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ON THE LOCAL TIME OF THE BROWNIAN MOTION

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In this paper explicit formulas are given for the distribution function, the density function and the moments of the local time of the reflecting Brownian motion process.

1. Introduction. Let $\{\xi(t),\ 0 \le t \le 1\}$ be a standard Brownian motion process. We have $\mathbf{P}\{\xi(t) \le x\} = \Phi(x/\sqrt{t})$, for $0 < t \le 1$, where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(\frac{-u^2}{2}\right) du \tag{1}$$

is the normal distribution function. Let us define

$$\tau(\alpha) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \text{ measure } \{t\colon\ \alpha \le \xi(t) < \alpha + \varepsilon,\ 0 \le t \le 1\} \tag{2}$$

for any real α. The limit (2) exists with probability 1 and $\tau(\alpha)$ is called the local time at level α. We have

$$\mathbf{P}\{\tau(\alpha) \le x\} = 2\Phi(|\alpha| + x) - 1 \tag{3}$$

for $x \ge 0$. The concept of local time was introduced by Lévy [9, 10]. See also Trotter [14] and Itô and McKean [6].

By (3) we obtain that

$$m_r(\alpha) = \mathbf{E}\{[\tau(\alpha)]^r\} = 2\alpha^{r+1} \int_1^{\infty} \varphi(\alpha x)(x-1)^r\, dx \tag{4}$$

for $\alpha > 0$ and $r \ge 1$, where

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right) \tag{5}$$

is the normal density function, or

$$m_r(\alpha) = 2r \int_0^{\infty} x^{r-1}[1 - \Phi(\alpha + x)]\, dx \tag{6}$$

for $\alpha > 0$ and $r \ge 1$, where $\Phi(x)$ is defined by (1). We shall prove later that $m_r(\alpha)$ can also be expressed in the form

$$m_r(\alpha) = \frac{\alpha r!}{2^{r/2}\Gamma(r/2+1)} \int_0^1 \varphi\left(\frac{\alpha}{\sqrt{t}}\right) \frac{(1-t)^{r/2}}{t^{3/2}}\, dt \tag{7}$$

if $\alpha > 0$ and $r \ge 1$.

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In this paper we consider the reflecting Brownian motion process $\{|\xi(t)|, 0 \le t \le 1\}$. Its local time at level $\alpha > 0$ is $\tau(\alpha) + \tau(-\alpha)$. We shall determine the distribution and the moments of $\tau(\alpha) + \tau(-\alpha)$. Our approach is based on a symmetric random walk $\{\zeta_r,\ r \ge 0\}$, where $\zeta_r = \xi_1 + \xi_2 + \cdots + \xi_r$ for $r \ge 1$, $\zeta_0 = 0$ and $\{\xi_r,\ r \ge 1\}$ is a sequence of independent and identically distributed random variables for which

$$\mathbf{P}\{\xi_r = 1\} = \mathbf{P}\{\xi_r = -1\} = 1/2. \tag{8}$$

By a result of Donsker [3], if $n \to \infty$, the process $\{\zeta_{[nt]}/\sqrt{n},\ 0 \le t \le 1\}$ converges weakly to the Brownian motion process $\{\xi(t),\ 0 \le t \le 1\}$. If we define

(9) $\quad \tau_n(a) =$ the number of subscripts $r = 1, 2, \ldots, n$ for which $\zeta_r = a$,

then by the results of Knight [7] we can draw the conclusion that

$$\lim_{n\to\infty} \mathbf{P}\left\{\frac{\tau_n([\alpha\sqrt{n}])}{\sqrt{n}} \le x\right\} = \mathbf{P}\{\tau(\alpha) \le x\} \tag{10}$$

for any α and $x > 0$, and also

$$\lim_{n\to\infty} \mathbf{P}\left\{\frac{\tau_n([\alpha\sqrt{n}]) + \tau_n(-[\alpha\sqrt{n}])}{\sqrt{n}} \le x\right\} = \mathbf{P}\{\tau(\alpha) + \tau(-\alpha) \le x\} \tag{11}$$

for $\alpha > 0$ and $x > 0$.

In this paper we shall determine the distribution and the moments of $\tau_n(a) + \tau_n(-a)$ and by a suitable limiting process we shall find the distribution function

$$\mathbf{P}\{\tau(\alpha) + \tau(-\alpha) \le x\} = L_\alpha(x) \tag{12}$$

and the moments

$$\mathbf{E}\{[\tau(\alpha) + \tau(-\alpha)]^r\} = M_r(\alpha) \tag{13}$$

for $\alpha > 0$ and $r \ge 0$.

2. A symmetric random walk. Let us recall some results for the symmetric random walk $\{\zeta_r,\ r \ge 0\}$ which we need in this paper. See Takács [13].

We have

$$\mathbf{P}\{\zeta_n = 2j - n\} = \binom{n}{j}\frac{1}{2^n}, \tag{14}$$

for $j = 0, 1, \ldots, n$, and by the central limit theorem,

$$\lim_{n\to\infty} \mathbf{P}\left\{\frac{\zeta_n}{\sqrt{n}} \le x\right\} = \Phi(x), \tag{15}$$

where $\Phi(x)$ is defined by (1). We have also

$$\lim_{n\to\infty} \mathbf{E}\left\{\left(\frac{\zeta_n}{\sqrt{n}}\right)^r\right\} = \int_{-\infty}^{\infty} x^r\, \varphi(x)\, dx \tag{16}$$

for $r = 0, 1, 2, \ldots$, where $\varphi(x)$ is defined by (5).

