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PERTURBATION OF NORMAL RANDOM VECTORS BY NONNORMAL TRANSLATIONS, AND AN APPLICATION TO HIV LATENCY TIME DISTRIBUTIONS[1]

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Let $\mathbf{Z}$ be a normal random vector in R^k and let $\mathbf{1}$ be the element of R^k with equal components 1. Let X be a random variable that is independent of $\mathbf{Z}$ and consider the sum $\mathbf{Z} + X\mathbf{1}$. The latter has a normal distribution in R^k if and only if X has a normal distribution in R^1. The first result of this paper is a formula for a uniform bound on the difference between the density function of $\mathbf{Z} + X\mathbf{1}$ and the density function in the case where X has a suitable normal distribution. This is applied to a problem in the theory of stationary Gaussian processes which arose from the author's work on a stochastic model for the CD4 marker in the progression of HIV.

1. Introduction and summary. Let $\mathbf{Z}$ be a random vector in R^k having a $N(\boldsymbol{\mu}, \Sigma)$ distribution where Σ is nonsingular. Let X and Y be random variables such that X and $\mathbf{Z}$ are independent and Y and $\mathbf{Z}$ are independent. Let $\mathbf{1}$ be the vector in R^k with equal components 1. The basic result of this paper is a formula for a bound on the absolute difference of the densities of $\mathbf{Z} + X\mathbf{1}$ and $\mathbf{Z} + Y\mathbf{1}$, stated in Theorem 2.1:

$$(1.1) \qquad (2\pi)^{-k/2}(\det \Sigma)^{-1/2} \int_{-\infty}^{\infty} |E \exp^{(i\lambda x X)} - E \exp^{(i\lambda x Y)}| \phi(x)\, dx,$$

where $\phi(x)$ is the standard normal density and $\lambda = (\mathbf{1}'\Sigma^{-1}\mathbf{1})^{1/2}$. Since $\phi(x)$ decreases rapidly for $|x| \to \infty$, the key to the closeness of the densities is in the closeness of the characteristic functions in a bounded interval containing the origin. In particular the bound (1.1) is relatively small if X and Y have the same set of moments up to a certain order or if they have a common mean and small variances (Theorem 2.3).

The main applications of (1.1) given in this paper are in the case where $EX = EY$, $EX^2 = EY^2$, $E|X|^3 < \infty$ and Y is assumed to have a normal distribution. In Theorem 2.4 we consider the bound (1.1) in the particular case where X is itself the normed sum of n i.i.d. random variables. It is shown that (1.1) is bounded by a Liapunov-type function involving the absolute moments up to order 3 and the sample size n.

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The motivation for (1.1) arose from the following general problem in the theory of stationary Gaussian processes: Let $Z(t)$, $-\infty < t < \infty$, be a real-valued stationary Gaussian process and let X be a random variable independent of the process. Then $Z(t) + X = X(t)$ is stationary, but is not Gaussian if X is not normally distributed. The question that led to (1.1) is: If X is close to having a normal distribution, how close is $X(t)$ to a Gaussian process? More exactly, what properties of the process $Z(t)$ based on its Gaussian nature hold approximately for $X(t)$ and what is the "error" of approximation? In applying (1.1), we take Y as a normally distributed random variable having the same mean and variance as X, and define $Y(t) = Z(t) + Y$, which is, of course, a stationary Gaussian process. Then (1.1) takes the form (2.5) and furnishes a uniform bound for the difference of the k-dimensional densities of the processes $X(t)$ and $Y(t)$, respectively. In the bound (2.5), the covariance matrix of the Gaussian k-vector enters the analytic expression only through the determinant in the denominator and the number $\mathbf{1}'\Sigma^{-1}\mathbf{1}$ = *sum of all the entries* of Σ^{-1} in the numerator. The mean vector has no role.

The particular problem that led to this more general question for stationary Gaussian processes is the refinement of the author's model for the CD4 marker in the progression of HIV [Berman (1990)]. In the original model, the logarithm of the CD4 marker of a seronegative (for HIV) person was, as a function of time, taken to be a stationary Gaussian process $Z(t)$. At the moment of seroconversion, a negative linear term $-\delta t$ is added to $Z(t)$, where $\delta > 0$ is a parameter, so that the process is changed to $Z(t) - \delta t$ and the Gaussian property is preserved. The model is now refined by incorporating the more recently observed fact that there is a nearly instantaneous drop of the CD4 level just following the time of seroconversion; see Levy (1988). Thus the stochastic process is transformed by the addition of a negative random variable X representing the CD4 drop, and which does not have an exactly normal distribution (because it is negative). The bound (2.5) is applied to the determination of the possible error that enters the calculation of the posterior density of the HIV latency time when the distribution of X is replaced by a corresponding normal distribution. This is analyzed in Section 4.

The main result of Berman (1990) is also extended in another direction. In the latter work, the posterior distribution of the latency time was a conditional distribution whose conditioning variable was the first CD4 measurement following seroconversion. The current work provides a posterior distribution defined as a conditional distribution with the conditioning set containing any number of CD4 measurements following seroconversion. By utilizing this extended posterior distribution, one can employ more observations and thereby obtain more information about the conditional distribution of the latency time. An interesting feature of this extension is that the extended posterior distribution is exactly of the same parametric form as that of the original posterior distribution, namely, it is a censored normal distribution. In the extended version the parameter values of the censored normal distribution depend on the set of all the conditioning variables.

