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SURVIVAL OF DISCRETE TIME GROWTH MODELS, WITH APPLICATIONS TO ORIENTED PERCOLATION[1]

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We prove survival for a class of discrete time Markov processes whose states are finite sets of integers. As applications, we obtain upper bounds for the critical values of various two-dimensional oriented percolation models. The technique of proof is based generally on that used by Holley and Liggett to prove survival of the one-dimensional basic contact process. However, the fact that our processes evolve in discrete time requires that we make substantial changes in the way this technique is used. When applied to oriented percolation on the two-dimensional square lattice, our result gives the following bounds: $p_c \leq 2/3$ for bond percolation and $p_c \leq 3/4$ for site percolation.

1. Introduction. The model we will study is a discrete time Markov chain A_n on the collection of finite subsets of the integers. There are two parameters in the definition of the chain, which we assume throughout satisfy $0 \leq p \leq q \leq 1$. Given A_n, the events $\{x \in A_{n+1}\}$ are conditionally independent, and

$$P(x \in A_{n+1} \mid A_n) = \begin{cases} q, & \text{if } |A_N \cap \{x, x+1\}| = 2, \\ p, & \text{if } |A_n \cap \{x, x+1\}| = 1, \\ 0, & \text{if } |A_n \cap \{x, x+1\}| = 0. \end{cases}$$

Here is an equivalent description of the chain, which will be somewhat more convenient in the sequel. To construct A_{n+1} from A_n, write A_n as a union of maximal subintervals

$$A_n = \bigcup_{i=1}^{k} I_i, \tag{1.1}$$

where $I_i = \{m_i + 1, m_i + 2, \dots, n_i\}$ and $m_i < n_i < m_{i+1}$. Then A_{n+1} is obtained by choosing points in $\{m_i + 1, m_i + 2, \dots, n_i - 1\}$ with probability q each, and points m_i and n_i with probability p each. The choices are made independently. The assumption that $p \leq q$ guarantees that the process is monotone: If $A_n \subset B_n$, then by appropriate coupling one can guarantee that $A_{n+1} \subset B_{n+1}$. We say that the chain survives or dies out according to whether $P(A_n \neq \varnothing \ \forall\ n)$ is positive or zero (for nonempty finite initial states). With these definitions, we can state our result:

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THEOREM 1. (a) *Suppose* $q < 2(1-p)$. *Then* A_n *dies out*.
(b) *Suppose* $\frac{1}{2} < p \le 1$ *and* $q \ge 4p(1-p)$. *Then* A_n *survives*.

Note that the extreme case $q = 1$ is easy to analyze directly, to see that $p = \frac{1}{2}$ (as given by the theorem) is the correct cutoff for survival. With $A_0 = \{0\}$, A_n is always an interval, whose length is transient if $p > \frac{1}{2}$ and recurrent otherwise.

Part (a) of the theorem is elementary and is given only for completeness and as a point of comparison for part (b). It could easily be improved—we are not looking for a particularly good bound here, just a simple one. To prove it, we couple A_n with an interval-valued process I_n so that $A_n \subset I_n$ for all n. In order to specify the coupling, suppose $A_n \subset I_n$ for a fixed n. To define the two processes at time $n+1$, first apply the transition mechanism to both A_n and I_n, leading to $A_{n+1} \subset I'_n$. This can be done by attractiveness (= monotonicity). Then let I_{n+1} be the smallest interval containing I'_n. The length L_n of I_n is dominated by a Markov chain Z_n on the nonnegative integers with the following transitions: $Z_{n+1} = Z_n + 1$ with probability p^2, $Z_{n+1} = (Z_n - X)^+$ with probability $2p(1-p)$ and $Z_{n+1} = (Z_n - X - Y - 1)^+$ with probability $(1-p)^2$. Here X and Y are independent geometric random variables with parameter q. For large values of Z_n, this is essentially a random walk with negative drift, provided that $q < 2(1-p)$.

Before discussing part (b) of the theorem, which is our main result, we will mention some results about oriented percolation. Oriented percolation is a probabilistic model which has been studied in the mathematics and physics literature for over three decades. The version which has been of greatest interest is obtained by letting the sites (or bonds) of Z^2 be labeled open or closed with probability α and $1-\alpha$, respectively. Percolation is said to occur if the probability is positive that there is an infinite oriented (in the positive x and y directions) path starting from the origin which passes only through open sites (or bonds). The critical value α_c is the infimum of all values of α for which percolation occurs. The following rigorous bounds have been obtained for oriented bond percolation on Z^2: $\alpha_c \le 0.84$ [Durrett (1984)], $\alpha_c \le 0.6863$ [Balister, Bollabas and Stacey (BBS) (1993)] and $\alpha_c \le 0.6735$ [Balister, Bollabas and Stacey (1994)]. For oriented site percolation, the corresponding known bounds are $\alpha_c \le 0.819$ [Durrett (1992)], $\alpha_c \le 0.762$ [Balister, Bollabas and Stacey (1993)] and $\alpha_c \le 0.7491$ [Balister, Bollabas and Stacey (1994)]. (The first BBS paper gives the site bound as 0.726; however, this is a misprint, as is explained at the end of their second paper.) These bounds were all obtained via some type of block argument, and the better bounds require substantial computer calculations in order to evaluate the probability of a particular event which depends on a large block of sites. Nonrigorous estimates of the critical values are 0.6447 in the bond case [Baxter and Guttmann (1988)] and 0.7055 in the site case [Onody and Neves (1992)]. Bollabas and Stacey (1995) give an upper bound for α_c of 0.647 in the bond case with "confidence" 99.99 + %.

At this point it may be worthwhile to comment briefly on why it is important to be able to prove good bounds on critical values for models of this sort. There is always, of course, the argument based on intellectual curiosity. The importance goes significantly beyond this, however. There have been a number of recent examples of results which assert that a particular kind of behavior occurs, whose proofs have been based on finding good estimates for critical values. Here are two examples.

EXAMPLE 1. The threshold voter model, which clusters in the nearest neighbor one dimensional case, was proved to coexist in all other cases—higher dimensions or one dimensional nonnearest neighbor—in Liggett (1994). The proof relies on the fact that an associated contact process survives for $\lambda = 1$. Because of the nature of the comparison between processes, knowing this for $\lambda = 1.01$ would not have helped at all. The Holley–Liggett technique works at $\lambda = 1$, and appears to work down to about $\lambda = 0.985$. Even techniques as good as those in the BBS papers are unlikely to do well enough to get the required result.

EXAMPLE 2. Pemantle (1992) proved that the contact process on the homogeneous tree in which each site has $n + 1$ neighbors can survive weakly (i.e., survive globally but die out locally) if $n \geq 3$. (This cannot occur if $n = 1$.) He did this by obtaining upper bounds on the critical value for global survival and lower bounds for the critical value for local survival which were sufficiently good to prove that these critical values are different. Liggett (1995) obtained better bounds, which allowed him to prove the result for $n = 2$ as well.