Let us define $\rho(a)$ as the first passage time through a, that is,

$$\rho(a) = \inf\{r\colon\ \zeta_r = a \text{ and } r \geq 0\}. \tag{17}$$

We have

$$\mathbf{P}\{\rho(a) = a + 2j\} = \frac{a}{a+2j}\binom{a+2j}{j}\frac{1}{2^{a+2j}} \tag{18}$$

for $a \geq 1$ and $j \geq 0$, or

$$\mathbf{P}\{\rho(a) \leq n\} = \mathbf{P}\{\zeta_n \geq a\} + \mathbf{P}\{\zeta_n > a\} \tag{19}$$

for $a \geq 1$ and $n \geq 0$.

By (18),

$$\sum_{j=0}^{\infty} \mathbf{P}\{\rho(a) = a + 2j\} w^j = \left(\frac{1 - \sqrt{1-w}}{w}\right)^a \tag{20}$$

for $a \geq 1$ and $|w| \leq 1$.

We note that

$$\sum_{j=0}^{n} \binom{a+j-1}{j} = \binom{a+n}{n} \tag{21}$$

for any a and $n \geq 0$, and

$$\sum_{j=0}^{\infty} \binom{a+j-1}{j} w^j = (1-w)^{-a} \tag{22}$$

for any a and $|w| < 1$. In particular, we have

$$\sum_{n=0}^{\infty} \binom{2n}{n}\frac{w^n}{2^{2n}} = \sum_{n=0}^{\infty} \binom{n-\frac{1}{2}}{n} w^n = (1-w)^{-1/2} \tag{23}$$

for $|w| < 1$. Finally, we note that the identity

$$\sum_{j=0}^{n} \mathbf{P}\{\rho(a) = j\}\mathbf{P}\{\rho(b) = n - j\} = \mathbf{P}\{\rho(a+b) = n\} \tag{24}$$

is valid for any $a \geq 1$, $b \geq 1$ and $n \geq 1$.

3. The distribution of $\tau_n(a)$. The distribution of $\tau_n(a)$ is determined by (18) and we have the following asymptotic results.

THEOREM 1. *If $\alpha > 0$ and $x > 0$, then*

$$\lim_{n\to\infty} \mathbf{P}\left\{\frac{\tau_n([\alpha\sqrt{n}])}{\sqrt{n}} \geq x\right\} = 2[1-\Phi(\alpha+x)] \tag{25}$$

and

$$\lim_{n\to\infty} \mathbf{E}\left\{\left[\frac{\tau_n([\alpha\sqrt{n}])}{\sqrt{n}}\right]^r\right\} = m_r(\alpha) \tag{26}$$

for $r \geq 1$, where $m_r(\alpha)$ is given by (4) *or by* (6).

PROOF. Let $a \geq 1$ and denote by $\theta_1, \theta_1+\theta_2, \ldots, \theta_1+\theta_2+\cdots+\theta_r, \ldots$ the successive subscripts $r = 1, 2, \ldots$, for which $\zeta_r = a$. Then $\theta_1, \theta_2, \ldots, \theta_r, \ldots$ are independent random variables. We have $\mathbf{P}\{\theta_1 = a+2j\} = \mathbf{P}\{\rho(a) = a+2j\}$ for $j = 0, 1, 2, \ldots$, and $\mathbf{P}\{\theta_r = 2j\} = \mathbf{P}\{\rho(1) = 2j-1\}$ for $j = 1, 2, \ldots$ and $r > 1$. By (24) we obtain that

$$\begin{aligned}\mathbf{P}\{\tau_n(a) \geq k\} &= \mathbf{P}\{\theta_1+\cdots+\theta_k \leq n\} = \mathbf{P}\{\rho(a+k-1) \leq n+1-k\} \\ &= \mathbf{P}\{\zeta_{n+1-k} \geq a+k-1\} + \mathbf{P}\{\zeta_{n+1-k} > a+k-1\}\end{aligned} \tag{27}$$

for $k \geq 1$. We note that

$$\mathbf{P}\{\tau_n(0) \geq k\} = \mathbf{P}\{\rho(k) \leq n-k\} \tag{28}$$

for $k \geq 1$. If in (27) we put $a = [\alpha\sqrt{n}\,]$, where $\alpha > 0$, and $k = [x\sqrt{n}\,]$, where $x > 0$, then by (15) we obtain (25). By (10), (25) proves (3) too. If we calculate the rth moment of $\tau_n(a)$ by (27) and replace a by $[\alpha\sqrt{n}\,]$, where $\alpha > 0$, then by (16) we obtain (26) for $r \geq 1$. □

Accordingly, if $\alpha > 0$, then $\tau(\alpha)$ has the same distribution as $[|\xi| - \alpha]^+$, where ξ is a random variable whose distribution function is given by (1). Consequently,

$$m_r(\alpha) = \mathbf{E}\{[\tau(\alpha)]^r\} \leq \mathbf{E}\{|\xi|^r\} \tag{29}$$

for $r \geq 1$ and $\alpha > 0$.

The following theorem gives an explicit expression for $m_r(\alpha)$.

THEOREM 2. *If $\alpha > 0$, then*

$$m_r(\alpha) = 2(-1)^r\{a_r(\alpha)[1-\Phi(\alpha)] - b_r(\alpha)\varphi(\alpha)\} \tag{30}$$

for $r = 1, 2, \ldots$, where

$$a_r(\alpha) = r! \sum_{j=0}^{[r/2]} \frac{\alpha^{r-2j}}{2^j j!(r-2j)!} \tag{31}$$

for $r \geq 1$, *and*

$$b_r(\alpha) = \sum_{j=0}^{[(r-1)/2]} \frac{\alpha^{r-1-2j}}{(r-1-2j)!} \sum_{\nu=0}^{j} \frac{(r-1+\nu-j)!}{2^\nu \nu!} \tag{32}$$

for $r \geq 1$.

