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# ASYMPTOTIC ANALYSIS OF TAIL PROBABILITIES BASED ON THE COMPUTATION OF MOMENTS

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Choudhury and Lucantoni recently developed an algorithm for calculating moments of a probability distribution by numerically inverting its moment generating function. They also showed that high-order moments can be used to calculate asymptotic parameters of the complementary cumulative distribution function when an asymptotic form is assumed, such as $F^c(x) \sim \alpha x^{\beta} e^{-\eta x}$ as $x \to \infty$. Moment-based algorithms for computing asymptotic parameters are especially useful when the transforms are not available explicitly as in models of busy periods or polling systems. Here we provide additional theoretical support for this moment-based algorithm for computing asymptotic parameters and new refined estimators for the case $\beta \neq 0$. The new refined estimators converge much faster (as a function of moment order) than the previous estimators, which means that fewer moments are needed, thereby speeding up the algorithm. We also show how to compute all the parameters in a multiterm asymptote of the form $F^c(x) \sim \sum_{k=1}^{m} \alpha_k x^{\beta-k+1} e^{-\eta x}$. We identify conditions under which the estimators converge to the asymptotic parameters and we determine rates of convergence, focusing especially on the case $\beta \neq 0$. Even when $\beta = 0$, we show that it is necessary to assume the asymptotic form for the complementary distribution function; the asymptotic form is not implied by convergence of the moment-based estimators alone. In order to get good estimators of the asymptotic decay rate $\eta$ and the asymptotic power $\beta$ when $\beta \neq 0$, a multiple-term asymptotic expansion is required. Such asymptotic expansions typically hold when $\beta \neq 0$, corresponding to the dominant singularity of the transform being a multiple pole ($\beta$ a positive integer) or an algebraic singularity (branch point, $\beta$ noninteger). We also show how to modify the moment generating function in order to calculate asymptotic parameters when all moments do not exist (the case $\eta = 0$).

**1. Introduction.** In many applied probability settings we are interested in small tail probabilities. For example, in the performance analysis of computer and telecommunication systems we might be interested in computing the 99.9th percentile of a critical delay or we might want to design a buffer that has a loss probability of $10^{-9}$. Quite often it turns out that the small tail probabilities are adequately approximated by the asymptote of the complementary cumulative distribution function. For example, Abate,

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Choudhury and Whitt (1994a, 1995a, 1996) show that the tail probabilities of the waiting time, sojourn time and workload in infinite-capacity queueing models with the first-come first-served discipline are often well approximated by an exponential asymptote for 80th percentiles or above. Indeed, there have been quite a few recent papers suggesting the use of asymptotes for computing small tail probabilities in the context of statistical multiplexers in communication networks; for example, see Botvich and Duffield (1995), Choudhury, Lucantoni and Whitt (1995), Duffield and O'Connell (1995) and Whitt (1993) and the references therein. The asymptotics are useful when exact computation is difficult (e.g., as in the case of models of statistical multiplexers with many sources) and even when exact computation is not difficult [e.g., when it is straightforward to perform numerical transform inversion, as in Abate and Whitt (1992)], because the simple formulas help convey understanding. In fact, the asymptotic analysis complements the direct numerical inversion, because the direct numerical inversion tends to have numerical difficulties far out in the tail, where the asymptotic analysis performs well. The two approaches also serve as checks on each other.

In this paper we present simple methods for computationally determining the asymptotic behavior of probability distributions based on moments. Choudhury and Lucantoni (1993, 1995) have shown that it is possible to compute moments (of high as well as low order) by numerically inverting the moment generating function, which requires the computation of the Laplace–Stieltjes transform or $z$-transform of the probability distribution at several complex values of the argument, but does not require knowledge of any of the properties of the transform (e.g., the location and type of its singularities). Our methods may be used with this algorithm or any other algorithm for computing moments.

Of course, asymptotic parameters can often be obtained directly by performing appropriate asymptotic analysis with the transforms, perhaps using symbolic mathematics programs such as MAPLE and MATHEMATICA, for example, as described in Chapter 5 and the Appendix of Wilf (1994). The moment-based algorithm is an attractive alternative either when the transform is not available explicitly or when someone is not familiar with asymptotic analysis. Examples in which transforms are not available explicitly, and for which we have applied the moment-based estimators of asymptotic parameters here, occur in the polling models in Choudhury and Whitt (1996) and the transient behavior of the BMAP/G/1 queue (with batch Markovian arrival process) in Lucantoni, Choudhury and Whitt (1994). Then the transforms are characterized implicitly via functional equations.

In addition to developing an algorithm to compute higher-order moments, Choudhury and Lucantoni (1993, 1995) developed estimators for the asymptotic parameters. Our primary purpose here is to prove that these estimators do indeed converge under suitable conditions to the asymptotic parameters as the moment index increases, and to obtain even better estimators in certain circumstances. We start with a *cumulative distribution function* (cdf) $F(x)$ on the nonnegative real line with associated *complementary cdf* $F^c(x) = 1 - F(x)$, $n$th *moment*

$$(1.1)\qquad m_n = \int_0^\infty x^n \, dF(x) = \int_0^\infty n x^{n-1} F^c(x)\, dx,$$

*Laplace–Stieltjes transform* (LST)

$$(1.2)\qquad \hat{f}(s) = \int_0^\infty e^{-sx}\, dF(x)$$

and *moment generating function* (mgf)

$$(1.3)\qquad M(z) = \hat{f}(-z) = \sum_{n=0}^{\infty} \frac{m_n z^n}{n!},$$

where $m_0 = 1$. The algorithm in Choudhury and Lucantoni (1993, 1995) computes the moments $m_n$ by numerically inverting the mgf $M(z)$ in (1.3).

The principal form of the asymptotics considered here is

$$(1.4)\qquad F^c(x) \sim \alpha x^{\beta} e^{-\eta x} \quad \text{as } x \to \infty,$$

where $f(x) \sim g(x)$ as $x \to \infty$ means that $f(x)/g(x) \to 1$ as $x \to \infty$. In (1.4), $\eta$ is the *asymptotic decay rate*, $\beta$ is the *asymptotic power* and $\alpha$ is the *asymptotic constant*. The parameters $\alpha$ and $\eta$ are assumed to be strictly positive, while $\beta$ can be positive, negative or 0. We discuss the case in which $\eta$ is 0 in Section 3.

In many contexts, we obtain the special case of (1.4) with $\beta = 0$. This corresponds to the Laplace transform $\hat{f}(s)$ in (1.2) having a dominant singularity that is a simple pole; see Section 5.2 of Wilf (1994). For example, this form often occurs with the steady-state waiting-time distribution in queueing models with the first-come first-served (FCFS) discipline; see Abate, Choudhury and Whitt (1994a, 1995a, 1996). On the other hand, nonzero $\beta$ often arises as well. Nonzero $\beta$ occurs when the dominant singularity of $\hat{f}(s)$ is a multiple pole (positive integer $\beta$) or an algebraic singularity or branch point (noninteger $\beta$); see Section 5.3 of Wilf (1994). Indeed, nonzero $\beta$ tends to be the rule rather than the exception for queueing models with non-FCFS service disciplines; for example, see Abate, Choudhury and Whitt (1995b) for last-come first-served (LCFS) and Choudhury and Whitt (1996) for polling models. This is primarily because the busy-period distribution has asymptotics with nonzero $\beta$; see page 156 of Cox and Smith (1960), page 167 of Abate and Whitt (1988) and Abate, Choudhury and Whitt (1995b). Hence, nonzero $\beta$ is also very much of interest and that will be our main concern.

When $\beta \neq 0$, we often have a stronger asymptotic form than (1.4); in particular,

$$(1.5)\qquad F^c(x) - \sum_{k=1}^{m-1} \alpha_k x^{\beta-k+1} e^{-\eta x} \sim \alpha_m x^{\beta-m+1} e^{-\eta x} \quad \text{as } x \to \infty$$

for $m \geq 1$. [For $m = 1$, (1.5) reduces to (1.4) with $\alpha_1 = \alpha$.] Asymptotic expansions of the form (1.5) are associated with Heavyside's theorem; see page 254 of Doetsch (1974) and page 139 of Van der Pol and Bremmer (1995). The successive terms in (1.5) have powers decreasing by 1. Variations of our methods apply to the case in which the powers decrease in a more general pattern, as on page 254 of Doetsch (1974), but (1.5) seems to be the form most commonly arising in applications. A familiar example is the M/M/1 busy-period distribution, for which $\beta = -3/2$; see page 167 of Abate and Whitt (1988). Choudhury and Lucantoni (1993, 1995) showed that their moment-based estimates of $\beta$ converge to $-3/2$ for this M/M/1 example. Cox and Smith (1960), page 156, showed that the M/G/1 busy-period distribution has the same asymptotic form (with $\beta = -3/2$). We should point out that numerical experience and the form of the asymptotics with a slower rate of convergence indicate that the quality of the approximation when $\beta \neq 0$ is not as spectacular as in the case $\beta = 0$. Nevertheless, from a practical standpoint, the asymptote is usually quite accurate for tail probabilities smaller than $10^{-3}$, so that it is important to be able to identify $\beta$ and the other asymptotic parameters.

