j(t)] = \sum_{r\in\mathscr{R}}\sum_{s\in S_r} P\{(\xi^*_0, \xi^*_1)\in\mathscr{S}, (\xi^*_1,\ldots,\xi^*_l) = r \,|\, \xi^*_s = j\} p_j(\mathscr{R}).
$$

Thus,

$$EX^*_{\mathscr{R}}(t) = E\left\{\sum_{j=1}^{m} E\left[\sum_{\nu=1}^{X_j^*(t)} U_{j\nu} \mid X_j^*(t)\right]\right\} \sum_{j=1}^{m} p_j(\mathscr{R}) EX_j^*(t).$$

This proves assertion (11).

To prove (12), first observe that we have the equality in distribution

$$N^*_{\mathscr{R}}(1) =_d \sum_{(j,k)\in\mathscr{S}} \sum_{\nu=1}^{N^*_{jk}(1)} U_{k\nu}, \tag{27}$$

where $U_{k1}, U_{k2}, \ldots$ are independent Bernoulli random variable with $P\{U_{k\nu} = 1\} = f_k(\mathscr{R})$, independent of $N^*_{jk}(1)$. Then taking expectations of the terms in (27) yields (12). The remaining assertions (13), (14) and (15) were established in (24), (22) and (23), respectively. □

5. Sojourn and travel times on overtake-free routes. This section describes the distributions of sojourn times and travel times on overtake-free routes. The following definition of an overtake-free route is a variant of that in [4], [6], [7], [15], [20], [25] and [31].

DEFINITION 5.1. A simple route $r = (r_1, \ldots, r_l)$ is *overtake-free* if the following conditions are satisfied:

(a) The $r_1, \ldots, r_l$ are distinct and each one of these nodes i serves units on a first-come-first-served basis with processing rate $\phi_i(n) = \mu_i$, independent of n.

(b) Each feasible path from r_s to any $i \in \{r_1, \ldots, r_l\}$ for any unit must pass through r_{s+1}, $s < l$ (i.e., if $p_{r_s j_1} p_{j_1 j_2} \cdots p_{j_\nu i} > 0$, then $r_{s+1} \in \{j_1, \ldots, j_\nu, i\}$).

(c) For each $s = 1, \ldots, l-1$, let B_s denote the set of nodes on all feasible paths from r_s to r_{s+1} that contains r_s, r_{s+1} only at the beginning and end nodes, respectively, and B_s contains r_s but not r_{s+1}. Think of B_s as the nodes between r_s and r_{s+1}. For each $j \in B_s$, the rate $\phi_j(n)$ is independent of $\{n_k\colon k \notin B_s\}$. And, for each $k \notin B_1 \cup \cdots \cup B_{l-1}$, the processing rate $\phi_k(n)$ is independent of $\{n_j\colon j \in B_1 \cup \cdots \cup B_{l-1}\}$.

A general route $\mathscr{R}$ is *overtake-free* if each $r \in \mathscr{R}$ is overtake-free.

Note that units traversing an overtake-free simple route r finish it in the same order in which they start it. Furthermore, a unit's sojourn time at any r_s is not affected, even indirectly, by the presence of units that start r later than it did. Units traversing a general overtake-free route $\mathscr{R}$ may overtake one another if they are traversing different simple routes in $\mathscr{R}$.

For the following result, we suppose that $\{X(t)\colon t \geq 0\}$ is the network process we have been studying with the additional assumption that $\phi_0(\cdot) = 1$ when X represents an open network. Assume that $r = (r_1, \ldots, r_l)$ is an overtake-free route for X. The equilibrium distribution of X is therefore

