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ON A FIRST PASSAGE PROBLEM FOR BRANCHING BROWNIAN MOTIONS

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Consider a (space-time) realization ω of a critical or subcritical one-dimensional branching Brownian motion. Let $Z_x(\omega)$ for $x \geq 0$ be the number of particles which are located for the first time on the vertical line through $(x, 0)$ and which do not have an ancestor on this line. In this note we study the process $Z = \{Z_x; x \geq 0\}$. We show that Z is a continuous-time Galton–Watson process and compute its creation rate and offspring distribution. Here we use ideas of Neveu, who considered a similar problem in a supercritical case. Moreover, in the critical case we characterize the continuous state branching processes obtained as weak limits of the processes Z under rescaling.

1. Introduction and basic definitions. Let $X = \{X_t; t \geq 0\}$, $X_0 = x$, be a branching Brownian motion in $\mathbf{R}$ with a constant creation rate α and the offspring distribution $p = \{p_k; k = 0, 1, 2, \dots\}$. It is assumed that $p_1 < 1$ and that X is (sub)critical, that is, $\sum_{k=0}^{\infty} k p_k \leq 1$. The canonical sample space of X is a space of marked trees which we now describe.

Consider the set

$$U := \bigcup_{n=1}^{\infty} \mathbf{N}_+^n \cup \{0\}, \qquad \mathbf{N}_+ := \{1, 2, \dots\}$$

and let ω be a subset of U with the properties (cf. Neveu [9]):

1. $0 \in \omega$,
2. $\forall\, u, v \in U$: $uv \in \omega \Rightarrow u \in \omega$,
3. $\forall\, u \in \omega\ \exists\, \nu^u(\omega) \in \mathbf{N}\ \forall\, j \in \mathbf{N}_+$: $uj \in \omega \Rightarrow 1 \leq j \leq \nu^u(\omega)$.

Such subsets ω are called trees and we denote the space of all trees by Ω. Elements in U are called particles. To explain the notation uv in (2), let $u = (i_1, \dots, i_s) \in U$, $v = (j_1, \dots, j_r) \in U$, then $uv = (i_1, \dots, i_s, j_1, \dots, j_r) \in U$. The variable ν^u in (3) gives the number of descendants of the particle u. A particle v is called an ancestor of a particle u, denoted $v \leq u$, if there exists $w \in U$ such that $u = vw$. Defining $0u = u$ and $u0 = u$, it is seen that for every $u \in U$ we have $u \leq u$ and $0 \leq u$.

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Let

$$M := \{(\zeta, \gamma); \zeta \in \mathbf{R}_+, \gamma\colon [0, \zeta) \to \mathbf{R}, \text{continuous}, \gamma(0) = 0\}.$$

A marked tree ω^0 (cf. Chauvin [3]) is defined as

$$\omega^0 := (\omega, \{(\zeta^u, \gamma^u); u \in \omega\}), \tag{1.1}$$

where $\omega \in \Omega$ and $(\zeta^u, \gamma^u) \in M$ for every $u \in \omega$. Let Ω^0 denote the set of all marked trees. For a given $\omega^0 \in \Omega^0$ let ω be the corresponding tree in (1.1). To emphasize the structure we often denote the marks in ω^0 with $(\zeta^u(\omega^0), \gamma^u(\omega^0))$.

The path of a particle $u \in \omega$ is defined as

$$\xi_t^u(\omega^0) = \begin{cases} x + \gamma_t^0(\omega^0), & \text{if } u = 0, t < \zeta^0, \\ \xi_{\zeta^v -}^v(\omega^0) + \gamma_t^u(\omega^0), & \text{if } u \neq 0, t < \zeta^u, \\ \Delta, & \text{if } t \geq \zeta^u, \end{cases} \tag{1.2}$$

where $u = vj$ for some j, $1 \leq j \leq \nu^v(\omega)$, that is, v is u's parent and Δ is a fictitious cemetery state. The parameter t in (1.2), when $t < \zeta^u$, is called the age of the particle u.

