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AFFINE PROCESSES AND APPLICATIONS IN FINANCE

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We provide the definition and a complete characterization of regular affine processes. This type of process unifies the concepts of continuous-state branching processes with immigration and Ornstein–Uhlenbeck type processes. We show, and provide foundations for, a wide range of financial applications for regular affine processes.

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1. Introduction. This paper provides a definition and complete characterization of *regular affine processes*, a class of time-homogeneous Markov processes that has arisen from a large and growing range of useful applications in finance, although until now without succinct mathematical foundations. Given a state space of the form $D = \mathbb{R}_+^m \times \mathbb{R}^n$ for integers $m \geq 0$ and $n \geq 0$, the key "affine" property, to be defined precisely in what follows, is roughly that the logarithm of the characteristic function of the transition distribution $p_t(x, \cdot)$ of such a process is affine with respect to the initial state $x \in D$. The coefficients defining this affine relationship are the solutions of a family of ordinary differential equations (ODEs) that are the essence of the tractability of regular affine processes. We classify these ODEs, "generalized Riccati equations," by their parameters, and state the precise set of admissible parameters for which there exists a unique associated regular affine process.

In the prior absence of a broad mathematical foundation for affine processes, but in light of their computational tractability and flexibility in capturing many of the empirical features of financial time series, it had become the norm in financial modeling practice to specify the properties of some affine process that would be exploited in a given problem setting, without assuredness that a uniquely well-defined process with these properties actually exists.

Strictly speaking, an affine process X in our set-up consists of an entire family of laws $(\mathbb{P}_x)_{x \in D}$, and is realized on the canonical space such that $\mathbb{P}_x[X_0 = x] = 1$, for all $x \in D$. This is in contrast to the finance literature, where an "affine process" usually is a single stochastic process defined (e.g., as the strong solution of a stochastic differential equation) on some filtered probability space. If there is no ambiguity, we shall not distinguish between these two notions and simply say "affine process" in both cases (see Theorem 2.12 below for the precise connection).

We show that a regular affine process is a Feller process whose generator is affine and vice versa. An affine generator is characterized by the affine dependence of its coefficients on the state variable $x \in D$. The parameters associated with the generator are in a one-to-one relation with those of the corresponding ODEs.

Regular affine processes include continuous-state branching processes with immigration (CBI) (e.g., [62]) and processes of the Ornstein–Uhlenbeck (OU) type (e.g., [79]). Roughly speaking, the regular affine processes with state space $\mathbb{R}_+^m$ are CBI, and those with state space $\mathbb{R}^n$ are of OU type. For any regular affine process $X = (Y, Z)$ in $\mathbb{R}_+^m \times \mathbb{R}^n$, we show that the first component Y is necessarily a CBI process. Any CBI or OU-type process is infinitely decomposable, as was well known and apparent from the exponential-affine form of the characteristic function of its transition distribution. We show that a regular (to be defined below) Markov process with state space D is infinitely decomposable if and only if it is a regular affine process.

We also show that a regular affine process X is (up to its lifetime) a semimartingale with respect to every $\mathbb{P}_x$, a crucial property in most financial applications because the standard model [53] of the financial gain generated by trading a security is a stochastic integral with respect to the underlying price process. We provide a one-to-one relationship between the coefficients of the characteristic function of a conservative regular affine process X and (up to a version) its semimartingale characteristics [57] (B, C, ν) (after fixing a truncation of jumps), of which B is the predictable component of the canonical decomposition of X, C is the "sharp-brackets" process and ν is the compensator of the random jump measure. The results justify, and clarify the precise limits of, the common practice in the finance literature of specifying an affine process in terms of its semimartingale characteristics. In particular, as we show, for any conservative regular affine process $X = (Y, Z)$ in $\mathbb{R}_+^m \times \mathbb{R}^n$, the sharp-brackets and jump characteristics of X depend only on the CBI component Y. This completes and extends the discussion in [31], where the authors provide sufficient conditions for affine diffusion (and hence continuous) Markov processes to be well defined and classify them by the number m of Y_i's that can enter the conditional variance matrix. We also provide conditions for the existence of (partial) higher order and exponential moments of X_t. An extension of the main results for the time-inhomogeneous case is given in [47].

Some common financial applications of the properties of a regular affine process X are following.

The term structure of interest rates. A typical model of the price processes of bonds of various maturities begins with a discount-rate process $\{L(X_t) : t \geq 0\}$ defined by an affine map $x \mapsto L(x)$ on D into $\mathbb{R}$. In Section 11, we examine conditions under which the discount factor

$$\mathbb{E}\left[\exp\left(-\int_s^t L(X_u)\,du\right)\middle| X_s\right]$$

is well defined, and is of the anticipated exponential-affine form in X_s. Some financial applications and pointers to the large theoretical and empirical literatures on affine interest-rate models are provided in Section 13.

The pricing of options. A put option, for example, gives its owner the right to sell a financial security at a prearranged exercise price at some future time t. Without going into details that are discussed in Section 13, the ability to calculate the market price of the option is roughly equivalent to the ability to calculate the probability that the option is exercised. In many applications, the underlying security price is affine with respect to the state variable X_t, possibly after a change of variables. Thus, the exercise probability can be calculated by inverting the characteristic function of the transition distribution $p_t(x, \cdot)$ of X. One can capture realistic empirical features such as jumps in price and stochastic return volatility, possibly of a high-dimensional type, by incorporating these features into the parameters of the affine process.

Credit risk. A recent spate of work, summarized in Section 13, on pricing and measuring default risk exploits the properties of a doubly stochastic counting process N driven by an affine process X. The stochastic intensity of N [16] is assumed to be of the form $\{\Lambda(X_{t-}) : t \geq 0\}$, for some affine $x \mapsto \Lambda(x)$. The time of default of a financial counterparty, such as a borrower or option writer, is modeled as the first jump time of N. The probability of no default by t, conditional on X_s and survival to s, is $\mathbb{E}[\exp(-\int_s^t \Lambda(X_u)\,du)|X_s]$. This is of the same form as the discount factor used in interest-rate modeling, and can be treated in the same manner. For pricing defaultable bonds, one can combine the effects of default and of discounting for interest rates.

The remainder of the paper is organized as follows. In Section 2 we provide the definition of a regular affine process X (Definitions 2.1 and 2.5) and the main results of this paper. In fact, we present three other equivalent characterizations of regular affine processes: in terms of the generator (Theorem 2.7), the semimartingale characteristics (Theorem 2.12) and by infinite decomposability (Theorem 2.15). We also show how regular affine processes are related to CBI and OU-type processes. In Section 2.1 we discuss the existence of moments of X_t.

The proofs of Theorems 2.7, 2.12 and 2.15 are divided into Sections 3–10. Section 3 is preliminary and provides some immediate consequences of the definiton of a regular affine process. At the end of this section, we sketch the strategy for the proof of Theorem 2.7 (which in fact is the hardest of the three). In Section 4 we prove a representation result for the weak generator of a Markov semigroup, which goes back to Venttsel' [92]. This is used in Section 5 to find the form of the ODEs (generalized Riccati equations) related to a regular affine process. In Section 6 we prove existence and uniqueness of solutions to these ODEs and give some useful regularity results. Section 7 is crucial for the existence result of regular affine processes. Here we show that the solution to any generalized Riccati equation yields a regular affine transtition function. In Section 8 we prove the Feller property of a regular affine process and completely specify the generator. In Section 9 we investigate conditions under which a regular affine process is conservative and give an example where these conditions fail. Section 10 completes the proof of the characterization results.

In Section 11 we investigate the behavior of a conservative regular affine process with respect to discounting. Formally, we consider the semigroup $Q_t f(x) = \mathbb{E}_x[\exp(-\int_0^t L(X_s)\,dt)\, f(X_t)]$, where L is an affine function on D. We use two approaches, one by the Feynman–Kac formula (Section 11.1), the other by enlargement of the state space D (Section 11.2). These results are crucial for most financial applications as was already mentioned.

In Section 12 we address whether the state space $D = \mathbb{R}_+^m \times \mathbb{R}^n$ that we choose for affine processes is canonical. We provide examples of affine processes that are well defined on different types of state spaces.

In Section 13 we show how our results provide a mathematical foundation for a wide range of financial applications. We provide a survey of the literature in the field. The common applications that we already have sketched are discussed in more detail.

The Appendix contains some useful results on the interplay between the existence of moments of a bounded measure on $\mathbb{R}^N$ and the regularity of its characteristic function.

1.1. *Basic notation.* For the stochastic background and notation we refer to [57] and [76]. Let $k \in \mathbb{N}$. We write

$$\mathbb{R}^k_+ = \{x \in \mathbb{R}^k \mid x_i \geq 0, \forall i\}, \qquad \mathbb{R}^k_{++} = \{x \in \mathbb{R}^k \mid x_i > 0, \forall i\},$$
$$\mathbb{C}^k_+ = \{z \in \mathbb{C}^k \mid \operatorname{Re} z \in \mathbb{R}^k_+\}, \qquad \mathbb{C}^k_{++} = \{z \in \mathbb{C}^k \mid \operatorname{Re} z \in \mathbb{R}^k_{++}\},$$

and analogously $\mathbb{R}^k_-$, $\mathbb{R}^k_{--}$, $\mathbb{C}^k_-$ and $\mathbb{C}^k_{--}$.

For $\alpha, \beta \in \mathbb{C}^k$ we write $\langle \alpha, \beta \rangle := \alpha_1\beta_1 + \cdots + \alpha_k\beta_k$ (notice that this is not the scalar product on $\mathbb{C}^k$). We let Sem^k be the convex cone of symmetric positive semidefinite $k \times k$ matrices.

Let U be an open set or the closure of an open set in $\mathbb{C}^k$. We write $\overline{U}$ for the closure, U^0 for the interior, $\partial U = \overline{U} \setminus U^0$ for the boundary and $U_\Delta = U \cup \{\Delta\}$ for the one-point compactification of U. Let us introduce the following function spaces:

1. $C(U)$ is the space of complex-valued continuous functions f on U.
2. bU is the Banach space of bounded complex-valued Borel-measurable functions f on U.
3. $C_{\mathrm{b}}(U)$ is the Banach space $C(U) \cap bU$.
4. $C_0(U)$ is the Banach space consisting of $f \in C(U)$ with $\lim_{x\to\Delta} f(x) = 0$.
5. $C_{\mathrm{c}}(U)$ is the Banach space consisting of $f \in C(U)$ with compact support.
6. $C^k(U)$ is the space of k times differentiable functions f on U^0 such that all partial derivatives of f up to order k belong to $C(U)$.
7. $C^k_{\mathrm{c}}(U) = C^k(U) \cap C_{\mathrm{c}}(U)$.
8. $C^\infty(U) = \bigcap_{k\in\mathbb{N}} C^k(U)$ and $C^\infty_{\mathrm{c}}(U) = \bigcap_{k\in\mathbb{N}} C^k_{\mathrm{c}}(U)$.

By convention, all functions f on U are extended to U_Δ by setting $f(\Delta) = 0$. Further notation is introduced in the text.

2. Definition and characterization of regular affine processes. We consider a time-homogeneous Markov process with state space $D := \mathbb{R}^m_+ \times \mathbb{R}^n$ and semigroup (P_t) acting on bD,

$$P_t f(x) = \int_D f(\xi)\, p_t(x, d\xi).$$

According to the product structure of D we shall write $x = (y, z)$ or $\xi = (\eta, \zeta)$ for a point in D. We assume $d := m + n \in \mathbb{N}$. Hence m or n may be 0. We do not demand that (P_t) is conservative, that is, we have

$$p_t(x, D) \le 1, \quad p_t(x, D_\Delta) = 1, \quad p_t(\Delta, \{\Delta\}) = 1 \qquad \forall (t, x) \in \mathbb{R}_+ \times D.$$

We let $(X, (\mathbb{P}_x)_{x\in D}) = ((Y, Z), (\mathbb{P}_x)_{x\in D})$ denote the canonical realization of (P_t) defined on $(\Omega, \mathcal{F}^0, (\mathcal{F}_t^0))$, where Ω is the set of mappings $\omega : \mathbb{R}_+ \to D_\Delta$ and $X_t(w) = (Y_t(\omega), Z_t(\omega)) = \omega(t)$. The filtration $(\mathcal{F}_t^0)$ is generated by X and $\mathcal{F}^0 = \bigvee_{t\in\mathbb{R}_+} \mathcal{F}_t^0$. For every $x \in D$, $\mathbb{P}_x$ is a probability measure on $(\Omega, \mathcal{F}^0)$ such that $\mathbb{P}_x[X_0 = x] = 1$ and the Markov property holds,

$$\begin{aligned} \mathbb{E}_x\big[f(X_{t+s})|\mathcal{F}_t^0\big] &= P_s f(X_t) \\ &= \mathbb{E}_{X_t}[f(X_s)], \qquad \mathbb{P}_x\text{-a.s. } \forall s, t \in \mathbb{R}_+, \ \forall f \in bD, \end{aligned} \tag{2.1}$$

where $\mathbb{E}_x$ denotes the expectation with respect to $\mathbb{P}_x$.

For $u = (v, w) \in \mathbb{C}^m \times \mathbb{C}^n$ we define the function $f_u \in C(D)$ by

$$f_u(x) := e^{\langle u,x\rangle} = e^{\langle v,y\rangle+\langle w,z\rangle}, \qquad x = (y, z) \in D.$$

Notice that $f_u \in C_\mathrm{b}(D)$ if and only if u lies in

$$\mathcal{U} := \mathbb{C}_-^m \times i\mathbb{R}^n,$$

and this is why the parameter set $\mathcal{U}$ plays a distinguished role. By dominated convergence, $P_t f_u(x)$ is continuous in $u \in \mathcal{U}$ for every $(t, x) \in \mathbb{R}_+ \times D$. We have

$$\mathcal{U}^0 = \mathbb{C}_{--}^m \times i\mathbb{R}^n \quad \text{and} \quad \partial\mathcal{U} = i\mathbb{R}^m \times i\mathbb{R}^n = i\mathbb{R}^d.$$

Observe that

$$\mathbb{R}^d \ni q \mapsto P_t f_{iq}(x)$$

is the characteristic function of the measure $p_t(x, \cdot)$, that is, the characteristic function of X_t with respect to $\mathbb{P}_x$.