(28) $$\pi(n) = c\Phi(n)\prod_{j=1}^{m} w_j^{n_j}, \qquad n \in E,$$

where

(29) $$\Phi(n) \equiv \Phi_I(n_I)\mu_{r_1}^{-n_{r_1}} \cdots \mu_{r_l}^{-n_{r_l}}$$

and $n_I \equiv (n_i: i \in I)$, $I \equiv M \setminus \{r_1, \dots, r_l\}$ with Φ_I defined as in (6) on the set of relevant n_I. We will also use the distribution

$$\pi'(n) = c'\Phi(n)\prod_{j=1}^{m} w_j^{n_j}, \qquad n \in E',$$

where E' is E with its capacity reduced by one unit [recall (Proposition 3.3)]. Note that $\pi' = c^{-1}c'\pi$ and $c' = c$ when X represents an open network with unlimited capacity ($E' = E$). In general, just think of π' as a distribution like π, only on the space E'.

For a unit traversing an overtake-free route r, let $W_{r_1}, \dots, W_{r_l}$ denote the unit's sojourn times at the respective nodes. The next result describes the joint distribution of these times under the Palm probability $P_{sr}^0(\cdot)$ of the process Y^* conditioned that a unit traversing r enters node r_s at time 0. Here $s = 1, \dots, l+1$ and r_{l+1} is an arbitrary, fixed node in M. We also describe the distribution of a travel time $T(\mathscr{R})$ on an overtake-free route $\mathscr{R}$, under the Palm probability P^0 of Y^* conditioned that a unit begins a traverse of $\mathscr{R}$ at time 0.

THEOREM 5.2. *Suppose that $r = (r_1, \dots, r_l)$ is an overtake-free route. Then the joint Laplace transform of $W_{r_1}, \dots, W_{r_l}$ under P_{sr}^0 is*

(30) $$E_{sr}^0\left[\exp(-z_1 W_{r_1} - \cdots - z_l W_{r_l})\right] = \sum_{n \in E'} \pi'(n)\left(\frac{\mu_{r_1}}{z_1 + \mu_{r_1}}\right)^{n_{r_1}+1} \cdots \left(\frac{\mu_{r_l}}{z_l + \mu_{r_l}}\right)^{n_{r_l}+1},$$

which is the same for any $s = 1, \dots, l+1$. Hence, if X represents an open network with unlimited capacity, then $W_{r_1}, \dots, W_{r_l}$ under P_{sr}^0 are independent exponential random variables with respective rates $\mu_{r_1} - w_{r_1}, \dots, \mu_{r_l} - w_{r_l}$. Furthermore, for an overtake-free route $\mathscr{R}$,

(31) $$P^0\{T(\mathscr{R}) \le t\} = \sum_{r \in \mathscr{R}} \alpha(r) P_{1r}^0\{W_{r_1} + \cdots + W_{r_l} \le t\} \Big/ \sum_{r' \in \mathscr{R}} \alpha(r'),$$

where the distribution of $W_{r_1}, \dots, W_{r_l}$ is as described in the preceding statements, and

(32) $$\alpha(r) = w_{r_1}^{-1} \sum_i w_i p_{i r_1} p(r) 1((i, r_1) \in \mathscr{S}).$$

REMARKS. The right side of (30) being independent of s means that a unit's sojourn times $W_{r_1}, \dots, W_{r_l}$ look the same to the unit at any one of its movements along the route. For instance, a unit entering the route ($s = 1$) sees its future sojourn times as being the same as when it finishes the route ($s = l + 1$) and looks back at the times and wonders what they were. We prove (30) by induction on l, similar in many respects to that in Kelly and Pollett [15]. The differences are that we exploit Palm probabilities explicitly rather than implicitly and use the independence of (30) on $s + 1$ as a shortcut in the induction. Also, Theorem 5.2 applies to networks whose nodes not on the route may have congestion-dependent service rates that satisfy (c) in Definition 5.1, and to networks with multiple types of units as described in Section 6. □

Our proof of Theorem 5.2 uses the following lemma. Associated with P_{sr}^0, we let $X^0 \equiv (X_1^0, \dots, X_m^0)$ denote the numbers of units at the respective nodes at time 0, excluding the moving unit that is traversing r and entering r_s at time 0. This X^0 is sometimes called the "disposition of the unmoved units" at time 0. The following result says that a unit traversing r and moving into r_s "sees" the rest of the units with distribution π' (the moving unit sees a time average). Similar MUSTA properties are in [27]; they are variants of arrivals seeing time averages (ASTA).

