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INEQUALITIES FOR THE OVERSHOOT[1]

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Let $X_1, X_2, \dots$ be independent and identically distributed positive random variables with $S_n = X_1 + \cdots + X_n$, and for nonnegative b define $R_b = \inf\{S_n - b \colon S_n > b\}$. Then R_b is called the *overshoot* at b. In terms of the moments of X_1, Lorden gave bounds for the moments of R_b that hold uniformly over all b. Using a coupling argument, we establish stochastic ordering inequalities that imply the moment inequalities of Lorden. In addition to simple new proofs of Lorden's inequalities, we provide new inequalities for the tail probabilities $P\{R_b > x\}$ and moments of R_b that improve upon those of Lorden. We also present conjectures for sharp moment inequalities and describe an application to the first ladder height of random walks.

1. Introduction. Let $X, X_1, X_2, \dots$ be independent and identically distributed positive random variables with finite mean μ. Let S_n denote the sum $X_1 + \cdots + X_n$, with S_0 defined to be 0. For $b \ge 0$, denote the stopping time $\inf\{n \colon S_n > b\}$ by τ_b or $\tau(b)$ and define $R_b := S_{\tau(b)} - b$ and $L_b := X_{\tau(b)}$. Then R_b is variously known as the *overshoot* or *excess* or *residual lifetime* at b, and L_b is called the *total lifetime* at b. Our main interest here is in investigating the behavior of the overshoot; the total lifetime will be an important ingredient in the analysis. The overshoot is among the fundamental objects of study in random walk and renewal theory and therefore plays an important role in a variety of fields of applied probability. Often the object is to show in some sense that the overshoot cannot be too large. In this context, the inequalities of Lorden (1970) are very useful; these are stated below in (1.5), (1.6) and (4.1). In this paper we provide simple new proofs, as well as sharper inequalities for those that are not sharp already.

Let R_∞ and L_∞ denote positive random variables having distributions

$$P\{R_\infty \in dx\} = \mu^{-1} P\{X > x\}\, dx \tag{1.1}$$

and

$$P\{L_\infty \in dx\} = \mu^{-1} x P\{X \in dx\}. \tag{1.2}$$

For $p > 0$, we have

$$E(R_\infty^p) = \mu^{-1}(p+1)^{-1} E(X^{p+1}) \tag{1.3}$$

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and

$$E(L_\infty^p) = \mu^{-1}E(X^{p+1}); \tag{1.4}$$

these moments are finite when $E(X^{p+1})$ is finite.

When X has a nonarithmetic distribution, a standard result says that R_b and L_b converge in distribution to R_∞ and L_∞, respectively, as $b \to \infty$. The well-known "length-biased sampling" or "inspection paradox" phenomenon arises from the fact that L_∞ (and indeed R_∞) may be stochastically much larger than X.

Such limiting considerations leave open the issue that is often the "real" question: At some given finite b, how large (measured in some stochastic sense such as moments or tail probabilities) could the overshoot R_b be? The problem of deriving bounds on the behavior of R_b that hold uniformly over all b was addressed by a beautiful paper of Lorden (1970), who established the following inequalities for the moments of R_b in terms of the moments of R_∞: For all nonnegative b and p,

$$E(R_b^p) \le (p+2)E(R_\infty^p). \tag{1.5}$$

Lorden also found a special result for the first moment: For all nonnegative b,

$$ER_b \le 2ER_\infty. \tag{1.6}$$

Stronger assertions can be made under special assumptions on the distribution of X. For example, Brown (1980) showed that if X has a decreasing failure rate distribution, then R_b is stochastically increasing in b. However, at their level of generality, Lorden's results have remained the best available.

Combining ideas of Lorden (1970) with a clever use of properties of the stationary renewal process, Carlsson and Nerman (1986) gave a simpler proof of the first moment inequality (1.6). They did not address the general moment inequalities (1.5).

In Section 3 we give a new proof of the general moment inequalities (1.5). The proof in fact shows that for all b, the overshoot R_b is with probability 1 bounded above by the maximum of two random variables, one of which is distributed as R_∞ and the other of which is stochastically bounded by L_∞. Lorden's inequalities (1.5) are then an immediate consequence. We also provide a new proof of (1.6) that shows that R_b is with probability 1 bounded above by the sum of two random variables, both of which are distributed as R_∞.