2. The bound for the difference of the densities.

THEOREM 2.1. *Let* $\mathbf{Z}$ *be a normal random vector in* R^k *with nonsingular covariance matrix* Σ *and let* X *and* Y *be real-valued random variables such that* $\mathbf{Z}$ *and* X *are independent and* $\mathbf{Z}$ *and* Y *are independent. Let* $\phi(z)$ *be the standard normal density, and define*

$$\lambda = (\mathbf{1}'\Sigma^{-1}\mathbf{1})^{1/2} \tag{2.1}$$

and

$$f(u) = Ee^{iuX}, \qquad g(u) = Ee^{iuY}. \tag{2.2}$$

Then the absolute difference between the densities of $\mathbf{Z} + X\mathbf{1}$ *and* $\mathbf{Z} + Y\mathbf{1}$ *is uniformly at most equal to*

$$(2\pi)^{-k/2}(\det \Sigma)^{-1/2} \int_{-\infty}^{\infty} |f(\lambda x) - g(\lambda x)|\phi(x)\, dx. \tag{2.3}$$

PROOF. By the form of the multivariate normal characteristic function and the inversion formula for an arbitrary density function in terms of the characteristic function, the absolute difference of the densities is at most equal to

$$(2\pi)^{-k} \int_{R^k}\!\!\cdots\!\int \exp(-\tfrac{1}{2}\mathbf{u}'\Sigma\mathbf{u})|E\exp(i\mathbf{u}'\mathbf{1}X) - E\exp(i\mathbf{u}'\mathbf{1}Y)|\, d\mathbf{u}$$

or

$$(2\pi)^{-k} \int_{R^k}\!\!\cdots\!\int \exp(-\tfrac{1}{2}\mathbf{u}'\Sigma\mathbf{u})|f(\mathbf{u}'\mathbf{1}) - g(\mathbf{u}'\mathbf{1})|\, d\mathbf{u}. \tag{2.4}$$

The expression (2.4) is, in fact, equal to (2.3). Indeed, it is true that for any bounded measurable function h,

$$(2\pi)^{-k/2}(\det \Sigma)^{1/2} \int_{R^k}\!\!\cdots\!\int \exp(-\tfrac{1}{2}\mathbf{u}'\Sigma\mathbf{u})h(\mathbf{u}'\mathbf{b})\, d\mathbf{u}$$
$$= \int_{-\infty}^{\infty} h\left(x(\mathbf{b}'\Sigma^{-1}\mathbf{b})^{1/2}\right)\phi(x)\, dx,$$

because the left-hand member represents $Eh(\boldsymbol{\xi}'\mathbf{b})$, where $\boldsymbol{\xi}$ is a $N(\mathbf{0}, \Sigma^{-1})$ random vector in R^k, so that $\boldsymbol{\xi}'\mathbf{b}$ is an $N(0, \mathbf{b}'\Sigma^{-1}\mathbf{b})$ random variable, and the right-hand member is a version of the same expected value. Applying the foregoing equation with $\mathbf{b} = \mathbf{1}$ and $h = |f - g|$, one obtains the equality of (2.3) and (2.4). □

COROLLARY 2.2. *Let* $\mathbf{T}$ *be an arbitrary random vector in* R^k *which is independent of* X, Y *and* $\mathbf{Z}$. *Then the bound* (2.3) *for the absolute difference of the densities continues to hold if* $\mathbf{Z}$ *is replaced by* $\mathbf{Z} + \mathbf{T}$.

PROOF. By the reasoning involving the inversion formula that led to the bound (2.3), the addition of the random vector $\mathbf{T}$ yields the modified bound

$$(2\pi)^{-k} \int_{R^k}\cdots\int \exp\left(-\tfrac{1}{2}\mathbf{u}'\Sigma\mathbf{u}\right)|Ee^{i\mathbf{u}'\mathbf{T}}|\,|f(\mathbf{u}'\mathbf{1}) - g(\mathbf{u}'\mathbf{1})|\,d\mathbf{u},$$

which is obviously at most equal to (2.3). □

If Y has a $N(\nu,\tau^2)$ distribution, then $g(u) = \exp(i\nu u - \frac{1}{2}u^2\tau^2)$ and (2.3) takes the form

$$\frac{\int_{-\infty}^{\infty}|\exp(-i\nu\lambda x)f(\lambda x) - \exp(-\frac{1}{2}x^2\lambda^2\tau^2)|\phi(x)\,dx}{(2\pi)^{k/2}(\det\Sigma)^{1/2}}. \tag{2.5}$$

THEOREM 2.3. *Assume, for some $m \ge 2$, that $E|X|^m < \infty$ and $E|Y|^m < \infty$ and $EX^j = EY^j$, for $j = 1,\dots,\max(2, m-1)$. Let z_ε be the upper 100ε percentile of the standard normal distribution. Then, for any $\varepsilon > 0$, the bound (2.3) is at most equal to*

$$(2\pi)^{-k/2}(\det\Sigma)^{-1/2}\left\{4\varepsilon + \frac{(z_\varepsilon\tau\lambda)^m}{m!}\left[E\left|\frac{X-EX}{\tau}\right|^m + E\left|\frac{Y-EY}{\tau}\right|^m\right]\right\}, \tag{2.6}$$

where $\tau^2 = \operatorname{Var} X = \operatorname{Var} Y$.