Even though the asymptotic parameters of primary interest already appear in (1.4), we show that the stronger multiterm asymptotic expansion in (1.5) is important for obtaining good moment-based estimates of the basic asymptotic parameters when $\beta \neq 0$. (Our "estimates" should not be confused with statistical estimates; we do not work with statistical data.) Here are candidate moment-based estimates of the asymptotic parameters in (1.4):

$$\hat{\eta}_n = \frac{n m_{n-1}}{m_n}, \tag{1.6}$$

$$\hat{\beta}_n = \frac{\eta m_n}{m_{n-1}} - n, \tag{1.7}$$

$$\hat{\alpha}_{1,n} = \frac{\eta^{\beta+n} m_n}{n! n^{\beta}}. \tag{1.8}$$

The idea is first to estimate $\eta$ using (1.6), then to estimate $\beta$ using (1.7) and the "known" $\eta$ and finally to estimate $\alpha_1 \equiv \alpha$ using (1.8) and the "known" values of $\eta$ and $\beta$. We will show that $\hat{\eta}_n$ and $\hat{\alpha}_{1,n}$ converge under assumption (1.4), but $\hat{\beta}_n$ does not. Instead, $\hat{\beta}_n$ converges under assumption (1.5).

The estimates in (1.6)–(1.8) are chosen for simplicity rather than numerical accuracy. They are convenient for deriving the asymptotic parameters analytically (by hand) when explicit expressions for the moments are available. To illustrate, we give an example from Abate and Whitt (1996). From Proposition 3.2 there, we know that there is an infinitely divisible probability distribution, called the *Caley–Einstein–Pólya* (*CEP*) *distribution*, with mean 1 and Laplace–Stieltjes transform $\hat{f}(s)$ which satisfies the equation

$$\hat{f}(s) = \exp(-s\hat{f}(s)) \tag{1.9}$$

and has moments

$$m_n = (n+1)^{n-1}, \qquad n \geq 1. \tag{1.10}$$

The CEP distribution is well known via (1.9), but evidently no closed-form expression is known. However, we can apply (1.10) to deduce that the estimates in (1.6)–(1.8) converge to proper limits as $n \to \infty$. Hence, the asymptotic parameters in (1.4) are

$$\eta = e^{-1}, \qquad \beta = -3/2, \qquad \alpha = \sqrt{e^5/2\pi}, \tag{1.11}$$

assuming that the asymptotic relations in (1.4) and (1.5) are valid.

However, for computational purposes (on the computer), the estimates in (1.6)–(1.8) tend not to be satisfactory. First, when $\beta \neq 0$, the estimate for $\eta$ in (1.6) tends to converge quite slowly, having an error of $O(n^{-1})$. Indeed, under (1.5) it follows from Theorem 5.3 below that

$$\begin{aligned} \hat{\eta}_n = \eta\big(1 + c_1 n^{-1} + c_2 n^{-2} \\ + \cdots + c_{m-2} n^{(m-2)} + O(n^{-(m-1)})\big) \quad \text{as } n \to \infty, \end{aligned} \tag{1.12}$$

where $c_1 = -\beta$. (The situation is usually much better when $\beta = 0$; see Theorem 4.3 below.) Hence, when $\beta \neq 0$ and we want an accurate estimate of $\eta$ based on relatively few moments (e.g., $n \leq 20$), it is often possible and necessary to do much better than (1.6). Fortunately, dramatic improvements can be obtained by exploiting *extrapolation* based on the asymptotic expansion in (1.12).

Given (1.12), we use a variant of *Richardson extrapolation*, sometimes referred to as the *Wimp–Salzer algorithm*. That is,

$$\hat{\eta}_n(k) = \sum_{j=0}^{k-1} \hat{\eta}_{n-j} w_j(n, k-1), \qquad k \geq 1, \tag{1.13}$$

where

$$w_j(n,k) = \frac{(-1)^j (n-j)^k}{j!(k-j)!}; \tag{1.14}$$

for example, see pages 35–38 and 67–75 of Wimp (1981), pages 375–378 of Bender and Orszag (1978) and page 231 of Smith and Ford (1979). For ease of use, it is significant that the weights in (1.14) do not depend on the sequence of estimates $\{\hat{\eta}_n\}$. Hence, we can use the extrapolation (1.13) with any sequence having the form (1.12). Given (1.12), the estimates $\hat{\eta}_n(k)$ in (1.13) have error of order $O(n^{-k})$. We call $\hat{\eta}_n(k)$ the $k$th-*order approximation* of $\eta$. [Note that $\hat{\eta}_n(1) = \hat{\eta}_n$.]

In some cases, it is sufficient to use the second-order approximation $\hat{\eta}_n(2)$ or a related estimate based on the reciprocal; that is, let $\hat{\xi}_n = \hat{\eta}_n^{-1}$, let $\hat{\xi}_n(k)$ be the extrapolation (1.13) applied to $\hat{\xi}_n$ and let

$$\begin{aligned} \eta_n^*(2) \equiv \hat{\xi}_n(2)^{-1} &= \big(n\hat{\xi}_n(1) - (n-1)\hat{\xi}_{n-1}(1)\big)^{-1} \\ &= \frac{m_{n-1} m_{n-2}}{m_n m_{n-2} - m_{n-1}^2} = (r_n - r_{n-1})^{-1} \end{aligned} \tag{1.15}$$

for $r_n \equiv m_n / m_{n-1}$. To obtain several digits' accuracy based on not too many moments, we have found that $\hat{\eta}_{20}(5)$ often works well, but in some cases it may be better to use lower-order estimates with more moments. Indeed, it is a good idea to print out and examine $\hat{\eta}_n(k)$ for all $k$ and $n$ with $k \leq 5$ and $n \leq 100$, say. (See the example in Section 7.)

Clearly, we need a good estimate of $\eta$ to estimate $\beta$. For example, we cannot estimate $\eta$ and $\beta$ with (1.6) and (1.7) for the same value of $n$; if we use $\hat{\eta}_n$ in (1.7), then we get $\beta_n = 0$. Here is an estimate of $\hat{\beta}$ that does not directly involve $\eta$:

$$\hat{\beta}_n^* = \frac{r_n}{r_n - r_{n-1}} - n \tag{1.16}$$

for $r_n \equiv m_n/m_{n-1}$ as above. The estimate (1.16) corresponds to the estimate (1.7) with $\hat{\eta}_n^*(2)$ in (1.15) used in place of $\eta$. In Theorem 5.3 below we show that (1.16) has $O(n^{-1})$ error under appropriate conditions. By the same reasoning leading to (1.12), we find that $\hat{\beta}_n$ in (1.7) and $\hat{\beta}_n^*$ in (1.16) also have expansions of the form (1.12). Hence, better estimates for $\beta$ can be obtained by extrapolating with (1.13), starting with (1.7) with the exact $\eta$ or with (1.16).

Turning to the asymptotic constant $\alpha$ in (1.4), we remark that in the denominator of (1.8), $n!n^\beta$ is an approximation for $n\Gamma(n+\beta)$, where $\Gamma(x)$ is the gamma function; that is, a more direct estimate of $\alpha$ is

$$\alpha_{1,n}^* = \eta^{\beta+n} m_n / (n\Gamma(n+\beta)). \tag{1.17}$$

When $\beta$ is not an integer, clearly (1.8) is preferable for analysis by hand, but the computer has no difficulty with the gamma function in (1.17). Both $\hat{\alpha}_{1,n}$ in (1.8) and $\alpha_{1,n}^*$ in (1.17) have expansions of the form (1.12), so that extrapolation also can be used to estimate $\alpha$.

Given the asymptotic expansion (1.5), we may also be interested in more terms than the first. We can estimate the higher-order asymptotic constants $\alpha_k$ in (1.5) in the same way. By essentially the same reasoning,

$$\hat{\alpha}_{k,n} = A_{k,n} B_{k,n}, \tag{1.18}$$

where

$$A_{k,n} = \frac{\alpha_1(n+\beta-1)(n+\beta-2)\cdots(n+\beta-k)}{\eta^k} \tag{1.19}$$

and

$$\begin{aligned} B_{k,n} = {} & \frac{\eta^{n+\beta} m_n}{\alpha_1 n\Gamma(n+\beta)} - 1 - \frac{\alpha_2 \eta}{\alpha_1(n+\beta-1)} \\ & - \cdots - \frac{\alpha_{k-1}\eta^{k-2}}{\alpha_1(n+\beta-1)\cdots(n+\beta-k+1)} \end{aligned} \tag{1.20}$$

is an estimate of $\alpha_k$ that has the asymptotic expansion in (1.12). Note that to use $\hat{\alpha}_{k,n}$ to estimate $\alpha_k$, we require that $\eta, \beta, \alpha_1, \ldots, \alpha_{k-1}$ be known. When these parameters are not known accurately, the estimate $\hat{\alpha}_{k,n}$ in (1.18) can experience numerical problems. If we use multiple terms in (1.5), then we lose much of the simplicity of (1.4), but gain numerical accuracy. Numerical accuracy may also be achieved by direct numerical transform inversion, but multiterm asymptotic formulas are useful because they give easily computable formulas for all large $x$, whereas direct numerical transform inversion becomes less accurate as $x$ increases. Multiterm formulas also reveal the accuracy of fewer terms.

Here is how the rest of this paper is organized. In Section 2 we show that it is necessary to *assume* (1.4); convergence of the estimates (1.6)–(1.8) does *not* by itself imply (1.4). In Section 3 we show how to treat cases in which not all moments exist, or $\eta = 0$ in (1.4), by first doing exponential damping. After making this transformation, all moments exist, and it is easy to extract the desired asymptotic parameters.