The approach taken in Section 3 gives additional insight into Lorden's inequalities. We establish almost sure or stochastic inequalities that are stronger than the moment inequalities, which then become immediate consequences. A coupling argument replaces the remarkably clever but mysterious tricks of integrating the function $t \mapsto ER_t$, several applications of Jensen's inequality, and solution of a polynomial inequality that appear in Lorden's original treatment. This is made possible by defining and using the residual R_b for all b rather than just for nonnegative b.

Inequality (1.6) is "sharp," in the sense that there is a case of equality; necessary and sufficient conditions for equality to hold are provided in

Section 3. It turns out that Lorden's general moment inequalities (1.5) are not sharp; one might anticipate this from a comparison of (1.5) and (1.6), which do not "match up" at $p = 1$. In Section 4 we obtain new general moment inequalities that are sharper than those in (1.5). Section 4 begins by developing bounds for the tail probability $P\{R_b > x\}$ that improve upon a bound established by Lorden. The sharpened general moment inequalities then follow as a crude consequence of the new tail probability bounds.

Section 5 introduces a conjecture that replacing the "$(p + 2)$" in the right side of (1.5) by "$(p + 1)$" gives sharp inequalities. The neat form of these conjectured inequalities leads to the interesting prospect that, if they indeed hold, a very nice probabilistic explanation might be waiting to be discovered. A counterexample shows that a natural first conjectured "explanation" is false.

Section 6 establishes a monotonicity property of the expected first ladder height of random walks and points out a connection with the conjecture of Section 5.

We denote stochastic ordering by $\leq_{\text{st}}$ and equality in distribution by $=_{\text{st}}$. That is, $W \leq_{\text{st}} Z$ means that $P\{W > x\} \leq P\{Z > x\}$ for all x, and $W =_{\text{st}} Z$ means that W and Z have the same distribution.

2. Almost sure inequalities. Here we present extensions of two inequalities that formed the basis of Lorden's paper. We begin by extending the definitions of $\tau(c)$ and R_c, given in the Introduction for nonnegative c, to all real c as follows: Take $\tau(c) = 0$ and $R_c = -c$ for $c < 0$. Note that these definitions are what one obtains by using the expressions for $\tau(c)$ and R_c given in the Introduction.

Next, for each real number a define the process $\{S_{a,n}: n \geq 0\}$ by $S_{a,n} := S_{\tau(a)+n} - S_{\tau(a)}$. Then for real c and nonnegative b, we denote by $\tau_a(c)$, $R_{a,c}$ and $L_{a,b}$ the quantities that stand in the same relation to the process $\{S_{a,n}\}$ as the quantities $\tau(c)$, R_c and L_b have with the process $\{S_n\}$. That is, define

$$\tau_a(c) = \inf\{n: S_{a,n} > c\},$$

$$R_{a,c} = S_{a,\tau_a(c)} - c$$

and

$$L_{a,b} = S_{a,\tau_a(b)} - S_{a,\tau_a(b)-1} = X_{\tau(a)+\tau_a(b)}.$$

Clearly, $R_{a,c} =_{\text{st}} R_c$ and $L_{a,b} =_{\text{st}} L_b$.

PROPOSITION 2.1. *With probability* 1, *the inequalities*

$$R_{a+c} \leq R_a + R_{a,c} \tag{2.1}$$

and

$$R_{a+b} \leq R_a \vee L_{a,b} \tag{2.2}$$

hold simultaneously for all $a \in \mathbb{R}$, $c \in \mathbb{R}$ *and* $b \geq 0$.

Lorden (1970) introduced Proposition 2.1 in the case of nonnegative a and c. We claim that (2.1) and (2.2) hold on the event $\{\sup S_n = \infty\}$. The verifications are a simple matter of checking cases. To see (2.2), for example, in the case of nonnegative a (and b) a picture makes it clear that

$$R_{a+b} > R_a \quad \text{implies} \quad R_{a+b} \le L_{a,b}, \tag{2.3}$$

so that (2.2) holds. For $a < 0$ and $a + b \ge 0$, (2.3) can be seen from a similar picture. Finally, for $a < 0$ and $a + b < 0$, (2.2) is trivial, since $R_{a+b} \le R_a$.