PROOF. We obtain (30) from (6) by repeated integrations by parts. In (30), $a_0(\alpha) = 1$, $a_1(\alpha) = \alpha$, $b_0(\alpha) = 0$, $b_1(\alpha) = 1$ and

$$a_r(\alpha) = \alpha a_{r-1}(\alpha) + (r-1)a_{r-2}(\alpha) \tag{33}$$

for $r \geq 2$, and

$$b_r(\alpha) = \alpha b_{r-1}(\alpha) + (r-1)b_{r-2}(\alpha) \tag{34}$$

for $r \geq 2$. Hence we obtain that

$$\sum_{r=0}^{\infty} \frac{a_r(\alpha)}{r!} x^r = \exp\left(\frac{x^2 + 2\alpha x}{2}\right) \tag{35}$$

and

$$\sum_{r=0}^{\infty} \frac{b_r(\alpha)}{r!} x^r = \exp\left(\frac{(x+\alpha)^2}{2}\right) \int_0^x \exp\left(\frac{-(u+\alpha)^2}{2}\right) du. \tag{36}$$

By expanding (35) and (36) into Taylor series, we get (31) and (32). □

We note that $a_n(\alpha) = H_n(i\alpha)/i^n$, for $n \geq 0$, where $H_n(x)$ is the nth Hermite polynomial defined by

$$H_n(x) = n! \sum_{j=0}^{[n/2]} \frac{(-1)^j x^{n-2j}}{2^j j!(n-2j)!} \tag{37}$$

for $n = 0, 1, 2, \ldots$. We have

$$H_n(x) = xH_{n-1}(x) - (n-1)H_{n-2}(x) \tag{38}$$

for $n \geq 2$, where $H_0(x) = 1$ and $H_1(x) = x$.

The moments of $\tau_n(a)$ can be determined explicitly by the following theorem.

THEOREM 3. *If* $a \geq 1$ *and* $r \geq 1$, *we have*

$$\mathbf{E}\left\{\binom{\tau_n(a)}{r}\right\} = \sum_{0 \leq j \leq (n-a-2r+2)/2} \binom{r/2 + [(n-a-2r-2j+2)/2]}{[(n-a-2r-2j+2)/2]} \times \mathbf{P}\{\rho(a+r-1) = a+r-1+2j\}, \tag{39}$$

where $\mathbf{P}\{\rho(a+r-1) = a+r-1+2j\}$ *is determined by* (18).

PROOF. If A_r denotes the event that $\zeta_r = a$, then the rth binomial moment of $\tau_n(a)$ can be expressed in the following way:

$$\mathbf{E}\left\{\binom{\tau_n(a)}{r}\right\} = \sum_{0\le i_1<i_2<\cdots<i_r\le(n-a)/2} \mathbf{P}\{A_{a+2i_1}A_{a+2i_2}\cdots A_{a+2i_r}\}. \tag{40}$$

It is easy to see that

$$\mathbf{P}\{A_{a+2j}\} = \sum_{0\le i\le j} \mathbf{P}\{\rho(a) = a+2i\}\binom{2j-2i}{j-i}\frac{1}{2^{2j-2i}} \tag{41}$$

for $j \ge 0$. Hence by (20) and (23),

$$\sum_{j=0}^{\infty}\mathbf{P}\{A_{a+2j}\}w^j = \left(\frac{1-\sqrt{1-w}}{w}\right)^a \frac{1}{\sqrt{1-w}} \tag{42}$$

for $|w| < 1$. Clearly,

$$\mathbf{P}\{A_{a+2i+2j}|A_{a+2i}\} = \binom{2j}{j}\frac{1}{2^{2j}} \tag{43}$$

for $j \ge 1$, and by (23),

$$\sum_{j=1}^{\infty}\mathbf{P}\{A_{a+2i+2j}|A_{a+2i}\}w^j = \frac{1-\sqrt{1-w}}{\sqrt{1-w}} \tag{44}$$

for $|w| < 1$.

Since the random walk $\{\zeta_r,\ r \ge 0\}$ possesses the Markov property, we can determine (40) by (41) and (43). If in the generating function

$$\begin{aligned}&\left(\frac{1-\sqrt{1-w}}{w}\right)^a\left(\frac{1-\sqrt{1-w}}{\sqrt{1-w}}\right)^{r-1}\frac{1}{\sqrt{1-w}}\\ &\quad= \left(\frac{1-\sqrt{1-w}}{w}\right)^{a+r-1}\frac{w^{r-1}}{(1-w)^{r/2}}\end{aligned} \tag{45}$$

we form the coefficient of w^k and sum these coefficients for every $k \le (n-a)/2$, then we obtain (40). Accordingly,

$$\begin{aligned}&\mathbf{E}\left\{\binom{\tau_n(a)}{r}\right\}\\ &\quad= \sum_{j+s\le(n-a-2r+2)/2}\binom{r/2+s-1}{s}\mathbf{P}\{\rho(a+r-1) = a+r-1+2j\},\end{aligned} \tag{46}$$

where $j \ge 0$ and $s \ge 0$. If in (46) we form the sum with respect to s for $0 \le s \le (n-a-2r+2-2j)/2$, then by (21) we obtain (39). □

Let us put $a = [\alpha\sqrt{n}\,]$, where $\alpha > 0$, and $j = [nt/2]$, where $0 < t < 1$, in (39), and let $n \to \infty$. Then we obtain that

$$\lim_{n\to\infty} \mathbf{E}\left\{\left[\frac{\tau_n([\alpha\sqrt{n}])}{\sqrt{n}}\right]^r\right\} = \frac{\alpha r!}{2^{r/2}\Gamma(r/2+1)} \int_0^1 \varphi\left(\frac{\alpha}{\sqrt{t}}\right)\frac{(1-t)^{r/2}}{t^{3/2}}\,dt \tag{47}$$

for $r \geq 1$. This proves (7). It is not obvious that formula (7) yields the same result as (4) or (6). A referee of this paper provided a simple probabilistic proof for formula (7). He pointed out that if in the process $\{\xi(t),\ 0 \leq t \leq 1\}$ the first passage through $\alpha > 0$ occurs at time $t \in (0,1)$, then under this condition $\tau(\alpha)$ has the same distribution as $(1-t)^{1/2}|\xi|$, where $\mathbf{P}\{\xi \leq x\} = \Phi(x)$ defined by (1). This observation implies (7).