In order to see the concrete connection between oriented percolation and the chain A_n, think of the "time" n as corresponding to the sites $(x, y) \in Z_+^2$ which satisfy $x + y = n$, and of A_n as the set of sites with this property which can be reached from the origin through open sites (or bonds). Then the identification is exact, provided that we take $q = p = \alpha$ in the site case and $q = \alpha(2 - \alpha)$, $p = \alpha$ in the bond case. Since $(p, q) = (\frac{2}{3}, \frac{8}{9})$ and $(p, q) = (\frac{3}{4}, \frac{3}{4})$ satisfy the assumptions of part (b) of Theorem 1, we obtain the upper bounds $\frac{2}{3}$ and $\frac{3}{4}$ for the critical values in these two cases. Note that these are very close to the bounds obtained by BBS. The advantage we see for our approach is that it is purely analytic and does not involve computer calculations. We should mention that the possibility of applying our technique to oriented percolation was anticipated by D. Williams, who is quoted on page 264 of Grimmett (1989) as conjecturing that $\alpha_c \leq \frac{2}{3}$ in the bond case could be proved, presumably in this way.

Our result also applies to mixed bond-site models, for which there do not appear to be good critical value bounds in the literature. Take sites in Z^2 to be open with probability α and bonds open with probability β. Say a path is open if all sites and bonds on the path are open. This percolation model corresponds to our chain if we let $p = \alpha\beta$, $q = \alpha\beta(2 - \beta)$. By the theorem, there is percolation provided that $4\alpha\beta \geq 2 + \beta$. Setting $\alpha = 1$ or $\beta = 1$, we recover the bond and site results, respectively. Estimates for the boundary of the percolation region in the mixed case can be found in Tretyakov and Inui (1995).

As pointed out to the author by L. Chayes, one can also apply Theorem 1 to oriented percolation on graphs other than Z^2. For example, consider the hexagonal lattice in Figure 1. The arrows give the orientation of the bonds. As before, the sites (or bonds) of this graph are labeled open or closed with probability γ and $1 - \gamma$, respectively. Percolation is said to occur if the probability is positive that there is an infinite oriented (in the directions of the arrows) path starting from a prescribed point which passes only through open sites (or bonds). In order to make the connection with the chain A_n, think of the vertical lines in the figure as corresponding to the time steps. Then A_n is the set of sites on the line labeled n which can be reached via an oriented open path from a given site on the line corresponding to the label 0. With this identification, we see that $q = p = \gamma^2$ in the site case and $p = \gamma^2$, $q = \gamma^2(2 - \gamma)$ in the bond case. So, these models correspond to the mixed bond-site models on Z^2 with $\alpha = \gamma^2$, $\beta = 1$ in the site case and $\alpha = \beta = \gamma$ in the bond case, and we get

$$\gamma_c \leq \frac{\sqrt{3}}{2} \quad \text{and} \quad \gamma_c \leq \frac{1 + \sqrt{33}}{8},$$

respectively.

Oriented bond percolation on Z^2 can be regarded as a discrete time version of the (continuous time) basic one-dimensional contact process. The best critical value upper bound for this process was obtained by Holley and Liggett (HL) (1978). Their argument is quite different in nature from block or

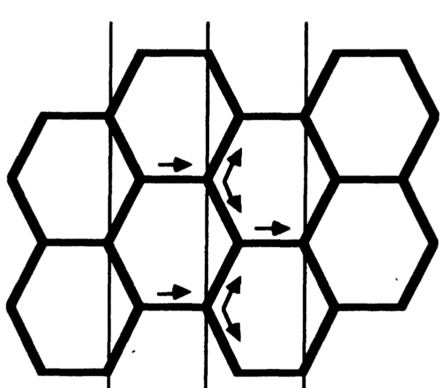

FIG. 1.

renormalization arguments, so it is natural to ask whether some version of the HL technique can be used in the discrete time context. This is our main motivation for the present paper—to see how the HL technique can be used in discrete time.

The outline of the proof of part (b) of Theorem 1 is essentially the same in the present context as in the case of the contact process:

STEP 1. Let ν be the stationary renewal measure on $\{0,1\}^Z$ corresponding to the density $f(n)$ on the positive integers with finite mean and tail probabilities

$$F(n) = \sum_{k=n}^{\infty} f(k), \qquad n \geq 1.$$

Define a function H on the finite subsets A of Z by

$$H(A) = \nu\{\eta: \eta(x) = 0 \quad \forall\, x \in A\}. \tag{1.2}$$

STEP 2. If $\frac{1}{2} < p < 1$ and $q \geq 4p(1-p)$, show that there exists a choice of f so that

$$H(A) = E^A H(A_1) \qquad \forall \text{ intervals } A. \tag{1.3}$$

STEP 3. Show that if f is chosen so that (1.3) holds, then

$$H(A) \geq E^A H(A_1) \qquad \forall \text{ finite sets } A. \tag{1.4}$$

Part (b) of Theorem 1 follows from these statements, since nonsurvival implies that

$$\lim_{n \to \infty} E^A H(A_n) = H(\varnothing) = 1,$$

while iteration of (1.4) implies that

$$E^A H(A_n) \leq H(A) < 1$$

for nonempty A.

While the outline is similar, the implementation of the HL technique in discrete time is not a straightforward adaptation of the proof in continuous time. One reason for this is that only one site can flip at a time in continuous time, while any number of sites can flip simultaneously in discrete time. This gives rise to more complicated expressions which have many more terms when the two sides of (1.4) are computed. A second reason is that the analogues of (1.3) and (1.4) can be expressed simply in terms of the one measure ν in the contact process case, but require two measures (essentially corresponding to time zero and time one) in discrete time; see the definition of ν^* below. The final reason is that, when written down explicitly, (1.3) becomes a convolution equation for F which can be solved explicitly in the contact process case, giving a form from which it is easy to read off properties which are needed in the proof. In the present context, while an explicit solution is possible, it is sufficiently more complicated that the derivation of needed properties requires more effort.

Next, we will derive the convolution equation for F which is equivalent to the set of equations (1.3). (Already in this computation, we will see that the discrete time analysis is significantly harder than the continuous time proof. The corresponding derivation in the case of the contact process takes one line.) First introduce ν^*, the measure obtained by thinning the renewal measure ν by $1-q$, that is, ν^* is the distribution of $\{\eta(i) \wedge \zeta(i),\ i \in Z\}$, where η is distributed according to ν and the $\zeta(i)$'s are i.i.d. (and independent of η) with $P(\zeta(i) = 1) = q$ and $P(\zeta(i) = 0) = 1 - q$. Then ν^* is also a renewal measure. Note that $\nu^*(1) = q\nu(1)$. Here, we have used the natural shorthand notation $\nu(1)$ for the cylinder probability $\nu\{\eta\colon \eta(x) = 1\}$, which is independent of x by translation invariance. Let f^* and F^* be the spacing density and tail probabilities corresponding to ν^*. Given A in the form (1.1), let A^* be the random set which contains the points in $\{m_i + 1, m_i + 2, \dots, n_i - 1\}$ with probability 1 each, and points m_i and n_i with probability $r = p/q$ each.