3. Lorden's moment inequalities. Throughout this section, let Y denote a random variable distributed as R_∞, with Y independent of $X_1, X_2, \ldots$.

PROPOSITION 3.1. *For all $b \ge 0$ we have $R_{b-Y} =_{\text{st}} R_\infty$.*

PROOF. Let R'_b denote the overshoot at time b of the delayed renewal process generated by $Y, X_1, X_2, \ldots$. That is, $R'_b = S'_{\tau'(b)} - b$, where $S'_n = Y + S_n$ for $n \ge 0$ and $\tau'(b) = \inf\{n: S'_n > b\}$ for $b \ge 0$. Then $R_{b-Y} = R'_b$. However, it is a standard fact [see, e.g., Asmussen (1987), pages 116 and 117] that the process $\{R'_b: b \ge 0\}$ is stationary, with $R'_b =_{\text{st}} R_\infty$ for all $b \ge 0$. □

Now we can imitate the clever trick Carlsson and Nerman (1986) used to prove (1.6). We also provide necessary and sufficient conditions for equality to hold.

PROPOSITION 3.2. *For all $b \ge 0$ we have $ER_b \le 2ER_\infty$. Equality holds if and only if $P\{X = \mu\} = 1$ and $b = k\mu$ for some nonnegative integer k.*

PROOF. Let Y_1 and Y_2 be independent of each other, independent of $X_1, X_2, \ldots$, and distributed as R_∞. Proposition 2.1 gives

$$R_b \le R_{b+Y_1-Y_2} + R_{b+Y_1-Y_2, Y_2-Y_1} \tag{3.1}$$

with probability 1. For $b \ge 0$, conditioning on Y_1 and applying Proposition 3.1 show that $R_{b+Y_1-Y_2} =_{\text{st}} R_\infty$. Next, the independence assumptions on Y_1 and Y_2, together with the fact that $R_{a,c} =_{\text{st}} R_c$ for all fixed a and c, imply that $R_{b+Y_1-Y_2, Y_2-Y_1} =_{\text{st}} R_{Y_2-Y_1}$. However, conditioning on Y_2 and applying Proposition 3.1, we obtain $R_{Y_2-Y_1} =_{\text{st}} R_\infty$. Thus, both $R_{b+Y_1-Y_2}$ and $R_{b+Y_1-Y_2, Y_2-Y_1}$ are distributed as R_∞, so that taking expected values in (3.1) proves Lorden's first-moment inequality (1.6).

For the case of equality, it is clear that the stated conditions are sufficient, since they give $ER_b = \mu = EL_\infty = 2ER_\infty$. To establish necessity, suppose that $ER_b = EL_\infty$ and observe that equality must then hold in (3.1) with probability 1. Conditioning on the random variable $Z = Y_2 - Y_1$, which has positive density (with respect to Lebesgue measure) at 0, we see that in particular

$$R_b = R_{b-z} + R_{b-z,z} \quad \text{with probability 1} \tag{3.2}$$

must hold for all sufficiently small positive z, except possibly for a set of Lebesgue measure 0. However, for positive z the equality in (3.2) implies that $R_{b-z} \le z$; indeed, if $R_{b-z} > z$, then $R_b = R_{b-z} - z < R_{b-z}$. Thus, letting

$z_1 > z_2 > \cdots$ with $z_n \downarrow 0$ be a sequence of values of z that satisfy (3.2), we have $R_{b-z_n} \le z_n$ with probability 1 for all n. That is, with probability 1, for all n there is a renewal in the interval $(b - z_n, b]$, so that there is a renewal at b with probability 1. This clearly implies the claimed necessary conditions if $b > 0$. On the other hand, if $b = 0$, then

$$EX = ER_0 = EL_\infty = \frac{E(X^2)}{EX},$$

so that $\operatorname{Var}(X) = 0$, and X is deterministic. □

To continue with the proof of the general moment inequalities (1.5), note that

$$R_b \le R_{b-Y} \vee L_{b-Y,Y} \tag{3.3}$$

with probability 1, by (2.2). We know that $R_{b-Y} =_{\text{st}} R_\infty$. Proposition 3.4 below will show that $L_{b-Y,Y} \le_{\text{st}} L_\infty$.