4. The distribution of $\tau(\alpha) + \tau(-\alpha)$. The distribution of $\tau_n(a) + \tau_n(-a)$ is determined by its binomial moments. We have

$$\mathbf{P}\{\tau_n(a) + \tau_n(-a) = k\} = \sum_{r=k}^{n} (-1)^{r-k}\binom{r}{k}\mathbf{E}\left\{\binom{\tau_n(a)+\tau_n(-a)}{r}\right\} \tag{48}$$

for $k = 0, 1, 2, \ldots$.

THEOREM 4. *If $r = 1, 2, \ldots$ and $a = 1, 2, \ldots$, we have*

$$\begin{aligned} &\mathbf{E}\left\{\binom{\tau_n(a)+\tau_n(-a)}{r}\right\} \\ &\quad = 2\sum_{\ell=1}^{r}\binom{r-1}{\ell-1}\mathbf{E}\left\{\binom{\tau_{n+\ell-1}((2\ell-1)a-\ell+1)}{r}\right\}, \end{aligned} \tag{49}$$

where the right-hand side is determined by (39).

PROOF. Let $C_i = A_i \cup B_i$, where $A_i = \{\zeta_i = a\}$ and $B_i = \{\zeta_i = -a\}$. Then

$$\mathbf{E}\left\{\binom{\tau_n(a)+\tau_n(-a)}{r}\right\} = \sum_{0 \leq i_1 < i_2 < \cdots < i_r \leq (n-a)/2} \mathbf{P}\{C_{a+2i_1}C_{a+2i_2}\cdots C_{a+2i_r}\}. \tag{50}$$

If $i \geq 0$ and $j \geq 0$, we have by symmetry

$$\mathbf{P}\{A_{3a+2i+2j}|B_{a+2i}\} = \mathbf{P}\{B_{3a+2i+2j}|A_{a+2i}\} \tag{51}$$

and

$$\mathbf{P}\{B_{3a+2i+2j}|A_{a+2i}\} = \sum_{0\leq s \leq j} \mathbf{P}\{\rho(2a) = 2a+2s\}\binom{2j-2s}{j-s}\frac{1}{2^{2j-2s}}. \tag{52}$$

By (20) and (23),

$$\sum_{j=0}^{\infty} \mathbf{P}\{B_{3a+2i+2j}|A_{a+2i}\}w^j = \left(\frac{1-\sqrt{1-w}}{w}\right)^{2a}\frac{1}{\sqrt{1-w}} \tag{53}$$

for $|w| < 1$. If in (50) we express each C_i as $A_i \cup B_i$, then the right-hand side of (50) can be expressed as a sum of 2^r probabilities. Each term is the probability of the occurrence of r events in succession. Among these 2^r probabilities there are $2\binom{r-1}{\ell-1}$ in which among the r events either an event A_i is followed by an event B_j or an event B_i is followed by an event A_j exactly $\ell - 1$ times, where $\ell = 1, 2, \dots, r$. If we use (42), (44) and (53), then (50) can be obtained in the following way: We form the generating function

$$(54)\quad \begin{aligned} &2\sum_{\ell=1}^{r}\binom{r-1}{\ell-1}\left(\frac{1-\sqrt{1-w}}{w}\right)^{a+2(\ell-1)a}\left(\frac{1}{\sqrt{1-w}}\right)^{\ell}\left(\frac{1-\sqrt{1-w}}{\sqrt{1-w}}\right)^{r-\ell} \\ &= 2\sum_{\ell=1}^{r}\binom{r-1}{\ell-1}\left(\frac{1-\sqrt{1-w}}{w}\right)^{(2\ell-1)a+r-\ell}\frac{w^{r-\ell}}{(1-w)^{r/2}}. \end{aligned}$$

In the ℓth term of this sum we form the coefficient of w^i and add these coefficients for every $i \le n - (2\ell - 1)a$ and also for every $\ell = 1, 2, \dots, r$. Then we obtain (50). Accordingly,

$$(55)\quad \begin{aligned} &\mathbf{E}\left\{\binom{\tau_n(a)+\tau_n(-a)}{r}\right\} \\ &= 2\sum_{\ell=1}^{r}\binom{r-1}{\ell-1}\sum_{s+j\le(n-(2\ell-1)a-2r+2\ell)/2}\binom{r/2+s-1}{s} \\ &\quad\times \mathbf{P}\{\rho((2\ell-1)a+r-\ell) = (2\ell-1)a+r-\ell+2j\}, \end{aligned}$$

where $s \ge 0$ and $j \ge 0$. If in (55) we form the summation with respect to s for $0 \le s \le (n - (2\ell - 1)a - 2r + 2\ell - 2j)/2$, we obtain that