DEFINITION 2.1. The Markov process $(X, (\mathbb{P}_x)_{x\in D})$, and (P_t), is called *affine* if, for every $t \in \mathbb{R}_+$, the characteristic function of $p_t(x, \cdot)$ has exponential-affine dependence on x. That is, if for every $(t, u) \in \mathbb{R}_+ \times \partial\mathcal{U}$ there exist $\phi(t, u) \in \mathbb{C}$ and $\psi(t, u) = (\psi^{\mathcal{Y}}(t, u), \psi^{\mathcal{Z}}(t, u)) \in \mathbb{C}^m \times \mathbb{C}^n$ such that

$$\begin{aligned} P_t f_u(x) &= e^{\phi(t,u)+\langle\psi(t,u),x\rangle} \\ &= e^{\phi(t,u)+\langle\psi^{\mathcal{Y}}(t,u),y\rangle+\langle\psi^{\mathcal{Z}}(t,u),z\rangle} \qquad \forall x = (y, z) \in D. \end{aligned} \tag{2.2}$$

REMARK 2.2. Since $P_t f_u \in bD$, for all $(t, u) \in \mathbb{R}_+ \times \mathcal{U}$, we easily infer from (2.2) that a fortiori $\phi(t, u) \in \mathbb{C}_-$ and $\psi(t, u) = (\psi^{\mathcal{Y}}(t, u), \psi^{\mathcal{Z}}(t, u)) \in \mathcal{U}$ for all $(t, u) \in \mathbb{R}_+ \times \partial\mathcal{U}$.

REMARK 2.3. Notice that $\psi(t,u)$ is uniquely specified by (2.2). But $\operatorname{Im}\phi(t,u)$ is determined only up to multiples of 2π. Nevertheless, by definition we have $P_t f_u(0) \neq 0$ for all $(t,u) \in \mathbb{R}_+ \times \partial\mathcal{U}$. Since $\partial\mathcal{U}$ is simply connected, $P_t f_u(0)$ admits a unique representation of the form (2.2)—*and we shall use the symbol $\phi(t,u)$ in this sense from now on*—such that $\phi(t,\cdot)$ is continuous on $\partial\mathcal{U}$ and $\phi(t,0)=0$.

DEFINITION 2.4. The Markov process $(X,(\mathbb{P}_x)_{x\in D})$, and (P_t), is called *stochastically continuous* if $p_s(x,\cdot) \to p_t(x,\cdot)$ weakly on D, for $s \to t$, for every $(t,x) \in \mathbb{R}_+ \times D$.

If $(X,(\mathbb{P}_x)_{x\in D})$ is affine then, by the continuity theorem of Lévy, $(X,(\mathbb{P}_x)_{x\in D})$ is stochastically continuous if and only if $\phi(t,u)$ and $\psi(t,u)$ from (2.2) are continuous in $t \in \mathbb{R}_+$ for every $u \in \partial\mathcal{U}$.

DEFINITION 2.5. The Markov process $(X,(\mathbb{P}_x)_{x\in D})$, and (P_t), is called *regular* if it is stochastically continuous and the right-hand derivative

$$\tilde{\mathcal{A}} f_u(x) := \partial_t^+ P_t f_u(x)|_{t=0}$$

exists for all $(x,u) \in D \times \mathcal{U}$, and is continuous at $u=0$ for all $x \in D$.

We call $(X,(\mathbb{P}_x)_{x\in D})$, and (P_t), simply *regular affine* if it is regular and affine.

If there is no ambiguity, we shall write indifferently X or (Y,Z) for the Markov process $(X,(\mathbb{P}_x)_{x\in D})$, and say shortly X is *affine*, *stochastically continuous*, *regular*, *regular affine* if $(X,(\mathbb{P}_x)_{x\in D})$ shares the respective property.

Before stating the main results of this paper, we need to introduce a certain amount of notation and terminology. Denote by $\{e_1,\dots,e_d\}$ the standard basis in $\mathbb{R}^d$, and write $\mathcal{I} := \{1,\dots,m\}$ and $\mathcal{J} := \{m+1,\dots,d\}$. We define the continuous truncation function $\chi = (\chi_1,\dots,\chi_d) : \mathbb{R}^d \to [-1,1]^d$ by

$$\chi_k(\xi) := \begin{cases} 0, & \text{if } \xi_k = 0, \\ (1 \wedge |\xi_k|)\dfrac{\xi_k}{|\xi_k|}, & \text{otherwise.} \end{cases} \tag{2.3}$$

Let $\alpha = (\alpha_{ij})$ be a $d \times d$-matrix, $\beta = (\beta_1,\dots,\beta_d)$ a d-tuple and $I, J \subset \{1,\dots,d\}$. Then we write α^T for the transpose of α, and $\alpha_{IJ} := (\alpha_{ij})_{i\in I,\, j\in J}$ and $\beta_I := (\beta_i)_{i\in I}$. Examples are $\chi_I(\xi) = (\chi_k(\xi))_{k\in I}$ or $\nabla_I := (\partial_{x_k})_{k\in I}$. Accordingly, we have $\psi^{\mathcal{Y}}(t,u) = \psi_{\mathcal{I}}(t,u)$ and $\psi^{\mathcal{Z}}(t,u) = \psi_{\mathcal{J}}(t,u)$ (since these mappings play a distinguished role we introduced the former, "coordinate-free" notation). We also write $\mathbf{1} := (1,\dots,1)$ without specifying the dimension whenever there is no ambiguity. For $i \in \mathcal{I}$ we define $\mathcal{I}(i) := \mathcal{I} \setminus \{i\}$ and $\mathcal{J}(i) := \{i\} \cup \mathcal{J}$, and let $\operatorname{Id}(i)$ denote the $m \times m$-matrix given by $\operatorname{Id}(i)_{kl} = \delta_{ik}\delta_{kl}$, where δ_{kl} is the Kronecker delta (δ_{kl} equals 1 if $k=l$ and 0 otherwise).

DEFINITION 2.6. The parameters $(a, \alpha, b, \beta, c, \gamma, m, \mu)$ are called *admissible* if

$$a \in \mathrm{Sem}^d \text{ with } a_{\mathcal{I}\mathcal{I}} = 0 \qquad (\text{hence } a_{\mathcal{I}\mathcal{J}} = 0 \text{ and } a_{\mathcal{J}\mathcal{I}} = 0), \tag{2.4}$$

$$\alpha = (\alpha_1, \dots, \alpha_m) \text{ with } \alpha_i \in \mathrm{Sem}^d \text{ and } \alpha_{i,\mathcal{I}\mathcal{I}} = \alpha_{i,ii}\mathrm{Id}(i) \text{ for all } i \in \mathcal{I}, \tag{2.5}$$

$$b \in D, \tag{2.6}$$

$$\begin{aligned} &\beta \in \mathbb{R}^{d\times d} \text{ such that } \beta_{\mathcal{I}\mathcal{J}} = 0 \text{ and } \beta_{i\mathcal{I}(i)} \in \mathbb{R}_+^{m-1} \text{ for all } i \in \mathcal{I} \\ &\qquad\qquad (\text{hence } \beta_{\mathcal{I}\mathcal{I}} \text{ has non-negative off-diagonal elements}), \end{aligned} \tag{2.7}$$

$$c \in \mathbb{R}_+, \tag{2.8}$$

$$\gamma \in \mathbb{R}_+^m, \tag{2.9}$$

$$\begin{aligned} &m \text{ is a Borel measure on } D \setminus \{0\} \text{ satisfying} \\ &\qquad M := \int_{D\setminus\{0\}} \big(\langle \chi_{\mathcal{I}}(\xi), \mathbf{1}\rangle + \|\chi_{\mathcal{J}}(\xi)\|^2\big) m(d\xi) < \infty, \end{aligned} \tag{2.10}$$

$$\begin{aligned} &\mu = (\mu_1, \dots, \mu_m) \text{ where every } \mu_i \text{ is a Borel measure on } D \setminus \{0\} \text{ satisfying} \\ &\qquad \mathcal{M}_i := \int_{D\setminus\{0\}} \big(\langle \chi_{\mathcal{I}(i)}(\xi), \mathbf{1}\rangle + \|\chi_{\mathcal{J}(i)}(\xi)\|^2\big) \mu_i(d\xi) < \infty. \end{aligned} \tag{2.11}$$

The following theorems contain the main results of this paper. Their proofs are provided in Sections 3–10. First, we state an analytic characterization result for regular affine processes.

THEOREM 2.7. *Suppose X is regular affine. Then X is a Feller process. Let $\mathcal{A}$ be its infinitesimal generator. Then $C_c^\infty(D)$ is a core of $\mathcal{A}$, $C_c^2(D) \subset \mathcal{D}(\mathcal{A})$, and there exist some admissible parameters $(a, \alpha, b, \beta, c, \gamma, m, \mu)$ such that, for $f \in C_c^2(D)$,*

$$\begin{aligned} \mathcal{A}f(x) = &\sum_{k,l=1}^d (a_{kl} + \langle \alpha_{\mathcal{I},kl}, y\rangle) \frac{\partial^2 f(x)}{\partial x_k \partial x_l} \\ &+ \langle b + \beta x, \nabla f(x)\rangle - (c + \langle \gamma, y\rangle) f(x) \\ &+ \int_{D\setminus\{0\}} \big(f(x+\xi) - f(x) - \langle \nabla_{\mathcal{J}} f(x), \chi_{\mathcal{J}}(\xi)\rangle\big) m(d\xi) \\ &+ \sum_{i=1}^m \int_{D\setminus\{0\}} \big(f(x+\xi) - f(x) - \langle \nabla_{\mathcal{J}(i)} f(x), \chi_{\mathcal{J}(i)}(\xi)\rangle\big) y_i \mu_i(d\xi). \end{aligned} \tag{2.12}$$

Moreover, (2.2) *holds for all* $(t,u) \in \mathbb{R}_+ \times \mathcal{U}$ *where* $\phi(t,u)$ *and* $\psi(t,u)$ *solve the* generalized Riccati equations,

$$\phi(t,u) = \int_0^t F\big(\psi(s,u)\big)\,ds, \tag{2.13}$$

$$\partial_t \psi^{\mathcal{Y}}(t,u) = R^{\mathcal{Y}}\big(\psi^{\mathcal{Y}}(t,u), e^{\beta^{\mathcal{Z}} t} w\big), \qquad \psi^{\mathcal{Y}}(0,u) = v, \tag{2.14}$$

$$\psi^{\mathcal{Z}}(t,u) = e^{\beta^{\mathcal{Z}} t} w \tag{2.15}$$

with

$$\begin{aligned} F(u) &= \langle au, u\rangle + \langle b, u\rangle - c \\ &\quad + \int_{D\setminus\{0\}} \big(e^{\langle u,\xi\rangle} - 1 - \langle u_{\mathcal{J}}, \chi_{\mathcal{J}}(\xi)\rangle\big) m(d\xi), \end{aligned} \tag{2.16}$$

$$\begin{aligned} R_i^{\mathcal{Y}}(u) &= \langle \alpha_i u, u\rangle + \langle \beta_i^{\mathcal{Y}}, u\rangle - \gamma_i \\ &\quad + \int_{D\setminus\{0\}} \big(e^{\langle u,\xi\rangle} - 1 - \langle u_{\mathcal{J}(i)}, \chi_{\mathcal{J}(i)}(\xi)\rangle\big) \mu_i(d\xi) \end{aligned} \tag{2.17}$$

for $i \in \mathcal{I}$, *and*

$$\beta_i^{\mathcal{Y}} := (\beta^T)_{i\{1,\dots,d\}} \in \mathbb{R}^d, \qquad i \in \mathcal{I}, \tag{2.18}$$

$$\beta^{\mathcal{Z}} := (\beta^T)_{\mathcal{J}\mathcal{J}} \in \mathbb{R}^{n\times n}. \tag{2.19}$$

Conversely, let $(a,\alpha,b,\beta,c,\gamma,m,\mu)$ *be some admissible parameters. Then there exists a unique, regular affine semigroup* (P_t) *with infinitesimal generator* (2.12), *and* (2.2) *holds for all* $(t,u) \in \mathbb{R}_+ \times \mathcal{U}$ *where* $\phi(t,u)$ *and* $\psi(t,u)$ *are given by* (2.13)–(2.15).

REMARK 2.8. To clarify the connections between the objects in Theorem 2.7 let us, for any $f \in C^2(D) \cap C_{\mathrm{b}}(D)$, define $\mathcal{A}^\sharp f(x)$ literally as the right-hand side of (2.12). Then we see that F, $R^{\mathcal{Y}} = (R_1^{\mathcal{Y}}, \dots, R_m^{\mathcal{Y}})$ and $\beta^{\mathcal{Z}}$ satisfy

$$\partial_t^+ P_t f_u(x)|_{t=0} = \big(F(u) + \langle R^{\mathcal{Y}}(u), y\rangle + \langle \beta^{\mathcal{Z}} w, z\rangle\big) f_u(x) = \mathcal{A}^\sharp f_u(x)$$

for $u = (v,w) \in \mathcal{U}$ and $x = (y,z) \in D$.

Equation (2.15) states that $\psi^{\mathcal{Z}}(t,u)$ depends only on (t,w). Hence, for $w = 0$, we infer from (2.2) that the characteristic function of Y_t with respect to $\mathbb{P}_x$,

$$P_t f_{(iq,0)}(x) = \int_D e^{i\langle q,\eta\rangle}\, p_t(x,d\xi) = e^{\phi(t,iq,0) + \langle \psi^{\mathcal{Y}}(t,iq,0), y\rangle}, \qquad q \in \mathbb{R}^m,$$

depends only on y. We obtain the following.

