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# THE EULER EQUATION: A UNIFORM NONSTANDARD CONSTRUCTION OF A GLOBAL FLOW, INVARIANT MEASURES AND STATISTICAL SOLUTIONS

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We present a simple nonstandard construction of a global Euler flow and some classes of measures invariant with respect to the flow, including examples of non-Gaussian ones. We also obtain existence of statistical solutions of the Euler equation for a wide class of initial measures.

**1. Introduction.** Albeverio and Cruzeiro [1] gave the first construction of a global flow for the two-dimensional Euler equation on a torus, such that the flow leaves a certain Gaussian measure $\mu_\gamma$ invariant. Here we give a simplified uniform approach to this question using nonstandard methods along the lines of [6]. We construct a single set $\Omega$ carrying a single flow that has a whole family of Gaussian and non-Gaussian invariant measures.

The idea is as follows. Take $\Omega = {}^*\mathbb{C}^K$ where $K$ is a hyperfinite set. For $\omega \in \Omega$ elementary methods show that there is an internal (nonstandard) solution $U(\tau, \omega)$ to the ${}^*$finite-dimensional Euler equation with $U(0, \omega) = \omega$. For almost all $\omega$ (with respect to each member of a family of nonstandard Gaussian measures $\nu_\gamma$) the standard part ${}^\circ U(\tau, \omega) = u(t, \omega)$, $t \approx \tau$, gives an individual solution to the standard Euler equation. Since each nonstandard measure $\nu_\gamma$ is invariant for the nonstandard flow (because of the invariance of enstrophy), it easily follows that the corresponding standard measures $\mu_\gamma$ defined on appropriate standard Hilbert space are invariant for this flow. Moreover, our flow has non-Gaussian invariant measures of the form

$$\mu = \int \mu_\gamma \, dq(\gamma),$$

where $q$ is any probability measure on $\mathbb{R}_+$. Such measures arise in a natural way when, in the nonstandard setting, we investigate evolving measures that are represented by densities on ${}^*\mathbb{C}^K$. We also consider the corresponding measures based on the invariance of energy.

Our framework also gives an easy construction of statistical solutions in the sense of Foias [11] to the Euler equation with a broad class of initial measures, in particular measures absolutely continuous with respect to any of the invariant ones.

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**2. The Euler equation.** We briefly review the setting of [1] with some minor changes of notation. The Euler equations for $v$: $[0,\infty) \times \mathbb{R}^2 \to \mathbb{R}^2$ are

$$\frac{\partial v}{\partial t} + \langle v, \nabla \rangle v + \nabla p = 0, \qquad (1)$$
$$\operatorname{div} v = 0,$$

where $\langle \cdot , \cdot \rangle$ is the scalar product in $\mathbb{R}^2$. We consider $x \in \mathbb{T}^2 = [0, 2\pi] \times [0, 2\pi]$ and impose periodic boundary conditions, which is equivalent to working on a two-dimensional torus. As is well known (see [2]), $v$ is a solution of (1) if and only if $v = \nabla^{\perp} \varphi$ for a scalar function $\varphi$ solving

$$\frac{\partial}{\partial t} \Delta \varphi = \langle \nabla^{\perp} \varphi, \nabla \Delta \varphi \rangle, \qquad (2)$$

where

$$\nabla^{\perp} = \left( - \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1} \right).$$

Now take an orthonormal basis of $L^2(\mathbb{T}^2)$ given by $e_k(x) = (1/2\pi)e^{i\langle k, x \rangle}$ for $k \in \mathbb{Z}^2$. The functions $e_k$ are the eigenfunctions of the operator $-\Delta$ with the eigenvalues $k^2 = \langle k, k \rangle$. The quest is for real solutions of (2) of the form

$$\varphi(t, x) = \sum_{k \in \mathbb{Z}^2} u_k(t) e_k(x)$$

with $u_k$: $[0, \infty) \to \mathbb{C}$. Since $\varphi(t, x) + g(t)$ is a solution for any function $g$, we may assume that $u_0(t) = 0$ for all $t$.

For $\varphi$ to be real we must impose the condition: for all $k$,

$$u_{-k}(t) = \overline{u_k(t)}. \qquad (3)$$

Defining $k > 0$ to mean $k_1 > 0$, or $k_1 = 0$ and $k_2 > 0$, we see that it is sufficient to find a family of functions $(u_k(t))_{k>0}$ such that (2) is satisfied by

$$\varphi(t, x) = \sum_{0 \neq k \in \mathbb{Z}^2} u_k(t) e_k(x),$$

where for $k > 0$ we *define* $u_{-k}(t) = \overline{u_k(t)}$. Substituting this in (2) gives the following system of equations for the family $(u_k(t))_{k>0}$:

$$\frac{du_k}{dt} = B_k(u), \qquad (4)$$

where

$$B_k(u) = - \frac{1}{2\pi k^2} \sum_{h+m=k} \langle h^{\perp}, m \rangle m^2 u_h u_m$$

and $h, m$ range over $\mathbb{Z}^2 \setminus \{0\}$.

Remark. It is easy to check that if $(u_k(t))_{k>0}$ solves (4), then $(u_k(t))_{k \neq 0}$ completed by means of (3) solves (4) for all $k$. This shows that some notational ambiguities in [1] do not in fact cause any problem.

The idea now (as developed in [1] and [2]) is to regard (4) as an evolution equation for the vector $u(t) = (u_k(t))_{k>0}$ in certain Hilbert spaces. The appropriate family is $H^p$, $p \in \mathbb{R}$, given by

$$H^p = \{(u_k)_{k>0} \colon \|u\|_p < \infty\},$$

where $\|u\|_p^2 = \sum_{k>0} k^{2p}|u_k|^2$. The elements of $H^p$ can be identified with the elements of the corresponding Sobolev spaces of functions defined on $\mathbb{T}^2$. In particular, we have $H^0 = L^2(\mathbb{T}^2)$ and $H^2 = \mathrm{dom}(\Delta)$.