In Section 4 we treat the relatively elementary problems involving the estimates of the asymptotic decay rate $\eta$ and the asymptotic constant $\alpha$ in (1.6), (1.8) and (1.17) based on only (1.4). Then in Section 5 we discuss the implications that can be obtained from stronger asymptotics such as in (1.5). Our proofs of the asymptotic relations in Sections 4 and 5 exploit elementary direct probabilistic arguments. An alternative approach is via classical asymptotic analysis, after recognizing that $m_n/n$ can be regarded as a Mellin transform of $F^c(x)$; that is, from (1.1),

$$\frac{m_n}{n} = \int_0^\infty x^{n-1} F^c(x)\, dx; \tag{1.21}$$

for example, see page 77 and Chapter 4 of Bleistein and Handelsman (1986). Using techniques from the asymptotic analysis of integrals depending on a parameter, it is possible to obtain the asymptotic expansion

$$\frac{m_n}{n} \sim \frac{\Gamma(n+\beta)}{\eta^{n+\beta}} \left\{ \sum_k \frac{a_k \eta^k}{(n+\beta)^k} \right\}, \tag{1.22}$$

where the constants $a_k$ depend on the asymptotic expansion of $e^x x^{-\beta} F^c(x)$ and its successive derivatives; see page 290 of Berg (1968). Hence, it is the application of the theorems rather than their statement and proof that constitutes the main contribution of our paper.

In Section 6 we discuss asymptotics of the form

$$F^c(x) \sim \alpha x^{\beta} e^{-\eta x^{\delta}} \quad \text{as } x \to \infty \tag{1.23}$$

for $\delta$ not necessarily equal to 1. In Section 7 we give an illustrative numerical example involving the time-dependent mean of *reflected Brownian motion* (RBM), for which $\beta = -3/2$; see Abate and Whitt (1987). We pick a relatively easy example with convenient explicit transform, so that we can verify our results. Harder examples for which there are no readily available alternative methods are the polling models in Choudhury and Whitt (1996). In Section 8 we consider long-tail examples with $\eta = 0$ in (1.4), drawing on Abate, Choudhury and Whitt (1994b). Finally, in Section 9 we state our conclusions.

We close this section by mentioning other related work. We have already mentioned the alternative asymptotic approach via Mellin transforms in (1.22). Another body of related literature studies the asymptotic analysis of an unknown function via the coefficients of its Taylor series. There is an extensive literature on this problem in mathematical physics, as can be seen from Hunter and Guerrieri (1980) and Guttman (1989). What we are doing here is closely related to the ratio method and its variants. With that theory, the asymptotic behavior of the moments, as coefficients of the moment generating function, can be used to obtain the asymptotic form of the Laplace transform. We would then use the asymptotic behavior of the Laplace transform to deduce the asymptotic form of the complementary cdf, as in Heaviside's theorem discussed above. However, it remains to identify and justify appropriate technical regularity conditions. Nevertheless, this is another potential route to our results.

**2. A counterexample to the converse.** For practical purposes, we consider the asymptotics in (1.4) justified, as well as the parameters identified, when we establish $\hat{\eta}_n \to \eta$, $\hat{\beta}_n \to \beta$ and $\hat{\alpha}_{1,n} \to \alpha$ as $n \to \infty$ via (1.6)–(1.8) or via an extrapolation. However, we actually need to *assume* the form (1.4). To make this clear, we now show that it is possible to have $\hat{\eta}_n \to \eta$, $\hat{\beta}_n \to \beta$ and $\hat{\alpha}_{1,n} \to \alpha$ as $n \to \infty$ without having (1.4).

For this purpose, we use a probability density function (pdf) with $\beta = 0$ and a sinusoidal component. In particular, let the pdf be

$$f(x) = 2e^{-x}(1 - \cos x) = (2\sin(x/2))^2 e^{-x}, \qquad x \ge 0, \tag{2.1}$$

with associated cdf

$$F(x) = 1 - e^{-x}(2 - \cos x + \sin x) \tag{2.2}$$

and Laplace transform

$$\hat{f}(s) = \int_0^\infty e^{-sx} f(x)\, dx = \frac{2}{(1+s)\left(1 + (1+s)^2\right)}; \tag{2.3}$$

see 29.3.27 on page 1023 of Abramowitz and Stegun (1972). From (7.16) on page 55 of Oberhettinger and Badii (1973),

$$\int_0^\infty e^{-\alpha x} x^n \cos x\, dx = \frac{n!}{(1+\alpha^2)^{(n+1)/2}} \cos\left((n+1)\arctan \alpha^{-1}\right), \tag{2.4}$$

so that the $n$th moment of $F$ is

$$m_n = 2(n!)\left(1 - 2^{-(n+1)/2}\cos((n+1)\pi/4)\right), \qquad n \ge 1. \tag{2.5}$$

From (2.5), it is elementary that $\hat{\eta}_n \to \eta = 1$, $\hat{\beta}_n \to \beta = 0$ and $\hat{\alpha}_{1,n} \to \alpha = 2$ as $n \to \infty$. However, we do not have (1.4) with these parameters; that is, we do *not* have $F^c(x) \sim 2e^{-x}$ as $x \to \infty$ because of the sinusoidal terms. From the perspective of the Laplace transform (2.3), the asymptotics is understandable, because the transform has three singularities for $s$ such that $\text{Re}(s) = -1$, namely, $-1$ and $-1 \pm i$. For a related example involving a Tauberian theorem, see Example 1 on page 107 of Abate, Choudhury and Whitt (1994a).

**3. When not all moments are finite.** To calculate the moments from the mgf, we need the mgf to be analytic at $z = 0$, which, in turn, requires that all moments be finite. However, we may not know if this condition is satisfied, or we may even know that the condition is not satisfied. For example, we may want to identify $\alpha$ and $\beta$ under the condition (1.4) with $\eta = 0$, that is, for a long-tail distribution, such as in the queueing examples in Abate, Choudhury and Whitt (1994b).

All these cases can be treated by first modifying the distribution so that it necessarily has all the desired properties. As in (5.2) of Abate, Choudhury and Whitt (1994b), starting with a probability density $f(x)$, we can construct a new probability density $f_u$ by *exponentially damping* $f$, that is, by letting

$$f_u(x) = e^{-ux}f(x)/\hat{f}(u). \tag{3.1}$$

Note that $f_u$ has Laplace transform

$$\hat{f}_u(s) = \hat{f}(s+u)/\hat{f}(u), \tag{3.2}$$

where $\hat{f}(s)$ is the Laplace transform of $f(x)$. If $\hat{f}$ has rightmost singularity at 0, then $\hat{f}_u$ has rightmost singularity at $-u$. Moreover, if

$$f(x) \sim \alpha x^{\beta} e^{-\eta x} \quad \text{as } x \to \infty, \tag{3.3}$$

then

$$f_u(x) \sim \frac{\alpha}{\hat{f}(u)} x^{\beta} e^{-(\eta+u)x} \quad \text{as } x \to \infty. \tag{3.4}$$

Given (3.4), the associated complementary cdf satisfies

$$F_u^c(x) \sim \frac{\alpha x^{\beta} e^{-(\eta+u)x}}{(\eta+u)\hat{f}(u)} \quad \text{as } x \to \infty; \tag{3.5}$$

see page 17 of Erdélyi (1956).

Hence, given the Laplace transform $\hat{f}(s)$, we can easily construct the new Laplace transform $\hat{f}_u(s)$ using (3.2), calculate its moments from $M_u(z) \equiv \hat{f}_u(-z)$, obtain its asymptotic parameters from (1.6)–(1.8) or via extrapolation and then obtain $\alpha$, $\beta$ and $\eta$ from (3.5). If $f$ satisfies (3.3) with $\beta < -1$ and $\eta = 0$, then the complementary cdf satisfies

$$F^c(x) \sim \frac{\alpha}{\beta+1} x^{\beta+1} \quad \text{as } x \to \infty. \tag{3.6}$$

Hence, we must multiply the asymptotic constant obtained for $F_u^c(x)$ by $u\hat{f}(u)/(\beta+1)$ to obtain the desired asymptotic constant for $F^c(x)$.

We close this section by pointing out that, once the asymptotic decay rate $\eta$ is known, exponential damping can be used in reverse to move the dominant singularity to the origin and produce asymptotics of the form (3.3) with $\eta = 0$. Instead of moments, we can then use Tauberian theorems as in Feller (1971) to derive the remaining asymptotic parameters $\alpha$ and $\beta$.

**4. The asymptotic decay rate and constant.** It is elementary to get the asymptotic decay rate $\eta$ from (1.6), and, given both the asymptotic decay rate and the asymptotic power, the asymptotic constant $\alpha$ from (1.8) given only (1.4). Throughout this paper we will use basic properties of the gamma functions $\Gamma(x)$ and the gamma distribution. The following is 6.1.46 of Abramowitz and Stegun (1972).

LEMMA 4.1. *For any real number* $\beta$, $\Gamma(n+\beta+1) \sim n! n^{\beta}$ *as* $n \to \infty$ *through the integers.*

Let $X(\eta, \nu)$ be a random variable with a gamma density

$$f(x) = \frac{1}{\Gamma(\nu)} \eta^{\nu} x^{\nu-1} e^{-\eta x}, \qquad x \geq 0, \tag{4.1}$$

with *scale parameter* $\eta$ and *shape parameter* $\nu$, which has mean $\nu/\eta$ and variance $\nu/\eta^2$. We will use a bound on the gamma tail probabilities; we omit the proof.

LEMMA 4.2. *For any* $\beta$, $x_0 > 0$ *and* $\eta > 0$, *there exist positive constants* $C_1$ *and* $C_2$ *such that*

$$P(X(\eta, n+\beta) < x_0) \leq C_1 e^{-C_2 n} \tag{4.2}$$

*for all suitably large* $n$.