COROLLARY 2.9. *Let $X = (Y, Z)$ be regular affine. Then $(Y, (\mathbb{P}_{(y,z)})_{y\in\mathbb{R}^m_+})$ is a regular affine Markov process with state space $\mathbb{R}^m_+$, independently of $z \in \mathbb{R}^n$.*

Theorem 2.7 generalizes and unifies two classical types of stochastic processes. For the notion of a *CBI process* we refer to [94, 62] and [85]. For the notion of an *OU-type process* see [79], Definition 17.2.

COROLLARY 2.10. *Let $X = (Y, Z)$ be regular affine. Then $(Y, (\mathbb{P}_{(y,z)})_{y\in\mathbb{R}^m_+})$ is a CBI process for every $z \in \mathbb{R}^n$. If $m = 0$ then X is an OU-type process.*

Conversely, every CBI and OU-type process is a regular affine Markov process.

REMARK 2.11. There exist affine Markov processes that are not stochastically continuous and for which Theorem 2.7 therefore does not hold. This is shown by the following example, taken from [62]. Let $x_0 \in D$. Then ($\delta_x =$ *Dirac measure at* x):

$$p_t(x, d\xi) = \begin{cases} \delta_x, & \text{if } t = 0, \\ \delta_{x_0}, & \text{if } t > 0, \end{cases}$$

is the transition function of an affine Markov process with

$$\phi(t, u) = \begin{cases} 0, & \text{if } t = 0, \\ \langle u, x_0\rangle, & \text{if } t > 0, \end{cases} \quad \text{and} \quad \psi(t, u) = \begin{cases} u, & \text{if } t = 0, \\ 0, & \text{if } t > 0, \end{cases}$$

which is obviously not of the form as stated in Theorem 2.7.

On the other hand, if $n = 0$ then a stochastically continuous affine Markov process is a fortiori regular; see [62], Lemmas 1.2 and 1.3. It is still an open problem whether this also holds true for $n \geq 1$.

Motivated by Theorem 2.7, we give in this paragraph a summary of some classical results for Feller processes. For the proofs we refer to [76], Chapter III.2. Let X be regular affine and hence, by Theorem 2.7, a Feller process. Since we deal with an entire family of probability measures, $(\mathbb{P}_x)_{x\in D}$, we make the convention that "a.s." means "$\mathbb{P}_x$-a.s. for all $x \in D$." Then X admits a cadlag modification, and from now on we shall only consider cadlag versions of a regular affine process X, still denoted by X. Let

$$\tau_X := \inf\{t \in \mathbb{R}_+ \mid X_{t-} = \Delta \text{ or } X_t = \Delta\}.$$

Then we have $X_\cdot = \Delta$ on $[\tau_X, \infty)$ a.s. Hence X is conservative if and only if $\tau_X = \infty$ a.s. Write $\mathcal{F}^{(x)}$ for the completion of $\mathcal{F}^0$ with respect to $\mathbb{P}_x$ and $(\mathcal{F}_t^{(x)})$ for the filtration obtained by letting $\mathcal{F}_t^{(x)}$ be the σ-algebra generated by the union of $\mathcal{F}_t^0$ and the $\mathbb{P}_x$-nullsets in $\mathcal{F}^{(x)}$. Define

$$\mathcal{F}_t := \bigcap_{x\in D} \mathcal{F}_t^{(x)}, \qquad \mathcal{F} := \bigcap_{x\in D} \mathcal{F}^{(x)}.$$

Then the filtrations $(\mathcal{F}_t^{(x)})$ and $(\mathcal{F}_t)$ are right continuous, and X is still a Markov process with respect to $(\mathcal{F}_t)$. That is, (2.1) holds for $\mathcal{F}_t^0$ replaced by $\mathcal{F}_t$, for all $x \in D$.

By convention, we call X a *semimartingale* if $(X_t \mathbb{1}_{\{t<\tau_X\}})$ is a semimartingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}_x)$, for every $x \in D$. For the definition of the *characteristics* of a semimartingale, refer to [57], Section II.2. We emphasize that the characteristics below are associated to the truncation function χ, defined in (2.3).

Let X' be a D_Δ-valued stochastic process defined on some probability space $(\Omega', \mathcal{F}', \mathbb{P}')$. Then $\mathbb{P}' \circ X'^{-1}$ denotes the law of X', that is, the image of $\mathbb{P}'$ by the mapping $\omega' \mapsto X'_\cdot(\omega') : (\Omega', \mathcal{F}') \to (\Omega, \mathcal{F}^0)$.

The following is a characterization result for regular affine processes in the class of semimartingales. An exposure of conservative regular affine processes is given in Section 9.

THEOREM 2.12. *Let X be regular affine and $(a, \alpha, b, \beta, c, \gamma, m, \mu)$ the related admissible parameters. Then X is a semimartingale. If X is conservative then it admits the characteristics (B, C, ν):*

$$B_t = \int_0^t (\tilde{b} + \tilde{\beta} X_s)\, ds, \tag{2.20}$$

$$C_t = 2\int_0^t \left(a + \sum_{i=1}^m \alpha_i Y_s^i\right) ds, \tag{2.21}$$

$$\nu(dt, d\xi) = \left(m(d\xi) + \sum_{i=1}^m Y_t^i \mu_i(d\xi)\right) dt \tag{2.22}$$

for every $\mathbb{P}_x$, where $\tilde{b} \in D$ and $\tilde{\beta} \in \mathbb{R}^{d\times d}$ are given by

$$\tilde{b} := b + \int_{D\setminus\{0\}} (\chi_{\mathcal{I}}(\xi), 0)\, m(d\xi), \tag{2.23}$$

$$\tilde{\beta}_{kl} := \begin{cases} \beta_{kl} + (1-\delta_{kl}) \displaystyle\int_{D\setminus\{0\}} \chi_k(\xi)\, \mu_l(d\xi), \\ \qquad\qquad \text{if } l \in \mathcal{I}, \\ \beta_{kl}, \qquad \text{if } l \in \mathcal{J}, \text{ for } 1 \le k \le d. \end{cases} \tag{2.24}$$

Moreover, let $X' = (Y', Z')$ be a D-valued semimartingale defined on some filtered probability space $(\Omega', \mathcal{F}', (\mathcal{F}_t'), \mathbb{P}')$ with $\mathbb{P}'[X_0' = x] = 1$. Suppose X' admits the characteristics (B', C', ν'), given by (2.20)–(2.22) *where X is replaced by X'. Then $\mathbb{P}' \circ X'^{-1} = \mathbb{P}_x$.*

REMARK 2.13. The notions (2.23) and (2.24) are not substantial and are only introduced for notational compatibility with [57]. In fact, we replaced $\langle \nabla_{\mathcal{J}} f(x), \chi_{\mathcal{J}}(\xi)\rangle$ and $\langle \nabla_{\mathcal{J}(i)} f(x), \chi_{\mathcal{J}(i)}(\xi)\rangle$ in (2.12) by $\langle \nabla f(x), \chi(\xi)\rangle$, which is compensated by replacing b and β by $\tilde{b}$ and $\tilde{\beta}$, respectively.

The second part of Theorem 2.12 justifies, and clarifies the limits of, the common practice in the finance literature of specifying an "affine process" in terms of its semimartingale characteristics.

There is a third way of characterizing regular affine processes, which generalizes [85]. Let $\mathbb{P}$ and $\mathbb{Q}$ be two probability measures on $(\Omega, \mathcal{F}^0)$. We write $\mathbb{P} * \mathbb{Q}$ for the image of $\mathbb{P} \times \mathbb{Q}$ by the measurable mapping $(\omega, \omega') \mapsto \omega + \omega' : (\Omega \times \Omega, \mathcal{F}^0 \otimes \mathcal{F}^0) \to (\Omega, \mathcal{F}^0)$. Let $\mathcal{P}_{\mathrm{RM}}$ be the set of all families $(\mathbb{P}'_x)_{x \in D}$ of probability measures on $(\Omega, \mathcal{F}^0)$ such that $(X, (\mathbb{P}'_x)_{x \in D})$ is a regular Markov process with $\mathbb{P}'_x[X_0 = x] = 1$, for all $x \in D$.

DEFINITION 2.14. We call $(\mathbb{P}_x)_{x \in D}$ *infinitely decomposable* if, for every $k \in \mathbb{N}$, there exists $(\mathbb{P}^{(k)}_x)_{x \in D} \in \mathcal{P}_{\mathrm{RM}}$ such that

$$\mathbb{P}_{x^{(1)} + \cdots + x^{(k)}} = \mathbb{P}^{(k)}_{x^{(1)}} * \cdots * \mathbb{P}^{(k)}_{x^{(k)}} \qquad \forall x^{(1)}, \ldots, x^{(k)} \in D. \tag{2.25}$$

THEOREM 2.15. *The Markov process $(X, (\mathbb{P}_x)_{x \in D})$ is regular affine if and only if $(\mathbb{P}_x)_{x \in D}$ is infinitely decomposable.*

We refer to Corollary 10.4 below for the corresponding *additivity property* of regular affine processes.

We have to admit that "infinitely decomposable" implies "regular affine," as stated in Theorem 2.15, only since our definition of "infinitely decomposable" includes regularity of $(\mathbb{P}^{(k)}_x)_{x \in D}$ appearing in (2.25). There exists, however, affine Markov processes which satisfy (2.25) but are not regular. As an example, consider the nonregular affine Markov process $(X, (\mathbb{P}_x)_{x \in D})$ from Remark 2.3. It is easy to see that, for any $k \in \mathbb{N}$,

$$p^{(k)}_t(x, d\xi) = \begin{cases} \delta_x, & \text{if } t = 0, \\ \delta_{x_0/k}, & \text{if } t > 0, \end{cases}$$

is the transition function of an affine Markov process $(X, (\mathbb{P}^{(k)}_x)_{x \in D})$ which satisfies (2.25), but is not regular.

We now give an intuitive interpretation of conditions (2.4)–(2.11) in Definition 2.6. Without going much into detail we remark that in (2.12) we can distinguish the three "building blocks" of any jump-diffusion process, the *diffusion matrix* $A(x) = a + y_1\alpha_1 + \cdots + y_m\alpha_m$, the *drift* $B(x) = b + \beta x$ and the *Lévy measure* (the compensator of the jumps) $M(x, d\xi) = m(d\xi) + y_1\mu_1(d\xi) + \cdots + y_m\mu_m(d\xi)$, minus the *killing rate* $C(x) = c + \langle \gamma, y \rangle$. An informal definition of an affine process could consist of the requirement that $A(x)$, $B(x)$, $C(x)$ and $M(x, d\xi)$ have affine dependence on x; see [40]. The particular kind of this affine dependence in the present setup is implied by the geometry of the state space D.

First, we notice that $A(x) \in \mathrm{Sem}^d$, $C(x) \geq 0$ and $M(x, D) \geq 0$, for all $x \in D$. Whence $A(x)$, $C(x)$ and $M(x, d\xi)$ cannot depend on z, and conditions (2.8) and (2.9) follow immediately. Now we consider the respective constraints on

drift, diffusion and jumps for the consistent behavior of X near the boundary of D. For $x \in D$, we define the *tangent cone* to D at $x \in D$,

$$T_D(x) := \{\xi \in \mathbb{R}^d \mid x + \varepsilon\xi \in D, \text{ for some } \varepsilon > 0\}.$$

Intuitively speaking, $T_D(x)$ consists of all "inward-pointing vectors at x." Write $I(x) := \{i \in \mathcal{I} \mid y_i = 0\}$. Then $x \in \partial D$ if and only if $I(x) \neq \varnothing$, and $\xi \in T_D(x)$ if and only if $\xi_i \geq 0$ for all $i \in I(x)$. The extreme cases are $T_D(0) = D$ and $T_D(x) = \mathbb{R}^d$ for $x \in D^0$. It is now easy to see that conditions (2.6) and (2.7) are equivalent to

$$B(x) \in T_D(x) \qquad \forall x \in D. \tag{2.26}$$

Conditions (2.4) and (2.5) yield the diagonal form of $A_{\mathcal{I}\mathcal{I}}(x)$,

$$A_{\mathcal{I}\mathcal{I}}(x) = \sum_{i \in \mathcal{I}} y_i \alpha_{i,ii} \mathrm{Id}(i) = \begin{pmatrix} y_1\alpha_{1,11} & & 0 \\ & \ddots & \\ 0 & & y_m\alpha_{m,mm} \end{pmatrix}. \tag{2.27}$$

Hence the diffusion component in the direction of $\operatorname{span}\{e_i \mid i \in I(x)\}$ is 0 at x. Conditions (2.10) and (2.11) express the integrability property

$$\int_{D\setminus\{0\}} \langle \chi_{I(x)}(\xi), \mathbf{1}\rangle \, M(x, d\xi) < \infty \qquad \forall x \in D. \tag{2.28}$$

This assures that the jump part in the direction of $\operatorname{span}\{e_i \mid i \in I(x)\}$ is of "finite variation at x." Theorem 2.7 suggests that (2.26)–(2.28) are the appropriate conditions for the invariance of D with respect to X.

2.1. *Existence of moments.* Criterions for the existence of first- or higher-order partial moments of X_t are of vital importance for all kinds of applications. They are provided by the following theorem. Examples are given in Sections 11 and 13 below.

THEOREM 2.16. *Suppose that X is conservative regular affine, and let $t \in \mathbb{R}_+$.*

(i) *Let $k \in \mathbb{N}$ and $1 \leq l \leq d$. If $\partial^{2k}_{\lambda_l}\phi(t, i\lambda)|_{\lambda=0}$ and $\partial^{2k}_{\lambda_l}\psi(t, i\lambda)|_{\lambda=0}$ exist, then*

$$\mathbb{E}_x\big[(X_t^l)^{2k}\big] < \infty \qquad \forall x \in D.$$

(ii) *Let U be an open convex neighborhood of* 0 *in $\mathbb{C}^d$. Suppose that $\phi(t, \cdot)$ and $\psi(t, \cdot)$ have an analytic extension on U. Then*

$$\mathbb{E}_x\big[e^{\langle q, X_t\rangle}\big] < \infty \qquad \forall q \in U \cap \mathbb{R}^d, \ \forall x \in D,$$

and (2.2) *holds for all $u \in U$ with* $\operatorname{Re} u \in U \cap \mathbb{R}^d$.