PROOF. Change the variable of integration in the integral in (2.3) to obtain

$$\int_{-\infty}^{\infty}\left|f\left(\frac{x}{\tau}\right) - g\left(\frac{x}{\tau}\right)\right|\phi\left(\frac{x}{\tau\lambda}\right)\frac{dx}{\tau\lambda}.$$

The latter is at most equal to

$$\int_{-z_\varepsilon\tau\lambda}^{z_\varepsilon\tau\lambda}\left|f\left(\frac{x}{\tau}\right) - g\left(\frac{x}{\tau}\right)\right|\phi\left(\frac{x}{\tau\lambda}\right)\frac{dx}{\tau\lambda} + 2\int_{|x|>z_\varepsilon\tau\lambda}\phi\left(\frac{x}{\tau\lambda}\right)\frac{dx}{\tau\lambda},$$

which, by the symmetry of ϕ and the definition of z_ε, is equal to

$$4\varepsilon + \int_{-z_\varepsilon\tau\lambda}^{z_\varepsilon\tau\lambda}\left|f\left(\frac{x}{\tau}\right) - g\left(\frac{x}{\tau}\right)\right|\phi\left(\frac{x}{\tau\lambda}\right)\frac{dx}{\tau\lambda}. \tag{2.7}$$

Put $\mu = EX = EY$ and note that

$$\left|f\left(\frac{x}{\tau}\right) - g\left(\frac{x}{\tau}\right)\right| = |E\exp(ix(X-\mu)/\tau) - E\exp(ix(Y-\mu)/\tau)|.$$

Thus, by the local expansion of a characteristic function [e.g., Loève (1963), page 199],

$$\left|E\exp\left(ix\frac{X-\mu}{\tau}\right) - \sum_{j=0}^{m-1}\frac{(ix)^j}{j!}E\left(\frac{X-\mu}{\tau}\right)^j\right| \le \frac{|x|^m}{m!}E\left|\frac{X-\mu}{\tau}\right|^m,$$

and the corresponding expansion with Y in place of X, we obtain, under the assumed equality of the sets of the first $m-1$ moments,

$$\left|f\left(\frac{x}{\tau}\right)-g\left(\frac{x}{\tau}\right)\right| \le \frac{|x|^m}{m!}\left[E\left|\frac{X-\mu}{\tau}\right|^m+E\left|\frac{Y-\mu}{\tau}\right|^m\right].$$

Apply the latter to the integrand in (2.7). Then the expression (2.7) is at most equal to

$$4\varepsilon+\frac{(z_\varepsilon\tau\lambda)^m}{m!}\left[E\left|\frac{X-\mu}{\tau}\right|^m+E\left|\frac{Y-\mu}{\tau}\right|^m\right]$$

and (2.6) follows. □

In the particular case $m = 2$, the bound (2.6) is simply

$$(2\pi)^{-k/2}(\det\Sigma)^{-1/2}\left[4\varepsilon+(z_\varepsilon\tau\lambda)^2\right].$$

THEOREM 2.4. *Let $X_1,\dots,X_n$ be i.i.d. random variables with $EX_1=\nu$, $\operatorname{Var}X_1=\tau^2$ and $E|X_1|^3<\infty$. Put $X=(X_1+\cdots+X_n)/\sqrt{n}$ and let Y be a random variable with a $N(\nu\sqrt{n},\tau^2)$ distribution. Then the bound* (2.5) *is at most equal to*

$$\frac{4\varepsilon+\left((z_\varepsilon\lambda)^3/(6\sqrt{n})\right)E\{|x_1-\nu|^3\}+(z_\varepsilon\tau\lambda)^4/(8n)}{(2\pi)^{k/2}(\det\Sigma)^{1/2}}. \tag{2.8}$$

PROOF. Put $h(x)=Ee^{ixX_1}$. Then

$$Ee^{i\lambda xX}=h^n(\lambda x/\sqrt{n})$$

and (2.5), with $\nu\sqrt{n}$ in place of ν, takes the form

$$\begin{aligned}(2\pi)^{-k/2}(\det\Sigma)^{-1/2}\int_{-\infty}^{\infty}\bigg|&\left(\exp\left(-\frac{i\nu\lambda x}{\sqrt{n}}\right)h\left(\frac{\lambda x}{\sqrt{n}}\right)\right)^n\\ &-\exp\left(-\frac{1}{2}x^2\lambda^2\tau^2\right)\bigg|\phi(x)\,dx.\end{aligned} \tag{2.9}$$

As in the proof of Theorem 2.3, the integral factor in (2.9) is at most equal to

$$4\varepsilon+\int_{|x|\le z_\varepsilon\tau\lambda}\left|\left(\exp\left(\frac{-ix\nu}{\tau\sqrt{n}}\right)h\left(\frac{x}{\tau\sqrt{n}}\right)\right)^n-\exp\left(-\frac{1}{2}x^2\right)\right|\phi\left(\frac{x}{\tau\lambda}\right)\frac{dx}{\tau\lambda}. \tag{2.10}$$

By the inequality $|w^n-\nu^n|\le n|w-\nu|$, for complex w and ν, $|w|\le 1$, $|\nu|\le 1$, the expression (2.10) is at most equal to

$$4\varepsilon+\int_{|x|\le z_\varepsilon\tau\lambda}n\left|\exp\left(\frac{-ix\nu}{\tau\sqrt{n}}\right)h\left(\frac{x}{\tau\sqrt{n}}\right)-\exp\left(-\frac{x^2}{2n}\right)\right|\phi\left(\frac{x}{\tau\lambda}\right)\frac{dx}{\tau\lambda}. \tag{2.11}$$