Now take a fixed finite set $A \subset Z$, η distributed according to ν, ζ and A^* as in the last paragraph and $\{\eta, \zeta, A^*\}$ independent. We can construct the state of the Markov chain at time 1 by

$$A_1 = \{k \in A^*\colon \zeta(k) = 1\}.$$

With this construction,

$$\begin{aligned} E^A H(A_1) &= E^A \nu\{\eta\colon \eta(i) = 0 \quad \forall\, i \in A_1\} \\ &= P[\eta(i) = 0 \quad \forall\, i \in A^* \ni \zeta(i) = 1] \\ &= P[\eta(i) \wedge \zeta(i) = 0 \quad \forall\, i \in A^*] \\ &= E\nu^*\{\eta\colon \eta(i) = 0 \quad \forall\, i \in A^*\}. \end{aligned} \tag{1.5}$$

Note that this is quite analogous to (1.2). When $A = \{1, 2, \dots, n\}$ is an interval with n points, A^* is a random interval which has $n-1$, n or $n+1$ points. Applying (1.3) with $n = 1$ gives

$$\nu(0) = (1-r)^2 + 2r(1-r)\nu^*(0) + r^2\nu^*(00),$$

where we have again used our shorthand notation for cylinder probabilities. Subtracting both sides of this identity from 1 leads to

$$\nu(1) = 2r(1-r)\nu^*(1) + r^2[\nu^*(11) + \nu^*(10) + \nu^*(01)].$$

Using $\nu^*(11) = \nu^*(1)f^*(1)$, $F^*(2) = 1 - f^*(1)$, $\nu^*(10) = \nu^*(1)F^*(2)$ and $\nu^*(1) = q\nu(1)$, this implies that

$$F^*(2) = \frac{1 - 2qr + qr^2}{qr^2}. \tag{1.6}$$

Taking differences of (1.3) for two consecutive values of n and arguing in a similar manner gives

$$F(n) = q\Big[r^2F^*(n+1) + 2r(1-r)F^*(n) + (1-r)^2F^*(n-1)\Big], \qquad n \geq 2. \tag{1.7}$$

On the other hand, for $n \geq 1$,

$$\begin{aligned}\nu^*(1)F^*(n) &= P[\eta(0) = \zeta(0) = 1, \eta(k) \wedge \zeta(k) = 0\ \forall\ 1 \leq k < n] \\ &= qP[\eta(0) = 1, \eta(k) = 0\ \forall\ 1 \leq k < n] \\ &\quad + q(1-q)\sum_{k=1}^{n-1} P[\eta(0) = \eta(k) = 1, \eta(j) = 0\ \forall\ 0 < j < k, \\ &\qquad\qquad \eta(i) \wedge \zeta(i) = 0\ \forall\ k < i < n],\end{aligned}$$

so that

$$F^*(n) = F(n) + (1-q)\sum_{k=1}^{n-1} f(k)F^*(n-k). \tag{1.8}$$

Next note that the right side of (1.7) is a linear combination of $F^*(n-1)$, $F^*(n)$ and $F^*(n+1)$. Taking the same linear combination of (1.8) for $n-1$, n and $n+1$ and using (1.7) yields

$$\begin{aligned}F(n) &= qr^2\left[F(n+1) + (1-q)\sum_{k=1}^{n} f(k)F^*(n+1-k)\right] \\ &\quad + 2qr(1-r)\left[F(n) + (1-q)\sum_{k=1}^{n-1} f(k)F^*(n-k)\right] \\ &\quad + q(1-r)^2\left[F(n-1) + (1-q)\sum_{k=1}^{n-2} f(k)F^*(n-1-k)\right].\end{aligned}$$

The F^* terms in the sums on the right of this expression appear in the right linear combination, so that (1.7) can be used again to write

$$\begin{aligned}F(n) &= qr^2F(n+1) + 2qr(1-r)F(n) + q(1-r)^2F(n-1) \\ &\quad + (1-q)\sum_{k=1}^{n-2} f(k)F(n-k) + qr^2(1-q)f(n) \\ &\quad + qr^2(1-q)f(n-1)F^*(2) + 2qr(1-r)(1-q)f(n-1).\end{aligned}$$

Using (1.6), $qr = p$ and the relation between f and F, this simplifies to

$$\begin{aligned}&(1-q)\sum_{k=1}^{n-1} f(k)F(n-k) \\ &\quad = (1 - 2p + 2p^2)F(n) \\ &\qquad - [q - p(2-p)]F(n-1) - p^2F(n+1),\end{aligned} \tag{1.9}$$

for $n > 1$. Writing $f(n) = F(n) - F(n+1)$ again, (1.9) can be rewritten as

$$\begin{aligned}(1-q)\sum_{k=1}^{n-1} F(k)F(n-k) + [q - p(2-p)]F(n-1) - p^2F(n)\\ = (1-q)\sum_{k=1}^{n} F(k)F(n+1-k)\\ +[q-p(2-p)]F(n) - p^2F(n+1),\end{aligned}$$

so the right side of this expression is independent of n. The right side is zero for $n = 1$ by (1.6), since

$$\begin{aligned}F(2) &= \frac{\nu(10)}{\nu(1)} = 1 - \frac{\nu(11)}{\nu(1)} = 1 - \frac{\nu^*(11)}{q\nu^*(1)}\\ &= 1 - \frac{1}{q}[1 - F^*(2)] = \left[\frac{1-p}{p}\right]^2,\end{aligned}$$

and hence it is zero for all $n \geq 1$. The conclusion is that

$$\begin{aligned}&p^2F(n+1)\\ \text{(1.10)}\qquad &= (1-q)\sum_{k=1}^{n-1} F(k)F(n+1-k) + (1-p)^2F(n),\end{aligned}$$

for $n \geq 1$. This is the explicit form of (1.3).

In the oriented bond percolation case on Z^2, $1 - q = (1-p)^2$, so the right side of (1.10) can be written as a single sum up to n. In fact, in this case, (1.10) is exactly the system of equations which must be solved in the contact process context. [See (1.20) on page 270 of Liggett (1985).] The solution of (1.10) is much simpler in this case than it is in general.

It is easy to see that the hypothesis of part (b) of Theorem 1 is a necessary condition for the existence of tail probabilities $F(n)$ which satisfy (1.10). First, note that (1.10) implies that $p^2F(n+1) \geq (1-p)^2F(n)$, so $p > \frac{1}{2}$ if $F(n) \downarrow 0$. Second, let

$$M = \sum_{n=1}^{\infty} F(n) < \infty$$

and sum (1.10) for $n \geq 1$, obtaining

$$p^2(M-1) = (1-q)M^2 + (q - 2p + p^2)M.$$

This is a quadratic in M with discriminant $q[q - 4p(1-p)]$, which must be nonnegative if (1.10) is to have a summable solution.

In the next section, we will show that the (trivially unique and positive) solution F of (1.10) is decreasing and summable if $\frac{1}{2} < p < 1$ and $q \geq 4p(1-p)$. The final part of the proof of Theorem 1 is to show that (1.4) holds. This is done in Section 3. The proof is considerably easier in the case $q = p$, and the reader might want to follow the proof in this case at first. When $q = p$, A^* is deterministic rather than random, and (1.7) simplifies to

$$F(n) = qF^*(n+1), \qquad n \geq 2, \tag{1.11}$$

which simplifies matters significantly. The remark following (3.15) indicates another place where the argument simplifies if $q = p$.