THEOREM 4.1. *If* (1.4) *holds with* $\eta > 0$ *and* $\alpha > 0$, *then*

$$m_n \sim \frac{\alpha n \Gamma(n+\beta)}{\eta^{n+\beta}} \sim \frac{\alpha (n!) n^{\beta}}{\eta^{n+\beta}} \quad \text{as } n \to \infty, \tag{4.3}$$

*so that* $\hat{\eta}_n \to \eta$, $\hat{\alpha}_{1,n} \to \alpha$ *and* $\alpha^*_{1,n} \to \alpha$ *for* $\hat{\eta}_n$ *in* (1.6), $\hat{\alpha}_{1,n}$ *in* (1.8) *and* $\alpha^*_{1,n}$ *in* (1.17).

PROOF. Relation (1.4) is equivalent to there being, for each $\varepsilon > 0$, an $x_0$ such that

$$F^c(x) \leq (1+\varepsilon)\alpha x^{\beta} e^{-\eta x} \quad \text{for } x \geq x_0$$

and

$$F^c(x) \geq (1-\varepsilon)\alpha x^{\beta} e^{-\eta x} \quad \text{for } x \geq x_0.$$

Then, using elementary properties of the gamma distribution, we obtain

$$\begin{aligned} m_n &= \int_0^{\infty} n x^{n-1} F^c(x)\, dx \\ &\leq \int_0^{x_0} n x^{n-1} F^c(x)\, dx + \alpha(1+\varepsilon) n \int_{x_0}^{\infty} x^{n+\beta-1} e^{-\eta x}\, dx \\ &\leq x_0^n + \frac{\alpha(1+\varepsilon) n \Gamma(n+\beta)}{\eta^{n+\beta}} P(X(\eta, n+\beta) > x_0) \\ &\leq x_0^n + \frac{\alpha(1+\varepsilon) n \Gamma(n+\beta)}{\eta^{n+\beta}}, \end{aligned} \tag{4.4}$$

while

$$m_n \geq \frac{\alpha(1-\varepsilon) n \Gamma(n+\beta)}{\eta^{n+\beta}} P(X(\eta, n+\beta) > x_0). \tag{4.5}$$

Since $x_0^n$ is negligible compared to $n\Gamma(n+\beta)\eta^{-(n+\beta)}$ and $P(X(\eta, n+\beta) > x_0) \to 1$ as $n \to \infty$ by Lemma 4.2, (4.4) and (4.5) imply that

$$m_n \sim \alpha n \Gamma(n+\beta)/\eta^{n+\beta} \quad \text{as } n \to \infty, \tag{4.6}$$

which is (4.3). The second asymptotic relation in (4.3) follows from Lemma 4.1. The limits for $\hat{\eta}_n$ and $\hat{\alpha}_{1,n}$ and $\alpha^*_{1,n}$ are elementary consequences. □

In order to get $\alpha$, we need to know $\eta$ and $\beta$. Even in the case $\beta = 0$, we cannot just usé any sequence of estimates of $\eta$ in our estimate of $\alpha$. The following elementary result supports this claim. It shows that it suffices to have $\overline{\eta}_n = \eta + o(n^{-1})$ and $\overline{\beta}_n = \beta + o(1/\log n)$ as $n \to \infty$ in order to have $\hat{\alpha}_{1,n}$ converge to $\alpha$ with estimates $\overline{\eta}_n$ and $\overline{\beta}_n$ instead of $\eta$ and $\beta$.

THEOREM 4.2. *If* $\overline{\eta}_n = \eta(1 + \gamma n^{-1} + o(n^{-1}))$ *and* $\overline{\beta}_n = \beta + \delta/\log n + o(1/\log n)$ *as* $n \to \infty$, *then*

$$\overline{\eta}_n^{n+\overline{\beta}_n} m_n / n! n^{\overline{\beta}_n} \to \alpha e^{\gamma - \delta} \quad as\ n \to \infty.$$

PROOF. To treat $\overline{\eta}_n$, use (4.3) with

$$\overline{\eta}_n^{n+\beta} = \eta^{n+\beta}\left(1 + \gamma n^{-1} + o(n^{-1})\right)^{n+\beta} \sim e^{\gamma}\eta^{n+\beta} \quad \text{as } n \to \infty.$$

To treat $\overline{\beta}_n$, note that $n^{\overline{\beta}_n - \beta} \to K$ if and only if $(\overline{\beta}_n - \beta)\log n \to \log K$, but, from the assumption, $(\overline{\beta}_n - \beta)\log n \to \delta$, so that $K = e^{\delta}$ □

We have seen in (1.12) that it is indeed natural to have $\hat{\eta}_n = \eta + O(n^{-1})$ for $\hat{\eta}_n$ in (1.6), but if we use $\hat{\eta}_n(k)$ in (1.13) for $k > 1$, then the error is $O(n^{-k})$ and so $o(n^{-1})$. Hence, with suitable estimates of $\eta$ and $\beta$, the estimator for $\alpha$ in (1.8) will converge.

It is easy to see that there is no difficulty in the case of a simple pole, where there is an exponential rate of convergence [see Section 5.2 of Wilf (1994)], provided the decay rate is not too small. The estimates $\hat{\eta}_n$ in (1.6) and $\hat{\eta}_n(2)$ in (1.15) are essentially equivalent in this case. We state the elementary result without proof.

THEOREM 4.3. *If*

$$F^c(x) - \alpha e^{-\eta x} \sim \gamma e^{-\phi x} \quad as\ x \to \infty$$

*for* $\phi > \eta$, *then*

$$\hat{\eta}_n \equiv \hat{\eta}_n(1) = \eta + O\left((\eta/\phi)^n\right) \quad as\ n \to \infty$$

*for* $\hat{\eta}_n$ *in* (1.6) *and*

$$\eta_n^*(2) = \eta + O\left(n(\eta/\phi)^n\right) \quad as\ n \to \infty$$

*for* $\eta_n^*(2)$ *in* (1.15), *so that*

$$\hat{\eta}_n(1)^n m_n/n! \to \alpha \quad and \quad \eta_n^*(2)^n m_n/n! \to \alpha \quad as\ n \to \infty.$$

**5. The asymptotic power and multiterm expansions.** Assuming only (1.4), it is relatively difficult to get a good estimator of the asymptotic power $\beta$. The following establishes an estimator that converges as $n \to \infty$ with an error that is $O(1/\log n)$, which is not good for numerical accuracy.

THEOREM 5.1. *If* (1.4) *holds, so that* (4.3) *holds, then*

$$\beta_n^{\#} \equiv \frac{\log(\eta^n m_n/n!)}{\log n} = \beta + \frac{\log(\alpha n^{-\beta})}{\log n} + o\left(\frac{1}{\log n}\right) \quad \text{as } n \to \infty. \tag{5.1}$$

PROOF. Use (4.3) and Lemma 4.1. □

It is important to note that an estimate of $\beta$ cannot be extracted directly from $1/\hat{\eta}_n$ in (1.6), assuming only (1.4), because $\beta$ appears in the $O(n^{-1})$ term of

$$\frac{1}{\hat{\eta}_n} = \frac{m_n}{n m_{n-1}} \sim \frac{n+\beta}{n\eta}; \tag{5.2}$$

that is, we cannot distinguish $n + \beta$ from $n$ on the right in (5.2) because of the asymptotic relation. For example, we evidently would have difficulty with (5.2) if we had

$$F^c(x) - \alpha x^{\beta} e^{-\eta x} \sim \frac{\gamma x^{\beta} e^{-\eta x}}{\log x} \quad \text{as } x \to \infty. \tag{5.3}$$

However, we often have a better rate of convergence than (5.3). Indeed, we often have (1.5) when $\beta \neq 0$. If we assume a stronger form of asymptotics corresponding to (1.5), then we can obtain a better estimate of the asymptotic power $\beta$ than (5.1). We can also justify convergence of $\eta_n^*(2)$ in (1.15). We start with a generalization of (1.5). For example, condition (5.4) below can arise in a mixture of two distributions satisfying (1.5) with the same asymptotic decay rate but different asymptotic power parameters. Condition (1.5) is the special case of (5.4) in which $\phi = 1$.

THEOREM 5.2. *If*

$$F^c(x) - \alpha x^{\beta} e^{-\eta x} - \gamma x^{\beta-\phi} e^{-\eta x} \sim \delta x^{\beta - 2\phi} e^{-\eta x} \quad \text{as } x \to \infty \tag{5.4}$$

*for strictly positive finite constants* $\alpha$, $\eta$ *and* $\phi$ *and for finite constants* $\beta$, $\gamma$ *and* $\delta$, *then*

$$m_n = \frac{\alpha n \Gamma(n+\beta)}{\eta^{n+\beta}}\left(1 + \frac{\gamma \eta^{\phi}}{\alpha n^{\phi}} + O(n^{-2\phi})\right) \quad \text{as } n \to \infty \tag{5.5}$$

*and*

$$r_n \equiv \frac{m_n}{m_{n-1}} = \frac{n}{\eta} + \frac{n\beta}{(n-1)\eta} + O(n^{1-\psi}) \quad \text{as } n \to \infty \tag{5.6}$$

*for* $\psi = \min\{1 + \phi, 2\phi\}$. *Hence, for* $\phi > 1/2$, $\hat{\beta}_n \to \beta$ *for* $\hat{\beta}_n$ *in* (1.7) *and* $\eta_n^*(2) \to \eta$ *for* $\eta_n^*(2)$ *in* (1.15).