PROOF. Combine Lemmas A.1, A.2 and A.4 in the Appendix. □

Notice that finiteness of moments of X_t with respect to $\mathbb{P}_x$ requires, strictly speaking, finiteness of X_t, $\mathbb{P}_x$-a.s. This is why we assume X to be conservative.

Explicit conditions for Theorem 2.16 in terms of the parameters of X are given in Lemmas 5.3 and 6.5 below.

3. Preliminary results. In this section we derive some immediate consequences of Definitions 2.1 and 2.5. First, we are extending the (t, u)-range of validity of (2.2), which is a priori $\mathbb{R}_+ \times \partial\mathcal{U}$. In Lemma 3.1 we fix $u \in \mathcal{U}$ and see how far we can go in the t-direction. In Lemma 3.2, conversely, we fix $t \in \mathbb{R}_+$ and explore the range of possible u in $\mathbb{C}^d$.

LEMMA 3.1. *Suppose X is regular affine. Then the set*

$$\mathcal{O} := \{(t, u) \in \mathbb{R}_+ \times \mathcal{U} \mid P_s f_u(0) \neq 0, \ \forall s \in [0, t]\} \tag{3.1}$$

is open in $\mathbb{R}_+ \times \mathcal{U}$, and there exists a unique continuous extension of $\phi(t, u)$ and $\psi(t, u)$ on $\mathcal{O}$ such that (2.2) *holds for all $(t, u) \in \mathcal{O}$.*

PROOF. Let $x \in D$. We claim that

$$P_s f_u(x) \to P_t f_u(x) \qquad \text{for } s \to t, \text{ uniformly in } u \text{ on compacts in } \mathcal{U}. \tag{3.2}$$

Although the proof of (3.2) is standard (see, e.g., [10], Lemma 23.7) we shall give it here, for the sake of completeness. Let $t \in \mathbb{R}_+$ and (t_k) a sequence with $t_k \to t$, and $\varepsilon > 0$. Since X is weakly continuous, the sequence $(p_{t_k}(x, \cdot))$ is tight. Hence there exists $\rho \in C_c(D)$ with $0 \le \rho \le 1$ and $\int_D (1 - \rho(\xi))\, p_{t_k}(x, d\xi) < \varepsilon$, for all $k \in \mathbb{N}$. Moreover, there exists $\delta' > 0$ such that for all $u, u' \in \mathcal{U}$ with $\|u - u'\| < \delta'$ we have

$$\sup_{\xi \in \operatorname{supp} \rho} |f_u(\xi) - f_{u'}(\xi)| \le \varepsilon.$$

Hence, for every $u, u' \in \mathcal{U}$ with $\|u - u'\| < \delta'$,

$$\begin{aligned}
\left|P_{t_k} f_u(x) - P_{t_k} f_{u'}(x)\right| &\le \int_D |f_u(\xi) - f_{u'}(\xi)| \rho(\xi)\, p_{t_k}(x, d\xi) \\
&\quad + \int_D |f_u(\xi) - f_{u'}(\xi)| \big(1 - \rho(\xi)\big)\, p_{t_k}(x, d\xi) \\
&\le 3\varepsilon \qquad \forall k \in \mathbb{N}.
\end{aligned}$$

We infer that the sequence $(P_{t_k} f_u(x))$ is equicontinuous in $u \in \mathcal{U}$. Thus there exists $\delta > 0$ such that

$$\begin{aligned}
&\left|\big(P_{t_k} f_u(x) - P_t f_u(x)\big) - \big(P_{t_k} f_{u'}(x) - P_t f_{u'}(x)\big)\right| \\
&\quad \le \left|P_{t_k} f_u(x) - P_{t_k} f_{u'}(x)\right| + |P_t f_u(x) - P_t f_{u'}(x)| \le \frac{\varepsilon}{2}
\end{aligned} \tag{3.3}$$

for all $u, u' \in \mathcal{U}$ with $\|u - u'\| \le \delta$, for all $k \in \mathbb{N}$. Now let $U \subset \mathcal{U}$ be nonempty and compact. Cover U with finitely many, say q, balls of radius δ whose centers are denoted by $u^{(1)}, u^{(2)}, \dots, u^{(q)}$. For any $u^{(i)}$ there exists a number $N^{(i)}$ such that

$$\left|P_{t_k} f_{u^{(i)}}(x) - P_t f_{u^{(i)}}(x)\right| \le \frac{\varepsilon}{2} \qquad \forall k \ge N^{(i)}. \tag{3.4}$$

Now let $u \in U$. Choose a ball, say with center $u^{(i)}$, that contains u. Combining (3.3) and (3.4) we obtain

$$\begin{aligned}\left|P_{t_k} f_u(x) - P_t f_u(x)\right| &\le \left|\left(P_{t_k} f_u(x) - P_t f_u(x)\right) - \left(P_{t_k} f_{u^{(i)}}(x) - P_t f_{u^{(i)}}(x)\right)\right| \\ &\quad + \left|P_{t_k} f_{u^{(i)}}(x) - P_t f_{u^{(i)}}(x)\right| \\ &\le \varepsilon \qquad \forall k \ge \max_i N^{(i)},\end{aligned}$$

which proves (3.2).

As a consequence of (3.2), $P_t f_u(x)$ is jointly continuous in $(t, u) \in \mathbb{R}_+ \times \mathcal{U}$. Hence $\mathcal{O}$ is open in $\mathbb{R}_+ \times \mathcal{U}$. Notice that $\mathcal{O} \supset \{0\} \times \mathcal{U}$, which is simply connected, and every loop in $\mathcal{O}$ is equivalent to its projection onto $\{0\} \times \mathcal{U}$. Hence $\mathcal{O}$ is simply connected.

Since X is affine, we have

$$P_t f_{(v,w)}(x) P_t f_{(v,w)}(\xi) = P_t f_{(v,w)}(x + \xi) P_t f_{(v,w)}(0) \qquad \forall x, \xi \in D, \tag{3.5}$$

for all $(t, v, w) \in \mathbb{R}_+ \times \partial\mathcal{U}$. By Lemma A.2 we see that the functions on both sides of (3.5) are analytic in $v \in \mathbb{C}^m_{--}$. By the Schwarz reflexion principle, (3.5) therefore holds for all $v \in \mathbb{C}^m_-$. Since $\mathcal{O}$ is simply connected, $\phi(t, u)$ has a unique continuous extension on $\mathcal{O}$ such that $P_t f_u(0) = \exp(\phi(t, u))$ for all $(t, u) \in \mathcal{O}$. Hence, for fixed $(t, u) \in \mathcal{O}$, the function $g(x) = \exp(-\phi(t, u)) P_t f_u(x)$ is measurable and satisfies the functional equation $g(x)g(\xi) = g(x + \xi)$. Consequently, there exists a unique continuous extension of $\psi(t, u)$ such that

$$e^{-\phi(t,u)} P_t f_u(x) = e^{\langle \psi(t,u), x\rangle} \qquad \forall x \in D, \forall (t, u) \in \mathcal{O},$$

whence the assertion follows. □

The following is a (less important) variation of Lemma 3.1. Let $\pi, \rho \in \mathbb{R}^d_+$, and define $V := \{q \in \mathbb{R}^d \mid -\pi_l \le q_l \le \rho_l,\ l = 1, \dots, d\}$ and the strip $S := \{u \in \mathbb{C}^d \mid \operatorname{Re} u \in V\} \supset \partial\mathcal{U}$.

LEMMA 3.2. *Let $t \in \mathbb{R}_+$. Suppose X is affine and*

$$\int_D e^{\langle q, \xi\rangle} p_t(x, d\xi) < \infty \qquad \forall q \in V, \forall x \in D. \tag{3.6}$$

Then $O(t) := \{u \in S \mid P_t f_u(0) \ne 0\}$ is open in S, and for every simply connected set $\partial\mathcal{U} \subset U \subset O(t)$ there exists a unique continuous extension of $\phi(t, \cdot)$ and $\psi(t, \cdot)$ on U such that (2.2) *holds for all $u \in U$.*

PROOF. Dominated convergence yields continuity of the function $S \ni u \mapsto P_t f_u(0)$. Hence $O(t)$ is open in S, and clearly $\partial \mathcal{U} \subset O(t)$. Now the assertion follows by the same arguments as in the proof of the second part of Lemmas 3.1 and A.2. Notice that we cannot assert continuity of $P_t f_u(x)$ in t since f_u is unbounded for $u \in S$, in general. $\square$

For the remainder of this section we assume that X is regular affine. As an immediate consequence of (2.2) and Remark 2.3 we have

$$\phi(0,u) = 0, \qquad \psi(0,u) = u \qquad \forall u \in \mathcal{U}. \tag{3.7}$$

Let $(t,u) \in \mathcal{O}$ and $s > 0$ such that $(t+s,u) \in \mathcal{O}$ and $(s, \psi(t,u)) \in \mathcal{O}$. By Lemma 3.1 and the Chapman–Kolmogorov equation,

$$\begin{aligned} &e^{\phi(t+s,u)+\langle \psi(t+s,u),x\rangle} \\ &\quad = \int_D p_s(x,d\xi) \int_D p_t(\xi,d\tilde{\xi}) f_u(\tilde{\xi}) \\ &\quad = e^{\phi(t,u)} \int_D p_s(x,d\xi)\, e^{\langle \psi(t,u),\xi\rangle} \\ &\quad = e^{\phi(t,u)+\phi(s,\psi(t,u))+\langle \psi(s,\psi(t,u)),x\rangle} \qquad \forall x \in D, \end{aligned}$$

hence

$$\phi(t+s,u) = \phi(t,u) + \phi\big(s, \psi(t,u)\big), \tag{3.8}$$

$$\psi(t+s,u) = \psi\big(s, \psi(t,u)\big). \tag{3.9}$$

In view of Definition 2.5 and Lemma 3.1, the following right-hand derivatives exist:

$$F(u) := \partial_t^+ \phi(0,u), \tag{3.10}$$

$$R(u) = \big(R^{\mathcal{Y}}(u), R^{\mathcal{Z}}(u)\big) := \partial_t^+ \psi(0,u), \tag{3.11}$$

and we have

$$\tilde{\mathcal{A}} f_u(x) = \big(F(u) + \langle R(u), x\rangle\big) f_u(x) \tag{3.12}$$

for all $u \in \mathcal{U}$, $x \in D$. Combining (3.8) and (3.9) with (3.10) and (3.11) we conclude that, for all $(t,u) \in \mathcal{O}$,

$$\partial_t^+ \phi(t,u) = F(\psi(t,u)), \tag{3.13}$$

$$\partial_t^+ \psi(t,u) = R(\psi(t,u)). \tag{3.14}$$

Equation (3.12) yields

$$F(u) = \tilde{\mathcal{A}} f_u(0), \tag{3.15}$$

$$R_i(u) = \frac{\tilde{\mathcal{A}} f_u(e_i)}{f_u(e_i)} - F(u), \qquad i = 1, \dots, d. \tag{3.16}$$

The strategy for the proof of Theorem 2.7 is now as follows. In the next two sections we specify F, $R^{\mathcal{Y}}$ and $R^{\mathcal{Z}}$, and show that they are of the desired form; see (2.15)–(2.17). In view of (3.15) and (3.16) it is enough to know $\tilde{\mathcal{A}} f_u$ on the coordinate axes in D. Then, in Section 8, we can prove that X shares the Feller property and that its generator is given by (2.12). Conversely, given some admissible parameters $(a, \alpha, b, \beta, c, \gamma, m, \mu)$, the generalized Riccati equations (2.13)–(2.15) uniquely determine some mappings $\phi(t, u)$ and $\psi(t, u)$ (see Section 6) which have the following property. For every $t \in \mathbb{R}_+$ and $x \in D$ fixed, the mapping $\mathbb{R}^d \ni q \mapsto e^{\phi(t,iq)+\langle\psi(t,iq),x\rangle}$ is the characteristic function of an infinitely divisible substochastic distribution, say $p_t(x, d\xi)$, on D (see Section 7). By the flow property of ϕ and ψ it follows that $p_t(x, d\xi)$ is the transition function of a Markov process on D, which by construction is regular affine.

4. A representation result for regular processes. Throughout this section we assume that X is regular.

LEMMA 4.1. *Let $i \in \mathcal{I}$ and $r \in \mathbb{R}_+$. Then there exist elements*

$$\alpha(i, r) = \big(\alpha_{kl}(i, r)\big)_{k,l \in \mathcal{J}(i)} \in \mathrm{Sem}^{n+1}, \tag{4.1}$$

$$\beta(i, r) \in \mathbb{R}^d \qquad \text{with } \beta_{\mathcal{I}(i)}(i, r) \in \mathbb{R}_+^{m-1}, \tag{4.2}$$

$$\gamma(i, r) \in \mathbb{R}_+, \tag{4.3}$$

and a non-negative Borel measure $\nu(i, r; d\xi)$ on $D \setminus \{re_i\}$ satisfying

$$\int_{D\setminus\{re_i\}} \big(\langle \chi_{\mathcal{I}(i)}(\xi - re_i), \mathbf{1}\rangle + \|\chi_{\mathcal{J}(i)}(\xi - re_i)\|^2\big)\nu(i, r; d\xi) < \infty, \tag{4.4}$$

such that for all $u \in \mathcal{U}$, we have

$$\begin{aligned} \frac{\tilde{\mathcal{A}} f_u(re_i)}{f_u(re_i)} &= \langle \alpha(i, r) u_{\mathcal{J}(i)}, u_{\mathcal{J}(i)}\rangle + \langle \beta(i, r), u\rangle - \gamma(i, r) \\ &\quad + \int_{D\setminus\{re_i\}} \big(e^{\langle u, \xi - re_i\rangle} - 1 - \langle u_{\mathcal{J}(i)}, \chi_{\mathcal{J}(i)}(\xi - re_i)\rangle\big)\nu(i, r; d\xi). \end{aligned} \tag{4.5}$$

PROOF. Fix $i \in \mathcal{I}$ and $r \in \mathbb{R}_+$, and let $u \in \mathcal{U}$. For simplicity we write $x = re_i$, $I = \mathcal{I}(i)$ and $J = \mathcal{J}(i)$. The proof, inspired by [88], Theorem 9.5.1, is divided into four steps.