This paper is devoted exclusively to the issue of survival of the "finite system," since it is this which is normally of interest in the percolation context. One could also ask about conditions for survival of the infinite system, in the sense that there is a nontrivial invariant measure for the Markov process A_n on the space of all (infinite) subsets of Z^1. One way to relate these two types of survival is via duality. To explain, suppose that B_n is another discrete time Markov process on this space, which evolves in the following way: Conditional on B_n,

$$B_{n+1} = \bigcup_{x \in B_n} J_x,$$

where $\{J_x,\ x \in Z^1\}$ are independent, and

$$J_x = \begin{cases} \emptyset, & \text{with probability } 1-q, \\ \{x\}, & \text{with probability } q-p, \\ \{x+1\}, & \text{with probability } q-p, \\ \{x, x+1\}, & \text{with probability } 2p-q, \end{cases}$$

where we assume $p \leq q \leq 2p$ (which is the interesting case). It is easy to check that $P^A(A_1 \not\ni x) = P(J_x \cap A = \emptyset)$, for every x, A, and hence $P^A(A_1 \cap B = \emptyset) = P^B(A \cap B_1 = \emptyset)$. Iterating this gives the duality relation

$$P^A(A_n \cap B = \emptyset) = P^B(A \cap B_n = \emptyset), \tag{1.12}$$

for all A, B and $n \geq 1$. In particular, one of these processes survives in the finite sense if and only if the other survives in the infinite sense. [Take $A = \{0\}$ and $B = Z^1$, or vice versa in (1.12).] In general, B_n is a different type of process than A_n. However, if $1 - q = (1-p)^2$ (which corresponds to oriented bond percolation), then they are the same process, so the finite system survives if and only if the infinite system survives. To prove survival of the infinite version of A_n in other cases, one would have to carry out the analysis of this paper for the corresponding (finite) B_n process. While we have not attempted to do so, we did the initial computations which suggest the values of (p, q) for which one can expect the argument to work. The analogue of (1.10) for the process B_n is

$$p^2 F(n+1) = (1-p)^2 \sum_{j=1}^{n-1} F(j) F(n+1-j) + \frac{p^2(1-q)}{2p-q} F(n),$$

for $n \geq 1$. The analogue of the condition $q \geq 4p(1-p)$ in Theorem 1 is

$$q \geq \frac{-2p + 9p^2 - 8p^3}{(1-p)(3p-1)}.$$

This is more stringent than the assumption in Theorem 1 if $p > \frac{2}{3}$, and less stringent otherwise.

2. The identities. This section is devoted to the analysis of the solution of (1.10).

PROPOSITION 2.1. *Suppose*

$$\tfrac{1}{2} < p < 1 \quad \textit{and} \quad q \geq 4p(1-p).$$

Let $F(n)$ be the unique solution of (1.10) *with* $F(1) = 1$. *Then*

$$F(n+1) \leq F(n),$$

for $n \geq 1$, and $\sum F(n) < \infty$.

PROOF. In order to solve (1.10), let

$$\phi(u) = \sum_{n=1}^{\infty} F(n) u^n$$

be the generating function of F. Multiplying (1.10) by u^{n+1} and summing from $n = 1$ to ∞ yields [recall $F(1) = 1$]

$$p^2[\phi(u) - u] = (1-q)[\phi(u) - u]\phi(u) + (1-p)^2 u\phi(u),$$

which simplifies to

$$(1-q)\phi^2(u) - \phi(u)\big(p^2 - [q - p(2-p)]u\big) + up^2 = 0.$$

In order to simplify the expression resulting from solving this quadratic, make the change of variables

$$a = \frac{1 - p + \sqrt{1-q}}{p}, \qquad b = \frac{1 - p - \sqrt{1-q}}{p}. \tag{2.2}$$

Note that the assumptions on p, q imply that $|b| \leq a \leq 1$. [The latter inequality comes from $1 - q \leq (2p - 1)^2$, which is equivalent to $q \geq 4p(1-p)$.] Inverting the relation in (2.2), we get

$$p = \frac{2}{2 + a + b} \quad \text{and} \quad 1 - q = \left[\frac{a - b}{2 + a + b}\right]^2.$$

Substituting, the quadratic becomes

$$(a-b)^2\phi^2(u) - 4(1 - abu)\phi(u) + 4u = 0.$$

Solving this quadratic for $\phi(u)$ gives

$$\phi(u) = \frac{2}{(a-b)^2}\left[1 - abu - \sqrt{(1 - a^2u)(1 - b^2u)}\right], \tag{2.3}$$

where the negative sign in front of the square root is used because $\phi(0) = 0$. Putting $u = 1$ in (2.3), we see that F is summable. Note also that $b = 0$ in the oriented bond percolation/contact process case, which means that the polynomial inside the square root in (2.3) is linear, rather than quadratic. It is that simplification which makes more explicit computations possible in that case.

Next, use

$$\sqrt{1-x} = -\sum_{k=0}^{\infty} c_k x^k, \qquad c_k = \frac{(2k)!}{4^k (k!)^2 (2k-1)} \tag{2.4}$$

to expand (2.3). After equating coefficients, the conclusion is that

$$F(n) = -\frac{2}{(a-b)^2} \sum_{j=0}^{n} c_j c_{n-j} a^{2j} b^{2(n-j)}, \qquad n > 1. \tag{2.5}$$

This is an identity in a and b (with F depending on a and b also), which is valid for $a \neq b$ (i.e., $q < 1$). Note from (1.3) that $F(n)$ is continuous in p, q away from $p = 0$ and that when $q = 1$,

$$F(n+1) = \left(\frac{1-p}{p}\right)^{2n} = a^{2n}, \qquad n \geq 0.$$

Multiply both sides of (2.5) by $(a-b)^2$ and take the limit as $b \uparrow a$ to conclude that the coefficients c_k satisfy

$$\sum_{j=0}^{n} c_j c_{n-j} = 0.$$

Writing this as

$$-2c_0 c_n = \sum_{0<j<n} c_j c_{n-j},$$

we see that

$$F(n) = \frac{1}{(a-b)^2} \sum_{0<j<n} c_j c_{n-j} [a^{2j} - b^{2j}][a^{2(n-j)} - b^{2(n-j)}], \tag{2.6}$$

for $n > 1$. Define

$$d_j = c_j \frac{a^{2j} - b^{2j}}{a-b},$$

for $j \geq 0$, $d_j = 0$ for $j \leq 0$, so that (2.6) can be written as $F(n) = \sum d_j d_{n-j}$, for $n > 1$. We will need to know that d_n satisfies a property which is slightly stronger than monotonicity:

$$2n d_{n+1} \leq (2n-1) d_n. \tag{2.7}$$

This is equivalent to the statement that the following sequence is bounded above by 1:

$$\frac{2n}{2n-1} \frac{d_{n+1}}{d_n} = \frac{2n}{2n-1} \frac{c_{n+1}}{c_n} \frac{a^{2n+2} - b^{2n+2}}{a^{2n} - b^{2n}} = \frac{n}{n+1} \frac{a^{2n+2} - b^{2n+2}}{a^{2n} - b^{2n}}.$$

This, in turn, is equivalent to

$$n[a^{2n+2} - b^{2n+2}] \leq (n+1)[a^{2n} - b^{2n}],$$

which is an exercise in calculus: $1 + nt^{n+1} - (n+1)t^n$ is decreasing on $[0,1]$ and is 0 at $t = 1$, and hence is greater than or equal to 0 on $[0,1]$. Now set $t = (b/a)^2$ and use $b^2 \leq a^2 \leq 1$.