PROOF. First assume that $\gamma$ and $\delta$ are nonnegative. The condition implies that for each $\varepsilon > 0$ there exists $x_0$ such that

$$F^c(x) - \alpha x^{\beta} e^{-\eta x} - \gamma x^{\beta-\phi} e^{-\eta x} \le (1+\varepsilon)\delta x^{\beta-2\phi} e^{-\eta x}$$

and

$$F^c(x) - \alpha x^{\beta} e^{-\eta x} - \gamma x^{\beta-\phi} e^{-\eta x} \ge (1-\varepsilon)\delta x^{\beta-2\phi} e^{-\eta x}$$

for all $x \ge x_0$. As in the proof of Theorem 4.1, we use properties of the gamma distribution. As before, let $X(\eta, \nu)$ be a gamma random variable with scale parameter $\eta$ and shape parameter $\nu$ with density in (4.1). We now establish an upper bound for $m_n$. Note that

$$\begin{aligned} m_n &= \int_0^\infty n x^{n-1} F^c(x)\,dx \\ &\le \int_0^{x_0} n x^{n-1} F^c(x)\,dx + \int_{x_0}^\infty n x^{n-1} F^c(x)\,dx \\ &\le x_0^n + \int_{x_0}^\infty \alpha n x^{n+\beta-1} e^{-\eta x}\,dx + \int_{x_0}^\infty \gamma n x^{n+\beta-\phi-1} e^{-\eta x}\,dx \\ &\quad + \int_{x_0}^\infty (1+\varepsilon)\delta n x^{n+\beta-2\phi-1} e^{\eta x}\,dx \\ &\le x_0^n + \frac{\alpha n \Gamma(n+\beta)}{\eta^{n+\beta}} + \frac{\gamma n \Gamma(n+\beta-\phi)}{\eta^{n+\beta-\phi}} \\ &\quad + \frac{(1+\varepsilon)\delta n \Gamma(n+\beta-2\phi)}{\eta^{n+\beta-2\phi}}, \end{aligned} \tag{5.7}$$

where the gamma distribution over the entire positive half line is used in the last step. Note that $x_0^n$ is negligible compared to $\Gamma(n+\beta)/\eta^{n+\beta}$ as $n$ grows. Since $\Gamma(n+b)/\Gamma(n+a) \sim n^{b-a}$ by Lemma 4.1,

$$m_n \le \frac{\alpha n \Gamma(n+\beta)}{\eta^{n+\beta}} \left(1 + \frac{\gamma \eta^{\phi}}{\alpha n^{\phi}} + O(n^{-2\phi})\right). \tag{5.8}$$

Similarly, as a lower bound we obtain

$$\begin{aligned} m_n &\ge \int_{x_0}^\infty n x^{n-1} F^c(x)\,dx \\ &\ge \frac{\alpha n \Gamma(n+\beta)}{\eta^{n+\beta}} P(X(\eta, n+\beta) > x_0) \\ &\quad + \frac{\gamma n \Gamma(n+\beta-\phi)}{\eta^{n+\beta-\phi}} P(X(\eta, n+\beta-\phi) > x_0) \\ &\quad + \frac{(1-\varepsilon)\delta n \Gamma(n+\beta-2\phi)}{\eta^{n+\beta-2\phi}} P(X(\eta, n+\beta-2\phi) > x_0). \end{aligned} \tag{5.9}$$

Hence, we can combine (5.9) and Lemma 4.2 to obtain

$$m_n \geq \frac{\alpha n \Gamma(n+\beta)}{\eta^{n+\beta}}\left(1 + \frac{\gamma\eta^{\phi}}{\alpha n^{\phi}} + O(n^{-2\phi})\right). \tag{5.10}$$

Combining (5.8) and (5.10), we obtain (5.5). From (5.5), we obtain

$$\begin{aligned}\frac{m_n}{m_{n-1}} &= \frac{n(n+\beta-1)}{(n-1)\eta}\left(\frac{1+\gamma\eta^{\phi}/\left[\alpha(n-1)^{\phi}\right]+O(n^{-2\phi})}{1+\gamma\eta^{\phi}/(\alpha n\phi)+O(n^{-2\phi})}\right)\\ &= \left(\frac{n}{\eta}+\left(\frac{n}{n-1}\right)\frac{\beta}{\eta}\right)(1+O(n^{-\psi}))\\ &= \frac{n}{\eta}+\left(\frac{n}{n-1}\right)\frac{\beta}{\eta}+O(n^{1-\psi})\end{aligned}$$

for $\psi = \min\{2\phi, 1+\phi\}$, which is (5.6).

At the outset, we assumed that $\gamma$ and $\delta$ are nonnegative. It is easy to modify the proof to cover the other cases. If $\gamma < 0$, then we can replace $\gamma$ by 0 in the upper bound and we can replace $P(X(\eta, n+\beta-\phi) > x_0)$ by 1 in the lower bound. If $\delta < 0$, then we change the role of $(1+\varepsilon)$ and $(1-\varepsilon)$ in the $\delta$ terms. Finally, the limits for $\hat{\beta}_n$ in (1.7) and $\eta_n^*(2)$ in (1.15) follow easily from (5.6). □

THEOREM 5.3. *If* (1.5) *holds for strictly positive finite constants* $\eta$ *and* $\alpha_1$ *and finite constants* $\beta, \alpha_2, \dots, \alpha_m$, *then*

$$\begin{aligned} m_n &= \frac{\alpha_1 n\Gamma(n+\beta)}{\eta^{n+\beta}}\\ &\quad\times\left(1+\frac{c_1}{n}+\frac{c_2}{n^2}+\cdots+\frac{c_{m-1}}{n^{m-1}}+O(n^{-m})\right) \quad \text{as } n\to\infty,\end{aligned} \tag{5.11}$$

*where*

$$\begin{aligned} c_1 &= \frac{\alpha_2\eta}{\alpha_1}, \qquad c_2 = \frac{\alpha_3\eta^2}{\alpha_1} - \frac{(\beta-1)\alpha_2\eta}{\alpha_1},\\ c_3 &= \frac{\alpha_4\eta^3}{\alpha_1} - \frac{(2\beta-3)\alpha_3\eta^2}{\alpha_1} + \frac{(\beta-1)^2\alpha_2\eta}{\alpha_1},\end{aligned} \tag{5.12}$$

*so that*

$$\begin{aligned} r_n &\equiv \frac{m_n}{m_{n-1}}\\ &= \frac{n}{\eta}+\frac{\beta}{\eta}+\frac{d_1}{n}+\frac{d_2}{n^2}+\cdots+\frac{d_{m-2}}{n^{m-2}}+O(n^{-(m-1)}) \quad \text{as } n\to\infty,\end{aligned} \tag{5.13}$$

*where*

$$d_1 = \frac{\beta}{\eta} - \frac{\alpha_2\eta}{\alpha_1},$$
$$(5.14)\qquad d_2 = \frac{\beta}{\eta} - \frac{\beta\alpha_2}{\alpha_1} + \frac{(2\beta-3)\alpha_2}{\alpha_1} + \frac{\alpha_2^2\eta}{\alpha_1^2} - \frac{2\alpha_3\eta}{\alpha_1}.$$

*Consequently, the expansion* (1.12) *holds for* $\hat{\eta}_n$ *in* (1.6), $\hat{\beta}_n$ *in* (1.7), $\hat{\alpha}_{1,n}$ *in* (1.8), $\beta_n^*$ *in* (1.16) $\alpha_{1,n}^*$ *in* (1.17) *and* $\hat{\alpha}_{k,n}$ *in* (1.18).

PROOF. We will use the expansion

$$(5.15)\qquad \frac{1}{n-x} = \frac{1}{n}\left(\frac{1}{1-(x/n)}\right) = \frac{1}{n}\left(1 + \frac{x}{n} + \left(\frac{x}{n}\right)^2 + \left(\frac{x}{n}\right)^3 + \cdots\right).$$

A minor modification of the proof of Theorem 5.2 yields

$$(5.16)\qquad m_n = \frac{\alpha_1 n\Gamma(n+\beta)}{\eta^{n+\beta}}X_n,$$

where

$$X_n = 1 + \frac{\alpha_2\eta}{\alpha_1(n+\beta-1)} + \frac{\alpha_3\eta^2}{\alpha_1(n+\beta-1)(n+\beta-2)}$$
$$(5.17)\qquad + \cdots + \frac{\alpha_m\eta^{m-1}}{\alpha_1(n+\beta-1)(n+\beta-2)\cdots(n+\beta-m+1)}$$
$$+ O(n^{-m}).$$

Combining (5.15) and (5.16) yields (5.11) with (5.12). Similarly,

$$(5.18)\qquad X_{n-1} = 1 + C_1 n^{-1} + C_2 n^{-2} + \cdots + C_{m-1}n^{-(m-1)} + O(n^{-m}),$$

where $C_1 = c_1$, $C_2 = c_1 + c_2$ and $C_3 = c_1 + 2c_2 + c_3$. Then, by 3.6.22 of Abramowitz and Stegun (1972),

$$(5.19)\qquad \frac{X_n}{X_{n-1}} = 1 + D_1 n^{-1} + D_2 n^{-2} + \cdots + D_{m-1}n^{-(m-1)} + O(n^{-m}),$$

where $D_1 = 0$, $D_2 = \alpha_2\eta/\alpha_1$ and

$$(5.20)\qquad D_3 = \frac{(2\beta-3)\alpha_2\eta}{\alpha_1} + \frac{\alpha_2^2\eta^2}{\alpha_1^2} - \frac{2\alpha_3\eta^2}{\alpha_1}.$$

Hence, by 3.6.21 of Abramowitz and Stegun (1972), we obtain (5.13) with (5.14) from

$$\frac{m_n}{m_{n-1}} = \frac{n(n+\beta-1)}{(n-1)\eta}\frac{X_n}{X_{n-1}}$$
$$(5.21)\qquad = \left(\frac{n}{\eta} + \frac{\beta}{\eta} + \frac{\beta}{\eta n} + \frac{\beta}{\eta n^2} + \cdots\right)$$
$$\times\left(1 + \frac{D_1}{n} + \cdots + \frac{D_{m-1}}{n^{m-1}} + O(n^{-m})\right).$$