Recall that the Gaussian probability measures $\mu_\gamma$, $\gamma > 0$, supported on $H^p$ for $p < 1$ are given by

$$d\mu_\gamma(u) = \prod_{k>0} \frac{\gamma k^4}{2\pi} \exp\left(-\frac{1}{2}\gamma k^4 |u_k|^2\right) du_k^1\, du_k^2,$$

where $u_k = u_k^1 + iu_k^2$.

For future use we define here finite dimensional approximations $A_k^n$ of $B_k$. Let $K_n = \{k \in \mathbb{Z}^2\colon k > 0,\ k^2 \le n\}$. For $k \in K_n$ and $v \in \mathbb{C}^{K_n}$ we put

$$A_k^n(v) = B_k(\tilde{v}),$$

where

$$\tilde{v}_k = \begin{cases} v_k, & \text{if } k^2 \le n, \\ 0, & \text{if } k^2 > n. \end{cases}$$

We need the following elementary identities.

LEMMA 2.1.

(a) $$\sum_{k^2 \le n} k^4 A_k^n(u)\bar{u}_k = 0,$$

(b) $$\sum_{k^2 \le n} k^2 A_k^n(u)\bar{u}_k = 0.$$

PROOF. We prove (a) only. The proof of (b) is similar:

$$\begin{aligned} -2\pi \sum_{k^2 \le n} k^4 A_k^n(u)\bar{u}_k &= \sum_{k^2 \le n} k^2 \sum_{\substack{h+m=k \\ h^2,\, m^2 \le n}} \langle m^\perp, h\rangle h^2 u_h u_m \bar{u}_k \\ &= \sum_{m^2 \le n} \sum_{\substack{k-h=m \\ k^2,\, h^2 \le n}} \langle m^\perp, h\rangle k^2 h^2 u_h u_m \bar{u}_k \\ &= -\sum_{m^2 \le n} \left( \sum_{\substack{k+j=m \\ k^2,\, j^2 \le n}} \langle j, k^\perp\rangle k^2 j^2 \bar{u}_j \bar{u}_k \right) u_m \end{aligned}$$

(putting $j = -h$, so that $u_h = \bar{u}_j$ and

$$\langle m^{\perp}, h \rangle = \langle k^{\perp} - h^{\perp}, h \rangle = \langle k^{\perp}, h \rangle = -\langle k^{\perp}, j \rangle).$$

Now using $\langle j, k^{\perp} \rangle = -\langle k, j^{\perp} \rangle$, we see that the matrix

$$\alpha_{kj} = \langle j, k^{\perp} \rangle k^2 j^2 \bar{u}_j \bar{u}_k$$

is antisymmetric, so the sum in the brackets is 0. □

REMARK. These identities show that the enstrophy $S = \Sigma k^4 |u_k|^2$ and the energy $E = \Sigma k^2 |u_k|^2$ are invariant; on the other hand, as in [1] for example, these identities can be seen as a consequence of the invariance of $S$ and $E$.

**3. Nonstandard preliminaries.** We assume the basics of nonstandard analysis and elementary Loeb theory (see, e.g., [3], [7], [8] or [12]). We denote the Loeb measure obtained from an internal measure $\nu$ by $\nu_L$.

Fix an infinite integer $N$ and let $\Omega = {}^*\mathbb{C}^K$, where

$$K = K_N = \{k \in {}^*\mathbb{Z}^2 \colon k > 0,\ k^2 \leq N\}.$$

This will be our basic nonstandard hyperfinite dimensional space used to represent the family of spaces $H^p$. We use $U, V$ to range over $\Omega$ and we put $\|U\|_p^2 = \Sigma_{k \in K} k^{2p} |U_k|^2$. The internal probabilities $\nu_\gamma$ corresponding to $\mu_\gamma$ are given by

$$d\nu_\gamma(U) = \prod_{k \in K} \frac{\gamma k^4}{2\pi} \exp\left(-\frac{1}{2}\gamma k^4 |U_k|^2\right) dU_k^1\, dU_k^2,$$

where $U_k = U_k^1 + iU_k^2$ and $dU_k^1, dU_k^2$ denote *Lebesgue measure.

We denote by ${}^\circ U = \operatorname{st} U$ the standard part of $U$ in the product topology; that is, if $U_k$ is nearstandard for all $0 < k \in \mathbb{Z}^2$, we then put ${}^\circ U = u$ where $u_k = {}^\circ U_k$. This standard part is consistent with that for the norms $\|\cdot\|_p$ as follows:

LEMMA 3.1. *Let $U \in \Omega$. Then $U$ is $\|\cdot\|_p$-nearstandard if and only if*

$$\|{}^\circ U\|_p \approx \|U\|_p < \infty$$

*and if this is the case then ${}^\circ U$ is the $\|\cdot\|_p$-standard part of $U$.*

For each $\gamma$ denote by $P_\gamma$ the Loeb measure $(\nu_\gamma)_L$. Note that all the measures $P_\gamma$ are supported on the single space $\Omega$.

The following is well known (see [3] or [9]).