LEMMA 3.3.

$$P\{L_\infty > x\} = \int_0^\infty P\{X > x \mid X > y\} P\{R_\infty \in dy\}.$$

PROOF. We have

$$\begin{aligned} P\{L_\infty > x\} &= \int_{z \in (x,\infty)} \mu^{-1} z P\{X \in dz\} \\ &= \mu^{-1} \int_{z \in (x,\infty)} \int_{y \in (0,z)} dy\, P\{X \in dz\} \\ &= \mu^{-1} \int_{y \in (0,\infty)} \int_{z \in (x \vee y,\infty)} P\{X \in dz\}\, dy \\ &= \mu^{-1} \int_{y \in (0,\infty)} P\{X > x, X > y\}\, dy \\ &= \mu^{-1} \int_{y \in (0,\infty)} P\{X > x \mid X > y\} P\{X > y\}\, dy, \end{aligned}$$

which is what we wanted to show, by (1.1). □

The previous lemma is very simple and presumably known, but it also has displayed a tendency to invite suspicion or misunderstanding at first sight. A simulation interpretation, which seems useful in clarifying both the statement of the lemma and the proof of the next proposition, states that one can generate a random variable distributed as L_∞ as follows. First generate a random variable Y distributed as R_∞. Then generate i.i.d. copies $X_1, X_2, \ldots$ of X until an X_k that exceeds Y is obtained. Such an X_k is distributed as L_∞. An apparently tempting but incorrect interpretation of the lemma is to equate $P\{L_\infty > x\}$ with $P\{X > x \mid X > R_\infty\}$, where X and R_∞ are independent. This would correspond to the following incorrect procedure for generating a random variable distributed as L_∞: Independently generate two i.i.d. sequences $X_1, X_2, \dots$ and $Y_1, Y_2, \dots$, where Y_1 is distributed as R_∞, and take the first X_k satisfying $X_k > Y_k$. To see that this is different from the correct simulation procedure mentioned above, let κ denote $\inf\{k: X_k > Y_k\}$ and observe that the distribution of Y_κ is not the same as that of R_∞; evidently Y_κ tends to be smaller.

PROPOSITION 3.4. *$L_{b-Y,Y} =_{\text{st}} L_Y$ for all $b \ge 0$, and $L_Y \le_{\text{st}} L_\infty$.*

PROOF. The first assertion follows from a simple conditioning argument using the independence of Y and the fact that $L_{b-y,y} =_{\text{st}} L_y$ for all nonnegative constants b and y. To prove the second assertion, define $L = X_\rho$, where $\rho = \inf\{r: X_r > Y\}$. It is easy to see that $P\{\rho < \infty\} = 1$. Now $L =_{\text{st}} L_\infty$ by Lemma 3.3 and clearly $L \ge L_Y$ on $\{\rho < \infty\}$. Indeed, if $L_Y > Y$, then L_Y must be the first X_r that exceeds Y, so that $L_Y = L$; otherwise, $L_Y \le Y < L$. Thus, we have displayed a random variable L distributed as L_∞ that satisfies $L \ge L_Y$ with probability 1, so we are done. □

Lorden's inequalities (1.5) now follow immediately.

COROLLARY 3.5. *For all nonnegative b and p we have*

$$E(R_b^p) \le (p+2)E(R_\infty^p).$$

PROOF. Using (3.3), Proposition 3.1, Proposition 3.4, (1.3) and (1.4), we obtain

$$E(R_b^p) \le E(R_{b-Y}^p) + E(L_{b-Y,Y}^p) \le E(R_\infty^p) + E(L_\infty^p) = (p+2)E(R_\infty^p). \quad \square$$

4. Tail probabilities and moment inequalities. Lorden's (1970) Theorem 4 develops a result for tail probabilities that may be stated as follows:

$$P\{R_b > x\} \le \left(\frac{b + EL_\infty}{b+x}\right)[P\{L_\infty > x\} + P\{R_\infty > x\}]. \tag{4.1}$$

The next two results provide improved tail inequalities. The first of these consists of a calculation that does not use results from the previous sections.