Next we introduce some relevant σ-fields in Ω^0 (cf. [3]). First, for every $u \in U$ define in the space $\Omega^{0,u} := \{\omega^0; u \in \omega\}$,

$$\mathscr{F}_t^u := \sigma\{\gamma_s^u(\omega^0); 0 \leq s < t \wedge \zeta^u(\omega^0), \omega^0 \in \Omega^{0,u}\}$$

and then, recursively,

$$\mathscr{G}^0 := \{\varnothing, \Omega^0\},$$

$$\mathscr{G}^u := \mathscr{G}^v \vee \mathscr{F}_\infty^v \wedge \sigma\{\nu^v(\omega); \omega^0 \in \Omega^{0,v}\},$$

where $u \neq 0$ and v is u's parent. Intuitively, $\mathscr{G}^u$ contains information on the branch leading to the particle u. To include the history of the particle itself, set

$$\mathscr{H}_t^u := \mathscr{G}^u \vee \mathscr{F}_t^u,$$

$$\mathscr{H}_\infty^u := \bigvee_{t>0} \mathscr{H}_t^u \wedge \sigma\{\nu^u(\omega); \omega^0 \in \Omega^{0,u}\}.$$

Finally, denote by $\mathscr{F}^0$ the smallest σ-field on Ω^0 which makes all marked trees measurable, and let $\mathbf{P}_x$ be the probability measure on $(\Omega^0, \mathscr{F}^0)$ associated with X, $X_0 = x$.

Let $\sigma_y^u\colon \Omega^{0,u} \to [0, +\infty]$ be the first hitting time for the particle u to the point y, that is,

$$\sigma_y^u := \begin{cases} \inf\{s; \xi_s^u(\omega^0) = y\}, & \text{if } \{\cdot\} \neq \varnothing, \\ +\infty, & \text{otherwise}. \end{cases}$$

Clearly, σ_y^u is for every u a stopping time with respect to $\mathscr{H} := \{\mathscr{H}_s^u; s \geq 0\}$.

Setting

$$\tau_y^u := \begin{cases} \sigma_y^u, & \text{if } \sigma_x^u < \infty \text{ and } \nexists v < u : \sigma_x^v < \infty, \\ +\infty, & \text{otherwise,} \end{cases}$$

the family $\tau_x = \{\tau_x^u : u \in U\}$ becomes a stopping line (cf. [3]) in the sense of:

DEFINITION 1. A stopping line τ is a family of nonnegative random variables $\tau^u : \Omega^{0,u} \to [0, \infty]$, such that:

(i) τ^u is a stopping time with respect to $\mathscr{H}^u$ for every $u \in U$,
(ii) the set $L_\tau(\omega^0) := \{u \in \omega;\ \tau^u(\omega^0) < \infty\}$ has the line property for every $\omega^0 \in \Omega^0$, that is,

$$u \in L_\tau(\omega^0) \Rightarrow \big(\nexists v < u : v \in L_\tau(\omega^0)\big).$$

REMARK 1. This definition differs slightly from that in [3], page 1197, because in our case τ^u may attain "the value" $+\infty$ and, therefore, $u \in L_\tau$ in the case $\tau^u = \zeta^u$. However, this is of no importance in the present case.

To introduce the first passage process, which is the main topic of this paper, assume that $X_0 = 0$, and for $x \geq 0$ define $L_x(\omega^0) := L_{\tau_x}(\omega^0)$ and

$$L_{x+}(\omega^0) := \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} L_{x+1/n}(\omega^0).$$

Then, clearly, $L_0 = L_{0+} = \{0\}$.

DEFINITION 2. The process

$$Z = \{Z_x := |L_{x+}|;\ x \geq 0\},$$

where $|\{\cdot\}|$ denotes the number of elements in $\{\cdot\}$, is called the (right-continuous) first passage process associated with X, $X_0 = 0$. The random times $(n = 1, 2, \dots)$

$$T_1(\omega^0) := \inf\{x > 0;\ L_x(\omega^0) \neq L_0(\omega^0)\},$$

$$T_{n+1}(\omega^0) := \inf\{x > T_n(\omega^0);\ L_x(\omega^0) \neq L_{T_n+}(\omega^0)\}$$

are called the splitting times of Z.

In this note it is shown that Z is a continuous-time Galton–Watson process (or a continuous-time Markov branching process in the terminology of Athreya and Ney [1]) and its creation rate and offspring distribution are computed. Further, in the critical case, we characterize the weak limiting behaviour of a sequence $\{Z^{(n)}\}$ of processes of the type Z. These arise from a sequence $\{X^{(n)}\}$ of branching Brownian motions, scaled to converge to the so-called super-Brownian motion. In particular, if the offspring distribution is independent of n and belongs to the domain of attraction of the $1 + \beta$-stable law, $0 < \beta \leq 1$, then the limiting process obtained from $\{Z^{(n)}\}$ is a random time change of a spectrally positive $1 + \beta/2$-stable process.