LEMMA 5.3. *Under the assumptions of Theorem* 5.2,

$$P_{sr}^0\{X^0 = n\} = \pi'(n), \qquad n \in E'. \tag{33}$$

PROOF. Associated with the stationary process Y^*, let $N(n)$ denote the number of units that enter node r_s in the interval $(0, 1]$ and see the other (unmoved) units in state n. Let $N_{sr}(n)$ denote the number of these units entering r_s that are traversing r. By the definition of P_{sr}^0,

$$P_{sr}^0\{X^0 = n\} = E[N_{sr}(n)] / \sum_{n' \in E} E[N_{sr}(n')]. \tag{34}$$

We can write $N_{sr}(n) = \sum_{\nu=1}^{N(n)} U_{k\nu}$, where $k \equiv r_s$ and $U_{k1}, U_{k2}, \dots$ are independent Bernoulli random variables independent of $N(n)$ and

$$\eta_k \equiv P\{U_{k\nu} = 1\} = \beta_k w_k^{-1} \sum_{(j, r_1) \in \mathscr{S}} w_j p_{jr_1} p(r).$$

Here $\beta_k = 1$ if k is inside $\{r_1, \dots, r_l\}$ and $\beta_k = P_{r_l k}$ if $k = r_{l+1}$. This η_k is the probability [like $p_k(\mathscr{R})$] that a unit entering k is traversing r completely. The structures of $N_{sr}(n)$ and $N(n)$ are such that

$$E[N_{sr}(n)] = \eta_k E[N(n)] = \eta_k \sum_{j \in M} \pi(T_{0j}n) q(T_{0j}n, T_{0k}n).$$

From (5), we know that

$$\pi(T_{0j}n) = c w_j \Phi(T_{0j}n) \prod_{i=1}^{m} w_i^{n_i}.$$

And from (6) [with $\phi_{r_1}(\cdot) = \mu_{r_1}$ and $T_{0j}n$ in place of n] and (29), it follows that

$$\Phi(T_{0j}n) = \mu_{r_1}\phi_j(T_{0j}n)^{-1}\Phi(T_{0r_1}n) = \phi_j(T_{0j}n)^{-1}\Phi(n).$$

Combining these observations with (34) yields the assertion (33). □

We are now ready to address the main issue.

PROOF OF THEOREM 5.2. We will prove (30) by induction on the route length l (for all network processes of this form). For simplicity, renumber the nodes such that $(r_1, \dots, r_l) = (1, \dots, l)$. The time W_1 is the sum of $X_1^0 + 1$ independent exponential service times for the X_1^0 units already at node 1 at time 0 plus the unit moving into 1. Consequently,

$$E_{1r}^0\{e^{-z_1 W_1} \mid X^0\} = \left(\frac{\mu_1}{z_1 + \mu_1}\right)^{X_1^0 + 1}.$$

Taking the expectation E_{1r}^0 of this and using Lemma 5.3, we obtain (30) for $l = s = 1$. For the case $l = 1$, $s = 2$, consider the reversed-time version $\{\overline{X}(t): t \geq 0\}$ of the process X. The transition function of $\overline{X}$ is

$$\begin{aligned}\bar{q}(n, T_{jk}n) &= \pi(n)^{-1}\pi(T_{jk}n)q(T_{jk}n, n)\\ &= \bar{\lambda}_{jk}\phi_j(n),\end{aligned}$$

where $\bar{\lambda}_{jk} = w_j^{-1}w_k\lambda_{kj}$. This $\overline{X}$ therefore has the same form as X, and its equilibrium distribution is the same as that for X, since $\overline{\Phi} = \Phi$ and $\overline{w}_j = w_j$. Now, by viewing X in reverse time, it follows that

$$E_{2r}^0(e^{-z_1 W_1}) = \overline{E}_{1r}^0(e^{-z_1 \overline{W}_1}), \tag{35}$$

where $\overline{W}_1$ is the sojourn time for the process $\overline{X}$. Since $\overline{X}$ is the same form as X, expression (30) for $l = s = 1$ applies to the right side of (35), and so $E_{2r}^0(e^{-z_1 W_1})$ equals the right side of (30).