The expansions for $\hat{\eta}_n$, $\hat{\beta}_n$, $\hat{\alpha}_{1,n}$, $\beta_n^*$ and $\alpha_{1,n}^*$ follow by similar arguments. To relate $\hat{\alpha}_{1,n}$ in (1.8) and $\alpha_{1,n}^*$ in (1.17), use the asymptotic expansion of the gamma function; see 6.1.37 and 6.1.47 of Abramowitz and Stegun (1972). □

**6. Weibull-like tails.** Suppose that, instead of (1.4), the complementary cdf has the tail behavior in (1.23), where $\delta$ is a positive constant. When $\beta = 0$, $F$ has the tail behavior of a Weibull distribution; see Chapter 20 of Johnson and Kotz (1970). The case (1.23) can be treated by reducing it to (1.4). In particular, if $X$ has cdf $F$ satisfying (1.23), then $X^\delta$ has complementary cdf of the form (1.4), that is,

$$P(X^\delta > x) \sim \alpha x^{(\beta/\delta)} e^{-\eta x} \quad \text{as } x \to \infty. \tag{6.1}$$

Hence, we can apply previous results with the moments of $X^\delta$; for example, Theorem 4.1 implies that

$$m_{\delta r} \sim \alpha \Gamma(r + (\beta/\delta) + 1)/\eta^{r+(\beta/\delta)} \quad \text{as } r \to \infty, \tag{6.2}$$

where $r$ is a positive real number.

However, the numerical inversion algorithm only computes the integer moments $m_n$. If $\delta$ is an integer (e.g., $\delta = 2$ for normal tails), then this approach can be applied directly. For example, instead of (1.6)–(1.8), we have

$$\hat{\eta}_n \equiv \frac{n m_{\delta(n-1)}}{m_{\delta n}} \to \eta \quad \text{as } n \to \infty, \tag{6.3}$$

$$\hat{\beta}_n \equiv \delta\left(\frac{\eta m_{\delta n}}{m_{\delta(n-1)}} - n\right) \to \beta \quad \text{as } n \to \infty, \tag{6.4}$$

$$\hat{\alpha}_n \equiv \frac{\eta^{n+(\beta/\delta)} m_{\delta n}}{n! n^{\beta/\delta}} \to \alpha \quad \text{as } n \to \infty, \tag{6.5}$$

assuming that (1.5) is valid. Moreover, the estimates (6.3)–(6.5) have the asymptotic form (1.12), so that we can extrapolate to greatly speed up the convergence.

More generally, we can estimate $\delta$ as well as $\eta$, $\beta$ and $\alpha$ by exploiting (6.2). The following result can be proved in the same way as previous results.

THEOREM 6.1. *If $X^\delta$ satisfies (1.5) with $\beta/\delta$ in place of $\beta$, then the estimators*

$$\hat{\delta}_n \equiv \frac{r_n}{n(r_{n+1} - r_n)}, \tag{6.6}$$

$$\hat{\eta}_n \equiv \frac{n}{r_n^\delta \delta}, \tag{6.7}$$

$$\hat{\beta}_n \equiv \delta\left((\eta\delta)^{1/\delta} r_n - n - 1\right) + 1, \tag{6.8}$$

$$\hat{\alpha}_n \equiv \frac{\eta^{(n+\beta)/\delta}}{\Gamma((n+\beta)/\delta)}, \tag{6.9}$$

*where* $r_n = m_n/m_{n-1}$, *converge to the appropriate limits as* $n \to \infty$ *and have the asymptotic form in* (1.12). *Consequently, we can extrapolate with* (6.6)–(6.9) *using* (1.13).

**7. An RBM example.** In this section we give a numerical example illustrating how the moment-based estimates of the asymptotic parameters perform. We use an elementary example here, for which we can calculate the moments and asymptotic parameters directly, so that it is easy to verify our results. We apply our methods to more difficult examples, polling models, in Choudhury and Whitt (1996).

In particular, here we consider the time-dependent mean of canonical *reflected Brownian motion* (RBM) starting off empty, which was considered in Abate and Whitt (1987, 1996). By canonical RBM, we mean that the drift is $-1$ and the diffusion coefficient is 1. If we divide the mean by the steady-state mean 1/2, then we obtain a bonafide cdf (cumulative distribution function) with mean 1/2, denoted by $H_1(x)$. We will further scale the distribution so that it has mean 1. Thus, we consider the cdf

$$(7.1)\quad H_1(x) = 1 - (x+2)\left[1 - \Phi\left(\sqrt{x/2}\right) + 2\sqrt{x/2}\,\phi\left(\sqrt{x/2}\right)\right], \qquad x \geq 0,$$

where $\Phi$ is the standard (mean 0, variance 1) normal cdf and $\phi$ is its density function. It is known that $H_1^c(x) \equiv 1 - H_1(x)$ has asymptotic form (1.4) with $\eta = 1/4$, $\beta = -3/2$ and $\alpha = 8/\sqrt{\pi}$; see Corollary 1.3.5 of Abate and Whitt (1987) and make the adjustment for the mean being increased from 1/2 to 1. It is also easy to show that $H_1^c(x)$ satisfies (1.5) by applying 26.2.12 of Abramowitz and Stegun (1972). The associated Laplace–Stieltjes transform of $H_1$ is

$$(7.2)\qquad \hat{h}_1(s) = \frac{2}{1 + \sqrt{1+4s}}$$

and the moments are

$$(7.3)\qquad m_n = (2n)!/(n+1)!$$

[see (10.12) and (10.15) of Abate and Whitt (1996)].

Using the explicit expression for the moments in (7.3) and the asymptotic expansion for $m_n$ in (5.11), we can also identify the asymptotic constants for all terms in (1.5). In particular,

$$(7.4)\qquad \alpha_k/\alpha_{k-1} = -(4k-2), \qquad k \geq 2.$$

Hence, in this case we know all the asymptotic constants appearing in (1.5) in advance. Moreover, from (7.4), we see that, for any fixed $x$, the asymptotic series on the right in (1.5) is actually *divergent*. This is a familiar phenomenon with asymptotic expansions. It implies, for each $x$, that additional terms in (1.5) will only help up to a point. (We illustrate this below.)

For this example, we can directly compute the estimators for $\eta$, $\beta$ and $\alpha$. First, for $\hat{\eta}_n$ in (1.6),

$$\begin{aligned}\hat{\eta}_n &\equiv \hat{\eta}_n(1) \equiv \frac{nm_{n-1}}{m_n} = \frac{n(n+1)}{2n(2n-1)} \\ &= \frac{1}{4}\frac{(1+1/n)}{(1-1/2n)} = \frac{1}{4} + \frac{3}{8n} + \frac{3}{16n^2} + \frac{3}{32n^3} + \cdots .\end{aligned} \tag{7.5}$$

Next, for $\hat{\beta}_n$ in (1.7),

$$\begin{aligned}\hat{\beta}_n &\equiv \frac{\eta m_n}{m_{n-1}} - n = \frac{1}{4}\left(\frac{4n^2-2n}{n+1}\right) - n \\ &= \frac{-3/2}{(1+1/n)} = \frac{-3}{2}\left(1 - \frac{1}{n} + \frac{1}{n^2} - \frac{1}{n^3} \pm \cdots\right).\end{aligned} \tag{7.6}$$

Then, for $\alpha_{1,n}^*$ in (1.17), using 6.1.12 in Abramowitz and Stegun (1972) for $\Gamma(n+1/2)$, we obtain

$$\begin{aligned}\alpha_{1,n}^* &= \frac{\eta^{n+\beta}m_n}{\Gamma(n+\beta+1)} = \frac{8}{\sqrt{\pi}}\left(\frac{2n-1}{2n+2}\right) \\ &= \frac{8}{\sqrt{\pi}}\frac{(1-(1/2n))}{(1+(1/n))} = \frac{8}{\sqrt{\pi}}\left(1 - \frac{3}{2n} + \frac{3}{2n^2} + \cdots\right).\end{aligned} \tag{7.7}$$

Now we want to see how the moment-based estimates of the asymptotic parameters perform, where we use only computed values (instead of the exact values) at each step. All computations were done with double precision. First, we calculated and stored the first 40 moments by numerically inverting the moment generating function $\hat{h}(-z)$ for $\hat{h}(s)$ in (7.2), using the algorithm described in Choudhury and Lucantoni (1993, 1995). In this case, computing the first 40 or first 200 moments is not difficult, but it is the biggest part of the computation. [For the polling problem in Choudhury and Whitt (1996), computing the first 100 moments is much more difficult, taking a few minutes on a SUN SPARC2 workstation, because the transform is not available explicitly.] Our computation of the $H_1$ moments consistently yielded good accuracy, with at least nine significant digits for each of the first 40 moments. (In general, the accuracy can be checked by doing the inversion for two different values of the roundoff-error control variable $l$.) This nine-digit accuracy is more than we usually care about for the moments themselves, but we will exploit it for estimating the asymptotic parameters.