In [10] Neveu considers the first passage problem as introduced above but to the lines $(\lambda t - x, t)$, $t \geq 0$, and for a (supercritical) binary branching Brownian motion (see also [3]). It is seen that Neveu's approach is applicable also in our (sub)critical case. In fact, to make the paper more self-contained, when computing the offspring distribution of Z in the next section, the basic facts in Neveu's approach are also recalled.

2. Characterization of the first passage process. In this section X is a (sub)critical branching Brownian motion with offspring generating function $F(u) = \Sigma p_k u^k$. Let $\mathscr{A}(u) := \alpha(F(u) - u)$ denote the infinitesimal generating function $(0 < u < 1)$. We have the following result.

THEOREM 1. *Let Z be the first passage process associated with X, as previously introduced. Then Z is a continuous-time Galton–Watson process with creation parameter $\gamma := \sqrt{2\alpha}$. Let $\{r_k;\ k = 1, 2, \dots\}$ denote its offspring distribution and let $G(v) = \Sigma r_k v^k$ and $\mathscr{B}(v) := \gamma(G(v) - v)$, $0 < u < 1$, denote the generating function and infinitesimal generating function, respectively. Then*

$$\mathscr{B}(v) = 2\left(\int_v^1 \mathscr{A}(u)\, du\right)^{1/2} \tag{2.1}$$

or, explicitly,

$$G(v) = \sqrt{2}\left(\sum p_k \frac{1}{k+1}(1 - v^{k+1}) - \frac{1}{2}(1 - v^2)\right)^{1/2} + v.$$

Further,

$$\mathbf{E}_0(Z_x) = \exp\left(-\sqrt{1 - F'(1)}\, x\right)$$

and, hence, Z is (sub)critical if and only if X is (sub)critical.

PROOF. We verify first that Z has the branching and Markov properties. This is done using the strong Markov property at the stopping line τ_x. We recall briefly this concept (see [3]): For $\tau := \tau_x$ introduce the stopped σ-field in Ω^0,

$$\mathscr{F}_\tau^{\,0} := \bigvee_{u \in U} \left\{\omega^0;\ u \notin D_\tau(\omega^0)\right\} \cap \mathscr{H}_{\tau^u}^u,$$

where

$$D_\tau(\omega^0) := \left\{u;\ \exists\ v\colon v < u,\ v \in L_\tau(\omega^0)\right\}.$$

Then we have

$$\mathbf{E}_0\left(\prod_{u \in L_x} f^u \circ \theta_{\tau^u}^u \middle| \mathscr{F}_\tau^{\,0}\right) = \prod_{u \in L_x} \mathbf{E}_x(f^u), \tag{2.2}$$

where f^u, $0 \le f^u < 1$, is for every $u \in U$ a $(\Omega^0, \mathscr{F}^0)$-measurable function and $\theta^u_{\tau^u}$ is the shift operator θ^u_s evaluated at $s = \tau^u$. For the definition of θ^u_s: $\Omega^{0,u} \cap \{\zeta^u > s\} \to \Omega^0$, see [3]. Informally, θ^u_s maps a marked tree ω^0 to the marked tree $\tilde{\omega}^0$, which is the subtree of ω^0 having the particle u at the age s as the first element.

Consider now (2.2) with $f^u(\omega^0) := f(\omega^0) := s^{Z_{y+x}(\omega^0)}$, where $0 < s < 1$ and $x, y \ge 0$. Then it is easily seen that

$$\begin{aligned}\mathbf{E}_0\big(s^{Z_{y+x}}|Z_x\big) &= \mathbf{E}_0\Big(\prod_{u \in L_x} s^{Z_{y+x}} \circ \theta^u_{\tau^u}|Z_x\Big)\\ &= \big(\mathbf{E}_x(s^{Z_{y+x}})\big)^{Z_x}\\ &= \big(\mathbf{E}_0\big(s^{Z_{y+x}}|Z_x = 1\big)\big)^{Z_x},\end{aligned}$$

which is the branching property of Z (cf. [3] Corollary 2.3). A similar computation combined with the spatial homogeneity of X gives the Markov property of Z. Consequently, taking into account the fact that the paths of Z are step functions, Z is a continuous-time Galton–Watson process (see [1], page 102). In particular, Z has the properties:

(i) $T_n - T_{n-1}$, $n = 1, 2, \dots$ $(T_0 = 0)$, are independent and, given $Z_{T_{n-1}} = k$, exponentially distributed with parameter γk for some $\gamma > 0$.

(ii) $Z_{T_n} - Z_{T_{n-1}}$, $n = 1, 2, \dots$ are i.i.d.