By the local expansion of the characteristic function

$$\left|\exp\left(-\frac{ix\nu}{\tau\sqrt{n}}\right)h\left(\frac{x}{\tau\sqrt{n}}\right)-1+\frac{x^2}{2n}\right|\le\frac{1}{6}\left|\frac{x}{\tau\sqrt{n}}\right|^3 E|X_1-\nu|^3$$

and the simple estimate

$$\left|\exp\left(-\frac{x^2}{2n}\right)-1+\frac{x^2}{2n}\right|\le\frac{x^4}{8n^2},$$

the expression (2.11) is at most equal to the numerator in (2.8). □

From the well-known relation

$$\int_x^\infty \phi(z)\,dz \sim \phi(x)/x, \qquad x\to\infty,$$

it follows that if $z^{(n)}$ is defined as $z^{(n)}=[\log n-2\log(2\pi\log n)^{1/2}]^{1/2}$, then $\int_{z^{(n)}}^\infty \phi(z)\,dz\sim n^{-1/2}$, for $n\to\infty$. Therefore, if ε in (2.8) is chosen as $n^{-1/2}$, then, for $n\to\infty$, the numerator of the expression (2.8) is asymptotically equal to

$$\frac{\tau^3E\{|x_1-\nu|^3\}(\log n)^{3/2}}{6\sqrt{n}}. \tag{2.12}$$

3. Extension of an earlier result on the censored normal distribution. In Berman (1990) the author showed that if Z and T are independent random variables such that Z has a standard normal distribution and T has an exponential distribution, then the conditional distribution of T, given $Z-T=x$, is a censored normal distribution having the density on the positive axis

$$\frac{\phi(t+m^{-1}+x)}{\int_{m^{-1}+x}^\infty\phi(y)\,dy}, \qquad t>0,$$

where $m=ET$. This is now generalized to an arbitrary normal random vector $\mathbf{Z}$.

LEMMA 3.1. *Let* $\mathbf{Z}$ *be a* $N(\boldsymbol{\mu},\Sigma)$ *random vector in* R^k *and* T *an independent random variable with the density function* $\theta e^{-\theta t}$, $t>0$, *for some fixed* $\theta>0$. *Define*

$$H(\mathbf{x})=\theta/\delta+\mathbf{1}'\Sigma^{-1}\mathbf{x} \tag{3.1}$$

for some fixed $\delta>0$. *Then the conditional density of* T *at* t, *given* $\mathbf{Z}-\delta T\mathbf{1}=\mathbf{x}$, *is*

$$\frac{\delta\lambda\phi(\delta\lambda t+\lambda^{-1}H(\mathbf{x}-\boldsymbol{\mu}))}{\int_{\lambda^{-1}H(\mathbf{x}-\boldsymbol{\mu})}^\infty\phi(z)\,dz}. \tag{3.2}$$

PROOF. It suffices to prove the result just for $\boldsymbol{\mu} = \mathbf{0}$ as the more general case follows by a simple extension.

Assume first $\delta = 1$. Put $\mathbf{R} = \Sigma^{-1}$ and define

$$\phi_{\mathbf{R}}(\mathbf{x}) = (2\pi)^{-k/2}(\det \mathbf{R})^{1/2} \exp\left(-\tfrac{1}{2}\mathbf{x}'\mathbf{R}\mathbf{x}\right). \tag{3.3}$$

The joint density of $\mathbf{Z} - T\mathbf{1}$ and T at $(\mathbf{x}, t)$ is

$$\theta e^{-\theta t}\phi_{\mathbf{R}}(\mathbf{x} + t\mathbf{1}). \tag{3.4}$$

It follows by simple algebra from (3.3) that

$$\phi_{\mathbf{R}}(\mathbf{x} + t\mathbf{1}) = \phi_{\mathbf{R}}(\mathbf{x})\exp\left(-t\mathbf{1}'\mathbf{R}\mathbf{x} - \tfrac{1}{2}t^2\mathbf{1}'\mathbf{R}\mathbf{1}\right),$$

so that (3.4) is equal to

$$\theta\phi_{\mathbf{R}}(\mathbf{x})\exp\left(-tH(\mathbf{x}) - \tfrac{1}{2}\lambda^2 t^2\right),$$

where $\delta = 1$ in the expression (3.1) for $H(\mathbf{x})$. By completion of the square, the foregoing displayed expression is equal to

$$\theta\phi_{\mathbf{R}}(\mathbf{x})\exp\left(\frac{1}{2}\frac{H^2(\mathbf{x})}{\lambda^2} - \frac{1}{2}\lambda^2\left(t + \frac{H(\mathbf{x})}{\lambda^2}\right)^2\right). \tag{3.5}$$

Integration of (3.5) over $t > 0$ yields

$$\theta\phi_{\mathbf{R}}(\mathbf{x})\exp\left(\frac{1}{2}\frac{H^2(\mathbf{x})}{\lambda^2}\right)\frac{\sqrt{2\pi}}{\lambda}\int_{H(\mathbf{x})/\lambda}^{\infty}\phi(y)\,dy. \tag{3.6}$$

The conditional density (3.2) is obtained by dividing (3.5) by (3.6). This completes the proof in the case $\delta = 1$.