$$(56)\quad \begin{aligned} &\mathbf{E}\left\{\binom{\tau_n(a)+\tau_n(-a)}{r}\right\} = 2\sum_{\ell=1}^{r}\binom{r-1}{\ell-1} \\ &\times \sum_{0\le j\le(n-(2\ell-1)a-2r+2\ell)/2}\binom{r/2+[(n-(2\ell-1)a-2r+2\ell-2j)/2]}{[(n-(2\ell-1)a-2r+2\ell-2j)/2]} \\ &\times \mathbf{P}\{\rho((2\ell-1)a+r-\ell) = (2\ell-1)a+r-\ell+2j\} \end{aligned}$$

for $r \ge 1$ and $a \ge 1$. A comparison of (39) and (56) proves (49). □

THEOREM 5. *If $\alpha > 0$ and $r \ge 1$, then the limit*

$$(57)\quad \lim_{n\to\infty}\mathbf{E}\left\{\left[\frac{\tau_n([\alpha\sqrt{n}])+\tau_n(-[\alpha\sqrt{n}])}{\sqrt{n}}\right]^r\right\} = M_r(\alpha)$$

exists and

$$(58)\quad M_r(\alpha) = 2\sum_{\ell=1}^{r}\binom{r-1}{\ell-1}m_r((2\ell-1)\alpha),$$

where $m_r(\alpha)$ is given by (4) or by (6).

PROOF. If in (49) we put $a = [\alpha\sqrt{n}\,]$, where $\alpha > 0$, and let $n \to \infty$, we get (57). □

THEOREM 6. *If $\alpha > 0$, then there exists a distribution function $L_\alpha(x)$ of a nonnegative random variable such that*

$$\lim_{n\to\infty} \mathbf{P}\left\{\frac{\tau_n([\alpha\sqrt{n}]) + \tau_n(-[\alpha\sqrt{n}])}{\sqrt{n}} \le x\right\} = L_\alpha(x) \tag{59}$$

in every continuity point of $L_\alpha(x)$. The distribution function $L_\alpha(x)$ is uniquely determined by its moments

$$\int_{-0}^{\infty} x^r \, dL_\alpha(x) = M_r(\alpha) \tag{60}$$

for $r \ge 0$, where $M_0(\alpha) = 1$ and $M_r(\alpha)$ for $r \ge 1$ is given by (58).

PROOF. Since

$$M_r(\alpha) \le 2^r \mathbf{E}\{|\xi|^r\} \tag{61}$$

for $r \ge 1$, where $\mathbf{P}\{\xi \le x\} = \Phi(x)$, the sequence of moments $\{M_r(\alpha)\}$ uniquely determines $L_\alpha(x)$, and $L_\alpha(x) = 0$ for $x < 0$. By the moment convergence theorem of Fréchet and Shohat [5] we can conclude that (60) implies (59). □

Theorems 5 and 6 imply (12) and (13). Formula (58) is a surprisingly simple expression for the rth moment of $\tau(\alpha) + \tau(-\alpha)$. If we know the rth moment of $\tau(\alpha)$ for $\alpha > 0$, then by (58) the rth moment of $\tau(\alpha) + \tau(-\alpha)$ can immediately be determined for $\alpha > 0$. Moreover, formula (58) makes it possible to determine $L_\alpha(x)$ explicitly.

THEOREM 7. *If $x \ge 0$ and $\alpha > 0$, we have*

$$\begin{aligned} L_\alpha(x) = 1 &- 4\sum_{\ell=1}^{\infty}(-1)^{\ell-1}[1 - \Phi((2\ell-1)\alpha + x)] \\ &+ 4\sum_{\ell=2}^{\infty}\sum_{j=1}^{\ell-1}\binom{\ell-1}{j}\frac{(-1)^{\ell-1}x^j}{j!}\varphi^{(j-1)}((2\ell-1)\alpha + x), \end{aligned} \tag{62}$$

and if $x > 0$ and $\alpha > 0$, we have

$$L'_\alpha(x) = 4\sum_{\ell=1}^{\infty}\sum_{j=0}^{\ell-1}\binom{\ell}{j+1}\frac{(-1)^{\ell-1}x^j}{j!}\varphi^{(j)}((2\ell-1)\alpha + x), \tag{63}$$

where

$$\varphi^{(j)}(x) = (-1)^j \varphi(x) H_j(x), \tag{64}$$

$\varphi(x)$ is defined by (5) *and $H_j(x)$ is an Hermite polynomial defined by* (37).

PROOF. For $\alpha > 0$ the Laplace–Stieltjes transform

$$\Psi_\alpha(s) = \int_{-0}^{\infty} e^{-sx}\, dL_\alpha(x) \tag{65}$$

can be expressed as

$$\Psi_\alpha(s) = \sum_{r=0}^{\infty} (-1)^r M_r(\alpha) s^r / r! \tag{66}$$

and the series is convergent on the whole complex plane. Here $M_r(\alpha)$ is given by (58). If we put (58) into (66), express $m_r(\alpha)$ by (6) and interchange summations with respect to r and ℓ, we obtain that

$$\begin{aligned} \Psi_\alpha(s) &= 1 + 4\sum_{\ell=1}^{\infty} \frac{1}{(\ell-1)!} \int_0^{\infty} \left(\frac{d^\ell e^{-sx}}{dx^\ell}\right) [1 - \Phi((2\ell-1)\alpha + x)] x^{\ell-1}\, dx \\ &= 1 + 4\sum_{\ell=1}^{\infty} (-1)^\ell [1 - \Phi((2\ell-1)\alpha)] \\ &\quad + 4\sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{(\ell-1)!} \int_0^{\infty} e^{-sx} \left(\frac{d^\ell [1 - \Phi((2\ell-1)\alpha + x)] x^{\ell-1}}{dx^\ell}\right) dx. \end{aligned} \tag{67}$$