Now, assume (30) holds for all routes of length $1, \dots, l-1$ for some l. Suppose r is a route of length l. We first consider (30) for the expectation E_{sr}^0 with $s = 2$; this pertains to the conditioning event that a unit traversing r moves from r_1 to r_2 at time 0. Let

$$J = B_1, \qquad K = B_2 \cup \cdots \cup B_{l-1} \cup \{l\}, \qquad L = \{1, \dots, m\} \setminus (J \cup K),$$

where B_s is the set of nodes between nodes s and $s+1$ (Definition 5.1). Let

$$Z = \exp\left(-\sum_{s=2}^{l} z_s W_s\right).$$

Since r is overtake-free, W_1 and Z conditioned on X^0 are independent under P_{2r}^0. Furthermore, W_1 depends on X^0 only through the values of $X_J^0 \equiv (X_j^0: j \in J)$, and Z depends on X^0 only through the values of X_K^0.

Therefore,

$$
\begin{aligned}
E_{2r}^{0}[e^{-z_1W_1}Z \mid X^0] &= E_{2r}^{0}[e^{-z_1W_1} \mid X_J^0]\, E_{2r}^{0}[Z \mid X_K^0] \\
&= g(X_J^0) h(X_K^0),
\end{aligned}
\tag{36}
$$

where g and h are nonrandom functions. Then using Lemma 5.3, we have

$$
\begin{aligned}
E_{2r}^{0}(e^{-z_1W_1}Z) &= E_{2r}^{0}[g(X_J^0) h(X_K^0)] \\
&= \sum_{n \in E'} \pi'(n) g(n_J) h(n_K).
\end{aligned}
\tag{37}
$$

The overtake-free property of r ensures that $\phi_j(n)$ is a function of only n_J if $j \in J$. Similar statements hold for processing rates at nodes in K and in L. Therefore, we can write

$$
\pi'(n) = c'' \pi_J(n_J) \pi_K(n_K) \pi_L(n_L),
\tag{38}
$$

where

$$
\pi_J(n_J) = c_J \Phi_J(n_J) \prod_{j \in J} w_j^{n_J}
$$

is a probability measure with normalizing constant c_J and Φ_J is defined as in (29); the π_K, π_L are defined similarly. Substituting (38) in (37), we have

$$
\begin{aligned}
&E_{2r}^{0}(e^{-z_1W_1}Z) \\
&\quad = c'' \sum_{n_J \in E_1} \pi_J(n_J) g(n_J) \sum_{n_K \in E_2} \pi_K(n_K) h(n_K) \sum_{n_L \in E_3} \pi_L(n_L),
\end{aligned}
\tag{39}
$$

where $E_1 = \{n_J \colon |n_J| \le \bar{\nu} - 1\}$,

$$
E_2 = \{n_K \colon |n_K| \le \bar{\nu} - |n_J| - 1\}, \quad E_3 = \{n_L \colon (n_J, n_K, n_L) \text{ is in } E'\}.
$$

Our next step is to apply the induction hypothesis to the first two sums in (39). To this end, let $\hat{X}_J$ denote an open network process on the nodes J with state space E_1 and transition rates

$$
\hat{q}(n_J, T_{jk} n_J) = \hat{\lambda}_{jk} \phi_j(n_J),
$$

where

$$
\hat{\lambda}_{jk} = \begin{cases} \sum_{i \notin J} w_i \lambda_{ik}, & \text{if } j = 0,\ k \in J, \\ \sum_{l \notin J} \lambda_{jl}, & \text{if } k = 0,\ j \in J, \\ \lambda_{jk}, & \text{otherwise.} \end{cases}
$$