To compute the parameter γ and the offspring distribution, that is, the distribution of Z_{T_1}, we need (cf. [10]) the following lemma. Unfortunately, we do not have an exact reference for this, but see Williams ([12], Theorems 4.7 and 4.9). In any case, the lemma can be proved using the reflection principle.

LEMMA 1. *Let $B = \{B_t: t \ge 0\}$, $B_0 = 0$, be a standard Brownian motion and $\tau \sim \exp(\alpha)$ independent of B. Then the random variables*

$$M := \sup_{t \le \tau} B_t \quad \text{and} \quad R := M - B_\tau$$

are independent and exponentially distributed with the parameter $\sqrt{2\alpha}$. Moreover, the time point for the occurrence of M is a.s. unique.

Let $M^0 := \sup\{\xi^0_s;\ 0 \le s < \zeta^0\}$. Then, by Lemma 1, $M^0 \sim \exp(\sqrt{2\alpha})$ and, from the definition of T_1, $T_1 = M^0$. Hence, $\gamma = \sqrt{2\alpha}$.

To proceed with the offspring distribution, it follows from the spatial homogeneity and the branching property of X that, in law,

$$Z_{T_1} = \begin{cases} Z^{(1)}_{R^0} + \cdots + Z^{(\nu^0)}_{R^0}, & \nu^0 \ge 1,\\ 0, & \nu^0 = 0,\end{cases} \tag{2.3}$$

where $Z^{(i)}$ are independent copies of Z evaluated at the independent exponential time $R^0 := M^0 - \xi^0_{\zeta^0-}$. Further, recall (see [1] page 106 or Neveu [10]):

LEMMA 2. *Let $N = \{N_t;\ t \geq 0\}$, $N_0 = 1$, be a continuous-time Galton–Watson process with the creation parameter β and offspring distribution $\{q_k;\ k = 0, 1, \dots\}$. For $0 < \psi_0 < 1$, let $\psi(t) := \mathbf{E}(\psi_0^{N_t})$, where $\mathbf{E}$ denotes the expectation operator associated with N. Then ψ is the solution of the initial value problem*

$$\psi' = \mathscr{G} \circ \psi,$$
$$\psi(0) = \psi_0,$$

where $\mathscr{G}$ is the infinitesimal generating function of N, that is,

$$\mathscr{G}(u) = \beta\Big(\sum q_k u^k - u\Big).$$

Moreover, in the (sub)critical case, ψ is increasing with $\lim_{t\to\infty}\psi(t) = 1$ and, hence,

$$\mathscr{G}(u) = \psi'(\psi^{-1}(u)).$$

By (2.3), for $0 < v < 1$,

$$\begin{aligned} G(v) = \mathbf{E}_0(v^{Z_{T_1}}) &= \mathbf{E}_0\big(v^{Z^{(1)}_{R^0} + \cdots + Z^{(\nu^0)}_{R^0}}\big) \\ &= \int_0^\infty \gamma e^{-\gamma s} \sum p_k \big(\mathbf{E}_0(v^{Z_s})\big)^k\, ds. \end{aligned}$$

Let $\phi(t) := \mathbf{E}_0(\phi_0^{Z_t})$, where ϕ_0, $0 < \phi_0 < 1$, is given, and $\bar{\phi}(t, v) := \mathbf{E}_0(v^{Z_t})$, when ϕ is considered as a function of two variables. Making use of the semigroup property $\phi(t+s) = \bar{\phi}(t, \phi(s))$ and Lemma 2 it is seen that

$$\begin{aligned} \phi'(t) = \mathscr{B}(\phi(t)) &= \gamma\left(\int_0^\infty \gamma e^{-\gamma s} F(\bar{\phi}(s, \phi(t)))\, ds - \phi(t)\right) \\ &= \gamma\left(\int_0^\infty \gamma e^{-\gamma s} F(\phi(s+t))\, ds - \phi(t)\right). \end{aligned}$$

Differentiating with respect to t, using $(d/dt)F(\phi(s+t)) = (d/ds)F(\phi(s+t))$ and integrating by parts,

$$\begin{aligned} \phi''(t) &= \gamma\left(\int_0^\infty \gamma e^{-\gamma s} \frac{d}{ds} F(\phi(t+s))\, ds - \phi'(t)\right) \\ &= -\gamma^2(F(\phi(t)) - \phi(t)) = -2\mathscr{A}(\phi(t)). \end{aligned}$$