Hence we can conclude that

$$L'_\alpha(x) = 4\sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{(\ell-1)!} \left(\frac{d^\ell [1 - \Phi((2\ell-1)\alpha + x)] x^{\ell-1}}{dx^\ell}\right) \tag{68}$$

for $x > 0$, and

$$L_\alpha(0) = 1 + 4\sum_{\ell=1}^{\infty} (-1)^\ell [1 - \Phi((2\ell-1)\alpha)] \tag{69}$$

for $\alpha > 0$. This proves (63) and (62) for $x = 0$. From (67) it follows also that

$$L_\alpha(x) = 1 + 4\sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{(\ell-1)!} \left(\frac{d^{\ell-1} [1 - \Phi((2\ell-1)\alpha + x)] x^{\ell-1}}{dx^{\ell-1}}\right) \tag{70}$$

for $x \geq 0$. This proves (62). □

Obviously,

$$L_\alpha(0) = \mathbf{P}\Big\{\sup_{0 \leq t \leq 1} |\xi(t)| \leq \alpha\Big\} \tag{71}$$

for $\alpha > 0$. By the results of Erdös and Kac [4] we have

$$\mathbf{P}\Big\{\sup_{0 \leq t \leq 1} |\xi(t)| \leq \alpha\Big\} = \frac{4}{\pi} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{2j-1} \exp\left(\frac{-(2j-1)^2 \pi^2}{8\alpha^2}\right) \tag{72}$$

for $\alpha > 0$. This formula provides an alternative expression for $L_\alpha(0)$.

The functions $M(\alpha) = M_1(\alpha)$, $D(\alpha) = M_2(\alpha) - [M_1(\alpha)]^2$ and $L_\alpha(0)$ are given in Tables 1, 2 and 3 and are depicted in Figures 1, 2 and 3.

TABLE 1
The expectation $M(\alpha)$

α	$M(\alpha)$	α	$M(\alpha)$	α	$M(\alpha)$
0.00	1.595769	0.85	0.439887	1.70	0.073151
0.05	1.497763	0.90	0.401725	1.75	0.064695
0.10	1.403741	0.95	0.366223	1.80	0.057102
0.15	1.313688	1.00	0.333262	1.85	0.050299
0.20	1.227579	1.05	0.302721	1.90	0.044217
0.25	1.145379	1.10	0.274478	1.95	0.038792
0.30	1.067045	1.15	0.248414	2.00	0.033963
0.35	0.992524	1.20	0.224410	2.05	0.029674
0.40	0.921755	1.25	0.202347	2.10	0.025873
0.45	0.854668	1.30	0.182112	2.15	0.022513
0.50	0.791186	1.35	0.163590	2.20	0.019548
0.55	0.731224	1.40	0.146673	2.25	0.016938
0.60	0.674691	1.45	0.131253	2.30	0.014646
0.65	0.621490	1.50	0.117227	2.35	0.012638
0.70	0.571518	1.55	0.104497	2.40	0.010882
0.75	0.524668	1.60	0.092968	2.45	0.009350
0.80	0.480829	1.65	0.082548	2.50	0.008017

TABLE 2
The variance $D(\alpha)$

α	$D(\alpha)$	α	$D(\alpha)$	α	$D(\alpha)$
0.00	1.453521	0.85	0.227301	1.70	0.048554
0.05	1.166538	0.90	0.215725	1.75	0.042835
0.10	0.938066	0.95	0.203610	1.80	0.037676
0.15	0.759520	1.00	0.191109	1.85	0.033043
0.20	0.622694	1.05	0.178399	1.90	0.028898
0.25	0.519946	1.10	0.165654	1.95	0.025203
0.30	0.444344	1.15	0.153039	2.00	0.021921
0.35	0.389769	1.20	0.140700	2.05	0.019017
0.40	0.350965	1.25	0.128758	2.10	0.016454
0.45	0.323547	1.30	0.117311	2.15	0.014201
0.50	0.303969	1.35	0.106433	2.20	0.012226
0.55	0.289458	1.40	0.096177	2.25	0.010500
0.60	0.277927	1.45	0.086576	2.30	0.008995
0.65	0.267875	1.50	0.077647	2.35	0.007688
0.70	0.258284	1.55	0.069393	2.40	0.006555
0.75	0.248519	1.60	0.061806	2.45	0.005577
0.80	0.238236	1.65	0.054868	2.50	0.004733

TABLE 3
The probability $L_\alpha(0)$

α	$L_\alpha(0)$	α	$L_\alpha(0)$	α	$L_\alpha(0)$
0.00	0.000000	0.85	0.230852	1.70	0.821739
0.05	0.000000	0.90	0.277614	1.75	0.839764
0.10	0.000000	0.95	0.324515	1.80	0.856279
0.15	0.000000	1.00	0.370777	1.85	0.871373
0.20	0.000000	1.05	0.415829	1.90	0.885134
0.25	0.000000	1.10	0.459269	1.95	0.897648
0.30	0.000001	1.15	0.500833	2.00	0.908999
0.35	0.000054	1.20	0.540358	2.05	0.919271
0.40	0.000570	1.25	0.577755	2.10	0.928542
0.45	0.002878	1.30	0.612990	2.15	0.936890
0.50	0.009157	1.35	0.646070	2.20	0.944386
0.55	0.021562	1.40	0.677027	2.25	0.951102
0.60	0.041362	1.45	0.705910	2.30	0.957104
0.65	0.068670	1.50	0.732785	2.35	0.962453
0.70	0.102674	1.55	0.757724	2.40	0.967210
0.75	0.142035	1.60	0.780806	2.45	0.971429
0.80	0.185242	1.65	0.802116	2.50	0.975161