Next, we calculated the $k$th-order estimates of $\eta$ for all $n$, $1 \le n \le 40$, and all $k$, $1 \le k \le 5$. [Here $\hat{\eta}_n(1)$ is $\hat{\eta}_n$ in (1.6).] We display the first-order and fifth-order estimates for several values of $n$ in Table 1. As can be seen from Table 1, the first-order estimate still has about 4% error at $n = 40$, whereas the fifth-order estimate already has four significant digits by $n = 6$. Since the fifth-order estimate $\hat{\eta}_n(5)$ uses the last six moments prior to $n$, we must have $n \ge 6$ to use the fifth-order estimate. We find that the fifth-order estimate $\eta_n(5)$ monotonically decreases in $n$ from $n = 6$ until $n = 17$, with $\hat{\eta}_{17}(5) =$

TABLE 1

*A comparison of first-order and fifth-order estimates of the six asymptotic parameters $\eta$, $\beta$, $\alpha_1$, $\alpha_2$, $\alpha_3$ and $\alpha_4$ in (1.5) as a function of the moment index n for the RBM first-moment cdf $H_1$ in (7.1) (in each case, the best fifth-order estimated values are used for previous asymptotic parameters)*

| **Estimators** | **Moment index *n*** | | | | | |
|---|---|---|---|---|---|---|
| | **6** | **10** | **15** | **20** | **40** | **∞ (exact)** |
| $\hat{\eta}_n(1)$ | 0.318 | 0.289 | 0.276 | 0.269 | 0.259 | |
| $\hat{\eta}_n(5)$ | 0.25007 | 0.2500011 | 0.25000007 | 0.25000005 | 0.2500023 | 0.25000 |
| $\hat{\beta}_n(1)$ | −1.29 | −1.36 | −1.41 | −1.43 | −1.46 | |
| $\hat{\beta}_n(5)$ | −1.4994 | −1.49997 | −1.499991 | −1.499994 | −1.5003 | |
| $\beta_n^*(1)$ | −1.11 | −1.24 | −1.32 | −1.36 | −1.43 | |
| $\beta_n^*(5)$ | −1.72 | −1.4996 | −1.49996 | −1.4992 | −1.476 | −1.5000 |
| $\hat{\alpha}_{1,n}(1)$ | 3.55 | 3.90 | 4.09 | 4.19 | 4.34 | $8/\sqrt{\pi}$ |
| $\hat{\alpha}_{1,n}(5)$ | 4.511 | 4.5132 | 4.5133 | 4.5133 | 4.5133 | = 4.5133 |
| $\hat{\alpha}_{2,n}(1)$ | −17.4 | −20.9 | −22.8 | −23.8 | −25.4 | $-48/\sqrt{\pi}$ |
| $\hat{\alpha}_{2,n}(5)$ | −27.05 | −27.074 | −27.075 | −27.075 | −27.073 | = −27.0811 |
| $\hat{\alpha}_{3,n}(1)$ | 135. | 184. | 211. | 226. | 247. | $480/\sqrt{\pi}$ |
| $\hat{\alpha}_{3,n}(5)$ | 270.2 | 270.5 | 270.5 | 270.5 | 270.2 | = 270.811 |
| $\hat{\alpha}_{4,n}(1)$ | −1351. | −2235. | −2718. | −2970. | −3363. | $-6720/\sqrt{\pi}$ |
| $\hat{\alpha}_{4,n}(5)$ | −3776. | −3777. | −3777. | −3777. | −3580. | = −3791.35 |

0.250000022, and thereafter $\hat{\eta}_n(5)$ oscillates and even degrades slightly as $n$ increases to 40. Evidently, numerical errors cause some loss of accuracy as $n$ gets larger. Hence, we look for the value of $n$ where $\hat{\eta}_n(5)$ stops being monotone and use that for our best numerical estimator of $\eta$. In this case, we use $\hat{\eta}_{17}(5)$ in all estimates of all remaining asymptotic parameters in which $\eta$ appears. Note that $\hat{\eta}_{17}(5)$ has seven significant digits, so we have two fewer significant digits than in our estimate of the moments themselves.

We also did a sensitivity analysis on the estimates of $\eta$. We first truncated all moments to six places and then all to three places. With these modifications, it is more difficult to pick out a good estimator of $\eta$. For example, the fifth-order estimate can perform poorly. However, it is easy to see that a reasonable estimator based on the first five orders has only three and one significant digits, respectively, when the moments have six and three significant digits.

In general, we have found, as one would expect, that the higher-order estimates converge more rapidly. For example, the best fifth-order estimate of $\eta$ occurs for $n = 17$, whereas the best $j$th-order estimate occurs for $n > 40$ for $j \leq 4$. However, we have also found that the accuracy of the best $j$th-order estimate can decline in $j$. Hence, for any given $n$, it is good to examine several orders of the estimates, say from 1 through 5, in order to locate the best estimate. For this $H_1$ example, the best estimate $\hat{\eta}_n(k)$ for $n \leq 40$ and $k \leq 5$ is $\eta_{17}(5)$.

Using $\hat{\eta}_{17}(5)$, we next estimate $\beta$ using $\hat{\beta}_n(k)$ based on $\hat{\beta}_n(1) \equiv \hat{\beta}_n$ in (1.7) and $\beta_n^*(k)$ based on $\beta_n^*(1) \equiv \beta_n^*$ in (1.16), where the $k$th-order estimate is obtained by extrapolation using (1.13). The estimators $\beta_n^*(k)$ are interesting because they do not directly involve $\eta$. We remark that the estimates of $\beta$ would be very bad if we used the first-order estimate $\hat{\eta}_n(1)$ even for $n = 40$ or $n = 100$.

We display the first-order estimates $\hat{\beta}_n(1)$ and $\beta_n^*(1)$ and the fifth-order estimates $\hat{\beta}_n(5)$ and $\beta_n^*(5)$ for several values of $n$ in Table 1. Again, the fifth-order estimates are much more accurate. Again, the fifth-order estimates monotonically improved until some point, which turns out to be $n = 16$ for $\hat{\beta}_n(5)$ and $n = 15$ for $\beta_n^*(5)$, and then oscillate and degrade as $n$ approaches 40. These best estimates have six and five significant digits, respectively. Hence, we have one less digit accuracy for $\beta$ than for $\eta$. These results indicate that it is somewhat better to estimate $\beta$ using a good estimate of $\eta$ than to try to estimate $\beta$ without using $\eta$.

We also did a sensitivity analysis on estimating $\beta$ based on $\eta$ when there is an error in $\eta$. We found that a 10% or 1% error in $\eta$ causes a serious problem in estimating $\beta$. For example, a 1% error in $\eta$ might lead to a 20% error in $\beta$. However, numerical evidence indicates that the overall tail probability match tends not to be quite so bad, because the error in $\beta$ tends to compensate somewhat for the error in $\eta$ (since the moments are given).

Next, we estimated the first four asymptotic constants $\alpha_1$, $\alpha_2$, $\alpha_3$ and $\alpha_4$. In each case we used the previous best estimates of $\eta$ and $\beta$, $\hat{\eta}_{17}(5)$ and $\hat{\beta}_{16}(5) = -1.4999924$. For $\alpha_2$, we also used the best estimate for $\alpha_1$ and so forth. We display the first-order estimates $\hat{\alpha}_{j,n}(1)$ and the fifth-order esti-

TABLE 2

*A comparison of the asymptotic approximations (based on the exact asymptotic parameters) with the exact values obtained by direct numerical transform inversion for the complementary cdf for the RBM example in Section 7*

| $x$ | Complementary cdf $F^c(x)$ | | | | |
|---|---|---|---|---|---|
| | **Exact** | **One term** | **Two terms** | **Four terms** | **10 terms** |
| 8 | 1.1537$E$-02 | 2.7$E$-02 | 6.70$E$-03 | -1.20$E$-02 | -5.59$E$+01 |
| 16 | 7.6564$E$-04 | 1.3$E$-03 | 8.07$E$-04 | 8.45$E$-04 | 2.80$E$-03 |
| 24 | 6.4916$E$-05 | 9.5$E$-05 | 7.13$E$-05 | 7.55$E$-05 | 7.03$E$-05 |
| 32 | 6.1804$E$-06 | 8.4$E$-06 | 6.80$E$-06 | 7.07$E$-06 | 7.10$E$-06 |
| 40 | 6.3033$E$-07 | 8.1$E$-07 | 6.88$E$-07 | 7.08$E$-07 | 7.11$E$-07 |
| 48 | 6.7304$E$-08 | 8.3$E$-08 | 7.30$E$-08 | 7.45$E$-08 | 7.47$E$-08 |
| 56 | 7.4270$E$-09 | 9.0$E$-09 | 8.00$E$-09 | 8.12$E$-09 | 8.13$E$-09 |
| 64 | 8.4012$E$-10 | 9.9$E$-10 | 9.00$E$-10 | 9.10$E$-10 | 9.11$E$-10 |
| 72 | 9.6891$E$-11 | 1.13$E$-10 | 1.03$E$-10 | 1.04$E$-10 | 1.04$E$-10 |
| 80 | 1.3499$E$-11 | 1.30$E$-11 | 1.203$E$-11 | 1.213$E$-11 | 1.213$E$-11 |

mates $\hat{\alpha}_{j,n}(5)$ for $j = 1,2,3,4$ and several values of $n$ in Table 1. Our best estimators of $\hat{\alpha}_{j,n}(5)$ were $\hat{\alpha}_{1,21}(5) = 4.51333$, $\hat{\alpha}_{2,16}(5) = -27.0747$, $\hat{\alpha}_{3,16}(5) = 270.514$ and $\hat{\alpha}_{4,15}(5) = -3777.1$. The number of significant digits in these best estimates is four for $j = 1$, three for $j = 2$, three for $j = 3$ and two for $j = 4$.