Step 1. *Decomposition*. Let $t > 0$ and write

$$\begin{aligned}\frac{P_t f_u(x) - f_u(x)}{t} &= \frac{1}{t}\int_D \big(f_u(\xi) - f_u(x) - \langle \nabla_J f_u(x), \chi_J(\xi - x)\rangle\big) p_t(x, d\xi) \\ &\quad + \frac{1}{t}\int_D \langle \nabla_J f_u(x), \chi_J(\xi - x)\rangle\, p_t(x, d\xi) \\ &\quad + \frac{1}{t}\big(p_t(x, D) - 1\big) f_u(x) \\ &= \frac{1}{t}\int_{D\setminus\{x\}} h_u(x,\xi) d(x,\xi)\, p_t(x, d\xi) \\ &\quad + \langle \beta_t(x), \nabla_J f_u(x)\rangle - \gamma_t(x) f_u(x),\end{aligned} \tag{4.6}$$

where

$$d(x,\xi) := \langle \chi_I(\xi - x), \mathbf{1}\rangle + \|\chi_J(\xi - x)\|^2, \tag{4.7}$$

$$h_u(x,\xi) := \frac{f_u(\xi) - f_u(x) - \langle \nabla_J f_u(x), \chi_J(\xi - x)\rangle}{d(x,\xi)} \tag{4.8}$$

and

$$\beta_t(x) := \frac{1}{t}\int_D \chi_J(\xi - x)\, p_t(x, d\xi) \in \mathbb{R}^{n+1},$$

$$\gamma_t(x) := \frac{1}{t}\big(1 - p_t(x, D)\big) \geq 0.$$

Notice that

$$0 \leq d(x,\xi) \leq d \qquad \forall \xi \in D, \qquad d(x,\xi) = 0 \iff \xi = x. \tag{4.9}$$

Hence we can introduce a new probability measure as follows. Set

$$\ell_t(x) := \frac{1}{t}\int_D d(x,\xi)\, p_t(x, d\xi) \geq 0.$$

If $\ell_t(x) > 0$, define

$$\mu_t(x, d\xi) := \frac{d(x,\xi)}{t\ell_t(x)}\, p_t(x, d\xi).$$

If $\ell_t(x) = 0$ we let $\mu_t(x, \cdot)$ be the Dirac measure at some point in $D \setminus \{x\}$. In both cases we have that $\mu_t(x, \cdot)$ is a probability measure on $D \setminus \{x\}$, and we can rewrite (4.6):

$$\begin{aligned}&\frac{P_t f_u(x) - f_u(x)}{t} \\ &\quad = \ell_t(x)\int_{D\setminus\{x\}} h_u(x,\xi)\, \mu_t(x, d\xi) + \langle \beta_t(x), \nabla_J f_u(x)\rangle - \gamma_t(x) f_u(x).\end{aligned} \tag{4.10}$$

Step 2. *Extension of* $h_u(x,\cdot)$. Notice that $h_u(x,\cdot) \in C_{\mathrm{b}}(D \setminus \{x\})$. But the value $\lim_{\xi \to x} h_u(x,\xi)$ depends on the direction from which ξ converges to x. Define the cuboid

$$Q(x) := \{\xi \in D \mid |\xi_k - x_k| \le 1,\ 1 \le k \le d\} \quad \text{and} \quad Q_0(x) := Q(x) \setminus \{x\}. \tag{4.11}$$

We shall construct a compactification of $Q_0(x)$ to which $h_u(x,\cdot)$ can be continuously extended.

We decompose the difference $\xi - x = (\xi - \xi^\perp) + (\xi^\perp - x)$ where

$$\xi_k^\perp := \begin{cases} x_k, & \text{if } k \in I, \\ \xi_k, & \text{if } k \in J. \end{cases}$$

Applying Taylor's formula twice we have

$$\begin{aligned} h_u(x,\xi) &= \frac{(f_u(\xi) - f_u(\xi^\perp)) + (f_u(\xi^\perp) - f_u(x) - \langle \nabla f_u(x), \xi^\perp - x\rangle)}{d(x,\xi)} \\ &= \left\langle \int_0^1 \nabla f_u\big(\xi^\perp + s(\xi - \xi^\perp)\big)\,ds, \frac{\xi - \xi^\perp}{d(x,\xi)} \right\rangle \\ &\quad + \sum_{k,l=1}^d \left(\int_0^1 \partial_{x_k}\partial_{x_l} f_u\big(x + s(\xi^\perp - x)\big)(1-s)\,ds \right) \\ &\qquad \times \frac{(\xi_k^\perp - x_k)(\xi_l^\perp - x_l)}{d(x,\xi)} \end{aligned} \tag{4.12}$$

for all $\xi \in Q_0(x)$.

We let $\mathbf{w}(x,\xi) := (\mathbf{w}_k(x,\xi))_{k\in I}$ and $\mathbf{a}(x,\xi) := (\mathbf{a}_{kl}(x,\xi))_{k,l\in J}$ be given by

$$\mathbf{w}_k(x,\xi) := \frac{\xi_k - x_k}{d(x,\xi)}, \qquad k \in I, \tag{4.13}$$

$$\mathbf{a}_{kl}(x,\xi) := \frac{(\xi_k - x_k)(\xi_l - x_l)}{d(x,\xi)}, \qquad k,l \in J. \tag{4.14}$$

Define the compact subset H of $[0,1]^{m-1} \times \mathrm{Sem}^{n+1}$ by

$$H := \left\{ (w,a) \in [0,1]^{m-1} \times \mathrm{Sem}^{n+1} \,\middle|\, \langle w, \mathbf{1}\rangle + \sum_{k\in J} a_{kk} = 1 \right\}. \tag{4.15}$$

Then it is easy to see that

$$\Gamma(x,\xi) := \big(\xi, \mathbf{w}(x,\xi), \mathbf{a}(x,\xi)\big) \in Q_0(x) \times H \qquad \forall \xi \in Q_0(x) \tag{4.16}$$

and that $\Gamma(x,\cdot) : Q_0(x) \to \Lambda(x) := \Gamma(x, Q_0(x)) \subset Q_0(x) \times H$ is a homeomorphism.

Now the function $\tilde{h}_u(x,\cdot) := h_u(x,\Gamma^{-1}(x,\cdot)) : \Lambda(x) \to \mathbb{C}$ can be continuously extended to the compact closure $\overline{\Lambda(x)}$. In fact, by (4.12), we have

$$\begin{aligned}\tilde{h}_u\big(x,\Gamma(x,\xi)\big) &= \sum_{k\in I} \mathbf{w}_k(x,\xi)\int_0^1 \partial_{x_k} f_u\big(\xi^\perp + s(\xi-\xi^\perp)\big)\,ds \\ &\quad + \sum_{k,l\in J} \mathbf{a}_{kl}(x,\xi)\int_0^1 \partial_{x_k}\partial_{x_l} f_u\big(x+s(\xi^\perp - x)\big)(1-s)\,ds \\ &\to \sum_{k\in I} w_k \partial_{x_k} f_u(x) + \tfrac{1}{2}\sum_{k,l\in J} a_{kl}\partial_{x_k}\partial_{x_l} f_u(x),\end{aligned} \tag{4.17}$$

if $\Gamma(x,\xi) \to (x,w,a) \in \overline{\Lambda(x)}$.

Denote by $\tilde{\mu}_t(x,\cdot)$ the image of $\mu_t(x,\cdot)$ by $\Gamma(x,\cdot)$. Then $\tilde{\mu}_t(x,\cdot)$ is a bounded measure on $\overline{\Lambda(x)}$ [giving mass zero to $\overline{\Lambda(x)} \setminus \Lambda(x)$], and we have

$$\int_{Q_0(x)} h_u(x,\cdot)\,d\mu_t(x,\cdot) = \int_{\overline{\Lambda(x)}} \tilde{h}_u(x,\cdot)\,d\tilde{\mu}_t(x,\cdot). \tag{4.18}$$

In particular,

$$\tilde{\mu}_t\big(x,\overline{\Lambda(x)}\big) + \mu_t\big(x, D\setminus Q(x)\big) = 1. \tag{4.19}$$

Notice that $h_u(x,\xi) = f_u(\xi) - f_u(x) - \langle \nabla_J f_u(x), \mathbf{1}\rangle$ for $\xi \in D\setminus Q(x)$. Hence we can rewrite (4.10):

$$\begin{aligned}\frac{P_t f_u(x) - f_u(x)}{t} &= \ell_t(x)\Big(\int_{\overline{\Lambda(x)}} \tilde{h}_u(x,\cdot)\,d\tilde{\mu}_t(x,\cdot) + \int_{D\setminus Q(x)} f_u\,d\mu_t(x,\cdot)\Big) \\ &\quad - \ell_t(x)\big(f_u(x) + \langle \nabla_J f_u(x),\mathbf{1}\rangle\big)\mu_t\big(x, D\setminus Q(x)\big) \\ &\quad + \langle \beta_t(x), \nabla_J f_u(x)\rangle - \gamma_t(x) f_u(x).\end{aligned} \tag{4.20}$$

Step 3. *Limiting*. We pass to the limit in (4.20). We introduce the non-negative numbers

$$\theta_j(x) := \ell_{1/j}(x) + \sum_{k\in J} |\beta^k_{1/j}(x)| + \gamma_{1/j}(x) \ge 0, \qquad j\in\mathbb{N}. \tag{4.21}$$

We have to distinguish two cases.

Case (i) $\liminf_{j\to\infty}\theta_j(x) = 0$. In this case there exists a subsequence of $(\theta_j(x))$ converging to 0. Because of (4.19) we conclude from (4.20) that $\tilde{\mathcal{A}} f_u(x) = 0$ for all $u \in \mathcal{U}$, and the lemma is proved.

Case (ii) $\liminf_{j\to\infty}\theta_j(x) > 0$. There exists a subsequence, denoted again by $(\theta_j(x))$, converging to $\theta(x) \in (0,\infty]$. By (4.21) the following limits exist:

$$\frac{1}{\theta_{1/j}(x)} \to \delta(x) \in \mathbb{R}_+, \qquad \frac{\ell_{1/j}(x)}{\theta_{1/j}(x)} \to \ell(x) \in [0,1],$$

$$\frac{\beta_{1/j}(x)}{\theta_{1/j}(x)} \to \beta(x) \in [-1,1]^{n+1}, \qquad \frac{\gamma_{1/j}(x)}{\theta_{1/j}(x)} \to \gamma(x) \in [0,1]$$

and satisfy

$$\ell(x) + \sum_{k \in J} |\beta^k(x)| + \gamma(x) = 1. \tag{4.22}$$

If $\ell(x) = 0$ then we have

$$\delta(x)\tilde{\mathcal{A}} f_u(x) = \langle \beta(x), \nabla_J f_u(x) \rangle - \gamma(x) f_u(x) \qquad \forall u \in \mathcal{U}. \tag{4.23}$$

Suppose now that $\ell(x) > 0$. After passing to a subsequence if necessary, the sequence $(\tilde{\mu}_{1/j}(x, \cdot))$ converges weakly to a bounded measure $\tilde{\mu}(x, \cdot)$ on $\overline{\Lambda(x)}$, and $\lim_{j\to\infty} \mu_{1/j}(x, D \setminus Q(x)) =: c(x) \in [0, 1]$ exists. Dividing both sides of (4.20) by $\theta_j(x)$ we get in the limit

$$\begin{aligned} &\lim_{j\to\infty} \int_{D\setminus Q(x)} f_u \, d\mu_{1/j}(x, \cdot) \\ &\quad = \frac{1}{\ell(x)} \big(\delta(x)\tilde{\mathcal{A}} f_u(x) - \langle \beta(x), \nabla_J f_u(x) \rangle + \gamma(x) f_u(x)\big) \\ &\qquad - \int_{\overline{\Lambda(x)}} \tilde{h}_u(x, \cdot) \, d\tilde{\mu}(x, \cdot) \\ &\qquad + c(x)\big(f_u(x) + \langle \nabla_J f_u(x), \mathbf{1} \rangle\big). \end{aligned} \tag{4.24}$$

Notice that (4.24) holds simultaneously for all $u \in \mathcal{U}$. Since X is regular, the right-hand side of (4.24) is continuous at $u = 0$ [see (3.12)]. The continuity theorem of Lévy (see, e.g., [44]) implies that the sequence of restricted measures $(\mu_{1/j}(x, \cdot \cap D \setminus Q(x)))$ converges weakly to a bounded measure $\mu'(x, \cdot)$ on D with support contained in $\overline{D \setminus Q(x)}$. In particular, $c(x) = \mu'(x, \overline{D \setminus Q(x)})$ and

$$\begin{aligned} &\lim_{j\to\infty} \int_{D\setminus Q(x)} f_u \, d\mu_{1/j}(x, \cdot) \\ &\quad = \int_{\overline{D\setminus Q(x)}} f_u \, d\mu'(x, \cdot) \qquad \forall u \in \mathcal{U}. \end{aligned} \tag{4.25}$$

Note that, by (4.19), we have

$$\tilde{\mu}\big(x, \overline{\Lambda(x)}\big) + \mu'\big(x, \overline{D \setminus Q(x)}\big) = 1. \tag{4.26}$$

We introduce the projections

$$\begin{aligned} W &: D \times H \to W(D \times H) \subset [0, 1]^{m-1}, \qquad & W(\xi, w, a) &:= w, \\ A &: D \times H \to A(D \times H) \subset \mathrm{Sem}^{n+1}, \qquad & A(\xi, w, a) &:= a \end{aligned}$$