For various values of α the functions $L_\alpha(x)$ and $L'_\alpha(x)$ can easily be calculated by utilizing the Wolfram Research program *Mathematica* [15]. In this program $\Phi(x)$ and $H_n(x)$ are built-in functions. Actually,

$$\Phi(x) = (1 + \mathrm{Erf}[x/\,\mathrm{Sqrt}[2]])/2, \tag{73}$$

$$1 - \Phi(x) = \mathrm{Erfc}[x/\,\mathrm{Sqrt}[2]]/2 \tag{74}$$

and

$$H_n(x) = \mathrm{HermiteH}\ [n, x/\,\mathrm{Sqrt}[2]]/2\hat{}(n/2). \tag{75}$$

We note that if in formula (62) we form the summation only for $1 \le \ell \le m$, the error $R_m(x)$ satisfies the inequality

$$|R_m(x)| < \frac{4\varphi((2m+1)\alpha + x)}{(2m+1)\alpha + x} + 4ke^{mx}\varphi\left(\frac{(2m-1)\alpha + x}{\sqrt{2}}\right) \tag{76}$$

for $(2m-1)\alpha^2 > 1 + x$ and $x \ge 0$. If in formula (63) we form the summation only for $1 \le \ell \le m$, the error $r_m(x)$ satisfies the inequality

$$|r_m(x)| < 4kme^{mx}\varphi\left(\frac{(2m-1)\alpha + x}{\sqrt{2}}\right) \tag{77}$$

for $(2m-1)\alpha^2 > 1 + x$ and $x \ge 0$. The inequalities (76) and (77) can be proved simply by using the following inequality of Cramér:

$$|H_n(x)| < k\sqrt{n!}\exp(x^2/4), \tag{78}$$

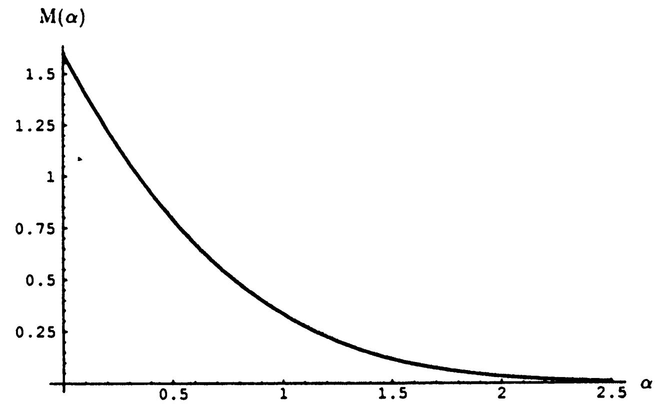

FIG. 1. *The expectation* $M(\alpha)$.

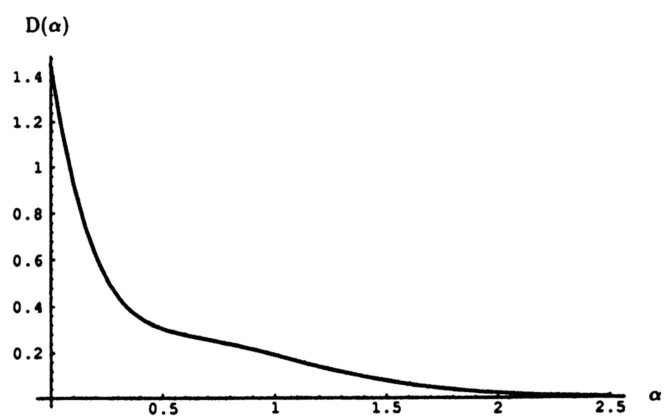

FIG. 2. *The variance* $D(\alpha)$.

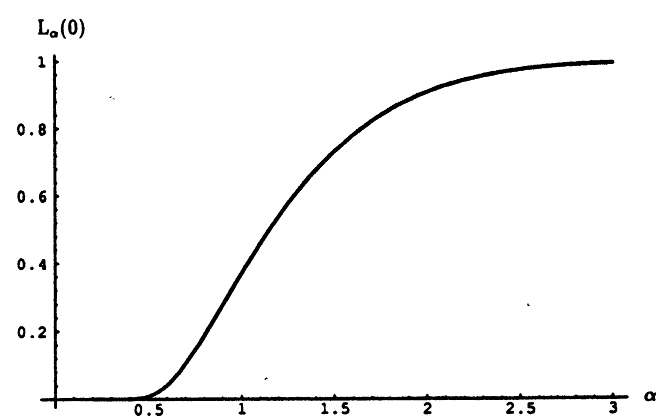

FIG. 3. *The probability* $L_\alpha(0)$.

where $k = 1.08643481121331\ldots$ or

$$k^2 = \frac{2}{\pi}\int_0^{\pi/2} \frac{du}{\sqrt{1-(1/2)\sin^2 u}}. \tag{79}$$

See Charlier ([2], pages 49–52).

Finally, we note that in the same way as we found $M_r(\alpha)$, we can prove that if $\alpha < \beta$, then

$$\begin{aligned}\mathbf{E}\{[\tau(\alpha)+\tau(\beta)]^r\} = \sum_{\ell=1}^{r}\binom{r-1}{\ell-1}\{&\mathbf{E}\{[\tau(|\alpha|+(\ell-1)(\beta-\alpha))]^r\}\\ &+\mathbf{E}\{[\tau(|\beta|+(\ell-1)(\beta-\alpha))]^r\}\}\end{aligned} \tag{80}$$

for $r \geq 1$. The distribution of $\tau(\alpha)+\tau(\beta)$ can be determined in the same way as the distribution of $\tau(\alpha)+\tau(-\alpha)$.