In the previous argument, let us now replace T by δT, which has an exponential density with mean δ/θ. It follows from what has just been established (in the case $\delta = 1$) that the conditional distribution of δT at t, given $\mathbf{Z} - \delta T\mathbf{1} = \mathbf{x}$, is given by

$$\frac{\lambda\phi\left(\lambda t + H(\mathbf{x})\lambda^{-1}\right)}{\int_{H(\mathbf{x})/\lambda}^{\infty}\phi(z)\,dz},$$

where $H(\mathbf{x})$ is defined as in (3.1). It then follows immediately that the conditional density of T itself is (3.2). □

4. An application to the distribution of HIV latency time based on CD4 levels. An individual who had not previously tested positive for the presence of HIV infection now undergoes such a test and is found to be seropositive. At the same time the level of CD4 cells in the blood of the individual is measured. In Berman (1990) it was shown, in terms of a stochastic model, how one could determine the distribution of the time since seroconversion took place on the basis of the CD4 level observed in the subject. It is well known that the CD4 level declines after seroconversion, and the analysis in Berman (1990) was based on a comparison of the lowered CD4 level with the normal level in the absence of HIV.

Let $X(t)$, where t is the time variable, be the function representing the CD4 level in a randomly selected seronegative individual, that is, one who is free of HIV antibodies. In the previous stochastic model we assumed that $X(t)$ is the sample function of a stationary stochastic process. The corresponding stochastic process for a seropositive individual is obtained from $X(t)$ (defined for seronegatives) by two successive transformations. Let t_0 represent the time of seroconversion. The first transformation is an instantaneous drop in CD4 at time t_0: The sample function $X(t)$ is replaced by $QX(t)$, for $t \geq t_0$, where Q is a random variable that is independent of the process $X(t)$ and assumes values in $[0, 1]$. The next transformation is the appending of an exponential damping factor: The sample function is changed to $QX(t)\exp(-\delta(t - t_0))$, for $t \geq t_0$, where $\delta > 0$ is a fixed number representing the rate of decay of CD4 after seroconversion. It follows that the value of the sample function t time units after seroconversion is $QX(t + t_0)e^{-\lambda t}$.

Since $X(\cdot)$ is stationary, the finite-dimensional distributions of the process are independent of t_0, and so we may put $t_0 = 0$ and write the process for seropositives as

$$QX(t)e^{-\lambda t}, \qquad t \geq 0.$$

Since the values are positive, we may write the process in exponential form as

$$\exp(Z(t) - \delta t + V),$$

where $V = \log Q$, and then consider the logarithm of the process

$$R(t) = Z(t) + V - \delta t, \qquad t \geq 0. \tag{4.1}$$

Berman (1990) assumed that $Z(t)$ is a stationary Gaussian process. The random variable V was absent from (4.1) in the original formulation. The assumption that $Z(t)$ is Gaussian was crucial in the calculation of the conditional distribution of the latency time. The addition of the random variable V disturbs the exact Gaussian nature of the process, and so the calculations involving the conditional distribution of the latency time are no longer strictly valid. However, the point of Theorem 2.1 is that, under a large variety of conditions, the underlying Gaussian nature of the process and, in particular, the joint density of finitely many observations, is only slightly modified, and so the conditional density of the latency time is altered by at most a small amount. The actual discrepancy will, in Theorem 4.2, be computed on the basis of the bound furnished by Theorem 2.1.

For a randomly selected seropositive individual, let T represent the length of the time interval from the moment of seroconversion to the time of the first HIV test when seropositivity is first discovered. The time T is considered to be the "latency time" of HIV for the selected individual. Its distribution is called the "prior" distribution of T, to distinguish it from a conditional distribution, to be defined, that is called the posterior distribution. The value of the logarithm of the CD4 level at time T is, by (4.1), $R(T) = Z(T) + V - \delta T$. For $k > 1$, put $s_1 = 0$ and let $s_2, \dots, s_k$ for $0 < s_2 < \cdots < s_k$ be defined as the time differences between the succeeding $k - 1$ CD4 level measurements and the first one at time T; thus there are k readings represented as the vector with components $R(T+s_i) = Z(T+s_i) + V - \delta T - \delta s_i$, $i = 1, \dots, k$. By the assumed stationarity of $Z(\cdot)$ and the independence of T, the preceding random vector has the same density as the vector with components

$$Z(s_1) + V - \delta T - \delta s_1, \dots, Z(s_k) + V - \delta T - \delta s_k. \tag{4.2}$$

As in Berman (1990), we define $EZ(t) = \mu$, $\operatorname{Var} Z(t) = \sigma^2$ and $\operatorname{Cov}(Z(0), Z(t)) = r(t)$. Now we also define $EV = \nu$ and $\operatorname{Var} V = \tau^2$. Let $q(t)$ be the (prior) density of T and let $G(v)$ be the distribution function of V. Then define

$$\mathbf{s} = \begin{pmatrix} s_i \\ \vdots \\ s_k \end{pmatrix}, \qquad \mathbf{Z} = \begin{pmatrix} Z(s_1) \\ \vdots \\ Z(s_k) \end{pmatrix},$$

$$\Sigma = \operatorname{matrix}\big(r(s_i - s_j)\big)$$

and let $\phi_{\mathbf{R}}(\mathbf{x})$ be the normal density (3.3). Then the observations (4.2) are represented as the k-component random vector

$$\mathbf{Z} + V\mathbf{1} - \delta T\mathbf{1} - \delta\mathbf{s}. \tag{4.3}$$

Under the assumption of the independence of $\mathbf{Z}$, V and T, the joint density of $(T, \mathbf{Z} + V\mathbf{1} - \delta T\mathbf{1} - \delta\mathbf{s})$ at $(t, \mathbf{x}) \in R^1 \times R^k$ is

$$q(t)\int_{-\infty}^{\infty} \phi_{\mathbf{R}}(\mathbf{x} - \mu\mathbf{1} - v\mathbf{1} + \delta t\mathbf{1} + \delta\mathbf{s})\, dG(v). \tag{4.4}$$