This process $\hat{X}_J$ has the same form as X and can be viewed as X on the node set J. An easy check shows that $\hat{w}_j = w_j$ and so the equilibrium distribution of $\hat{X}_J$ is π_J. For this process $\hat{X}_J$, consider the route $r' = \{1\}$ of length 1. Clearly

$$
\hat{E}_{2r'}^{0}(e^{-z_1\hat{W}_1} \mid \hat{X}_J^0) = g(\hat{X}_J^0),
$$

where g is as in (36). Then using Lemma 5.3 for $\hat{X}_J$ and the induction hypothesis, we have

$$\sum_{n_J \in E_1} \pi_J(n_J) g(n_J) = \hat{E}^0_{2r'}\big[g(\hat{X}^0_J)\big] = \hat{E}^0_{2r'}(e^{-z_1 \hat{W}_1}) \tag{40}$$
$$= \sum_{n_J \in E_1} \pi_J(n_J) \left(\frac{\mu_1}{z_1 + \mu_1} \right)^{n_1 + 1}.$$

We can define a similar process $\hat{X}_K$ on E_2 and, for the route $r' = (2, \dots, l)$, we get, under the induction hypothesis,

$$\sum_{n_K \in E_2} \pi_K(n_K) h(n_K) = \hat{E}^0_{1r'}\big[h(\hat{X}^0_K)\big] = \hat{E}^0_{1r'}(Z) \tag{41}$$
$$= \sum_{n_K \in E_2} \pi_K(n_K) \prod_{s=2}^{l} \left(\frac{\mu_s}{z_s + \mu_s} \right)^{n_s + 1}.$$

Substituting (40) and (41) in (39) and combining terms using (38), we obtain (30) for $s = 2$.

To complete the induction, we will show that the left-hand side of (30) is the same for each s. The times $W_1, \dots, W_l$ are determined by the disposition X^0 of the unmoved units at time 0 and related service times, independently of the node r_s into which the moving unit enters. Consequently,

$$P^0_{sr}\{W_1 \le t_1, \dots, W_l \le t_l \,|\, X^0\} = H(X^0),$$

where H is a function on E' independent of s. Hence, by Lemma 5.3,

$$P^0_{sr}\{W_1 \le t_1, \dots, W_l \le t_l\} = E^0_{sr}\big[H(X^0)\big] = \sum_{n \in E'} \pi'(n) H(n)$$

and this is independent of s as suggested.

We now prove the second assertion of Theorem 5.2. Under the hypothesis that X represents an open network with unlimited capacity, we have $\pi' = \pi$. Then using expression (28) for π in (30), we can write

$$E^0_{sr}[\exp(-z_1 W_1 - \cdots - z_l W_l)]$$
$$= c \sum_{n_I} \Phi_I(n_I) \prod_{i \in I} w_i^{n_i} \prod_{j=1}^{l} \mu_j (z_j + \mu_j)^{-1} \sum_{n_j = 0}^{\infty} \left(\frac{w_j}{z_j + \mu_j} \right)^{n_j}.$$

The first sum is over $n_i = 0, 1, \dots$ for $i \in I$. Recall that the normalizing constant c is defined by

$$c^{-1} = \sum_{n_I} \Phi_I(n_I) \prod_{i \in I} w_i^{n_i} \prod_{j=1}^{l} \sum_{n_j = 0}^{\infty} (w_j / \mu_j)^{n_j}.$$

Combining these expressions with their geometric series summed yields

$$E^0_{sr}[\exp(-z W_1 - \cdots - z_l W_l)] = \prod_{j=1}^{l} \left(\frac{\mu_j - w_j}{z_j + \mu_j - w_j} \right). \tag{42}$$

This proves that the $W_1, \dots, W_l$ under P^0_{sr} are independent exponential random variables with respective rates $\mu_1 - w_1, \dots, \mu_l - w_l$.