5. The local time of the Brownian bridge. Let $\{\eta(t),\ 0 \leq t \leq 1\}$ be a standard Brownian bridge. We have $\mathbf{P}\{\eta(t) \leq x\} = \Phi(x/\sqrt{t(1-t)})$ for $0 < t < 1$, where $\Phi(x)$ is defined by (1). Let

$$\tau^*(\alpha) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\text{ measure }\{t\colon\ \alpha \leq \eta(t) < \alpha+\varepsilon,\ 0 \leq t \leq 1\} \tag{81}$$

for any real α. The limit (81) exists with probability 1 and $\tau^*(\alpha)$ is called the local time at level α. Each theorem proved in this paper for the local time of the Brownian motion has a corresponding version for the Brownian bridge and can be proved in the same way. In particular, we have

$$\mathbf{P}\{\tau^*(\alpha) \leq x\} = 1-\exp\left(\frac{-(2|\alpha|+x)^2}{2}\right) \tag{82}$$

for $x \geq 0$.

For the local time of the reflecting Brownian bridge $\{|\eta(t)|,\ 0 \leq t \leq 1\}$ we have the following results: If $\alpha > 0$ and $r \geq 1$, then

$$\mathbf{E}\{[\tau^*(\alpha)+\tau^*(-\alpha)]^r\} = 2\sum_{\ell=1}^{r}\binom{r-1}{\ell-1}\mathbf{E}\{[\tau^*(\ell\alpha)]^r\} \tag{83}$$

and, by (82),

$$\mathbf{E}\{[\tau^*(\alpha)]^r\} = r\int_0^\infty x^{r-1}\exp(-(2\alpha+x)^2/2)\,dx. \tag{84}$$

By (83) we can determine the distribution function

$$\mathbf{P}\{\tau^*(\alpha)+\tau^*(-\alpha) \leq x\} = T_\alpha(x) \tag{85}$$

for $\alpha > 0$. If $x \geq 0$ and $\alpha > 0$, then

$$T_\alpha(x) = 1-2\sqrt{2\pi}\sum_{\ell=1}^{\infty}\sum_{j=0}^{\ell-1}\binom{\ell-1}{j}\frac{(-1)^{\ell-1}x^j}{j!}\varphi^{(j)}(2\ell\alpha+x), \tag{86}$$

and if $x > 0$ and $\alpha > 0$, then

$$T'_\alpha(x) = 2\sqrt{2\pi}\sum_{\ell=1}^{\infty}\sum_{j=1}^{\ell}\binom{\ell}{j}\frac{(-1)^\ell x^{j-1}}{(j-1)!}\varphi^{(j)}(2\ell\alpha + x), \tag{87}$$

where $\varphi^{(j)}(x)$ is given by (64). If $\alpha > 0$, then $T_\alpha(0) = K(\alpha)$, where

$$K(\alpha) = 1 - 2\sum_{\ell=1}^{\infty}(-1)^{\ell-1}\exp(-2\ell^2\alpha^2) = \frac{\sqrt{2\pi}}{\alpha}\sum_{j=1}^{\infty}\exp\left(\frac{-(2j-1)^2\pi^2}{8\alpha^2}\right) \tag{88}$$

is the Kolmogorov distribution function. See Kolmogorov [8].

The distribution function $T_\alpha(x)$ was determined by Smirnov [12] in 1939 in the context of order statistics. His results can be stated as follows: Let $\xi_1, \xi_2, \dots, \xi_n$ be mutually independent random variables having a common continuous distribution function $F(x)$. Let $G_n(x)$ be the empirical distribution function of the sample $(\xi_1, \xi_2, \dots, \xi_n)$, that is, $G_n(x)$ is defined as the number of variables less than or equal to x divided by n. Let $\nu_n(a)$ be the number of jumps of $G_n(x)$ over $F(x) + a/n$, that is, the number of values of x for which

$$G_n(x-0) \le F(x) + \frac{a}{n} < G_n(x). \tag{89}$$

Smirnov [12] proved that

$$\lim_{n\to\infty}\mathbf{P}\left\{\frac{\nu_n(\alpha\sqrt{n})}{\sqrt{n}} \le x\right\} = 1 - \exp\left(\frac{-(2\alpha + x)^2}{2}\right) \tag{90}$$

and

$$\lim_{n\to\infty}\mathbf{P}\left\{\frac{\nu_n(\alpha\sqrt{n}) + \nu_n(-\alpha\sqrt{n})}{\sqrt{n}} \le x\right\} = T_\alpha(x) \tag{91}$$

if $x > 0$ and $\alpha > 0$, where $T_\alpha(x)$ is given by (86). The limit theorems (90) and (91) imply (82) and (85).

In 1973, in the context of random mappings, Proskurin [11] also found the distribution function $T_\alpha(x)$. He considered a random mapping of the set $\{1, 2, \dots, n\}$ into itself. There are n^n possible mappings and they are considered equally probable. The graph of a mapping contains n vertices labeled $1, 2, \dots, n$, and in the graph two vertices i and j are joined by an edge (i, j) if and only if i is mapped into j. Each component of the graph contains only one cycle. Let us choose a mapping at random and denote by $\omega_n(m)$ the number of vertices of the graph at distance m from the nearest vertex in a cycle of the graph. Proskurin [11] proved that if $\alpha > 0$ and $x > 0$, then

$$\lim_{n\to\infty}\mathbf{P}\left\{\frac{2\omega_n([2\alpha\sqrt{n}])}{\sqrt{n}} \le x\right\} = T_\alpha(x), \tag{92}$$

where $T_\alpha(x)$ is given by (86).

The referee who proved (7) also called my attention to a forthcoming paper of Aldous and Pitman [1] in which the above result of Proskurin [11] is proved by using a Brownian bridge approach.

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