THEOREM 3.2. *Let $p < 1$. Then $P_\gamma$-almost all $U$ are $\|\cdot\|_p$-nearstandard and*

$$\mu_\gamma(X) = P_\gamma \circ \mathrm{st}^{-1}(X)$$

*for $X \subseteq H^p$.*

PROOF. From [9], Theorem 3.3, we simply have to observe that

$$\sum_{k>0} \frac{1}{\gamma k^4} k^{2p} < \infty,$$

which is the case for $p < 1$. □

**4. Construction of the flow and invariant measures.** An evolution equation for $U(\tau) \in \Omega$, which is a nonstandard counterpart of (4), is given by

$$\frac{dU_k}{d\tau} = A_k(U(\tau)), \qquad k \in K,\ \tau \in {}^*(0,\infty), \tag{5}$$

where

$$A_k(U) = -\frac{1}{2\pi k^2} \sum_{\substack{h+m=k \\ h^2,\, m^2 \le N}} \langle h^\perp, m\rangle m^2 U_h U_m = {}^*B_k(\tilde{U}),$$

where $U_{-k} = \overline{U}_k$ for $k < 0$ and

$$\tilde{U}_k = \begin{cases} U_k, & \text{if } k^2 \le N, \\ 0, & \text{if } k^2 > N. \end{cases}$$

In fact $A_k = A_k^N$, bearing in mind the notation introduced in Section 2; we drop $N$ since it is fixed. It is not hard to check that we also have

$$A_k(U) = E_\gamma\big({}^*B_k | u_k = U_k,\ k^2 \le N\big).$$

Equation (5) is in fact the Galerkin approximation of (4) in hyperfinite dimension $N$. The transfer of standard theory (cf. [1], Lemma 2.2.1) tells us that:

LEMMA 4.1. *For each $\omega \in \Omega$ there is a unique internal solution $U(\tau)$ to (5) with $U(0) = \omega$.*

Denote this solution by $U(\tau, \omega)$. The aim now is to show that in an appropriate sense ${}^\circ U(\tau, \omega)$ gives a flow which leaves each $\mu_\gamma$ invariant. First note that:

LEMMA 4.2. *For each $\gamma > 0$ the internal measure $\nu_\gamma$ is invariant for the internal flow $U(\tau, \omega)$, that is, for internal $Y \subseteq \Omega$ and $\tau \ge 0$,*

$$\nu_\gamma(Y) = \nu_\gamma(\{\omega : U(\tau, \omega) \in Y\}).$$

PROOF. It is sufficient to show that the right-hand side is constant in time since both sides agree for $\tau = 0$.

$$\nu_\gamma(\{\omega\colon U(\tau,\omega) \in Y\})$$
$$= \int_{\{\omega\colon U(\tau,\omega)\in Y\}} \prod_{k\in K} \frac{\gamma k^4}{2\pi} \exp\left(-\frac{1}{2}\gamma k^4 |U_k|^2\right) dU_k^1\, dU_k^2$$
$$= \int_Y \prod_{k\in K} \frac{\gamma k^4}{2\pi} \exp\left(-\frac{1}{2}\gamma k^4 |U_k(-\tau,\omega)|^2\right) \left|\frac{dU(\tau,\omega)}{d\omega}\right|^{-1} d\omega_k^1\, d\omega_k^2.$$

The Jacobian $|dU(\tau,\omega)/d\omega|$ is given by the formula

$$\left|\frac{dU(\tau,\omega)}{d\omega}\right| = \exp\left(\int_0^\tau \sum_{k\in K} \frac{\partial A_k(U(\sigma,\omega))}{\partial U_k}\, d\sigma\right) = 1,$$

where by $\partial g/\partial U_k$ for complex $g = g^1 + ig^2$ we mean $\partial g^1/\partial U_k^1 + \partial g^2/\partial U_k^2$. Next,

$$\frac{d}{d\tau} \prod_{k\in K} \frac{\gamma k^4}{2\pi} \exp\left(-\frac{1}{2}\gamma k^4 |U_k(\tau,\omega)|^2\right)$$
$$= \left(-\operatorname{Re} \sum_{k\in K} \gamma k^4 \frac{dU_k(\tau,\omega)}{d\tau} \overline{U}_k(\tau,\omega)\right) \cdot \prod_{k\in K} \frac{\gamma k^4}{2\pi} \exp\left(-\frac{1}{2}\gamma k^4 |U_k(\tau,\omega)|^2\right),$$

which is 0 because

$$\sum_{k\in K} \gamma k^4 \frac{dU_k(\tau,\omega)}{d\tau} \overline{U}_k(\tau,\omega) = \sum_{k\in K} \gamma k^4 A_k(U(\tau,\omega)) \overline{U}_k(\tau,\omega) = 0$$

by Lemma 2.1. □

The next (standard) lemma plays a crucial role in [1] and is also important for the development here.

LEMMA 4.3. *Let $(A_k^n)_{k^2 \le n}$ be the standard finite dimensional approximations to $B$ defined in Section 2 and let $\gamma > 0$. Then*:

(a) *for $p < 1$ and $k \in \mathbb{Z}_+^2$, $A_k^n \to B_k$ in $L^2(H^p, \mu_\gamma)$ and there are constants $c_k$ such that*

$$E_{\mu_\gamma}\left(|B_k(u)|^2\right) \le \frac{c_k}{\gamma^2};$$

(b) *for $p < -1/2$, $B \in L^2(H^p, \mu_\gamma)$.*

PROOF. See Lemma 1.3.2 of [1]. □

For our use we derive the following information from this lemma.

LEMMA 4.4. *Let $\gamma > 0$ be noninfinitesimal.*

(a) *For all finite $k > 0$ we have*

$$E_{\nu_\gamma}\left(|A_k(U)|^2\right) < \infty$$

*and*

$$A_k(U) \approx B_k({}^\circ U) < \infty$$

*for $P_\gamma$-a.a. $U \in \Omega$;*

(b) *For $p < -1/2$,*

$$E_{\nu_\gamma}\left(\|A(U)\|_p^2\right) < \infty,$$

*where $A(U) = (A_k(U))_{k \in K} \in \Omega$.*

PROOF. (a) Fix finite $k > 0$ and any $p < 1$. From Lemma 4.2(a) we have (note that $A_k = A_k^N$)

$$E_{{}^*\mu_\gamma}\left(|{}^*B_k(u) - A_k(u)|\right)^2 \approx 0,$$

where $A_k(u) = A_k(u|_K)$. Thus ${}^*B_k(u) \approx A_k(u)$ for a.a. $u \in {}^*H^p$ with respect to $({}^*\mu_\gamma)_L$. Now Anderson's Lusin theorem (see [3] or [7]) gives ${}^*B_k(u) \approx B_k({}^\circ u) < \infty$ for a.a. $u$ [since $B_k \in L^2(H^p, \mu_\gamma)$]. So $A_k(u) \approx B_k({}^\circ u)$ for a.a. $u$. Here each side depends only on $u|_K \in \Omega$; now the projection of ${}^*\mu_\gamma$ on $\Omega$ is precisely $\nu_\gamma$ so (a) follows.