Therefore, the conditional density of T at t, given $\mathbf{Z} + V\mathbf{1} - \delta t\mathbf{1} - \delta\mathbf{s} = \mathbf{x}$, is equal to

$$\frac{q(t)\int_{-\infty}^{\infty}\phi_{\mathbf{R}}(\mathbf{x} - v\mathbf{1} - \mu\mathbf{1} + \delta t\mathbf{1} + \delta\mathbf{s})\, dG(v)}{\int_0^{\infty} q(w)\int_{-\infty}^{\infty}\phi_{\mathbf{R}}(\mathbf{x} - v\mathbf{1} - \mu\mathbf{1} + \delta w\mathbf{1} + \delta\mathbf{s})\, dG(v)\, dw}. \tag{4.5}$$

For future reference, the formula (4.5) is recorded here for the case $q(t) = \theta e^{-\theta t}$:

$$\frac{\theta e^{-\theta t}\int_{-\infty}^{\infty}\phi_{\mathbf{R}}(\mathbf{x} - v\mathbf{1} - \mu\mathbf{1} + \delta t\mathbf{1} + \delta\mathbf{s})\, dG(v)}{\int_0^{\infty}\theta e^{-\theta w}\int_{-\infty}^{\infty}\phi_{\mathbf{R}}(\mathbf{x} - v\mathbf{1} - \mu\mathbf{1} + \delta w\mathbf{1} + \delta\mathbf{s})\, dG(v)\, dw}. \tag{4.6}$$

The denominator in (4.6) is reducible to the product of a single integral over v and an explicit function of $\mathbf{x}$:

$$\begin{aligned} &\frac{\sqrt{2\pi}}{\lambda}\int_{-\infty}^{\infty}\frac{\theta}{\delta}\exp\left(-\frac{\theta v}{\delta}\right)\left(\int_{H(\mathbf{y} - v\mathbf{1})/\lambda}^{\infty}\phi(y)\,dy\right) dG(v) \\ &\qquad \times \phi_{\mathbf{R}}(\mathbf{y})\exp\left(\frac{\theta^2}{2\lambda^2\delta^2} + \frac{(\mathbf{1}'\mathbf{R}\mathbf{y})^2}{2\lambda^2} + \frac{\theta}{\delta}\frac{\mathbf{1}'\mathbf{R}\mathbf{y}}{\lambda^2}\right), \end{aligned} \tag{4.7}$$

where

$$\mathbf{y} = \mathbf{x} - \mu\mathbf{1} + \delta\mathbf{s}. \tag{4.8}$$

For the proof of (4.7), we integrate in the denominator of (4.6), first with respect to w with v fixed:

$$\int_0^\infty \theta e^{-\theta w} \phi_{\mathbf{R}}(\mathbf{z} + \delta w \mathbf{1})\, dw,$$

where $\mathbf{z} = \mathbf{x} - v\mathbf{1} - \mu\mathbf{1} + \delta\mathbf{s}$. The integral is transformed to

$$\int_0^\infty (\theta/\delta) e^{-(\theta/\delta)w} \phi_{\mathbf{R}}(\mathbf{z} + w\mathbf{1})\, dw,$$

which, by the calculations leading from (3.4) to (3.6), is equal to

$$\left(\frac{\theta}{\delta}\right) \phi_{\mathbf{R}}(\mathbf{z}) \exp\left(\frac{1}{2}\frac{H^2(\mathbf{z})}{\lambda^2}\right) \frac{\sqrt{2\pi}}{\lambda} \int_{H(\mathbf{z})/\lambda}^\infty \phi(y)\, dy. \tag{4.9}$$

Take $\mathbf{y}$ as in (4.8), so that $\mathbf{z} = \mathbf{y} - v\mathbf{1}$. Then, by (3.1) and (3.3),

$$\begin{aligned} &\phi_{\mathbf{R}}(\mathbf{y} - v\mathbf{1}) \exp\left(\frac{1}{2}\frac{H^2(\mathbf{y} - v\mathbf{1})}{\lambda^2}\right) \\ &\quad = \phi_{\mathbf{R}}(\mathbf{y}) \exp\left(v\mathbf{1}'\mathbf{R}\mathbf{y} - \frac{1}{2}v^2\lambda^2 + \frac{1}{2\lambda^2}\left[\frac{\theta}{\delta} + 1'Ry - v\lambda^2\right]^2\right). \end{aligned} \tag{4.10}$$