[see (4.15)]. In view of (4.17), we have

$$\begin{aligned}\int_{\overline{\Lambda(x)}} \tilde{h}_u \, d\tilde{\mu}(x,\cdot) &= \int_{\overline{\Lambda(x)}\setminus\Lambda(x)} \tilde{h}_u \, d\tilde{\mu}(x,\cdot) + \int_{\Lambda(x)} \tilde{h}_u \, d\tilde{\mu}(x,\cdot) \\ &= \sum_{k\in I}\Big(\int_{\overline{\Lambda(x)}\setminus\Lambda(x)} W_k \, d\tilde{\mu}(x,\cdot)\Big)\partial_{x_k} f_u(x) \\ &\quad + \tfrac{1}{2}\sum_{k,l\in J}\Big(\int_{\overline{\Lambda(x)}\setminus\Lambda(x)} A_{kl} \, d\tilde{\mu}(x,\cdot)\Big)\partial_{x_k}\partial_{x_l} f_u(x) \\ &\quad + \int_{Q_0(x)} h_u \, d\tilde{\mu}\big(x,\Gamma(x,\cdot)\big).\end{aligned} \tag{4.27}$$

Define the bounded measure $\mu(x,\cdot)$ on $D\setminus\{x\}$ by

$$\mu(x,\cdot) := \tilde{\mu}\big(x,\Gamma\big(x,Q_0(x)\cap\cdot\big)\big) + \mu'(x,\cdot). \tag{4.28}$$

Combining (4.24), (4.25) and (4.27), we conclude that

$$\begin{aligned}\delta(x)\tilde{\mathcal{A}} f_u(x) &= \frac{\ell(x)}{2}\sum_{k,l\in J}\Big(\int_{\overline{\Lambda(x)}\setminus\Lambda(x)} A_{kl} \, d\tilde{\mu}(x,\cdot)\Big)\partial_{x_k}\partial_{x_k} f_u(x) \\ &\quad + \ell(x)\sum_{k\in I}\Big(\int_{\overline{\Lambda(x)}\setminus\Lambda(x)} W_k \, d\tilde{\mu}(x,\cdot)\Big)\partial_{x_k} f_u(x) + \langle\beta(x),\nabla_J f_u(x)\rangle \\ &\quad - \gamma(x) f_u(x) + \ell(x)\int_{D\setminus\{x\}} h_u \, d\mu(x,\cdot) \qquad \forall u\in\mathcal{U}.\end{aligned}$$

Hence

$$\begin{aligned}&\delta(x)\frac{\tilde{\mathcal{A}} f_u(x)}{f_u(x)} \\ &\quad = \langle\alpha(x)u_J,u_J\rangle + \langle\tilde{\beta}(x),u_I\rangle + \langle\beta(x),u_J\rangle - \gamma(x) \\ &\qquad + \int_{D\setminus\{x\}}\big(e^{\langle u,\xi-x\rangle} - 1 - \langle u_J,\chi_J(\xi-x)\rangle\big)\nu(x,d\xi) \qquad \forall u\in\mathcal{U},\end{aligned} \tag{4.29}$$

where

$$\begin{aligned}\alpha(x) &:= \frac{\ell(x)}{2}\int_{\overline{\Lambda(x)}\setminus\Lambda(x)} A \, d\tilde{\mu}(x,\cdot) \in \mathrm{Sem}^{n+1}, \\ \tilde{\beta}(x) &:= \ell(x)\int_{\overline{\Lambda(x)}\setminus\Lambda(x)} W \, d\tilde{\mu}(x,\cdot) \in \mathbb{R}_+^{m-1}, \\ \nu(x,d\xi) &:= \frac{\ell(x)}{d(x,\xi)}\mu(x,d\xi).\end{aligned}$$

Step 4. *Consistency*. It remains to verify that $\delta(x) > 0$. Then we can divide (4.23) and (4.29) by $\delta(x)$, and the lemma follows also for Case (ii).

We show that the right-hand side of (4.29) is not the zero function in u. Assume that $\beta(x) = 0$, $\gamma(x) = 0$ and $\nu(x, D \setminus \{x\}) = 0$. Equality (4.22) implies that $\ell(x) = 1$. Hence we have $\mu(x, D \setminus \{x\}) = 0$, and by (4.26) and (4.28) therefore, $\tilde{\mu}(x, \overline{\Lambda(x)} \setminus \Lambda(x)) = 1$. It follows from (4.15) that

$$\langle \tilde{\beta}(x), \mathbf{1} \rangle + 2 \sum_{k \in J} a_{kk}(x) = \int_{\overline{\Lambda(x)} \setminus \Lambda(x)} \left(\langle W, \mathbf{1} \rangle + \sum_{k \in J} A_{kk} \right) d\tilde{\mu}(x, \cdot) = 1.$$

Hence $\alpha(x)$ and $\tilde{\beta}(x)$ cannot both be 0 at the same time. But the representation of the function in u on the right-hand side of (4.29) by $\alpha(x)$, $\tilde{\beta}(x)$, $\beta(x)$, $\gamma(x)$ and $\nu(x, d\xi)$ is unique (see [79], Theorem 8.1). Thus it does not vanish identically in u and therefore $\delta(x) = 0$ is impossible. The same argument applies for (4.23). □

The fact that only $u_{\mathcal{J}(i)}$ appears in the quadratic term in (4.5) and that (4.2)–(4.4) hold is due to the geometry of D, which makes $d(x, \cdot)$ a measure for the distance from x [see (4.9) and (4.7)]. By a slight variation of the preceding proof we can derive the following lemma.

LEMMA 4.2. *Let $j \in \mathcal{J}$ and $s \in \mathbb{R}$. Then there exist elements*

$$\alpha(j, s) = \big(\alpha_{kl}(j, s)\big)_{k,l \in \mathcal{J}} \in \mathrm{Sem}^n, \tag{4.30}$$

$$\begin{aligned} \beta(j, s) &\in D, \\ \gamma(j, s) &\in \mathbb{R}_+, \end{aligned} \tag{4.31}$$

and a non-negative Borel measure $\nu(j, s; d\xi)$ on $D \setminus \{se_j\}$ satisfying

$$\int_{D \setminus \{se_j\}} \big(\langle \chi_{\mathcal{I}}(\xi - se_j), \mathbf{1} \rangle + \|\chi_{\mathcal{J}}(\xi - se_j)\|^2 \big) \nu(j, s; d\xi) < \infty, \tag{4.32}$$

such that, for all $u \in \mathcal{U}$, we have

$$\begin{aligned} \frac{\tilde{\mathcal{A}} f_u(se_j)}{f_u(se_j)} &= \langle \alpha(j, s) u_{\mathcal{J}}, u_{\mathcal{J}} \rangle + \langle \beta(j, s), u \rangle - \gamma(j, s) \\ &\quad + \int_{D \setminus \{se_j\}} \big(e^{\langle u, \xi - se_j \rangle} - 1 - \langle u_{\mathcal{J}}, \chi_{\mathcal{J}}(\xi - se_j) \rangle \big) \nu(j, s; d\xi). \end{aligned} \tag{4.33}$$

PROOF. Fix $j \in \mathcal{J}$ and $s \in \mathbb{R}$. Write $x = se_j$, $I = \mathcal{I}$ and $J = \mathcal{J}$. Now the lemma follows line by line as in the proof of Lemma 4.1. □

5. The mappings $F(u)$ and $R(u)$. Let $\beta_i^{\mathcal{Y}} \in \mathbb{R}^d$, $i \in \mathcal{I}$, and $\beta^{\mathcal{Z}} \in \mathbb{R}^{n \times n}$. Then (2.18) and (2.19) together with $\beta_{\mathcal{I}\mathcal{J}} := 0$ uniquely define a matrix $\beta \in \mathbb{R}^{d \times d}$.

DEFINITION 5.1. The parameters $(a, \alpha, b, \beta^{\mathcal{Y}}, \beta^{\mathcal{Z}}, c, \gamma, m, \mu)$ are called *admissible* if $(a, \alpha, b, \beta, c, \gamma, m, \mu)$ are admissible. Hence

$$\beta^{\mathcal{Y}} = (\beta_1^{\mathcal{Y}}, \dots, \beta_m^{\mathcal{Y}}) \text{ with } \beta_i^{\mathcal{Y}} \in \mathbb{R}^d \text{ and } \beta_{i,\mathcal{I}(i)}^{\mathcal{Y}} \in \mathbb{R}_+^{m-1} \qquad \text{for all } i \in \mathcal{I}, \tag{5.1}$$

$$\beta^{\mathcal{Z}} \in \mathbb{R}^{n \times n}. \tag{5.2}$$

Combining (3.15) and (3.16) and Lemmas 4.1 and 4.2, we can now calculate $F(u)$, $R^{\mathcal{Y}}(u)$ and $R^{\mathcal{Z}}(u)$ [see (3.10) and (3.11)].

PROPOSITION 5.2. *Suppose X is regular affine. Then $F(u)$ and $R^{\mathcal{Y}}(u)$ are of the form* (2.16) *and* (2.17), *respectively, and*

$$R^{\mathcal{Z}}(u) = \beta^{\mathcal{Z}} w, \tag{5.3}$$

where $(a, \alpha, b, \beta^{\mathcal{Y}}, \beta^{\mathcal{Z}}, c, \gamma, m, \mu)$ are admissible parameters.

PROOF. We derive (2.16), (2.17) and (5.3) separately in three steps.

Proof of (2.16). For $m = 0$ the assertion follows directly from (3.15) and Lemma 4.2. Suppose $(m, n) = (1, 0)$. We already know from (3.15) and Lemma 4.1 that there exist $\tilde{a}, c \in \mathbb{R}_+$, $\tilde{b} \in \mathbb{R}$ and a non-negative Borel measure $m(d\eta)$ on $\mathbb{R}_{++}$, integrating $|\chi(\eta)|^2$, such that

$$F(v) = \tilde{a}v^2 + \tilde{b}v - c + \int_{\mathbb{R}_{++}} \big(e^{v\eta} - 1 - v\chi(\eta)\big) m(d\eta) \qquad \forall\, v \in \mathbb{C}_-. \tag{5.4}$$

It remains to show that $\tilde{a} = 0$ and

$$\int_{\mathbb{R}_{++}} \chi(\eta)\, m(d\eta) \le \tilde{b}.$$

Since F is analytic on $\mathbb{C}_{--}$ (this follows from Lemma A.2) and by uniqueness of the representation (5.4) (see [79], Theorem 8.1), it is enough to consider $v \in \mathbb{R}_-$. But then we have

$$F(v) = \lim_{t\to 0} \frac{e^{\phi(t,v)} - 1}{t} = \lim_{t\to 0} \left(\int_{\mathbb{R}_+} (e^{v\eta} - 1) \frac{p_t(0, d\eta)}{t} + \frac{p_t(0, \mathbb{R}_+) - 1}{t} \right).$$

It is well known that, for fixed t, the function in v on the right-hand side is the exponent of the Laplace transform of an infinitely divisible substochastic measure on $\mathbb{R}_+$ (see [79], Section 51). Hence $e^{F(v)}$, being the pointwise limit of such Laplace transforms, is itself the Laplace transform of an infinitely divisible substochastic measure on $\mathbb{R}_+$. Thus $F(v)$ is of the desired form.

Suppose now that $m \ge 1$ and $(m, n) \ne (1, 0)$. By (3.15), (4.5) and (4.33) we have

$$\begin{aligned} F(u) &= \langle \alpha(i,0) u_{\mathcal{J}(i)}, u_{\mathcal{J}(i)} \rangle + \langle \tilde{\beta}(i,0), u \rangle - \gamma(i,0) \\ &\quad + \int_{D\setminus\{0\}} \big(e^{\langle u,\xi\rangle} - 1 - \langle u, \chi(\xi)\rangle\big) \nu(i, 0; d\xi) \end{aligned} \tag{5.5}$$

$$\begin{aligned} &= \langle \alpha(j,0) u_{\mathcal{J}}, u_{\mathcal{J}} \rangle + \langle \tilde{\beta}(j,0), u \rangle - \gamma(j,0) \\ &\quad + \int_{D\setminus\{0\}} \big(e^{\langle u,\xi\rangle} - 1 - \langle u, \chi(\xi)\rangle\big) \nu(j, 0; d\xi) \qquad \forall\, u \in \mathcal{U}, \end{aligned} \tag{5.6}$$

for all $(i, j) \in \mathcal{I} \times \mathcal{J}$, where $\tilde{\beta}(i, 0), \tilde{\beta}(j, 0) \in \mathbb{R}^d$ are given by

$$\tilde{\beta}_k(i,0) := \begin{cases} \beta_k(i,0) + \int_{D\setminus\{0\}} \chi_k(\xi)\, \nu(i,0;d\xi) \in \mathbb{R}_+, & \text{if } k \in \mathcal{I}(i), \\ \beta_k(i,0), & \text{if } k \in \mathcal{J}(i), \end{cases}$$

$$\tilde{\beta}_k(j,0) := \begin{cases} \beta_k(j,0) + \int_{D\setminus\{0\}} \chi_k(\xi)\, \nu(j,0;d\xi) \in \mathbb{R}_+, & \text{if } k \in \mathcal{I}, \\ \beta_k(j,0), & \text{if } k \in \mathcal{J}. \end{cases}$$

By uniqueness of the representation in (5.5) and (5.6), we obtain

$$\begin{aligned} \alpha_{ik}(i,0) &= \alpha_{ki}(i,0) = 0 && \forall k \in \mathcal{J}(i), \\ \alpha_{kl}(i,0) &= \alpha_{kl}(j,0) =: a_{kl} && \forall k, l \in \mathcal{J}, \\ \tilde{\beta}(i,0) &= \tilde{\beta}(j,0) =: b, \\ \gamma(i,0) &= \gamma(j,0) =: c, \\ \nu(i,0;d\xi) &= \nu(j,0;d\xi) =: m(d\xi) \end{aligned}$$

for all $(i, j) \in \mathcal{I} \times \mathcal{J}$, and the assertion is established.