Further, since $\phi'(+\infty) = 0$,

$$\phi'(t)^2 = -2\int_t^{+\infty} \phi'(s)\phi''(s)\, ds = 4\int_t^\infty \phi'(s)\mathscr{A}(\phi(s))\, ds.$$

By Lemma 2, $t \mapsto \phi(t)$ is increasing and therefore, because $\phi(+\infty) = 1$,

$$\phi'(t) = \gamma\left(2\int_{\phi(t)}^1 (F(u) - u)\, du\right)^{1/2},$$

which gives the basic relationship (2.1). Now observe that

$$G'(1) = 1 - \sqrt{1 - F'(1)}\,.$$

Note also that $G''(1) = +\infty$. From this the remaining statements are easily obtained and the proof of the theorem is complete. □

REMARK 2. More generally, let $Z_x(0, t)$, $x > 0$, $t > 0$, be the number of first hits in a family tree to the level x during the time interval $(0, t)$. To convince the reader that the process Z is the "natural" first passage process we point out that in the critical case one can prove

$$\mathbf{E}_0(Z_x(0,t)) = \mathbf{P}_0^B(\tau_x < t),$$

where $\mathbf{P}^B$ is the measure associated with a standard one-dimensional Brownian motion and τ_x is the first hitting time of the point x. This should be compared with the relation

$$\mathbf{E}_0(N_t(x,y)) = \mathbf{P}_0^B(B_t \in (x,y))$$

for $N_t(x, y)$ the number of particles in the interval (x, y) at time t. Furthermore, there is an analogue of Theorem 1 for supercritical branching. We intend to study these topics in a forthcoming paper.

EXAMPLE 1. Consider the family of critical offspring distributions for X given by

$$\mathscr{A}(u) = \frac{\alpha}{1+\beta}(1-u)^{1+\beta}, \qquad 0 < \beta \le 1.$$

For this particular family the offspring distributions can be given explicitly. With $\beta = 1$ this is the binary branching model $p_0 = p_2 = 1/2$. For $0 < \beta < 1$,

$$p_k = \frac{1}{1+\beta}\binom{1+\beta}{k}(-1)^k, \qquad k \ne 1,$$
$$p_1 = 0.$$

Note that the variance is finite only for $\beta = 1$. The offspring distribution of Z is now of the same type with the infinitesimal generating function

$$\mathscr{B}(v) = \frac{\gamma}{\sqrt{(1+\beta)(1+\beta/2)}}(1-v)^{1+\beta/2}$$

and offspring probabilities

$$r_k = \frac{1}{\sqrt{(1+\beta)(1+\beta/2)}}\binom{1+\beta/2}{k}(-1)^k, \qquad k \ne 1,$$
$$r_1 = 1 - \sqrt{\left(\frac{1+\beta/2}{1+\beta}\right)}\,.$$

Here, r_0 is the probability that the whole tree lies on the left-hand side of the line determined by the rightmost maximum of the initial particle until the first branching. Furthermore, for $0 < \phi_0 < 1$,

$$\phi(x) := \mathbf{E}_0(\phi_0^{Z_t}) = 1 - \frac{1-\phi_0}{(1+cx)^{2/\beta}}, \qquad c := \frac{\gamma\beta(1-\phi_0)^{\beta/2}}{\sqrt{(1+\beta)(1+\beta/2)}}.$$

From this formula, setting $\phi_0 = 0$ we obtain the probability $\mathbf{P}(Z_x = 0)$ that the whole tree lies on the left-hand side of the line (x, t), $t \geq 0$ (cf. Walsh [11], Proposition 8.14, which has a misprint).

3. Diffusion approximation, critical case. In this section we restrict attention to the critical case and consider a sequence of branching Brownian motions with the interpretation that each particle is assigned a decreasingly small mass and life length, whereas this is balanced by an increasing density of particles. More exactly, we consider a scaling which in a weak limit leads to the distribution of "mass" on the real line known as super-Brownian motion. We characterize the continuous state branching processes (CB processes) obtained in this limit for the associated sequence of first passage processes.