PROPOSITION 4.1. *For all nonnegative b and all $x \ge ER_b$,*

$$P\{R_b > x\} \le \left(\frac{b + ER_b}{b+x}\right)P\{L_\infty > x\}. \tag{4.2}$$

PROOF. Let F denote the distribution of X, so that $P\{X \in dz\} = F(dz)$ and $P\{L_\infty \in dz\} = \mu^{-1}zF(dz)$, and let U denote the renewal function, defined by $U(x) = \sum_{n=0}^{\infty} P\{S_n \le x\}$. Then

$$\begin{aligned}
P\{R_b > x\} &= \int_{[0,b]} P\{X > b + x - y\} U(dy) \\
&= \int_{[0,b]} \int_{(b+x-y,\infty)} F(dz) U(dy) \\
&= \int_{(x,b+x]} \int_{(b+x-z,b]} U(dy) F(dz) + \int_{(b+x,\infty)} \int_{[0,b]} U(dy) F(dz) \\
&= \int_{(x,b+x]} [U(b) - U(b+x-z)] F(dz) + U(b) \int_{(b+x,\infty)} F(dz).
\end{aligned}$$

We now use Wald's equation $U(c) = (c + ER_c)/\mu$ to treat the two summands in the last expression. For the first summand, we obtain

$$\begin{aligned}
&\int_{(x,b+x]} [U(b) - U(b+x-z)] F(dz) \\
&\quad = \mu^{-1} \int_{(x,b+x]} (z - x + ER_b - ER_{b+x-z}) F(dz) \\
&\quad \le \mu^{-1} \int_{(x,b+x]} (z + ER_b - x) F(dz) \\
&\quad \le \mu^{-1} \left(1 + \frac{ER_b - x}{b + x}\right) \int_{(x,b+x]} z F(dz) \\
&\quad = \left(\frac{b + ER_b}{b + x}\right) P\{L_\infty \in (x, b + x]\},
\end{aligned}$$

where the last inequality uses the assumption that $x \ge ER_b$. For the second summand,

$$\begin{aligned}
U(b) \int_{(b+x,\infty)} F(dz) &= \mu^{-1}(b + ER_b) \int_{(b+x,\infty)} F(dz) \\
&\le \mu^{-1} \left(\frac{b + ER_b}{b + x}\right) \int_{(b+x,\infty)} z F(dz) \\
&= \left(\frac{b + ER_b}{b + x}\right) P\{L_\infty \in (b + x, \infty)\}.
\end{aligned}$$

The desired result is obtained by adding the last two displays. □

By (1.6), Proposition 4.1 improves (4.1) when $x \ge ER_b$, and hence when $x \ge EL_\infty$. On the other hand, the inequality

$$P\{R_b > x\} \le P\{L_\infty > x\} + P\{R_\infty > x\},$$

which is trivial from the results of the previous section [see the paragraph containing (3.3)], improves (4.1) when $x < EL_\infty$. The next proposition gives a somewhat better inequality.

PROPOSITION 4.2. *For all nonnegative b and x,*

$$P\{R_b > x\} \le P\{L_\infty > x\} + P\{R_\infty > x\}P\{X \le x\}. \tag{4.3}$$

PROOF. Begin with (3.3) and Propositions 3.1 and 3.4, which imply

$$P\{R_b > x\} \le P\{R_\infty > x\} + P\{L_\infty > x\} - P\{R_{b-Y} > x, L_{b-Y,Y} > x\}. \tag{4.4}$$

Next write

$$\begin{aligned} P\{R_{b-Y} > x, L_{b-Y,Y} > x\} &= \int P\{R_{b-y} > x, L_{b-y,y} > x\}P\{Y \in dy\} \\ &= \int P\{R_{b-y} > x\}P\{L_y > x\}P\{Y \in dy\}. \end{aligned}$$

However, defining the age $A_y = \inf\{y - S_n : S_n \le y\}$, we have

$$P\{L_y > x \mid A_y = z\} = P\{X > x \mid X > z\} \ge P\{X > x\}$$

for all z such that $P\{X > z\} > 0$. It follows that $P\{L_y > x\} \ge P\{X > x\}$. Thus, using Proposition 3.1 again,

$$\begin{aligned} P\{R_{b-Y} > x, L_{b-Y,Y} > x\} &\ge P\{X > x\} \int P\{R_{b-y} > x\}P\{Y \in dy\} \\ &= P\{X > x\}P\{R_\infty > x\}. \end{aligned}$$

The proof is completed by substituting the last display into (4.4). □

The tail probability bounds in Propositions 4.1 and 4.2 contain enough information to give new moment inequalities. As an example, the next proposition applies these tail probability bounds in a straightforward and rather crude way to obtain a strict improvement on Lorden's inequalities (1.5) in all nontrivial cases, that is, cases in which the stated upper bound is finite and X is not deterministic.