(b) This is immediate from Lemma 4.2 (b). □

LEMMA 4.5. *Let $\gamma > 0$ be noninfinitesimal and $0 < k < \infty$. For $P_\gamma$-a.a. $\omega$, $A_k(U(\tau, \omega))$ is $S$ integrable on $[0, T]$ for finite $T$, and hence $U_k(\tau, \omega)$ is $S$ continuous for $\tau < \infty$.*

PROOF. Fix finite $T$; using invariance of $\nu_\gamma$ we have

$$\int_0^T E_{\nu_\gamma}\left(A_k(U(\tau, \omega))^2\right) d\tau = T \cdot E_{\nu_\gamma}\left(A_k(\omega)^2\right),$$

which is finite by Lemma 4.3. By Lindstrøm's lemma [7], $A_k(U(\tau, \omega))$ is $S$ integrable over ${}^*[0, T] \times \Omega$ and so for a.a. $\omega$, $A_k(U(\tau, \omega))$ is $S$ integrable on ${}^*[0, T]$. Now

$$U_k(\sigma, \omega) = U_k(0, \omega) + \int_0^\sigma A_k(U(\tau, \omega))\, d\tau,$$

which shows that $U_k(\cdot, \omega)$ is $S$ continuous as required. □

Let $\Omega_0 \subseteq \Omega$ be the set for which $U_k(\tau, \omega)$ is $S$ continuous for $\tau < \infty$ for all finite $k$. Note that $\Omega_0$ does not depend in any way on $\gamma$ or $p$ and it is a full set for each of the measures $P_\gamma$.

For $\omega \in \Omega_0$ define $u_k(t, \omega)$ for standard $t$ and $k$ by

$$u_k(t, \omega) = {}^\circ U_k(\tau, \omega), \qquad \tau \approx t.$$

Then $u_k(t, \omega)$ is continuous in $t$ for all $\omega \in \Omega_0$. Now let us state our main theorem concerning the flow $(u_k(t, \omega))_{k>0}$ thus constructed. Let a.a. $\omega$ abbreviate $P_\gamma$-a.a. $\omega$.

THEOREM 4.6. *Fix a standard* $\gamma > 0$.

(a) *For all* $t$, $u(t, \omega) \in \bigcap_{p<1} H^p$ *a.s. and for* $X \subseteq \bigcap_{p<1} H^p$,

$$P_\gamma(\{\omega: u(t, \omega) \in X\}) = \mu_\gamma(X) \quad \text{for all } k,$$

*so* $\mu_\gamma$ *is invariant for the flow* $u(t, \omega)$.

(b) *For a.a.* $\omega$ *and all* $k$:

$$\text{(i)} \qquad B_k(u(\cdot, \omega)) \in L^2[0, T], \quad \text{all } T < \infty,$$

$$\text{(ii)} \qquad u_k(t, \omega) = u_k(0, \omega) + \int_0^t B_k(u(s, \omega))\, ds, \quad \text{all } t < \infty.$$

(c) *For a.a.* $\omega$:

$$\text{(i)} \qquad u(\cdot, \omega) \in L^2(0, T; H^p) \quad \text{for all } T < \infty,\ p < 1,$$

$$\text{(ii)} \qquad u(\cdot, \omega) \in C([0, \infty), H^p) \quad \text{for } p < -\tfrac{1}{2}$$

(*in fact* $u(\cdot, \omega)$ *is Hölder continuous with exponent* $\alpha$ *for any* $\alpha < 1/2$).

PROOF. (a) From Theorem 3.2 and Lemma 4.2 for all $\tau$, $U(\tau, \omega)$ is $\|\cdot\|_p$ nearstandard for a.a. $\omega$. Moreover,

$$\begin{aligned} P_\gamma(\{\omega: u(t, \omega) \in X\}) &= P_\gamma(\{\omega: U(t, \omega) \in \mathrm{st}^{-1}(X)\}) \\ &= P_\gamma(\mathrm{st}^{-1}(X)) \qquad \text{(by Lemma 4.2)} \\ &= \mu_\gamma(X) \qquad \text{(by Theorem 3.2)}. \end{aligned}$$

(b) (i) We have

$$\begin{aligned} E_{P_\gamma} \int_0^T |B_k(u(t))|^2\, dt &= \int_0^T E_{P_\gamma} |B_k(u(t))|^2\, dt \\ &= T \cdot E_{P_\gamma} |B_k(u)|^2 \\ &< \infty \end{aligned}$$

by Lemma 4.3, and the result follows by Fubini's theorem.

(ii) Let $T < \infty$. From Lemmas 4.2 and 4.4, for all $\tau \le T$,

$$^\circ A_k(U(\tau, \omega)) = B_k(^\circ U(\tau, \omega)) = B_k(u(^\circ\tau, \omega)) \quad \text{a.a. } \omega.$$

Then by Fubini's theorem, for a.a. $\omega$ this also holds for a.a. $\tau \in [0, T]$.

Combining this with Lemma 4.5 we have for a.a. $\omega$ and $t < \infty$,

$$\begin{aligned} u_k(t,\omega) &= {}^\circ U_k(t,\omega) \\ &= {}^\circ U_k(0,\omega) + {}^\circ\int_0^t A_k(U(\tau,\omega))\,d\tau \\ &= u_k(0,\omega) + \int_0^t B_k(u({}^\circ\tau,\omega))\,d_L\tau \\ &= u_k(0,\omega) + \int_0^t B_k(u(s,\omega))\,ds, \end{aligned}$$

where $d_L\tau$ denotes the Loeb measure of *Lebesgue measure $d\tau$ on $^*[0,T]$.