For each $n \geq 1$, let $X^{(n)}$ be a critical branching Brownian motion with creation rate α_n, generating function F_n and infinitesimal generating function $\mathscr{A}_n(u) = \alpha_n(F_n(u) - u)$. We assume that the processes $X^{(n)}$ all start with one particle at the origin. Let $\{X^{(i,n)}; 1 \leq i \leq n\}$ be independent copies of $X^{(n)}$ and let $Z^{(n)}$ and $Z^{(i,n)}$ be the corresponding first passage processes. According to Theorem 1 the characteristics of $Z^{(n)}$ are given by $\sqrt{2\alpha_n}$, G_n and $\mathscr{B}_n$, say. Introduce the processes

$$Y^{(n)} = \left\{ Y_t^{(n)} = \frac{1}{n}\sum_{i=1}^{n} Z_t^{(i,n)}; t \geq 0 \right\}, \qquad n = 1, 2, \dots .$$

It is well known that a possible limiting process of $Y^{(n)}$ as $n \to \infty$ can be expressed in terms of a random time change of a spectrally positive Lévy process. This is due to Lamperti [8] for discrete time processes and it was pointed out by Helland that the result also applies to the continuous-time branching processes. Indeed, to see this relation let

$$H_n(v) := G_n(e^{-v/n})e^{v/n},$$

and denote by $A^{(n)}$, $A_0^{(n)} = 1$, a compound Poisson process with

$$\log \mathbf{E}\left[\exp\left(-v\left(A_{t+s}^{(n)} - A_s^{(n)}\right)\right)\right] = \sqrt{2\alpha_n}\, n\left(H_n(v) - 1\right)t. \tag{3.1}$$

Then, $Y^{(n)}$ converges in finite-dimensional distributions to a CB process Y, $Y_0 = 1$ if and only if $A^{(n)}$ converges weakly to a spectrally positive Lévy process A, $A_0 = 1$.

Further, if the process Y does not explode (for a definition of this see the proof below), then $Y^{(n)}$ converges also weakly to Y. See Helland ([6], Theorem 6.1). (The quoted result refers to a slightly different but equivalent setting emphasizing the scaling $t \mapsto nt$; cf. Remark 4.)

In order to study the sequence $\{Y^{(n)}\}$ via the relation (3.1) we recall some results for the sequence $\{X^{(n)}\}$. For $u \geq 0$, let

$$\mathscr{A}_*(u) := cu^2 + \int_0^\infty (e^{-su} - 1 + su)\nu(ds), \tag{3.2}$$

where $c \geq 0$ and ν is a measure on $\mathbf{R}_+$ such that

$$\int_0^\infty (s \wedge s^2)\nu(ds) < \infty.$$

For each $m > 0$ and $n \geq m$ put

$$\varepsilon_n(m) := \sup_{u \leq m} \left| n\mathscr{A}_n\left(1 - \frac{u}{n}\right) - \mathscr{A}_*(u) \right|.$$

We say that the branching mechanism of $X^{(n)}$ is in the domain of attraction of a branching exponent $\mathscr{A}_*(u)$ if

$$\lim_{n \to \infty} \varepsilon_n(m) = 0, \quad \text{for all } m > 0. \tag{3.3}$$

Under this assumption it is known that

$$\frac{1}{n} \sum_{i=1}^n X^{(i,n)} \Rightarrow \text{super-Brownian motion with branching exponent } \mathscr{A}_*$$

[weak convergence, e.g., in $\mathscr{D}([0, +\infty), \mathscr{M})$, where $\mathscr{M}$ is the set of finite measures on $\mathbf{R}$]. See Ethier and Kurtz ([5], Section 9.4). In particular, if $N^{(n)}$ ($N^{(i,n)}$) denotes the number of particles in $X^{(n)}$ ($X^{(i,n)}$), then $N^{(n)}$, $n \geq 1$, is a sequence of continuous-time Galton–Watson processes such that the total mass process has a weak limit

$$\frac{1}{n} \sum_{i=1}^n N^{(i,n)} \Rightarrow N, \qquad t \geq 0,$$

with $h(t) := -\log \mathbf{E}[e^{-\theta N_t}]$ the unique positive solution of

$$h'(t) = -\mathscr{A}_*(h(t)), \qquad h(0) = \theta.$$

We now present an analogous result for the first passage process Z.