We now use (2.7) to complete the proof of the proposition:

$$\begin{aligned}
F(n+1) &= \sum_j d_j d_{n+1-j} \\
&\le \sum_j \left\{ \frac{j-1}{n-1}\left[\frac{2j-3}{2j-2} d_{j-1}\right] d_{n+1-j} + \frac{n-j}{n-1} d_j \left[\frac{2n-2j-1}{2n-2j} d_{n-j}\right]\right\} \\
&= \sum_j \left[\frac{2n-2j-1}{2n-2} + \frac{2j-1}{2n-2}\right] d_j d_{n-j} = F(n),
\end{aligned}$$

so that F is decreasing. □

In the next section, we will need to use a relation between the density f and the renewal sequence u corresponding to it, which is defined by $u(0) = 1$, and

$$u(n) = \sum_{k=1}^{n} f(k) u(n-k), \tag{2.8}$$

for $n \ge 1$. To obtain the desired relation, first take the difference of (1.9) for two successive values of n to obtain

$$\begin{aligned}
(1-q)&\sum_{j=1}^{n-1} f(j) f(n-j) \\
&= (2-q-2p+2p^2) f(n) \\
&\quad -[q-p(2-p)] f(n-1) - p^2 f(n+1),
\end{aligned} \tag{2.9}$$

for $n > 1$. Next, take $n \ge 2$ and use (2.8) twice and (2.9) once to write

$$\begin{aligned}
(1-q)u(n) &= (1-q)\sum_{k=1}^{n} f(k) u(n-k) \\
&= (1-q)\sum_{k=1}^{n-1} f(k) \sum_{j=1}^{n-k} f(j) u(n-k-j) + (1-q) f(n) \\
&= (1-q)\sum_{\substack{j,k\ge 1\\ k+j\le n}} f(k) f(j) u(n-k-j) + (1-q) f(n) \\
&= (1-q)\sum_{1\le j<i\le n} f(i-j) f(j) u(n-i) + (1-q) f(n) \\
&= (1-q)\sum_{i=2}^{n} u(n-i) \sum_{j=1}^{i-1} f(j) f(i-j) + (1-q) f(n) \\
&= \sum_{i=2}^{n} u(n-i)\big[(2-q-2p+2p^2) f(i) \\
&\qquad\qquad -[q-p(2-p)] f(i-1) - p^2 f(i+1)\big] \\
&\quad + (1-q) f(n).
\end{aligned}$$

Now apply (2.8) to each of the three convolutions of u and f which appear on the right, obtaining

$$\begin{aligned}(1-q)u(n) = {} & [2-q-2p+2p^2][u(n)-f(1)u(n-1)] \\ & -[q-p(2-p)]u(n-1) \\ & -p^2[u(n+1)-f(1)u(n)-f(2)u(n-1)] \\ & +(1-q)f(n).\end{aligned}$$

Using the values

$$f(1) = \frac{2p-1}{p^2} \quad \text{and} \quad f(2) = \frac{(1-p)^2(2p+q-2)}{p^4},$$

which come from (1.10), the above identity becomes

$$p^2[u(n+1)-2u(n)+u(n-1)] = (1-q)f(n).$$

Rewrite this as

$$\begin{aligned}p^2[u(n)-u(n+1)] - (1-q)F(n+1) \\ = p^2[u(n-1)-u(n)] - (1-q)F(n).\end{aligned}$$

This implies that the right side is independent of n for $n \geq 2$, and direct computation shows that it is zero for $n = 2$. It follows that it is zero for all $n \geq 2$, and hence that

$$p^2[u(n-1)-u(n)] = (1-q)F(n), \qquad n \geq 2. \tag{2.10}$$

In particular, we see that u is decreasing.

3. The inequalities. This section is devoted to the proof of (1.4). We assume throughout that p and q satisfy the assumptions of part (b) of Theorem 1 and that F is the solution to (1.10). Define functions L, R, L^* and R^* by the conditional probabilities

$$\begin{aligned}L(i) &= \nu\{\eta(j) = 0 \quad \forall j \in A \cap (-\infty, i) \mid \eta(i) = 1\}, \\ R(i) &= \nu\{\eta(j) = 0 \quad \forall j \in A \cap (i, \infty) \mid \eta(i) = 1\}, \\ L^*(i) &= E\nu^*\{\eta(j) = 0 \quad \forall j \in A^* \cap (-\infty, i) \mid \eta(i) = 1\}\end{aligned}$$

and

$$R^*(i) = E\nu^*\{\eta(j) = 0 \quad \forall j \in A^* \cap (i, \infty) \mid \eta(i) = 1\}.$$

Then using (1.2), (1.5), the renewal property and a decomposition according to the locations of the first 1's to the left and right of $n + 0.5$, we may write, for any n,

$$\begin{aligned}H(A) &= \nu\{\eta = 0 \text{ on } A\} \\ &= \sum_{\substack{j \leq n < l \\ j, l \notin A}} \nu\{\eta = 0 \text{ on } A \cup (j,l), \eta(j) = \eta(l) = 1\} \\ &= \nu(1) \sum_{\substack{j \leq n < l \\ j, l \notin A}} L(j) f(l-j) R(l)\end{aligned} \tag{3.1}$$

and

$$
\begin{aligned}
E^A H(A_1) &= E\nu^*\{\eta = 0 \text{ on } A^*\} \\
&= E \sum_{\substack{j \le n < l \\ j, l \notin A^*}} \nu^*\{\eta = 0 \text{ on } A^* \cup (j,l), \eta(j) = \eta(l) = 1\} \\
&= \nu^*(1) E \sum_{\substack{j \le n < l \\ j, l \notin A^*}} L^*(j) f^*(l-j) R^*(l).
\end{aligned} \tag{3.2}
$$

In order to prove that $H(A) \ge E^A H(A_1)$, we will do an interpolation between the right sides of (3.1) and (3.2). Write A in the form (1.1). For $1 \le i \le k$, define

$$S_i = cE \sum_{\substack{j \le m_i,\, j \notin A \\ l \ge n_i,\, l \notin A^*}} L(j) g(l-j) R^*(l),$$

where c is a constant and g is a probability density on the positive integers which will be determined shortly. Let G be the tail probabilities corresponding to g. Putting $n = m_k$ in (3.1) and noting that $R(l) = R^*(l) = 1$, for $l \ge n_k$, we see that

$$H(A) = \nu(1) \sum_{\substack{j \le m_k \\ j \notin A}} L(j) F(n_k - j + 1). \tag{3.3a}$$

Putting $i = k$ in the definition of S_i gives

$$
\begin{aligned}
S_k &= c \sum_{\substack{j \le m_k \\ j \notin A}} L(j) \sum_{l \ge n_k} g(l-j) P(l \notin A^*) \\
&= c \sum_{\substack{j \le m_k \\ j \notin A}} L(j) [G(n_k - j) - rg(n_k - j)].
\end{aligned} \tag{3.3b}
$$

We have used here the fact that $P(n_k \in A) = r$ and $P(l \in A) = 0$ for $l > n_k$. Similarly, using (3.2) instead of (3.1) (with $n = m_1$) and $i = 1$ in the definition of S_i,

$$E^A H(A_1) = \nu^*(1) E \sum_{\substack{l \ge n_1 \\ l \notin A^*}} R^*(l) [F^*(l - m_1) - rf^*(l - m_1)] \tag{3.4a}$$

and

$$S_1 = cE \sum_{\substack{l \ge n_1 \\ l \notin A^*}} R^*(l) G(l - m_1). \tag{3.4b}$$