(c) (i) is immediate from (a) and the fact that $E_{\mu_\gamma}\|u\|_p^2 < \infty$.

(ii) For any $\tau < \tau' \in {}^*[0,\infty)$ and $k \in K$ we have

$$\begin{aligned} E_{\nu_\gamma}\left(|U_k(\tau',\omega) - U_k(\tau,\omega)|^2\right) &= E_{\nu_\gamma}\left(\left|\int_\tau^{\tau'} A_k(U(\sigma,\omega))\,d\sigma\right|^2\right) \\ &\le (\tau'-\tau)E_{\nu_\gamma}\left(\int_\tau^{\tau'} |A_k(U(\sigma,\omega))|^2\,d\sigma\right) \\ &= (\tau'-\tau)\int_\tau^{\tau'} E_{\nu_\gamma}\left(|A_k(U(\sigma,\omega))|^2\right)d\sigma \\ &= (\tau'-\tau)^2 E_{\nu_\gamma}\left(|A_k(\omega)|^2\right). \end{aligned}$$

Thus

$$E_{\nu_\gamma}\left(\|U(\tau',\omega) - U(\tau,\omega)\|_p^2\right) \le (\tau'-\tau)^2 E_{\nu_\gamma}\left(\|A(\omega)\|_p^2\right) = c(\tau'-\tau)^2,$$

where $c < \infty$ by Lemma 4.4 (given that $p < -1/2$ now). For any real $\alpha < 1/2$, the Kolmogorov continuity theorem (see [15] or [3]) now shows that for a.a. $\omega$, $U(\tau,\omega)$ is Hölder $S$ continuous with power $\alpha$ on the finite *dyadic rationals. Now take real $t < t' < \infty$ and *dyadic rationals $\tau \approx t$, $\tau' \approx t'$. Then

$$\|u(t',\omega) - u(t,\omega)\|_p = \|{}^\circ U(\tau',\omega) - {}^\circ U(\tau,\omega)\|_p \le {}^\circ\|U(\tau',\omega) - U(\tau,\omega)\|_p,$$

which gives the result. □

We now consider the question of other invariant probability measures for the flow we have constructed. Again let us emphasize that the flow $u_k(t,\omega)$ defined for $\omega \in \Omega_0 \subseteq \Omega$ is constructed without reference to any measure on $\Omega$ or any particular space $H^p$.

Observe first that the family of measures $\mu_{\beta,\gamma}$ discussed in [2] and [1] is easily seen to be invariant for the flow we have constructed. They are obtained from the internal measures $\nu_{\beta,\gamma}$ on $\Omega$ given by

$$d\nu_{\beta,\gamma}(U) = \prod_{k\in K}\frac{\beta k^2 + \gamma k^4}{2\pi}\exp\left(-\frac{1}{2}(\beta k^2 + \gamma k^4)|U_k|^2\right)dU_k^1\,dU_k^2$$

for $\gamma > 0$ and $\beta > -\gamma$. (In [2] and [1], $\beta$ was restricted to be positive.) Let

$$P_{\beta,\gamma} = (\nu_{\beta,\gamma})_L,$$

then Theorem 3.2 applies also to $P_{\beta,\gamma}$ since

$$\sum_{k>0} \frac{1}{\beta k^2 + \gamma k^4} k^{2p} < \infty$$

if $p < 1$ and $\mu_{\beta,\gamma} = P_{\beta,\gamma} \circ \mathrm{st}^{-1}$. It is routine to check that all the results for $P_\gamma$ and $\mu_\gamma$ extend to $P_{\beta,\gamma}$ and $\mu_{\beta,\gamma}$.

All invariant measures mentioned so far are Gaussian. A non-Gaussian family is easily obtained as follows: let $q$ be a Borel probability measure on $[0,\infty)$ and define $\mu_q$ on $\bigcap_{p<1} H^p$ by

$$\mu_q(X) = \int \mu_{\alpha^{-1}}(X)\, dq(\alpha),$$

where we make the natural definition $\mu_\infty = \delta_0$, the Dirac measure concentrated at $0 \in H^p$. (We remark that $\alpha = \gamma^{-1}$ is a better variable to index the family $\mu_\gamma$, but we have followed the convention adopted in [1].)

The internal counterpart of $\mu_q$ is $\nu_q$ on $\Omega$ given by

$$\nu_q(Y) = \int \nu_{\alpha^{-1}}(Y)\, d^*q(\alpha).$$

Putting $P_q = (\nu_q)_L$, it is routine to see that the results above apply to the measures $\mu_q$ and $P_q$; in particular $\mu_q$ is a non-Gaussian invariant probability for the flow $u(t, \omega)$ (unless $q$ is Dirac).

We can similarly obtain further invariant measures from the family $\mu_{\beta,\gamma}$ by means of a probability $\pi$ on $\{(\alpha, \beta) \in \mathbb{R}^2\colon \alpha \geq 0,\ \alpha\beta > -1\}$. On this set we have $\beta > -1/\alpha = -\gamma$, say, and we define

$$\mu_\pi(X) = \int \mu_{\beta,\alpha^{-1}}(X)\, d\pi(\beta, \alpha),$$

where $\mu_{\beta,\infty} = \delta_0$, all $\beta$. Then the measure $\mu_\pi$ is also invariant.

**5. Invariant densities.** The internal measures $\nu_\gamma$ and $\nu_{\beta,\gamma}$ on $\Omega$, invariant for the internal flow $U(\tau, \omega)$, have explicit densities against *Lebesgue measure on $\Omega = {}^*\mathbb{C}^K$. Here we pursue the idea of searching for invariant measures having (nonstandard) invariant densities, using the methods of [4] and [5].