THEOREM 2. *Assume that the branching mechanism of $X^{(n)}$ is in the domain of attraction of a branching exponent $\mathscr{A}_*$ as in* (3.3). *Then the sequence of processes $Y^{(n)}$, $n = 1, 2, \dots$, converges weakly to a CB process Y, such that its cumulant generating function $h(t) := -\log \mathbf{E}[e^{-\theta Y_t}]$ is the unique positive solution of the equation*

$$h'(t) = -\mathscr{B}_*(h(t)), \qquad h(0) = \theta, \tag{3.4}$$

where

$$\mathscr{B}_*(v) = 2\left(\int_0^v \mathscr{A}_*(u)\, du\right)^{1/2}.$$

Equivalently, Y is a random time change of a spectrally positive Lévy process A, $A_0 = 1$, with the Laplace transform given by

$$\log \mathbf{E}\big[\exp(-v(A_{t+s} - A_s))\big] = \mathscr{B}_*(v)t, \qquad v \geq 0. \tag{3.5}$$

PROOF. We first establish that $Y^{(n)}$ converges in distribution to a process Y which is defined as a random time change of a spectrally positive Lévy process A characterized via (3.5). More specifically, let τ_t be the right continuous inverse of the additive functional

$$\alpha_t := \int_0^{t \wedge T} \frac{ds}{A_s}, \qquad T := \inf\{t\colon A_t = 0\},$$

and set $Y_t := A_{\tau_t}$ in the case $\tau_t < T$. Because A is spectrally positive, $A_t > 0$ for $t < T$ and $A_{T-} = 0$. Further, note that if $T = \infty$ and $\alpha_\infty < \infty$, then τ_t is not defined for $t > \alpha_\infty$. In this case we say that Y has exploded. If $T < \infty$ and $\alpha_T < \infty$, we set $Y_t = 0$ for $t \geq \alpha_T$, and say that Y has become extinct.

Now by Theorem 6.1 in [6], it is enough to show that the sequence $\{A^{(n)}\}$, where $A^{(n)}$ is as in (3.1), converges in finite-dimensional distributions to the process A. Since all processes have stationary independent increments, this is equivalent to the convergence in one-dimensional distribution, that is,

$$\log \mathbf{E}\big[\exp\big(-v\big(A_{t+s}^{(n)} - A_s^{(n)}\big)\big)\big] = \sqrt{2\alpha_n}\, n(H_n(v) - 1)t \to \mathscr{B}_*(v)t, \qquad n \to \infty.$$

However,

$$\sqrt{2\alpha_n}\, n(H_n(v) - 1) = n e^{v/n} \mathscr{B}_n(e^{-v/n}),$$

so the desired relation follows from

$$\lim_{n \to \infty} \left| n\mathscr{B}_n\left(1 - \frac{v}{n}\right) - \mathscr{B}_*(v) \right| = 0.$$

But for each $n \geq 1$,

$$\mathscr{B}_n(v) = 2\left(\int_v^1 \mathscr{A}_n(u)\, du\right)^{1/2}$$

according to Theorem 1. Hence, for fixed v and $n > v$,

$$\begin{aligned}
\left| n\mathscr{B}_n\left(1 - \frac{v}{n}\right) - \mathscr{B}_*(v) \right| &= 2\left| n\left(\int_{1-v/n}^1 \mathscr{A}_n(u)\, du\right)^{1/2} - \left(\int_0^v \mathscr{A}_*(u)\, du\right)^{1/2} \right| \\
&= 2\left| \left(\int_0^v n\mathscr{A}_n\left(1 - \frac{u}{n}\right) du\right)^{1/2} - \left(\int_0^v \mathscr{A}_*(u)\, du\right)^{1/2} \right| \\
&\leq 2\left(\int_0^v \left| n\mathscr{A}_n\left(1 - \frac{u}{n}\right) - \mathscr{A}_*(u) \right| du\right)^{1/2} \\
&\leq 2(v\varepsilon_n(v))^{1/2} \to 0, \qquad n \to \infty,
\end{aligned}$$

where the assumption (3.2) is used at the last step.

We now verify that Y does not explode. For this it is enough to show that $T < \infty$ a.s., that is, the process A hits zero a.s. A result due to Zolotarev says that

$$\mathbf{E}[\exp(-\theta T)] = \exp(-\mathscr{B}_*^{-1}(\theta)), \qquad \theta \geq 0,$$

where $\mathscr{B}_*^{-1}$ is the inverse of the continuous and increasing function $\mathscr{B}_*$ (for a nice proof, see Bingham [2]). Hence

$$\mathbf{P}(T < \infty) = \exp(-\mathscr{B}_*^{-1}(0)) = 1.$$

Finally, because Y is a random time change of a Lévy process, it is strong Markov and has cadlag sample paths. Consequently, its cumulant generating function is the unique solution of (3.4); see Kawazu and Watanabe [7]. The proof is complete. □

We next consider some examples of offspring distributions.