Simple algebra yields

$$\begin{aligned} &v\mathbf{1}'\mathbf{R}\mathbf{y} - \frac{1}{2}v^2\lambda^2 + \frac{1}{2\lambda^2}\left(\frac{\theta}{\delta} + \mathbf{1}'\mathbf{R}\mathbf{y} - v\lambda^2\right)^2 \\ &\quad = \frac{\theta^2}{2\lambda^2\delta^2} + \frac{(\mathbf{1}'\mathbf{R}\mathbf{y})^2}{2\lambda^2} + \frac{\theta}{\delta}\frac{\mathbf{1}'\mathbf{R}\mathbf{y}}{\lambda^2} - \frac{\theta}{\delta}v. \end{aligned}$$

Therefore, the right-hand member of (4.10) is equal to

$$e^{-\theta v/\delta} \phi_{\mathbf{R}}(\mathbf{y}) \exp\left(\frac{\theta^2}{2\lambda^2\delta^2} + \frac{(\mathbf{1}'\mathbf{R}\mathbf{y})^2}{2\lambda^2} + \frac{\theta}{\delta}\frac{\mathbf{1}'\mathbf{R}\mathbf{y}}{\lambda^2}\right). \tag{4.11}$$

Therefore, substituting $\mathbf{y} - v\mathbf{1}$ for $\mathbf{z}$ in (4.9) and multiplying by $dG(v)$ and integrating over v, we find that the denominator in (4.6) is equal to the product in (4.7).

THEOREM 4.1. *Suppose that V has a normal distribution and that T has an exponential distribution with mean $1/\theta$. Define these analogues of λ and $H(\mathbf{x})$:*

$$\begin{aligned} \lambda_\tau &= \left[\mathbf{1}'(\Sigma + \tau^2\mathbf{1}\mathbf{1}')^{-1}\mathbf{1}\right]^{1/2}, \\ H_\tau(\mathbf{x}) &= \theta/\delta + \mathbf{1}'(\Sigma + \tau^2\mathbf{1}\mathbf{1}')^{-1}\mathbf{x}. \end{aligned} \tag{4.12}$$

Then the posterior density of T at t, that is, its conditional density given $Z + V\mathbf{1} - \delta t\mathbf{1} - \delta\mathbf{s} = \mathbf{x}$, is equal to the expression (3.2) *with λ_τ and H_τ in the places of λ and H, respectively, and $(\mu + \nu)\mathbf{1} - \delta\mathbf{s}$ in the role of $\boldsymbol{\mu}$.*

PROOF. Since $\mathbf{Z}$ has a $N(\mu\mathbf{1}, \Sigma)$ distribution, $\mathbf{Z} + V\mathbf{1} - \delta\mathbf{s}$ has a $N((\mu + \nu)\mathbf{1} - \delta\mathbf{s}, \Sigma + \tau^2\mathbf{1}\mathbf{1}')$ distribution, and the statement of the theorem is a direct consequence of Lemma 3.1 and the definitions (4.12). □

In the following theorem we furnish a bound for the absolute difference

$$\begin{aligned}&|\text{posterior density (4.6) with normal } G(v)\\ &\quad - \text{posterior density (4.6) with general } G(v)|.\end{aligned} \tag{4.13}$$

THEOREM 4.2. *The absolute difference* (4.13) *is at most equal to the product of the factors*

$$\frac{\textit{expression } (2.5)}{\textit{denominator of } (4.6)} \tag{4.14}$$

and

$$1 + \textit{density } (3.2) \tag{4.15}$$

with λ_τ *and* H_τ *in the places of* λ *and* H, *and* $(\mu + \nu)\mathbf{1} - \delta\mathbf{s}$ *in the role of* $\boldsymbol{\mu}$.

PROOF. Let x, x', y and y' be positive numbers. Then

$$\left|\frac{x}{y} - \frac{x'}{y'}\right| \le \frac{\max(|x - x'|, |y - y'|)}{y'}\left(1 + \frac{x}{y}\right). \tag{4.16}$$

For the proof observe that

$$\left|\frac{x}{y} - \frac{x'}{y'}\right| = \frac{1}{yy'}|x(y' - y) + y(x - x')| \le \frac{\max(|x - x'|, |y - y'|)}{y'}\left(1 + \frac{x}{y}\right).$$

We apply (4.16) to (4.13) with

$$\begin{aligned} x &= \text{numerator in (4.6) with normal } G,\\ y &= \text{denominator in (4.6) with normal } G,\\ x' &= \text{numerator in (4.6) with general } G,\\ y' &= \text{denominator in (4.6) with general } G.\end{aligned}$$

Then (4.13) is representable as the left-hand member of (4.16). The product of (4.14) and (4.15), as a consequence of Theorem 2.1, Corollary 2.2, formula (2.5) and Theorem 4.1, represents a bound for the right-hand member of (4.16) □

5. Approximations. It is clear in the formulation of the general problem stated in Section 1 that the normal distribution of $\mathbf{Z}$ is transformed into a normal distribution with mean $\boldsymbol{\mu} + \nu\mathbf{1}$ if X has a distribution that is degenerate at ν. If X is "close in probability" to ν, then one expects the distribution of $\mathbf{Z} + X\mathbf{1}$ to be close to normal, and the posterior density of T, considered in Section 4, to be close to the censored normal density. In this section, we illustrate precise estimates by means of the results of Section 4.