Suppose that $\Lambda$ is an internal probability on $\Omega$ and define the evolving family $\Lambda_\tau$ by

$$\Lambda_\tau(X) = \Lambda\big(U(\tau, \cdot)^{-1}(X)\big),$$

so $\Lambda_0 = \Lambda$. The internal equation satisfied by $\Lambda_\tau$ (analogous to that introduced by Foias [11] for the Navier–Stokes equations) reads

$$\frac{d}{d\tau}\left(\int \psi(U)\, d\Lambda_\tau(U)\right) = \int \sum_{k\in K} \psi_k'(U) A_k(U)\, d\Lambda_\tau(U), \tag{6}$$

where $\psi\colon \Omega \to {}^*\mathbb{R}$ is ${}^*$differentiable and bounded.

If $\Lambda_\tau$ have densities $f(\tau, U)$ against ${}^*$Lebesgue measure on $\Omega$, then as in [5] it is easy to derive the following *density equation*:

$$\frac{\partial f}{\partial \tau} + \sum_{k\in K} \frac{\partial}{\partial U_k}(A_k(U) f) = 0,$$

where by $\partial g/\partial U_k$ for complex $g = g^1 + ig^2$ we understand $\partial g^1/\partial U_k^1 + \partial g^2/\partial U_k^2$. Since by the definition of $A_k$, $(\partial/\partial U_k)A_k(U) = 0$, the density equation reduces to

$$\frac{\partial f}{\partial \tau} + \sum_{k\in K} A_k(U)\frac{\partial}{\partial U_k} f = 0,$$

where $A_k\partial/\partial U_k = A_k^1\partial/\partial U_k^1 + A_k^2\partial/\partial U_k^2$. Hence the internal equation for an invariant density $f(U)$ takes the form

$$\sum_{k\in K} A_k(U)\frac{\partial}{\partial U_k} f = 0. \tag{7}$$

As is well known, a density $f(U)$ satisfies (7) if and only if it is constant on the solutions of (5). In particular the densities of the measures $\nu_\gamma$ and $\nu_{\beta,\gamma}$ satisfy (7) as is shown in the proof of Lemma 4.2. Their invariance stems from the fact that energy $E(U) = (1/2)\Sigma_{k\in K}k^2|U_k|^2$ and enstrophy $S(U) = (1/2)\Sigma_{k\in K}k^4|U_k|^2$ are invariants of the motion, which is essentially the content of Lemma 2.1. In fact any density of the form $f(U) = r(E(U), S_h(U))$ is an invariant density where

$$S_h(U) = \int_{{}^*\mathbb{T}} h\left(\sum_{k\in K} k^2 U_k e^{i\langle k,x\rangle}\right) dx$$

also is an invariant of motion for any continuous function $h$, as is shown in [2].

For a while let us restrict our attention to densities $f$ that are constant on surfaces of the form $\Sigma_{k\in K}k^4|U_k|^2 = \text{constant}$, which we call "enstrophy shells." Such densities take the form

$$f(U) = r\left(\sum_{k\in K} k^4|U_k|^2\right). \tag{8}$$

From the point of view of the standard flow $u(t,\omega)$ obtained from $U$ we get an invariant measure only if the internal invariant density is nearstandardly concentrated. We now show that in the case of the densities of the form (8) we then obtain standard measures of the form $\mu_q$ as in the previous section.

To this end we first prove a theorem concerned with measures uniformly distributed on single enstrophy shells. Recall that we define $\mu_\infty = \delta_0$.

THEOREM 5.1. *Let $M = 2|K|$ where $|K|$ denotes the number of elements in $K$. Let $\theta$ be the uniform probability on the surface*

$$S_c = \left\{ V \in \Omega \colon \sum_{k \in K} k^4 |V_k|^2 = c \right\}.$$

*Then $V$ is $\theta_L$-a.s. $\|\cdot\|_p$ nearstandard for real $p < 1$ if and only if $c/M < \infty$, and in this case $\theta_L \circ st^{-1} = \mu_\gamma$ where $\gamma^{-1} = {}^\circ(c/M)$.*

PROOF. Consider first the case $c = M$. Let $\hat{V}_k = k^2 V_k$; then under $\theta$, $\hat{V}$ is uniformly distributed on the sphere

$$S^{M-1}(M^{1/2}) = \{W \colon |W|^2 = M\} \subseteq \Omega,$$

where $|W|^2 = \Sigma |W_k|^2$.

Now consider $U$ in $\Omega$ under the Gaussian distribution $\nu = \nu_1$. Then under $\nu$, $\hat{U}_k^i$, $i = 1, 2$, are i.i.d. $N(0,1)$ and $(M^{1/2}/|\hat{U}|)\hat{U}$ is uniformly distributed on $S^{M-1}(M^{1/2})$. Thus $V(U) = (M^{1/2}/|\hat{U}|)U$ has distribution $\theta$.

By direct calculation we have $E_\nu(|\hat{U}|^2/M) = 1$ and $E_\nu((|\hat{U}|^2/M - 1)^2) = 2/M \approx 0$ so the normalising factor $M^{1/2}/|\hat{U}| \approx 1$, $\nu_L$ a.s. Now $U$ is a.s. $\|\cdot\|_p$-nearstandard, hence so is $V(U)$, and ${}^\circ V(U) = {}^\circ U$ a.s. (all under $\nu_L$). Therefore, $V(U)$ is $\theta_L$-a.s. $\|\cdot\|_p$-nearstandard and for $X \subseteq H^p$ we have

$$\begin{aligned} \theta_L(\mathrm{st}^{-1}(X)) &= \nu_L(\{U \colon {}^\circ V(U) \in X\}) \\ &= \nu_L(\{U \colon {}^\circ U \in X\}) \\ &= \mu_1(X) \end{aligned}$$

by Theorem 3.2.

For general $c$ with $\gamma^{-1} = {}^\circ(c/M) < \infty$ the result follows by rescaling.