Proof of (2.17). Let $i \in \mathcal{I}$. For $r \in \mathbb{R}_+$, we define $\tilde{\alpha}(i, r) \in \mathrm{Sem}^d$ by

$$\tilde{\alpha}_{kl}(i,r) := \begin{cases} \alpha_{kl}(i,r), & \text{if } k, l \in \mathcal{J}(i), \\ 0, & \text{else} \end{cases}$$

[see (4.1)]. Combining (3.16), (4.5) and (2.16), we obtain

$$\begin{aligned} R_i^{\mathcal{Y}}(u) &= \frac{\tilde{\mathcal{A}} f_u(e_i)}{f_u(e_i)} - F(u) \\ &= \big\langle \big(\tilde{\alpha}(i,1) - a\big)u, u\big\rangle + \langle \beta(i,1) - b, u\rangle - \big(\gamma(i,1) - c\big) \\ &\quad + \int_{\tilde{D}\setminus\{0\}} \big(e^{\langle u,\xi\rangle} - 1 - \big\langle u_{\mathcal{J}(i)}, \chi_{\mathcal{J}(i)}(\xi)\big\rangle\big)\, d\nu(i,1;e_i+\xi) \\ &\quad - \int_{D\setminus\{0\}} \big(e^{\langle u,\xi\rangle} - 1 - \langle u_{\mathcal{J}}, \chi_{\mathcal{J}}(\xi)\rangle\big) m(d\xi) \\ &= \langle \alpha_i u, u\rangle + \langle \beta_i^{\mathcal{Y}}, u\rangle - \gamma_i \\ &\quad + \int_{\tilde{D}\setminus\{0\}} \big(e^{\langle u,\xi\rangle} - 1 - \big\langle u_{\mathcal{J}(i)}, \chi_{\mathcal{J}(i)}(\xi)\big\rangle\big) \mu_i(d\xi), \end{aligned} \tag{5.7}$$

where

$$
\begin{aligned}
\alpha_i &:= \tilde{\alpha}(i,1) - a,\\
\beta^{\mathcal{Y}}_{i,k} &:= \begin{cases} \beta(i,1)_i - b_i - \displaystyle\int_{D\setminus\{0\}} \chi_i(\xi)\, m(d\xi), & \text{if } k = i,\\ \beta(i,1)_k - b_k, & \text{if } k \in \{1,\dots,d\}\setminus\{i\}, \end{cases}\\
\gamma_i &:= \gamma(i,1) - c,\\
\tilde{D} &:= D - e_i,\\
\mu_i(\cdot) &:= \nu(i,1; e_i + \cdot) - m(D \cap \cdot),
\end{aligned}
\tag{5.8}
$$

which is a priori a signed measure on $\tilde{D} \setminus \{0\}$. On the other hand, by (3.12), we have

$$
\frac{\tilde{\mathcal{A}} f_u(re_i)}{f_u(re_i)} = F(u) + R^{\mathcal{Y}}_i(u) r \qquad \forall r \in \mathbb{R}_+. \tag{5.9}
$$

Insert (5.7) in (5.9) and compare with (4.5) to conclude that, for all $r \in \mathbb{R}_+$,

$$
\begin{aligned}
\tilde{\alpha}(i,r) &= a + r\alpha_i,\\
\beta(i,r) &= b + r\beta^{\mathcal{Y}}_i,\\
\gamma(i,r) &= c + r\gamma_i,\\
\nu(i,r; re_i + \cdot) &= m(D \cap \cdot) + r\,\mu_i(\cdot) \qquad \text{on } \tilde{D} \setminus \{0\}.
\end{aligned}
$$

By letting $r \to \infty$ we obtain conditions (2.5), (5.1) and (2.9) from (4.1)–(4.3), and that μ_i is non-negative. Letting $r \to 0$ yields $\mu_i(\tilde{D} \setminus D) = 0$, and (2.11) is a consequence of (4.4), (2.10) and (5.8).

Proof of (5.3). Let $j \in \mathcal{J}$. For $s \in \mathbb{R}$, we define $\tilde{\alpha}(j,s) \in \mathrm{Sem}^d$ by

$$
\tilde{\alpha}_{kl}(j,s) := \begin{cases} \alpha_{kl}(j,s), & \text{if } k,l \in \mathcal{J},\\ 0, & \text{else} \end{cases}
$$

[see (4.30)]. Combining (3.16), (4.33) and (2.16), we obtain

$$
\begin{aligned}
R^{\mathcal{Z}}_{j-m}(u) &= \frac{\tilde{\mathcal{A}} f_u(e_j)}{f_u(e_j)} - F(u)\\
&= \big\langle \big(\tilde{\alpha}(j,1) - a\big)u, u\big\rangle + \langle \beta(j,1) - b, u\rangle - \big(\gamma(j,1) - c\big)\\
&\quad + \int_{D\setminus\{0\}} \big(e^{\langle u,\xi\rangle} - 1 - \langle u_{\mathcal{J}}, \chi_{\mathcal{J}}(\xi)\rangle\big)\mu_j(d\xi),
\end{aligned}
\tag{5.10}
$$

where we write $\mu_j(\cdot) := \nu(j,1; e_j + \cdot) - m(\cdot)$, which is a priori a signed measure on $D \setminus \{0\}$ (notice that $D - e_j = D$). But $R^{\mathcal{Z}}_{j-m}(u) \in i\mathbb{R}$; therefore the right-hand side of (5.10) is purely imaginary. This yields immediately $\tilde{\alpha}(j,1) - a = 0$,

$(\beta(j,1)-b)_{\mathcal{I}}=0$ and $\gamma(j,1)-c=0$. On the other hand, by (3.12), we have

$$\frac{\tilde{\mathcal{A}} f_u(se_j)}{f_u(se_j)} = F(u) + R^{\mathcal{Z}}_{j-m}(u)s \qquad \forall s \in \mathbb{R}. \tag{5.11}$$

Insert (5.10) in (5.11) and compare with (4.33) to conclude that, for all $s \in \mathbb{R}$,

$$\nu(j,s;se_j+\cdot) = m(\cdot) + s\,\mu_j(\cdot) \qquad \text{on } D\setminus\{0\}.$$

But this is possible only if $\mu_j = 0$. Hence $R^{\mathcal{Z}}_{j-m}(v,w) = \langle(\beta(j,1)-b)_{\mathcal{J}}, w\rangle$, and the proposition is established. □

We end this section with a regularity result. Let $Q_0(0) = Q(0)\setminus\{0\}$ where $Q(0)$ is given by (4.11). Decompose the integral term, say $I(u)$, in $F(u)$ as follows:

$$I(u) = \int_{Q_0(0)} \big(e^{\langle u,\xi\rangle} - 1 - \langle u_{\mathcal{J}}, \xi_{\mathcal{J}}\rangle\big)\, m(d\xi) + H(u) - (1+\langle u_{\mathcal{J}}, \mathbf{1}\rangle)\, m\big(D\setminus Q(0)\big),$$

where $H(u) := \int_{D\setminus Q(0)} e^{\langle u,\xi\rangle}\, m(d\xi)$ [see (2.16)]. In view of Lemma A.2, the first integral on the right-hand side is analytic in $u \in \mathbb{C}^d$, and so is the last term. Hence the degree of regularity of $I(u)$, and thus of $F(u)$, is given by that of $H(u)$. The same reasoning applies for $R^{\mathcal{Y}}_i$ [see (2.17)]. From Lemmas A.1 and A.2 we now obtain the following result.

LEMMA 5.3. *Let $k \in \mathbb{N}$ and $i \in \mathcal{I}$.*

(i) *$F(\cdot,w)$ and $R^{\mathcal{Y}}_i(\cdot,w)$ are analytic on $\mathbb{C}^m_{--}$, for every $w \in i\mathbb{R}^n$.*

(ii) *If*

$$\int_{D\setminus Q(0)} \|\xi\|^k\, m(d\xi) < \infty \quad \textit{and} \quad \int_{D\setminus Q(0)} \|\xi\|^k\, \mu_i(d\xi) < \infty,$$

then $F \in C^k(\mathcal{U})$ and $R^{\mathcal{Y}}_i \in C^k(\mathcal{U})$, respectively.

(iii) *Let $V \subset \mathbb{R}^d$ be open. If*

$$\int_{D\setminus Q(0)} e^{\langle q,\xi\rangle}\, m(d\xi) < \infty \quad \textit{and} \quad \int_{D\setminus Q(0)} e^{\langle q,\xi\rangle}\, \mu_i(d\xi) < \infty \qquad \forall q \in V, \tag{5.12}$$

then F and $R^{\mathcal{Y}}_i$ are analytic on the open strip $S = \{u \in \mathbb{C}^d \mid \operatorname{Re} u \in V\}$, respectively.

REMARK 5.4. A sufficient condition for Lemma 5.3(iii) is, for example,

$$\int_{D\setminus Q(0)} \exp\left(\sum_{l=1}^d \rho_l |\xi_l|\right) m(d\xi) < \infty,$$

for some $\rho \in \mathbb{R}^d_{++}$. Then $V = \{q \in \mathbb{R}^d \mid |q_l| < \rho_l,\ l = 1,\dots,d\}$ and (5.12) holds for $m(d\xi)$, and analogously for $\mu_i(d\xi)$.

6. Generalized Riccati equations. Let $(a, \alpha, b, \beta^{\mathcal{Y}}, \beta^{\mathcal{Z}}, c, \gamma, m, \mu)$ be admissible parameters, and let $F(u)$ and $R(u) = (R^{\mathcal{Y}}(u), R^{\mathcal{Z}}(u))$ be given by (2.16), (2.17) and (5.3). In this section we discuss the generalized Riccati equations

$$\partial_t \Phi(t, u) = F(\Psi(t, u)), \qquad \Phi(0, u) = 0, \tag{6.1}$$

$$\partial_t \Psi(t, u) = R(\Psi(t, u)), \qquad \Psi(0, u) = u. \tag{6.2}$$

Observe that (6.1) is a trivial differential equation. A solution of (6.1) and (6.2) is a pair of continuously differentiable mappings $\Phi(\cdot, u)$ and $\Psi(\cdot, u) = (\Psi^{\mathcal{Y}}(\cdot, u), \Psi^{\mathcal{Z}}(\cdot, u))$ from $\mathbb{R}_+$ into $\mathbb{C}$ and $\mathbb{C}^d$, respectively, satisfying (6.1) and (6.2) or, equivalently,

$$\Phi(t, u) = \int_0^t F(\Psi(s, u))\, ds, \tag{6.3}$$

$$\partial_t \Psi^{\mathcal{Y}}(t, u) = R^{\mathcal{Y}}\big(\Psi^{\mathcal{Y}}(t, u), e^{\beta^{\mathcal{Z}} t} w\big), \qquad \Psi^{\mathcal{Y}}(0, u) = v, \tag{6.4}$$

$$\Psi^{\mathcal{Z}}(t, u) = e^{\beta^{\mathcal{Z}} t} w, \tag{6.5}$$

for $u = (v, w) \in \mathbb{C}^d$. We shall see in Lemma 9.2 and Example 9.3 below that $R^{\mathcal{Y}}(u)$ may fail to be Lipschitz continuous at $u \in \partial \mathcal{U}$. The next proposition evades this difficulty.

PROPOSITION 6.1. *For every $u \in \mathcal{U}^0$ there exists a unique solution $\Phi(\cdot, u)$ and $\Psi(\cdot, u)$ of* (6.1) *and* (6.2) *with values in $\mathbb{C}_-$ and $\mathcal{U}^0$, respectively. Moreover, Φ and Ψ are continuous on $\mathbb{R}_+ \times \mathcal{U}^0$.*

PROOF. There is nothing to prove if $m = 0$. Hence suppose that $m \geq 1$.

Since (6.5) is decoupled from (6.4), we only have to focus on the latter equation. For every fixed $w \in i\mathbb{R}^n$, (6.4) should be regarded as an inhomogeneous ODE for $\Psi^{\mathcal{Y}}(\cdot, v, w)$, with $\Psi^{\mathcal{Y}}(0, v, w) = v$. Notice that the mapping

$$(t, v, w) \mapsto R^{\mathcal{Y}}\big(v, e^{\beta^{\mathcal{Z}} t} w\big) : \mathbb{R} \times \mathcal{U} \to \mathbb{C}^m \tag{6.6}$$

is analytic in $v \in \mathbb{C}^m_{--}$ with jointly, on $\mathbb{R} \times \mathcal{U}^0$, continuous v-derivatives; see Lemma 5.3. In particular, (6.6) is locally Lipschitz continuous in $v \in \mathbb{C}^m_{--}$, uniformly in (t, w) on compact sets. Therefore, for any $u = (v, w) \in \mathcal{U}^0$, there exists a unique $\mathbb{C}^m_{--}$-valued local solution $\Psi^{\mathcal{Y}}(t, u)$ to (6.4). Its maximal lifetime in $\mathbb{C}^m_{--}$ is

$$T_u := \liminf_{n \to \infty} \big\{ t \mid \|\Psi^{\mathcal{Y}}(t, u)\| \geq n \text{ or } \Psi^{\mathcal{Y}}(t, u) \in i\mathbb{R}^m \big\} \leq \infty.$$

We have to show that $T_u = \infty$.