Let $G(x)$ be an arbitrary distribution function such that $\int_{-\infty}^{\infty}|x|^3\,dG(x) < \infty$, $\int_{-\infty}^{\infty} x\,dG(x) = 0$ and $\int_{-\infty}^{\infty} x^2\,dG(x) = 1$. For arbitrary real ν and $\tau > 0$, consider $G((v-\nu)/\tau)$, a distribution with mean ν and variance τ^2. If $f(u)$ represents the characteristic function of $G(x)$, then $e^{iu\nu}f(u\tau)$ is the characteristic function of $G((v-\nu)/\tau)$, and the bound (2.5) takes the form

$$\frac{\int_{-\infty}^{\infty}|f(\lambda\tau x) - \exp(-\frac{1}{2}x^2\lambda^2\tau^2)|\phi(x)\,dx}{(2\pi)^{k/2}(\det\Sigma)^{1/2}}. \tag{5.1}$$

By employing for f the local expansion of the characteristic function used in the proof of Theorem 2.3, together with the inequality $|\exp(-\frac{1}{2}x^2) - 1 + \frac{1}{2}x^2| \leq \frac{1}{8}x^4$, one finds that the numerator of (5.1) is at most equal to

$$\frac{1}{6}\tau^3\lambda^3\int_{-\infty}^{\infty}|y|^3\,dG(y)\frac{4}{\sqrt{2\pi}} + \frac{1}{8}\tau^4\lambda^4, \tag{5.2}$$

where we have used the known relations

$$\int|u|^3\phi(u)\,du = 4/\sqrt{2\pi} \quad\text{and}\quad \int u^4\phi(u)\,du = 3.$$

It follows that the bound (2.5) is on the order τ^3 for $\tau \to 0$, corresponding to the first term in (5.2).

Next we show that the order τ^3 is preserved for the posterior density of T. On the basis of Theorem 4.2 it suffices to show that the denominator in (4.14), that is, the expression (4.7), converges to a positive limit for $\tau \to 0$, and that the expression (4.15) converges to a finite limit. These limits are explicit, and their numerical values are deducible from the tables of the normal distribution and density.

The expression (4.7) depends on τ only through the integral

$$\int_{-\infty}^{\infty}\frac{\theta}{\delta}e^{-\theta v/\delta}\left(\int_{H(\mathbf{y}-v\mathbf{1})/\lambda}^{\infty}\phi(y)\,dy\right)dG\left(\frac{v-\nu}{\tau}\right).$$

This converges, for $\tau \to 0$, to the positive limit

$$\frac{\theta}{\delta}e^{-\nu\theta/\delta}\int_{H(\mathbf{y}-\nu\mathbf{1})/\lambda}^{\infty}\phi(y)\,dy.$$

In the expression (4.15), the density (3.2) with λ_τ and H_τ is, by the definitions (4.12), a continuous function of τ and converges, for $\tau \to 0$, to the density (3.2) with the given λ and H. This completes the proof of the assertion about the rate of convergence of the posterior density of T.

Next we consider the following more general problem. Let (X_n) be a sequence of random variables with $EX_n = \nu_n$ and $\operatorname{Var} X_n = \tau_n^2$. Suppose that $\nu_n \to \nu$ and $\tau_n \to \tau$, and that $X_n \to Y$ in distribution, where $\mathbf{Y}$ is $N(\nu, \tau^2)$. Let Δ_n be the bound corresponding to (2.5):

$$\Delta_n = \frac{\int_{-\infty}^{\infty}|\exp(-i\nu_n\lambda x)f_n(\lambda x) - \exp(-\frac{1}{2}x^2\lambda^2\tau_n^2)|\phi(x)\,dx}{(2\pi)^{k/2}(\det\Sigma)^{1/2}}, \tag{5.3}$$

where f_n is the characteristic function of X_n. Then $\Delta_n \to 0$ for $n \to \infty$.

Let us show that the difference between the posterior density of T, under the (generally nonnormal) distribution of X_n, and the posterior density of T, under the normal distribution of Y, is on the order Δ_n in (5.3). It suffices to show that the denominator in (4.14), that is, the expression (4.7), converges to a positive limit and that (4.15) converges to a finite limit.

In evaluating the limit of (4.7), we let $G_n(v) = G(v)$ be the distribution function of X_n. By the assumed complete convergence of G_n to G, it follows that (4.7) converges to the same expression with a $N(\nu, \tau^2)$ distribution in the place of G. This is clearly positive.

In evaluating the limit of (4.15), we again refer to (3.2) with

$$\lambda_n = \left(\mathbf{1}'\left(\Sigma + \tau_n^2\mathbf{1}\mathbf{1}'\right)^{-1}\mathbf{1}\right)^{1/2},$$

$$H_n(\mathbf{x}) = \theta/\delta + \mathbf{1}'\left(\Sigma + \tau_n^2\mathbf{1}\mathbf{1}'\right)^{-1}\mathbf{x}$$

and with $(\mu + \nu_n)\mathbf{1} - \delta\mathbf{s}$ in the role of $\boldsymbol{\mu}$. These sequences converge under the hypothesis $\nu_n \to \nu$ and $\tau_n \to \tau$ to the corresponding limits λ_τ, H_τ and $(\mu + \nu)\mathbf{1} - \delta\mathbf{s}$, respectively. The limit (4.15) is finite.

This application demonstrates that the particular estimates obtained for the bound (2.5) in the expressions (2.8) and (2.12), for $\varepsilon \to 0$ and $n \to \infty$, can be suitably extended to the bound for the error in the posterior density of T.

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