Now consider $c$ with $c/M = H$ infinite. Again by rescaling, the argument above shows that under $\theta_L$, ${}^\circ(V_k^i/H^{1/2})$ is Gaussian distributed with variance $1/k^4$. Thus for finite $k$, ${}^\circ(V_k/H^{1/2}) \neq 0$ a.s., and so $V_k$ is a.s. infinite. □

REMARK. This result (and its proof) is closely related to the main result of [10] which shows that uniform measure on $S^{M-1}(1)$ gives Wiener measure. The intuition underlying this goes back to Wiener [16], building on the observation of Poincaré [14] that the normal distribution is obtained as the limit of projections of uniform measure on $S^{n-1}(\sqrt{n})$ onto the 1-axis. Wiener intuitively thought of uniform measure on $S^\infty(\sqrt{\infty})$, which he called differential space (see McKean [13] for a discussion). Nonstandard methods allow a rigorous treatment of $S^\infty(\sqrt{\infty})$ in the guise $S^{N-1}(\sqrt{N})$, or equivalently (as in [10]) the scaled version $S^{N-1}(1)$.

We can now prove the main result of this section.

THEOREM 5.2. *Let $p < 1$. Suppose that $f(U) = r(\sum_{k\in K} k^4|U_k|^2)$ is a nonstandard density with corresponding internal measure $\nu$ on $\Omega$, that is, $\nu(Y) = \int_Y f(U)\,dU$. Let $Q$ be the internal probability on ${}^*[0,\infty)$ given by*

$$Q({}^*[0,a]) = \int_{\{\sum k^4|U_k|^2 \le Ma\}} f(U)\,dU.$$

*Then $\nu$ is $\|\cdot\|_p$-nearstandardly concentrated if and only if $Q$ is nearstandardly concentrated, and then*

$$\nu_L \circ st^{-1} = \mu_q,$$

*where $q = Q_L \circ st^{-1}$.*

PROOF. Let $\theta_c$ be the uniform probability on $S_c$. Then

$$\nu(Y) = \int_0^{*\infty} \theta_{Ma}(Y \cap S_{Ma})\,dQ(a)$$

for internal $Y$ and a similar equality holds for the corresponding Loeb measures. Now let $\Omega_p$ be the set of $\|\cdot\|_p$ nearstandard points in $\Omega$. Then, from Theorem 5.1, $(\theta_c)_L(\Omega_p \cap S_c) = 1$ if and only if $a = c/M$ is finite. So $\nu_L(\Omega_p) = 1$ if and only if $Q$ is nearstandardly concentrated, that is, $Q_L(F) = 1$ where $F$ denotes the set of finite elements of ${}^*[0,\infty)$. If it is so, then for $X \subseteq H^p$ we have

$$\begin{aligned}\nu_L(st^{-1}(X)) &= \int_F (\theta_{Ma})_L(\mathrm{st}^{-1}(X) \cap S_{Ma})\,dQ_L(a)\\ &= \int_F \mu_{\circ a^{-1}}(X)\,dQ_L(a) \qquad \text{(by Theorem 5.1)}\\ &= \int \mu_{a^{-1}}(X)\,dq(a). \qquad \square\end{aligned}$$

By considering densities constant on the surfaces of the form

$$\beta \sum_{k\in K} k^2|U_k|^2 + \gamma \sum_{k\in K} k^4|U_k|^2 = c,$$

we obtain in the same way the invariant measures $\mu_{\beta,\gamma}$ and their integrals $\mu_\pi$.

REMARK. We do not know which (if any) standard invariant measure is obtained by taking an internal invariant measure uniformly distributed on a surface

$$\left\{\sum_{k\in K} k^2|U_k|^2 = c_1\right\} \cap \left\{\sum_{k\in K} k^4|U_k|^2 = c_2\right\}.$$

Do such measures fit into the spectrum of invariant measures already obtained? Are they Gaussian?

**6. Statistical solutions to the Euler equation.** We conclude this paper by showing that the flow $u(t,\omega)$ defined earlier gives an easy existence proof for statistical solutions of the Euler equation for a wide class of initial measures $\lambda_0$. Following Foias [11], by a *statistical solution of the Euler equation* we mean a family of measures $\lambda_t$ defined on one of the Hilbert spaces $H^p$ and satisfying

$$(9)\quad \int \psi(u)\,d\lambda_t(u) = \int \psi(u)\,d\lambda_0(u) + \int_0^t \int_H \langle B(u), \psi'(u)\rangle_p \,d\lambda_s(u)\,ds$$

for all $t$ and all *test* functionals $\psi$ on $H^p$ of the form $\psi(u) = \exp(i\langle u, v\rangle_p)$, $v = (v_k)_{k^2 \le m}$ for some $m$, where by $\langle\ ,\ \rangle_p$ we denote the scalar product in $H^p$.

THEOREM 6.1. *Take a probability measure $\lambda$ on $H^p$ and put $P = ({}^*\lambda^K)_L$, where ${}^*\lambda^K$ is the projection of ${}^*\lambda$ on $\Omega = {}^*\mathbb{C}^K$. Suppose that:*

(i) *the mapping $(t,\omega) \mapsto B_k(u(t,\omega))$ is $L^1([0,T] \times \Omega)$ for all $k$ and $T < \infty$;*
(ii) *$u_k(t,\omega) = u_k(0,\omega) + \int_0^t B_k(u(s,\omega))\,ds$ for all $k, t < \infty$ and $P$-a.a. $\omega$.*

*Then the family of measures*

$$\lambda_t(X) = P(\{\omega: u(t,\omega) \in X\})$$

*is a statistical solution to the Euler equation with $\lambda_0 = \lambda$.*

PROOF. Basic Loeb theory shows that $\lambda_0 = \lambda$. Fix a test functional $\psi$ as above. From (ii) for $P$-a.a. $\omega$,

$$\begin{aligned}\frac{d}{dt}\psi(u(t,\omega)) &= \langle B(u(t,\omega)), \psi'(u(t,\omega))\rangle_p \\ &= \sum_{k^2 \le m} k^{2p} v_k \psi(u(t,\omega)) B_k(u(t,\omega)),\end{aligned}$$

which is integrable on $[0,T] \times \Omega$. Next we integrate in time from 0 to $t$ and then with respect to $P$:

$$\begin{aligned}\int \psi(u(t,\omega))\,dP(\omega) &= \int \psi(u(0,\omega))\,dP(\omega) \\ &\quad + \int\int_0^t \langle B(u(t,\omega)), \psi'(u(t,\omega))\rangle_p \,ds\,dP(\omega).\end{aligned}$$