EXAMPLE 2. Suppose that the variance σ^2 of the offspring distribution is finite. Take $\alpha_n = \alpha n$ and $F_n = F$. Then

$$n\mathscr{A}_n\left(1 - \frac{u}{n}\right) = n^2\alpha\left(F\left(1 - \frac{u}{n}\right) - \left(1 - \frac{u}{n}\right)\right) \approx \frac{\alpha\sigma^2 u^2}{2} + o\left(\frac{1}{n}\right);$$

hence $\nu(ds) \equiv 0$ and $\mathscr{A}_*(u) = \alpha\sigma^2 u^2/2$. The basic case is binary branching with $F(u) - u = (1-u)^2/2$. We obtain $\mathscr{B}_*(v) = \sqrt{(2\alpha/3)}\,\sigma v^{3/2}$. Thus Y is a CB process of index $3/2$. The corresponding Lévy process is spectrally positive $3/2$-stable.

EXAMPLE 3. A standard example to illustrate the effect of infinite variance is the case

$$c = 0, \qquad \nu(ds) = \beta\Gamma(1-\beta)^{-1}s^{-(2+\beta)}\,ds, \qquad 0 < \beta < 1,$$

in (3.2). Then

$$\mathscr{A}_*(u) = \frac{\alpha}{1+\beta}u^{1+\beta}, \qquad 0 < \beta < 1.$$

If a sequence (α_n, F_n) is in the domain of attraction of this stable $1 + \beta$ type exponent in the sense of (3.3), then for the first passage process

$$-\log \mathbf{E}[e^{-\theta Y_t}] = \frac{\theta}{\left(1 + \gamma d_\beta(\beta/2)\theta^{\beta/2}t\right)^{2/\beta}}, \qquad \theta > 0, \gamma = \sqrt{2\alpha},$$

and

$$\log \mathbf{E}[\exp(-v(A_{t+s} - A_s))] = \gamma d_\beta v^{1+\beta/2}t, \qquad d_\beta^{-1} = \sqrt{(1+\beta)(1+\beta/2)}.$$

Here, moments of order less than $1 + \beta$ are finite. Note that Example 1 with $\alpha_n = \alpha n^\beta$ and $\mathscr{A}_n(u) = \alpha_n(F_n(u) - u)$ provides a case which (trivially) yields this limit.

REMARK 3. Suppose that the branching Brownian motions are such that the limiting branching law has the exponent

$$\mathscr{A}_*(u) = \frac{\alpha}{1+\beta} u^{1+\beta}, \qquad 0 < \beta \le 1. \tag{3.6}$$

The case $\beta = 1$ is Example 2 with $\sigma = 1$, and $0 < \beta < 1$ corresponds to Example 3. Let $a_n \sim [n^{\beta/2}]$. Then for fixed α,

$$\frac{1}{a_n} \sum_{i=1}^{a_n} Z_{nt}^{(i)} \Rightarrow Y_t.$$

Still another equivalent scaling for α independent of n is

$$\frac{1}{n} \sum_{i=1}^{n} Z_{n^{\beta/2}t}^{(i)} \Rightarrow Y_t.$$

EXAMPLE 4. To obtain the general form of $\mathscr{A}_*$ in (3.2) define, for a given branching exponent $\mathscr{A}_*$, $\alpha_n = \mathscr{A}'_*(n)$ and $F_n(u) = u + \mathscr{A}_*(n(1-u))/n\alpha_n$. Then $\mathscr{A}_n(u) = \alpha_n(F_n(u) - u)$ defines an approximating branching mechanism such that (3.2) holds. See Dawson and Perkins ([4], Lemma 3.4c).

EXAMPLE 5. Consider again the situation in (3.6). The extinction probability for the total mass process N is given by

$$-\log \mathbf{P}(N_t = 0) = \lim_{\theta \to \infty} \frac{\theta}{\left(1 + \alpha\beta(1+\beta)^{-1}\theta^\beta t\right)^{1/\beta}} = ((1+\beta)/\alpha\beta t)^{1/\beta}.$$

Similarly,

$$-\log \mathbf{P}(Y_t = 0) = (2/\gamma\beta d_\beta t)^{2/\beta}, \qquad \gamma = \sqrt{2\alpha}$$

where d_β, $0 < \beta < 1$, was given in Example 3 and $d_1 = 1/\sqrt{3}$. For example, if $\beta = 1$, the asymptotic relation

$$t\mathbf{P}(N_t > 0) \sim 2/\alpha, \qquad t \to \infty,$$

has the counterpart

$$t^2 \mathbf{P}(Y_t > 0) \sim 12/\gamma^2, \qquad t \to \infty,$$

which gives a quadratic rate of extinction for the number of particles reaching far out to the right.

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