PROPOSITION 4.3. *For all nonnegative b and p,*

$$E(R_b^p) \le E(L_\infty^p) + P\{X \le EL_\infty\}E\left[(R_\infty \wedge EL_\infty)^p\right].$$

PROOF. In fact, we have

$$\begin{aligned} E(R_b^p) &= \int_0^\infty px^{p-1}P\{R_b > x\}\,dx \\ &\le \int_0^{ER_b} px^{p-1}[P\{L_\infty > x\} + P\{X \le x\}P\{R_\infty > x\}]\,dx \\ &\quad + \int_{ER_b}^\infty px^{p-1}P\{L_\infty > x\}\,dx \\ &\le E(L_\infty^p) + P\{X \le ER_b\}E\left[(R_\infty \wedge ER_b)^p\right]. \end{aligned}$$

From this, the stated inequality is a consequence of Lorden's first-moment inequality $ER_b \le EL_\infty$. □

Tighter bounds are possible: the preceding proof proceeds crudely in pursuit of expediency and a relatively neat-looking bound. First, for $x \le ER_b$, the proof in effect applies only the consequence

$$P\{R_b > x\} \le P\{L_\infty > x\} + P\{R_\infty > x\}P\{X \le ER_b\}$$

of inequality (4.3). Using (4.3) "as is" gives the better, but not quite as neat-looking, uniform bound

$$E(R_b^p) \le E(L_\infty^p) + \int_0^{EL_\infty} px^{p-1}P\{X \le x\}P\{R_\infty > x\}\,dx.$$

In some cases, even using (4.3) for $x \le ER_b$ is clearly wasteful. For example, if F has a positive density $f(0)$ at 0, then the bound given by (4.3) is of the form $1 + f(0)x + o(x)$ as $x \downarrow 0$, which is greater than 1 for small x. Finally, for $x \ge ER_b$, the proof just bounds $P\{R_b > x\}$ by $P\{L_\infty > x\}$, ignoring the extra factor $(b + ER_b)/(b + x)$ that is permitted by (4.2).

5. Conjectures. I would conjecture that the following inequalities hold:

CONJECTURE 5.1. *For all nonnegative b and p,*

$$E(R_b^p) \le (p+1)E(R_\infty^p) = E(L_\infty^p).$$

Note that if these inequalities hold, it is clear that they are sharp: equality holds when X has a deterministic distribution $X = \mu$, say, in which case $L_\infty = \mu = R_b$ whenever b is an integer multiple of μ. Proposition 4.3 makes partial progress toward the goal of eliminating the extra $E(R_\infty^p)$ in Lorden's bound by reducing that $E(R_\infty^p)$ to $P\{X \le EL_\infty\}E[(R_\infty \wedge EL_\infty)^p]$. Since $P\{X \le EL_\infty\}$ may range over the whole interval from 0 to 1, the improvement contributed by the factor $P\{X \le EL_\infty\}$ may range from great to negligible. Certainly for large p the improvement is significant—it is easy to see that if X is not deterministic, then $E[(R_\infty \wedge EL_\infty)^p]$ becomes negligibly small compared to $E(R_\infty^p)$ as $p \to \infty$.

A natural reaction to Conjecture 5.1 is to ask whether the stochastic inequality $R_b \le_{\text{st}} L_\infty$ holds for all b. Momentary encouragement is provided by Proposition 4.1, which implies the inequality $P\{R_b > x\} \le P\{L_\infty > x\}$ for all $x \ge ER_b$. However, these lovely thoughts are spoiled by simple counterexamples. For example, let X have the distribution

$$X = \begin{cases} 1, & \text{with probability } m/(m+1), \\ m, & \text{with probability } 1/(m+1), \end{cases}$$

for a given positive integer m. Then L_∞ takes on the values 1 and m with probabilities $1/2$ each, while

$$P\{R_{m-2} = 1\} = P\{X_1 = X_2 = \cdots = X_{m-1} = 1\} = \left(\frac{m}{m+1}\right)^{m-1}.$$

Thus, if m is chosen large enough, $P\{R_{m-2} > 1\} = 1 - P\{R_{m-2} = 1\}$ is close to $1 - e^{-1}$, which is larger than $0.5 = P\{L_\infty > 1\}$, so that we do not have $R_{m-2} \le_{\text{st}} L_\infty$. (Here $m = 5$ is large enough to give a counterexample.)