Now apply Fubini's theorem to the last term. The definition of the measures $\lambda_t$ means that $\int g(u(t,\omega))\,dP(\omega) = \int g(u)\,d\lambda_t(u)$ for any $g$ so we have the required result. □

REMARK. We have considered a weak version of the notion of statistical solution which is reflected by the fact that the class of test functionals $\psi$ is narrow. However, this class is sufficiently large for the integrals $\int\psi(u)\,d\lambda$ of all test functionals to determine the measure $\lambda$. To get stronger statistical solutions we would need to impose a condition such as $B(u(\cdot,\cdot)) \in L^1([0,T]\times\Omega, H^p)$, which is not satisfied by the members of a natural class of measures considered below.

THEOREM 6.2. *Let $\lambda \ll \mu_q$ where $q$ is a probability on $[0,\infty)$ and $\mu_q$ is defined on $H^p$, $p<1$, as in Section 4. If $d\lambda/d\mu_q \in L^2(\mu_q)$ and $\int\alpha^2\,dq(\alpha)<\infty$, then there is a solution $\lambda_t$ of (9) with $\lambda_0 = \lambda$.*

PROOF. We write $P_q = ({}^*\mu_q^K)_L = (\nu_q)_L$ and $P = ({}^*\lambda^K)_L$. Let $R\colon \Omega \to {}^*\mathbb{R}$ be any $SL^2$ lifting of $d\lambda/d\mu_q$. Then $P \ll P_q$ with $dP/dP_q = {}^\circ R \in L^2(P_q)$. We know that condition (ii) of Theorem 6.1 is satisfied by $P_q$ (see Theorem 4.6) and hence by $P$. To check condition (i) we have

$$\begin{aligned}\left(E_P\int_0^T |B_k(u(t,\omega))|\,dt\right)^2 &= \left(E_{P_q}\left({}^\circ R(\omega)\int_0^T |B_k(u(t,\omega))|\,dt\right)\right)^2\\ &\le E_{P_q}\left({}^\circ R(\omega)^2\right)E_{P_q}\left(\int_0^T |B_k(u(t,\omega))|\,dt\right)^2\\ &\le E_{P_q}\left({}^\circ R(\omega)^2\right)TE_{P_q}\left(\int_0^T |B_k(u(t,\omega))|^2\,dt\right)\\ &< \infty\end{aligned}$$

since $\mu_q$ is invariant and

$$E_{\mu_q}\left(|B_k(u)|^2\right) \le c_k\int\alpha^2\,dq(\alpha)$$

by Lemma 4.3 (a). □

REMARK. It is routine to extend this result to measures $\lambda \ll \mu_\pi$ provided $d\lambda/d\mu_\pi \in L^2$ and $\alpha/(1+\alpha\beta) \in L^2(\pi)$.

## REFERENCES

[1] ALBEVERIO, S. and CRUZEIRO, A.-B. (1990). Global flows with invariant (Gibbs) measures for Euler and Navier–Stokes two dimensional fluids. *Comm. Math. Phys.* **129** 431–444.

[2] ALBEVERIO, S., RIBEIRO DE FARIA, M. and HØEGH-KROHN, R. (1979). Stationary measures for the periodic Euler flow in two dimensions. *J. Statist. Phys.* **20** 585–595.

[3] ALBEVERIO, S., FENSTAD, J. E., HØEGH-KROHN, R. and LINDSTRØM, T. (1986). *Nonstandard Methods in Stochastic Analysis and Mathematical Physics*. Academic, London.

[4] CAPIŃSKI, M. and CUTLAND, N. J. (1991). Statistical solutions of PDEs by nonstandard densities. *Monatsh. Math.* **111** 99–117.

[5] CAPIŃSKI, M. and CUTLAND, N. J. (1991). Statistical solutions of Navier–Stokes equations by nonstandard densities. *Math. Models and Methods Appl. Sci.* **1** 447–460.

[6] CAPIŃSKI, M. and CUTLAND, N. J. (1992). A simple proof of existence of weak and statistical solutions of Navier–Stokes equations. *Proc. Roy. Soc. London Ser. A* **436** 1–11.

[7] CUTLAND, N. J. (1983). Nonstandard measure theory and its applications. *J. London Math. Soc.* **15** 529–589.

[8] CUTLAND, N. J. (1988). *Nonstandard Analysis and Its Applications*. Cambridge Univ. Press.

[9] CUTLAND, N. J. (1991). On large deviations in Hilbert space. *Proc. Edinburgh Math. Soc.* **34** 487–495.

[10] CUTLAND, N. and NG, S.-A. (1993). The Wiener sphere and Wiener measure. *Ann. Probab.* **21** 1–13.

[11] FOIAŞ, C. (1973). Statistical study of Navier–Stokes equations I. *Rend. Sem. Math. Univ. Padova* **48** 219–348.

[12] HURD, A. E. and LOEB, P. A. (1985). *An Introduction to Nonstandard Real Analysis*. Academic, New York.

[13] MCKEAN, H. P. (1973). Geometry of differential space. *Ann. Probab.* **1** 197–206.

[14] POINCARÉ, H. (1912). *Calcul des Probabilités*. Gauthier-Villars, Paris.

[15] SIMON, B. (1979). *Functional Integration and Quantum Physics*. Academic, New York.

[16] WIENER, N. (1923). Differential space. *J. Math. Phys.* **2** 132–174.

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