The expressions on the right of (3.3a) and (3.3b) agree if

$$\nu(1)F(i+1) = c[G(i) - rg(i)], \qquad i \ge 1, \tag{3.5}$$

while the expressions on the right of (3.4a) and (3.4b) agree if

$$\nu^*(1)[F^*(i) - rf^*(i)] = cG(i), \qquad i \ge 1. \tag{3.6}$$

We want to solve (3.5) and (3.6) for c and $G(\cdot)$. Recall that $G(1) = 1$, so c is determined by (3.6) with $i = 1$:

$$c = \nu^*(1)[1 - rf^*(1)] = \frac{q(1-p)\nu(1)}{p}.$$

Then (3.6) for the other i's can be solved to give

$$\begin{aligned} g(i) &= G(i) - G(i+1) \\ &= \frac{\nu^*(1)}{c}[(1-r)f^*(i) + rf^*(i+1)], \qquad i \ge 1. \end{aligned} \tag{3.7}$$

By (1.7) and (3.7) [recalling $\nu^*(1) = q\nu(1)$],

$$\begin{aligned} \nu(1)&F(i+1) - c[G(i) - rg(i)] \\ &= \nu(1)qr^2F^*(i+2) + 2\nu(1)qr(1-r)F^*(i+1) \\ &\quad + \nu(1)q(1-r)^2F^*(i) - \nu^*(1)(1-r)F^*(i) \\ &\quad - \nu^*(1)rF^*(i+1) + \nu^*(1)r(1-r)f^*(i) \\ &\quad + \nu^*(1)r^2f^*(i+1) \\ &= 0, \end{aligned}$$

so that (3.5) is automatically satisfied. It follows from (3.3) and (3.4) that, with this choice,

$$H(A) = S_k \quad \text{and} \quad E^A H(A_1) = S_1,$$

so (1.4) will follow from the appropriate monotonicity of S_i's in i: $S_{i+1} \ge S_i$, $1 \le i < k$. So, take $1 \le i < k$ and use the definition of the S_i's to write

$$\begin{aligned} &\frac{S_{i+1} - S_i}{c} \\ &\quad = E \sum_{\substack{j \le m_{i+1},\, j \notin A \\ l \ge n_{i+1},\, l \notin A^*}} L(j)g(l-j)R^*(l) \\ &\qquad - E \sum_{\substack{j \le m_i,\, j \notin A \\ l \ge n_i,\, l \notin A^*}} L(j)g(l-j)R^*(l) \\ &\quad = E \sum_{\substack{j \le m_i,\, j \notin A \\ l \ge n_{i+1},\, l \notin A^*}} L(j)g(l-j)R^*(l) \end{aligned}$$

$$
\begin{aligned}
(3.8)\qquad &+ E \sum_{\substack{n_i < j \le m_{i+1} \\ l \ge n_{i+1},\, l \notin A^*}} L(j) g(l-j) R^*(l) \\
&- E \sum_{\substack{j \le m_i,\, j \notin A \\ l \ge n_{i+1},\, l \notin A^*}} L(j) g(l-j) R^*(l) \\
&- E \sum_{\substack{j \le m_i,\, j \notin A \\ n_i \le l \le m_{i+1},\, l \notin A^*}} L(j) g(l-j) R^*(l) \\
= {} & E \sum_{\substack{n_i < j \le m_{i+1} \\ l \ge n_{i+1},\, l \notin A^*}} L(j) g(l-j) R^*(l) \\
&- E \sum_{\substack{j \le m_i,\, j \notin A \\ n_i \le l \le m_{i+1},\, l \notin A^*}} L(j) g(l-j) R^*(l).
\end{aligned}
$$

We need to show that the right side of (3.8) is nonnegative. In order to do so, we exploit some relations which the functions L and R^* satisfy, as a consequence of the renewal properties of ν and ν^*:

$$L(j) = \sum_{\substack{l<j \\ l \notin A}} L(l) f(j-l), \tag{3.9a}$$

$$R^*(l) = \sum_{j>l} P(j \notin A^*) f^*(j-l) R^*(j), \tag{3.9b}$$

$$L(j) = 1 - \sum_{\substack{l<j \\ l \in A}} u(j-l) L(l), \tag{3.9c}$$

$$R^*(l) = 1 - q \sum_{j>l} u(j-l) R^*(j) P(j \in A^*). \tag{3.9d}$$

To prove (3.9c), for example, use a decomposition according to the location of the leftmost 1 to write

$$
\begin{aligned}
1 - L(j) &= \nu\{\eta(l) = 1 \text{ for some } l \in A \cap (-\infty, j) \mid \eta(j) = 1\} \\
&= \sum_{\substack{l<j \\ l \in A}} \nu\{\eta(l) = 1, \eta(i) = 0 \text{ for all } i \in A \cap (-\infty, l) \mid \eta(j) = 1\} \\
&= \sum_{\substack{l<j \\ l \in A}} u(j-l) L(l).
\end{aligned}
$$

For (3.9d), the argument is similar, but uses also the fact that $u^*(n) = qu(n)$ for $n \ge 1$. For (3.9a), use a decomposition according to location of the rightmost 1.

Using (3.7) to write g in terms of f^* in the first and last steps below, and using (3.9b) in the middle step, we see that the first term on the right of (3.8) is

$$\frac{\nu^*(1)}{c} \sum_{n_i < j \le m_{i+1}} L(j) \Bigg[(1-r) \sum_{l \ge n_{i+1}} P(l \notin A^*) f^*(l-j) R^*(l) + r \sum_{l \ge n_{i+1}} P(l \notin A^*) f^*(l-j+1) R^*(l) \Bigg]$$

$$(3.10) \qquad \begin{aligned} &= \frac{\nu^*(1)}{c} \sum_{n_i < j \le m_{i+1}} L(j) \Bigg[(1-r) R^*(j) - (1-r) \\ &\qquad \times \sum_{j < l \le m_{i+1}} f^*(l-j) R^*(l) P(l \notin A^*) + r R^*(j-1) \\ &\qquad - r \sum_{j \le l \le m_{i+1}} f^*(l-j+1) R^*(l) P(l \notin A^*) \Bigg] \\ &= \frac{\nu^*(1)}{c} \sum_{n_i < j \le m_{i+1}} L(j) [(1-r) R^*(j) + r R^*(j-1) \\ &\qquad - r f^*(1) R^*(j) P(j \notin A^*)] \\ &\quad - \sum_{n_i < j < l \le m_{i+1}} L(j) g(l-j) R^*(l) P(l \notin A^*). \end{aligned}$$

Therefore, the right side of (3.8) becomes

$$(3.11) \qquad \begin{aligned} &\frac{\nu^*(1)}{c} \sum_{n_i < j \le m_{i+1}} L(j) [(1-r) R^*(j) + r R^*(j-1) \\ &\qquad - r f^*(1) R^*(j) P(j \notin A^*)] \\ &\quad - \sum_{\substack{j < l,\, j \notin A \\ n_i \le l \le m_{i+1}}} L(j) g(l-j) R^*(l) P(l \notin A^*). \end{aligned}$$