The previous example has also been useful in ruling out some other conjectures. Despite such disappointments, it is most tempting to continue to search for other probabilistic assertions that would "explain" why the moments of R_b might be bounded by the moments of L_∞. Of course, also enlightening would be a counterexample to the conjecture, which is supported by computer experiments and perhaps a certain amount of wishful thinking.

6. Application to the first ladder height. In many applications of Lorden's inequalities, the multiplicative constant is not critical: the $(p+2)$ of (1.5) will do as well as the $(p+1)$ of Conjecture 5.1. In this section we discuss an example of a problem in which the distinction becomes important.

Dropping the assumption that X is positive, suppose the distribution F has mean 0. We exclude the trivial case where $X = 0$ with probability 1. Define the moment generating function ϕ by $\phi(\theta) = \int e^{\theta x} F(dx)$ and assume that the set $\Theta = \{\theta\colon \phi(\theta) < \infty\}$ contains a neighborhood of 0. Then in fact Θ is an interval having endpoints θ_1 and θ_2, say, with $-\infty \le \theta_1 < 0 < \theta_2 \le \infty$. Define the exponential family of distributions $\{F_\theta\colon \theta \in \Theta\}$ by

$$F_\theta(dx) = e^{\theta x - \psi(\theta)} F(dx),$$

where $\psi(\theta) = \log \phi(\theta)$. We will assume that $X_1, X_2, \dots$ are independent and identically distributed having a distribution F_θ that is a member of the exponential family just described. In this situation, P_θ and E_θ will denote probability and expectation.

The process $\{S_n\colon n \ge 0\}$ is a random walk. The residual R_0, the first positive value taken by the random walk, is known as the *first ladder height*, and τ_0 is called the *first ladder epoch*.

PROPOSITION 6.1. *The first moment $E_\theta R_0$ is a nondecreasing function of θ for nonnegative θ.*

PROOF. In fact, defining $h(\theta) = E_\theta R_0$, we will show that the function h has a nonnegative derivative for $0 < \theta < \theta_2$. Define $\mu = E_\theta X = \psi'(\theta)$. Then Siegmund (1979) shows that h is differentiable in $(0, \theta_2)$ with

$$h'(\theta) = E_\theta R_0^2 - \mu E_\theta(\tau_0 R_0)$$

and also, letting M denote $\min\{S_n\colon n \ge 0\}$, that

$$\mu E_\theta(\tau_0 R_0) = E_\theta R_0 \int_{[0,\infty)} E_\theta(R_x) P_\theta\{-M \in dx\}.$$

Applying Lorden's first moment inequality in the form

$$E_\theta(R_x) \le \frac{E_\theta(R_0^2)}{E_\theta(R_0)}$$

to the last integral gives the result. □

In view of the case of equality in Proposition 3.2, it is easy to see that $E_\theta R_0$ is actually a strictly increasing function of θ for $\theta \in (0, \theta_2)$ unless R_0 is constant with P_0-probability 1, that is, unless there is a positive c such that $P_0\{X \in \{\dots, -2c, -c, 0, c\}\} = 1$.

Although apparently not noticed before, Proposition 6.1 is proved by a simple combination of known results. Extending the simple proof to other moments would require Conjecture 5.1. To see this, define $h_p(\theta) = E_\theta(R_0^p)$. Then

$$h'_p(\theta) = E_\theta(R_0^{p+1}) - \mu E_\theta(\tau_0 R_0^p)$$

and

$$\mu E_\theta(\tau_0 R_0^p) = E_\theta R_0 \int_{[0,\infty)} E_\theta(R_x^p) P_\theta\{-M \in dx\}.$$

Thus, if

$$E_\theta(R_x^p) \le \frac{E_\theta(R_0^{p+1})}{E_\theta(R_0)}$$

holds, then $h'_p(\theta) \ge 0$. This is a case where we need the $(p+1)$ in Conjecture 5.1; the $(p+2)$ in Lorden's inequality (1.5) is too large.

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