We also computed the exact values of the tail probabilities $H^c(x)$ for this example by numerically inverting $[1 - \hat{h}_1(s)]/s$ for $\hat{h}_1(s)$ in (7.2) using the algorithm EULER from Abate and Whitt (1992). We compare the asymptotic approximations to these exact values in Table 2 and Figure 1. In the asymptotic approximations we use the exact values of all the asymptotic parameters, but that makes negligible difference.

In Table 2 we show the multiterm asymptotic expansions in (1.5) with 2, 4 and 10 terms as well as the one-term asymptote in (1.4) for 10 values of $x$. Over these values of $x$, the tail probabilities range from about 0.01 to $10^{-11}$. First, the convergence as $x$ increases is slow. The one-term asymptote has errors of 134%, 46% and 15% at $x = 8$, $x = 24$ and $x = 80$, respectively; the four-term asymptote has errors of 16% and 7% at $x = 24$ and $x = 80$, respectively. However, for very small tail probabilities, it is more appropriate to consider the probabilities in log scale. Figure 1 shows that the one-term and four-term asymptotes provide excellent approximations in log scale for $x \geq 20$.

Next, the cases $x = 8$ and $x = 16$ clearly show that having more terms does not necessarily help; for $x = 8$, having 4 terms or 10 terms yields negative values. More generally, the improvement provided by additional terms is not impressive. Overall, because of its simplicity, the one-term asymptote in (1.4) seems most valuable.

**8. A long-tail example.** In this section we illustrate the exponential damping approach to computing the asymptotic parameters of long-tail distributions described in Section 3. For this purpose, we consider a *Pareto mixture*

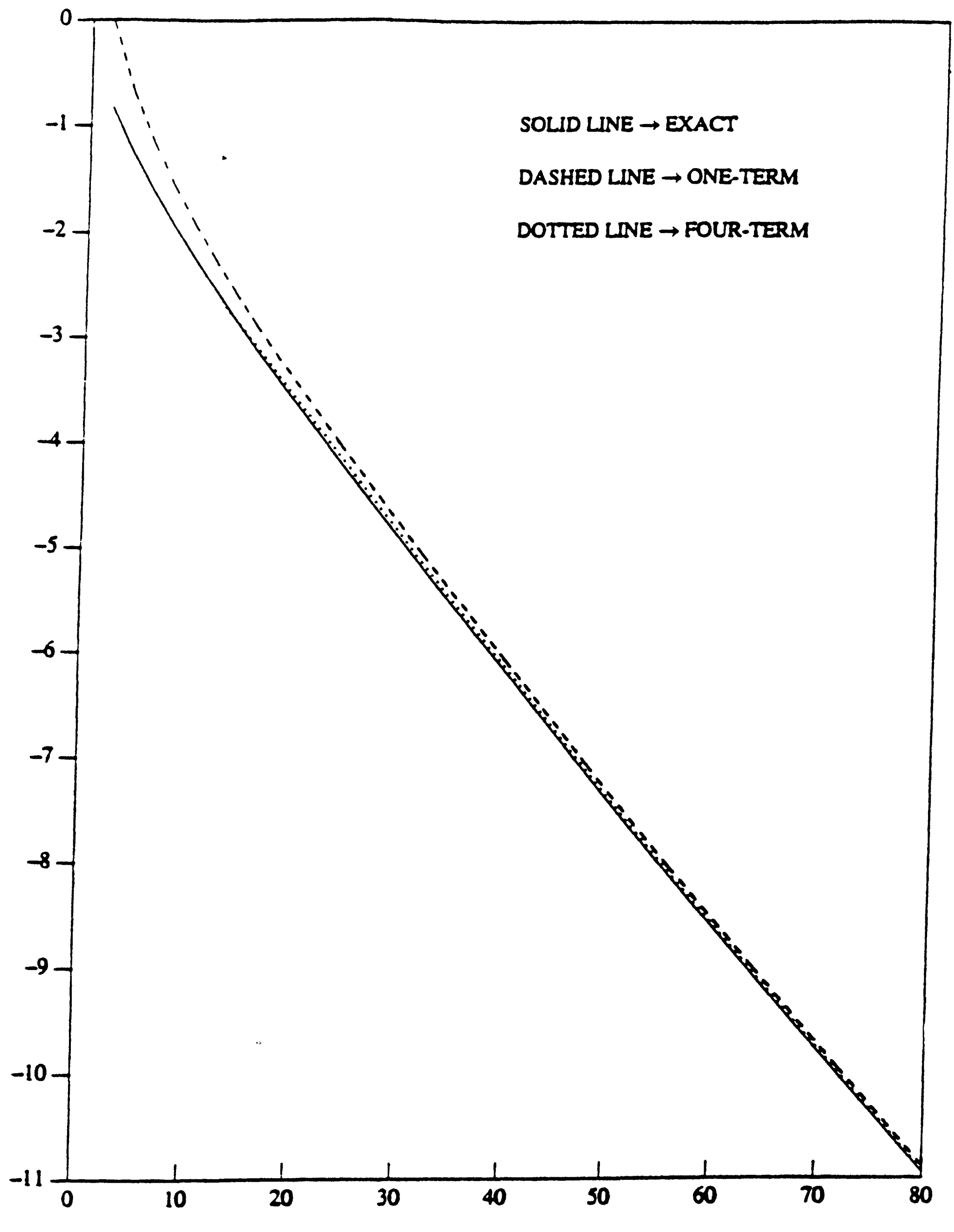


FIG. 1. *A comparison of the asymptotic approximations with exact values of the complementary cdf in log scale for the RBM example in Section 7.*

*of exponential* (*PME*) *distributions* from Section 2 of Abate, Choudhury and Whitt (1994b); see (2.14) and (2.19) there. The specific distribution has Laplace transform

$$\hat{g}_4(s) = 1 - s + \frac{9}{8}s^2 - \frac{27}{16}s^3 + \frac{81}{64}s^4 \ln\left(1 + \frac{4}{3s}\right), \tag{8.1}$$

density

$$g_4(x) = \frac{243}{8x^5}\left(1 - \left(1 + \frac{4x}{3} + \frac{8x^2}{9} + \frac{128x^3}{9} + \frac{32x^4}{243}\right)e^{-4x/3}\right), \tag{8.2}$$

complementary cdf

$$G_4^c(x) = \frac{243}{32x^4}\left(1 - \left(1 + \frac{4x}{3} + \frac{8x^2}{9} + \frac{32x^3}{81}\right)e^{-4x/3}\right), \tag{8.3}$$

mean 1 and squared coefficient of variation $c^2 = 5/4$. We also consider the distribution of the steady-state waiting time $W$ in the M/G/1 queue with arrival rate $\rho$ and this PME distribution as a service-time distribution, which has transform

$$\hat{w}(s) \equiv \int_0^\infty e^{-sx}\, dP(W \le x) = \frac{1-\rho}{1 - \rho\hat{g}_e(s)}, \tag{8.4}$$

where

$$\hat{g}_e(s) = 1 - \hat{g}_4(s). \tag{8.5}$$

The asymptotic behavior of the tail probabilities $P(W > x)$ is described in Section 3 of Abate, Choudhury and Whitt (1994b). The first term is

$$P(W > x) \sim \frac{\rho}{(1-\rho)}\int_x^\infty G_4^c(y)\, dy \sim \frac{81\rho}{32x^3(1-\rho)} \quad \text{as } x \to \infty. \tag{8.6}$$

We first calculated $\hat{f}_u(s)$ in (3.2) for $u = 1$ associated with $\hat{f}(s) = \hat{g}_4(s)$ in (8.1). We directly obtain 0.9992, $-4.999998$ and 59.575 for the asymptotic decay rate, power and constant for $F_u^c(x)$, based on the first 40 moments using extrapolation. Since $\hat{g}_4(1) = 0.509861$, we obtain $-$ 0.0008, $-3.999998$ and 7.59375 for the asymptotic parameters of $G_4^c(x)$, agreeing exceptionally well with (8.3).

We next calculated $\hat{f}_u(s)$ in (3.2) for $u = 1$ associated with $\hat{f}(s) = \hat{w}(s)$ in (8.3). In the case $\rho = 0.2$ the waiting-time distribution proved to be substantially more difficult, but we obtained reasonably good estimates based on 150 moments. The second-order estimate yielded an asymptotic decay rate of 0.999. Assuming $\eta = 1$, the second-order estimate yielded asymptotic power of $-3.99$ and asymptotic constant 2.14. Here $\hat{f}_u(1) = 0.88694522$, so that our final estimate of the asymptotic power and constant are $-2.99$ and 0.6327, which agree with (8.6) to three digits. Taking out the atom at 0 by looking at the conditional distribution $P(W \le x | W > 0)$ improved the numerics somewhat but not greatly.

**9. Conclusions.** In this paper we presented both simple and refined estimates of asymptotic parameters of a complementary distribution function based on its moments when the asymptotic form is as in (1.4), (1.5) or (1.23), focusing especially on the case with an asymptotic power ($\beta \neq 0$). We have shown that these estimates converge to the true asymptotic parameters under suitable conditions and we have determined the rate of convergence. Unfortunately, the conditions will often not be directly verifiable in many applications. Nevertheless, the theorems here provide useful background. For practical purposes, convergence of the estimates as the moment index increases will confirm that the conditions are satisfied (even though Section 2 shows that this is not strictly valid). In many cases it will also be possible to confirm the asymptotics by directly computing some of the tail probabilities, as in Table 2 and Figure 1 here. For example, this is done for the polling models in Choudhury and Whitt (1996). Computational efficiency is obtained by exploiting extrapolation as in (1.13). Extrapolation often makes it possible to obtain very good estimates of asymptotic parameters with remarkably few moments. Even in difficult cases, such as the waiting-time example in Section 8, the accuracy seems to be sufficient for most engineering applications.

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