We now need to write g in terms of f, so that we can use (3.9a) to rewrite the last part of this expression. Taking differences in (3.5), we see that

$$\nu(1) f(i+1) = c[(1-r) g(i) + r g(i+1)], \qquad i \ge 1.$$

Rewrite this as

$$g(i+1) = \frac{\nu(1)}{rc} f(i+1) + \frac{r-1}{r} g(i)$$

and iterate (i.e., show by induction on k) to get

$$g(n) = \frac{\nu(1)}{rc} \sum_{j=0}^{k} f(n-j) \left[\frac{r-1}{r} \right]^j + g(n-k-1) \left[\frac{r-1}{r} \right]^{k+1}, \qquad 0 \le k \le n-2.$$

Use this for $k = n - 2$ to conclude that

$$g(n) = \frac{1}{1-p}\sum_{j=0}^{n-1} f(n-j)\left[\frac{r-1}{r}\right]^j + \frac{q-p-pq}{p(1-p)}\left[\frac{r-1}{r}\right]^{n-1}, \qquad n \geq 1. \tag{3.12}$$

In the last step of this computation, we have used the following values, which come from (1.10), (1.6), (1.7) and (3.6):

$$f(1) = \frac{2p-1}{p^2}, \qquad pc = q\nu(1)(1-p) \quad \text{and} \quad g(1) = \frac{p+pq-1}{p^2}. \tag{3.13}$$

Put $\lambda = (r-1)/r \in (-1, 0]$ (since $2p \geq q \geq p$) and, for the moment, fix $l \in [n_i, m_{i+1}]$. Then using (3.12) in the first step and (3.9a) in the second gives

$$\begin{aligned}
\sum_{\substack{j<l\\ j\notin A}} L(j)g(l-j) &= \frac{1}{1-p}\sum_{m\geq 0}\lambda^m \sum_{\substack{j\leq l-m-1\\ j\notin A}} L(j)f(l-j-m) \\
&\quad + \frac{q-p-pq}{p(1-p)}\sum_{\substack{j<l\\ j\notin A}} L(j)\lambda^{l-j-1} \\
&= \frac{1}{1-p}\sum_{j\leq l} L(j)\lambda^{l-j} - \frac{q-p-pq}{(q-p)(1-p)}\sum_{\substack{j<l\\ j\notin A}} L(j)\lambda^{l-j} \\
&= \frac{1}{1-p}L(l) + \frac{pq}{(q-p)(1-p)}\sum_{j<l} L(j)\lambda^{l-j} \\
&\quad + \frac{q-p-pq}{(q-p)(1-p)}\sum_{\substack{j<l\\ j\in A}} L(j)\lambda^{l-j}.
\end{aligned} \tag{3.14}$$

Using this in (3.11) and the fact that

$$\frac{\nu^*(1)}{c} = \frac{q\nu(1)}{c} = \frac{p}{1-p} \quad \text{and} \quad f^*(1) = \frac{2p-1}{pr}$$

by (3.13) and (1.6), respectively, we see that, except for a common factor of

$1-p$ in the denominator, the right side of (3.8) becomes

$$\begin{aligned} & p(1-r) \sum_{n_i<j \leq m_{i+1}} L(j) R^*(j)-2 p \sum_{n_i<j \leq m_{i+1}} L(j) R^*(j) P(j \notin A^*) \\ & \quad+p r \sum_{n_i<j \leq m_{i+1}} L(j) R^*(j-1) \\ & \quad-\frac{p q}{q-p} \sum_{n_i \leq l \leq m_{i+1}, j<l} R^*(l) P(l \notin A^*) L(j) \lambda^{l-j} \\ & \quad-\frac{q-p-p q}{q-p} \sum_{\substack{n_i \leq l \leq m_{i+1} \\ j<l, j \in A}} R^*(l) P(l \notin A^*) L(j) \lambda^{l-j} \\ & \quad-(1-r) L(n_i) R^*(n_i) . \end{aligned} \tag{3.15}$$

It is this expression which we must show is nonnegative.

We pause at this point to note that in case $p=q$ (i.e., $r=1$ and hence $\lambda=0$), (3.15) simplifies to

$$\begin{aligned} & -2 p \sum_{n_i<j<m_{i+1}} L(j) R^*(j)+p \sum_{n_i<j \leq m_{i+1}} L(j) R^*(j-1) \\ & \quad+p \sum_{n_i+1<j<m_{i+1}} L(j-1) R^*(j) . \end{aligned}$$

[The $q-p$ in the denominators of the fourth and fifth terms of (3.15) cancel with a $q-p$ in $\lambda=(p-q)/p$ before taking $p=q$.] This expression can be rewritten as

$$p \sum_{n_i \leq k<m_{i+1}}\left[L(k+1)-L(k) 1_{\{k>n_i\}}\right]\left[R^*(k)-R^*(k+1) 1_{\{k<m_{i+1}-1\}}\right] .$$

This is seen to be nonnegative, once one has some monotonicity for the functions L and R^*. [See (3.21) below for the required fact about R^*. The monotonicity of L is similar.] For general p, q, (3.15) involves many more terms, including $L(j)$ for j's which are in different subintervals of A^c. (The monotonicity of L and R^* does not extend across points in A.) Handling them requires more analysis, which is given next.

Use (3.9c) to rewrite the first four terms in (3.15) in terms of the values of L on A. The terms in the resulting expression which do not contain a factor of L are

$$\begin{aligned} & p(1-r) \sum_{n_i<j \leq m_{i+1}} R^*(j)-2 p \sum_{n_i<j \leq m_{i+1}} R^*(j) P(j \notin A^*) \\ & \quad+p r \sum_{n_i<j \leq m_{i+1}} R^*(j-1)-\frac{p q}{q-p} \sum_{n_i \leq l \leq m_{i+1}} R^*(l) P(l \notin A^*) \sum_{j<l} \lambda^{l-j} . \end{aligned}$$

Summing the geometric series in the last term explicitly [the sum is $\lambda/(1-\lambda)=r-1$] and using the fact that $P(j \notin A^*)=1$ for $n_i<j<m_{i+1}$ and $P(j \notin A^*)=1-r$ for $j=n_i, m_{i+1}$, it follows that all the above terms cancel except $pR^*(n_i)$.

Next, collect all the terms that do involve a factor of L, noting that $l < j \le m_{i+1}$, $l \in A$ implies $l \le n_i$. It is easy to see what the contributions from the first three sums of (3.15) are. The contributions from the last three terms give the following multiple of $L(l)R^*(j)P(j \notin A^*)$, for $l \le n_i \le j \le m_{i+1}$, $l \in A$:

$$\frac{pq}{q-p} \sum_{l<m<j} u(m-l)\lambda^{j-l} - \frac{q-p-pq}{q-p}\lambda^{j-l}1_{\{l<j\}} - 1_{\{j=l=n_i\}},$$

which can be rewritten as

$$\frac{pq}{q-p} \sum_{l \le m > j} u(m-l)\lambda^{j-l} - \lambda^{j-l}.$$

The result of these observations is that (3.15) can be expressed as

$$\begin{aligned} pR^*(n_i) + \sum_{l \le n_i,\, l \in A} L(l)\Bigg[&\sum_{n_i < j \le m_{i+1}} pu(j-l)\big(2R^*(j)P(j \notin A^*) \\ &\qquad\qquad -(1-r)R^*(j) - rR^*(j-1)\big) \\ &+ \sum_{n_i \le j \le m_{i+1}} R^*(j)P(j \notin A^*) \\ &\qquad \times \left(\frac{pq}{q-p} \sum_{l \le m < j} u(m-l)\lambda^{j-m} - \lambda^{j-l}\right)\Bigg]. \end{aligned} \tag{3.16}$$