Our last task is to prove (31). For $r \in \mathscr{R}$, let A_r denote the event that the unit beginning a traverse of $\mathscr{R}$ at time 0 selects $r \in \mathscr{R}$ for its traverse. From the transition rates of the process Y in Section 4, it follows that

$$P^0\{A_r\} = \alpha(r) / \sum_{r' \in \mathscr{R}} \alpha(r').$$

The $\alpha(r)$ is the probability that, in equilibrium, a unit at node r_1 came from some node i with $(i, r_1) \in \mathscr{S}$ and the unit will traverse r completely. Then

$$P^0\{T(\mathscr{R}) \le t\} = \sum_{r \in \mathscr{R}} P^0\{T(\mathscr{R}) \le t \mid A_r\} P^0\{A_r\},$$

which equals the right-hand side of (31). □

6. Networks with multiple types of units. In this section, we point out how the results above also apply to certain networks with multiple types of units. This is accomplished by simply keeping track of each unit's type via another subscript on the processes.

Consider the network as above with the additional feature that each unit carries an attribute (or class label) from a finite set A. A unit's attribute may change over time, possibly depending on the unit's location or its transitions. We represent the network by the process

$$X(t) = \{X_{\alpha j}(t) \colon \alpha \in A,\ j \in \{1, \dots, m\}\}, \qquad t \ge 0,$$

where $X_{\alpha j}(t)$ denotes the number of α units at node j at time t. We assume that X is a Markov process with features consistent with those above. If X is in state $n = \{n_{\alpha j} \colon \alpha \in A,\ j \in \{1, \dots, m\}\}$, a typical transition is triggered by an α unit at node j moving to node k and entering there as a β unit. The rate of this transition is

$$q(n, T_{\alpha j, \beta k} n) = \lambda_{\alpha j, \beta k} \phi_{\alpha j}(n).$$

We make all the assumptions in Section 2 and at the beginning of Section 3 for this new process. The only difference between this process and the one above is that here we are using the double subscript αj instead of simply j. This type of network process is a generalization of the BCMP and Kelly networks in which the attribute α may determine a routing status or processing rate at a node.

A route $\mathscr{R}$ is defined in the natural way with a simple route $r = (r_1, \dots, r_l)$ being a vector with elements of the form $r_s \equiv \alpha_s j_s \in A \times M$. Viewing each pair αj in $A \times M$ as a "node," one talks of each unit moving in $A \times M$ according to a Markov chain $\xi_0, \xi_1, \dots$ with transition probabilities

$$p_{\alpha j, \beta k} = \lambda_{\alpha j, \beta k} / \sum_{\gamma l} \lambda_{\alpha j, \gamma l}.$$

The results in Section 3 and 5 apply to the new setting; just replace $j, k, l, \dots$

throughout by $\alpha j, \beta k, \gamma l, \dots$. For instance, (10) would read

$$E^0[T(\mathscr{R})] = \sum_{\alpha j} p_{\alpha j}(\mathscr{R}) EX^*_{\alpha j} / \sum_{(\alpha j, \beta k) \in \mathscr{S}} f_{\beta k}(\mathscr{R}) EN^*_{\alpha j, \beta k}.$$

The attribute α adds another dimension to the description of a route. For example, as in Corollary 3.4, one might be interested in the time T a unit spends in I as an A_I unit (i.e., its attribute is in $A_I \subset A$) while traveling in L between J and K. An additional restriction might be that the unit starts from J as an A_J unit, travels in L as an A_L unit ($A_L \supset A_I$), and ends in K as an A_K unit. In this case, the $E^0(T)$ is as in Corollary 3.4 with J, K, L replaced by $A_J \times J$, $A_k \times K$ and $A_L \times L$, respectively.

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