Use (3.9c) again (with $j = n_i$; recall that $n_i \in A$) to write the first term in (3.16) in terms of the values of L on A, so that (3.16) becomes

$$\begin{aligned} \sum_{\substack{l \le n_i \\ l \in A}} L(l)\Bigg[&pR^*(n_i)u(n_i - l) \\ &+ \sum_{n_i < j \le m_{i+1}} pu(j-l)\big(2R^*(j)P(j \notin A^*) \\ &\qquad\qquad -(1-r)R^*(j) - rR^*(j-1)\big) \\ &+ \sum_{n_i \le j \le m_{i+1}} R^*(j)P(j \notin A^*) \\ &\qquad \times \left(\frac{pq}{q-p} \sum_{l \le m < j} u(m-l)\lambda^{j-m} - \lambda^{j-l}\right)\Bigg]. \end{aligned} \tag{3.17}$$

Recall that we are trying to show that (3.8) is nonnegative. We now see that it is sufficient to show that the coefficients of the L's in (3.17) are nonnegative. Rewriting these coefficients to put together those terms for which R^* has a given argument, we see that it is enough to show that the following is nonnegative for $l \le n_i$:

$$\begin{aligned} &R^*(n_i)\left[p \sum_{l\le m\le n_i} u(m-l)\lambda^{n_i-m} - rpu(n_i+1-l) - (1-r)\lambda^{n_i-l}\right] \\ &\quad + \sum_{n_i<j<m_{i+1}} R^*(j)\left[\frac{pq}{q-p}\sum_{l\le m\le j} u(m-l)\lambda^{j-m}\right. \\ &\qquad\qquad \left. -rpu(j+1-l) - \frac{p^3}{q(q-p)}u(j-l) - \lambda^{j-l}\right] \\ &\quad + R^*(m_{i+1})\left[p\sum_{l\le m\le m_{i+1}} u(m-l)\lambda^{m_{i+1}-m}\right. \\ &\qquad\qquad \left. -rpu(m_{i+1}-l) - (1-r)\lambda^{m_{i+1}-l}\right]. \end{aligned} \tag{3.18}$$

In this computation, we have used $P(j \notin A^*) = 1 - r$ for $j = n_i, m_{i+1}$ and the relation $(1-r)q = q - p$.

To simplify (3.18), define

$$\sigma(n) = p \sum_{\substack{j+l=n \\ j,l\ge 0}} u(j)\lambda^l - pru(n+1) + r\lambda^{n+1},$$

for $n \ge 0$. Recalling that $r\lambda + (1-r) = 0$, we see that (3.18) becomes

$$\begin{aligned} &R^*(n_i)\sigma(n_i - l) + \sum_{n_i<j<m_{i+1}} R^*(j)\left[\frac{q}{q-p}\sigma(j-l)\right. \\ &\qquad\qquad \left. -\frac{p^2 r}{q-p}[u(j-l) - u(j-l+1)]\right] \\ &\quad + R^*(m_{i+1})[\sigma(m_{i+1}-l) - pr[u(m_{i+1}-l) - u(m_{i+1}+1-l)]]. \end{aligned}$$

A further simplification occurs if we use the following relation:

$$\begin{aligned} &(1-r)\sigma(n) + r\sigma(n+1) \\ &\quad = p[1-r+r\lambda]\sum_{j=0}^{n} u(j)\lambda^{n-j} + rpu(n+1) \\ &\qquad - pr(1-r)u(n+1) - pr^2u(n+2) + r[1-r+r\lambda]\lambda^{n+1} \\ &\quad = pr^2[u(n+1) - u(n+2)]. \end{aligned} \tag{3.19}$$

Replacing the differences of u's by σ's, the result is that (3.18) equals

$$\begin{aligned} &R^*(n_i)\sigma(n_i - l) + \sum_{n_i<j<m_{i+1}} R^*(j)[\sigma(j-l) - \sigma(j-l-1)] \\ &\qquad - \frac{q-p}{p} R^*(m_{i+1})\sigma(m_{i+1} - l - 1) \\ &\qquad = \sum_{n_i<j<m_{i+1}} [R^*(j-1) - R^*(j)]\sigma(j-l-1) \\ &\qquad\quad + \left[R^*(m_{i+1}-1) - \frac{q-p}{p}R^*(m_{i+1})\right]\sigma(m_{i+1} - l - 1). \end{aligned} \tag{3.20}$$

The equality in (3.20) comes from summing by parts.

To complete the proof that (3.20) is nonnegative, it now suffices to show that

$$\begin{aligned} R^*(l-1) &\ge R^*(l) \quad \text{for } n_i < l < m_{i+1}, \\ R^*(m_{i+1}-1) &\ge \frac{q-p}{p} R^*(m_{i+1}) \end{aligned} \tag{3.21}$$

and

$$\sigma(n) \ge 0 \quad \text{for } n \ge 0. \tag{3.22}$$

Inequalities (3.21) follow from (3.9d) and (2.10), which implies that u is decreasing. Here are the details. Use (3.9d) to write

$$R^*(l) = 1 - q \sum_{j \ge m_{i+1}} u(j-l)R^*(j)P(j \in A^*),$$

for $n_i \le l < m_{i+1}$, which immediately gives the first part of (3.21) by the monotonicity of u. For the second part, write

$$R^*(m_{i+1}) = 1 - q \sum_{j>m_{i+1}} u(j - m_{i+1})R^*(j)P(j \in A^*)$$

and

$$\begin{aligned} R^*(m_{i+1}-1) = 1 &- q \sum_{j>m_{i+1}} u(j - m_{i+1} + 1)R^*(j)P(j \in A^*) \\ &- qu(1)R^*(m_{i+1})(p/q) \end{aligned}$$

and use $u(1) = (2p-1)/(p^2)$ [see (3.13)] and the monotonicity of u again to conclude that

$$R^*(m_{i+1}) \le R^*(m_{i+1}-1) + \frac{2p-1}{p} R^*(m_{i+1}),$$

and hence that

$$R^*(m_{i+1} - 1) \geq \frac{1-p}{p} R^*(m_{i+1}) \geq \frac{q-p}{p} R^*(m_{i+1}).$$

To prove (3.22), use (3.19), (2.10), (3.5) and (3.13) successively to show that

$$\begin{aligned}
&(1-r)\sigma(n) + r\sigma(n+1) \\
&\quad = pr^2[u(n+1) - u(n+2)] \\
&\quad = \frac{r^2(1-q)}{p} F(n+2) \\
&\quad = \frac{r^2(1-q)c}{p\nu(1)} [(1-r)G(n+1) + rG(n+2)] \\
&\quad = \frac{(1-q)(1-p)}{q} [(1-r)G(n+1) + rG(n+2)],
\end{aligned}$$

for $n \geq 0$. Since

$$\sigma(0) = p - pru(1) + r\lambda = \frac{(1-q)(1-p)}{q},$$

it follows that

$$\sigma(n) = \frac{(1-q)(1-p)}{q} G(n+1) \geq 0,$$

for $n \geq 0$ as required.

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