An easy calculation yields

$$\begin{aligned} \operatorname{Re} R_i^{\mathcal{Y}}(u) &= \alpha_{i,ii} (\operatorname{Re} v_i)^2 - \langle \alpha_i \operatorname{Im} u, \operatorname{Im} u \rangle + \langle \beta_{i,\mathcal{I}}^{\mathcal{Y}}, \operatorname{Re} v \rangle - \gamma_i \\ &\quad + \int_{D \setminus \{0\}} \big(e^{\langle \operatorname{Re} v, \eta \rangle} \cos\langle \operatorname{Im} u, \xi \rangle - 1 - \operatorname{Re} v_i \chi_i(\xi) \big) \mu_i(d\xi). \end{aligned} \tag{6.7}$$

We recall definition (4.11) and write $Q_0(0) := Q(0) \setminus \{0\}$. From conditions (2.5), (5.1) and (6.7), we deduce

$$
\begin{aligned}
\operatorname{Re} R_i^{\mathcal{Y}}(u) &\le \alpha_{i,ii}(\operatorname{Re} v_i)^2 + \beta_{i,i}^{\mathcal{Y}} \operatorname{Re} v_i - \gamma_i \\
&\quad + \int_{D\setminus\{0\}} \big(e^{\langle \operatorname{Re} v, \eta\rangle} \cos\langle \operatorname{Im} u, \xi\rangle - e^{\operatorname{Re} v_i \eta_i}\big)\mu_i(d\xi) \\
&\quad + \int_{D\setminus\{0\}} \big(e^{\operatorname{Re} v_i \eta_i} - 1 - \operatorname{Re} v_i \chi_i(\xi)\big)\mu_i(d\xi) \\
&\le \alpha_{i,ii}(\operatorname{Re} v_i)^2 + \beta_{i,i}^{\mathcal{Y}} \operatorname{Re} v_i - \gamma_i \\
&\quad + (\operatorname{Re} v_i)^2 \int_{Q_0(0)} \bigg(\int_0^1 (1-t) e^{t \operatorname{Re} v_i \eta_i}\, dt\bigg) \eta_i^2\, \mu_i(d\xi) \\
&\quad - \mu_i\big(D \setminus Q(0)\big) \operatorname{Re} v_i \\
&\le C_i\big((\operatorname{Re} v_i)^2 - \operatorname{Re} v_i\big) - \gamma_i,
\end{aligned}
\tag{6.8}
$$

where $C_i \ge 0$ does not depend on u. The first decomposition of the integral is justified and the second inequality in (6.8) follows since

$$
I(u,\xi) := e^{\langle \operatorname{Re} v, \eta\rangle} \cos\langle \operatorname{Im} u, \xi\rangle - e^{\operatorname{Re} v_i \eta_i} \le 0 \qquad \forall \xi \in D \setminus \{0\}
$$

and

$$
\begin{aligned}
|I(u,\xi)| &\le \big|e^{\langle \operatorname{Re} v_{\mathcal{I}(i)}, \eta_{\mathcal{I}(i)}\rangle} - 1\big| + |\cos\langle \operatorname{Im} u, \xi\rangle - 1| \\
&\le C\big(|\operatorname{Re} v_{\mathcal{I}(i)}||\eta_{\mathcal{I}(i)}| + |\langle \operatorname{Im} u, \xi\rangle|^2\big)
\end{aligned}
$$

for $\|\xi\|$ small enough, for some C; see condition (2.11). Hence we have shown that $\operatorname{Re} \Psi_i^{\mathcal{Y}}(t,u)$ satisfies the differential inequality, for $t \in (0, T_u)$,

$$
\begin{aligned}
&\partial_t \operatorname{Re} \Psi_i^{\mathcal{Y}}(t,u) \le C_i\big((\operatorname{Re} \Psi_i^{\mathcal{Y}}(t,u))^2 - \operatorname{Re} \Psi_i^{\mathcal{Y}}(t,u)\big) - \gamma_i, \\
&\operatorname{Re} \Psi_i^{\mathcal{Y}}(0,u) = \operatorname{Re} v_i.
\end{aligned}
\tag{6.9}
$$

A comparison theorem (see [12]) and condition (2.9) yield

$$
\operatorname{Re} \Psi_i^{\mathcal{Y}}(t,u) \le g_i(t,u) \qquad \forall t \in [0, T_u),
\tag{6.10}
$$

where

$$
\begin{aligned}
&\partial_t g_i(t,u) = C_i\big((g_i(t,u))^2 - g_i(t,u)\big), \\
&g_i(0,u) = \operatorname{Re} v_i \qquad (<0).
\end{aligned}
\tag{6.11}
$$

However, $-\infty < g_i(t,u) < 0$ for all $t \in \mathbb{R}_+$. Thus $\Psi^{\mathcal{Y}}(t,u)$ never hits $i\mathbb{R}^m$ and

$$
T_u = \liminf_{n\to\infty} \big\{t \mid \|\Psi^{\mathcal{Y}}(t,u)\| \ge n\big\}.
$$

It remains to derive an upper bound for $\|\Psi^{\mathcal{Y}}(t,u)\|$. For every $t \in (0, T_u)$, we have

$$\partial_t \|\Psi^{\mathcal{Y}}(t,u)\|^2 = 2\operatorname{Re}\big\langle \overline{\Psi^{\mathcal{Y}}(t,u)}, R^{\mathcal{Y}}\big(\Psi^{\mathcal{Y}}(t,u), e^{\beta^{\mathcal{Z}} t} w\big)\big\rangle. \tag{6.12}$$

A calculation shows that

$$\begin{aligned} \operatorname{Re}\big(\overline{v}_i R_i^{\mathcal{Y}}(u)\big) &= \alpha_{i,ii}|v_i|^2 \operatorname{Re} v_i + K(u) \\ &\quad + \operatorname{Re}\bigg(\overline{v}_i \int_{D\setminus\{0\}} \big(e^{\langle u,\xi\rangle} - 1 - \langle u_{\mathcal{J}(i)}, \chi_{\mathcal{J}(i)}(\xi)\rangle\big)\mu_i(d\xi)\bigg), \end{aligned} \tag{6.13}$$

where

$$K(u) := \operatorname{Re} v_i \langle \alpha_{i,\mathcal{J}\mathcal{J}} w, w\rangle + \operatorname{Re}\big(\overline{v}_i(\langle \beta_i^{\mathcal{Y}}, u\rangle - \gamma_i)\big).$$

Hence

$$|K(u)| \le C\big(\|v\|\|w\|^2 + \|v\|^2 + \|v\|\|w\| + \|v\|\big) \qquad \forall\, u = (v,w) \in \mathcal{U}. \tag{6.14}$$

The first term on the right-hand side in (6.13) which could cause growth problems has the right sign. Hence, combining (6.13) and (6.14) with Lemma 6.2 below, we derive

$$\operatorname{Re}\big(\overline{v}_i R_i^{\mathcal{Y}}(u)\big) \le C(1 + \|w\|^2)(1 + \|v\|^2) \qquad \forall\, u = (v,w) \in \mathcal{U}. \tag{6.15}$$

We insert (6.15) in (6.12) and obtain

$$\partial_t \|\Psi^{\mathcal{Y}}(t,u)\|^2 \le C\big(1 + \|e^{\beta^{\mathcal{Z}} t} w\|^2\big)\big(1 + \|\Psi^{\mathcal{Y}}(t,u)\|^2\big) \qquad \forall\, t \in (0, T_u).$$

Gronwall's inequality (see [1]) yields

$$\begin{aligned} \|\Psi^{\mathcal{Y}}(t,u)\|^2 &\le \bigg(\|v\|^2 + C\int_0^t \big(1 + \|e^{\beta^{\mathcal{Z}} s} w\|^2\big)\,ds\bigg) \\ &\quad \times \exp\bigg(C\int_0^t \big(1 + \|e^{\beta^{\mathcal{Z}} s} w\|^2\big)\,ds\bigg) \qquad \forall\, t \in [0, T_u). \end{aligned} \tag{6.16}$$

Thus the solution cannot explode and we have $T_u = \infty$.

The continuity of Φ and Ψ on $\mathbb{R}_+ \times \mathcal{U}^0$ is a standard result; see [12], Chapter 6. $\square$

LEMMA 6.2. *For every $i \in \mathcal{I}$ and $u = (v,w) \in \mathcal{U}$, we have*

$$\begin{aligned} &\operatorname{Re}\bigg(\overline{v}_i \int_{D\setminus\{0\}} \big(e^{\langle u,\xi\rangle} - 1 - \langle u_{\mathcal{J}(i)}, \chi_{\mathcal{J}(i)}(\xi)\rangle\big)\mu_i(d\xi)\bigg) \\ &\qquad \le C(1 + \|v\|^2)(1 + \|w\|^2), \end{aligned} \tag{6.17}$$

where C only depends on μ_i.

PROOF. Let $i \in \mathcal{I}$. We recapture the notation of Steps 1 and 2 in the proof of Lemma 4.1. Write $I = \mathcal{I}(i)$ and $J = \mathcal{J}(i)$, and let

$$d(\xi) := d(0,\xi) = \langle \chi_I(\xi), \mathbf{1}\rangle + \|\chi_J(\xi)\|^2,$$
$$h_u(\xi) := h_u(0,\xi) = \frac{f_u(\xi) - 1 - \langle u_J, \chi_J(\xi)\rangle}{d(\xi)}$$

[see (4.7) and (4.8)]. By condition (2.11), $\overline{\mu}_i(d\xi) := d(\xi)\mu_i(d\xi)$ is a bounded measure on $D \setminus \{0\}$. Now the integral in (6.17) can be written as

$$\int_{D\setminus\{0\}} h_u(\xi)\,\overline{\mu}_i(d\xi). \tag{6.18}$$

We proceed as in (4.12), and perform a convenient Taylor expansion,

$$\begin{aligned} h_u(\xi) &= \frac{1}{d(\xi)}\big(e^{\langle u,\xi\rangle} - e^{\langle u_J,\xi_J\rangle} + e^{v_i\eta_i}(e^{\langle w,\zeta\rangle} - 1 - \langle w,\zeta\rangle) \\ &\qquad + \langle w,\zeta\rangle(e^{v_i\eta_i} - 1) + e^{v_i\eta_i} - 1 - v_i\eta_i\big) \\ &= e^{\langle u_J,\xi_J\rangle}\bigg(\int_0^1 e^{t\langle u_I,\xi_I\rangle}\,dt\bigg)\langle u_I, \mathbf{w}(\xi)\rangle \\ &\quad + e^{v_i\eta_i}\bigg(\int_0^1 (1-t)e^{t\langle w,\zeta\rangle}\,dt\bigg)\langle \mathbf{a}_{\mathcal{J}\mathcal{J}}(\xi)w, w\rangle \\ &\quad + \bigg(\int_0^1 e^{tv_i\eta_i}\,dt\bigg)v_i\langle \mathbf{a}_{i\mathcal{J}}(\xi), w\rangle \\ &\quad + \bigg(\int_0^1 (1-t)e^{tv_i\eta_i}\,dt\bigg)\mathbf{a}_{ii}(\xi)(v_i)^2 \qquad \forall\,\xi = (\eta,\zeta) \in Q_0(0), \end{aligned} \tag{6.19}$$

where we have set $\mathbf{w}(\xi) := \mathbf{w}(0,\xi)$ and $\mathbf{a}_{kl}(\xi) := \mathbf{a}_{kl}(0,\xi)$; see (4.13) and (4.14). Now we compute

$$\operatorname{Re}\big(\overline{v}_i h_u(\xi)\big) = K(u,\xi) + L(v_i,\eta_i)\mathbf{a}_{ii}(\xi)|v_i|^2, \qquad \xi = (\eta,\zeta) \in Q_0(0), \tag{6.20}$$

where we have set

$$\begin{aligned} L(v_i,\eta_i) &:= \int_0^1 (1-t)\operatorname{Re}(v_i e^{tv_i\eta_i})\,dt \\ &= \int_0^1 (1-t)e^{t\operatorname{Re} v_i\eta_i}\big(\operatorname{Re} v_i\cos(t\operatorname{Im} v_i\eta_i) \\ &\qquad - \operatorname{Im} v_i\sin(t\operatorname{Im} v_i\eta_i)\big)\,dt \end{aligned} \tag{6.21}$$

and $K(u,\xi)$ satisfies, in view of (4.16),

$$\begin{aligned} |K(u,\xi)| &\le C\big(\|v\| + \|w\|^2 + \|v\|^2\|w\|\big) \\ &\qquad \forall\, u = (v,w) \in \mathcal{U},\ \forall\,\xi \in Q_0(0). \end{aligned} \tag{6.22}$$

We claim that

$$L(v_i, \eta_i) \le 0 \qquad \forall\, v_i \in \mathbb{C}_-,\ \forall\, \eta_i \in [0, 1]. \tag{6.23}$$

Indeed, since $L(v_i, \eta_i)$ is symmetric in $\operatorname{Im} v_i$, we may assume that $\operatorname{Im} v_i$ in (6.21) is non-negative. Now (6.23) follows by Lemma 6.3 below.

On the other hand we have, by inspection,

$$\begin{aligned} |v_i h_u(\xi)| &\le C(1 + \|v\|^2 + \|v\|\|w\|) \\ &\qquad\qquad \forall\, u = (v, w) \in \mathcal{U},\ \forall\, \xi \in D \setminus Q(0). \end{aligned} \tag{6.24}$$

Combining (6.20) and (6.22)–(6.24), we finally derive

$$\int_{D\setminus\{0\}} \operatorname{Re}\big(\overline{v}_i h_u(\xi)\big)\, \overline{\mu}_i(d\xi) \le C(1 + \|v\|^2)(1 + \|w\|^2) \qquad \forall\, u = (v, w) \in \mathcal{U},$$

which is (6.17). □

LEMMA 6.3. *For all $p, q \in \mathbb{R}_+$, we have*

$$\int_0^1 (1 - t)e^{-pt} \cos(qt)\, dt \ge 0, \tag{6.25}$$

$$\int_0^1 (1 - t)e^{-pt} \sin(qt)\, dt \ge 0. \tag{6.26}$$

PROOF. Estimate (6.26) is trivial, since $\int_0^s f(t) \sin(t)\, dt \ge 0$ for all $t \in \mathbb{R}_+$, for any non-negative nonincreasing function f. Similarly, it is easy to see that (6.25) holds for all $q \in [0, \pi]$. It remains to prove (6.25) for $q > \pi$. Straightforward verification shows that

$$\begin{aligned} &\int_0^s (1 - t)e^{-pt} \cos(qt)\, dt \\ &\quad = \frac{e^{-ps}}{(p^2 + q^2)^2}\big(e^{ps}\big(p^2(p - 1) + q^2 + pq^2\big) \\ &\qquad\qquad + \big((s - 1)(p^3 + pq^2) + p^2 - q^2\big)\cos(qs) \\ &\qquad\qquad\quad + \big((1 - s)(p^2 + q^2) - 2p\big)q \sin(qs)\big). \end{aligned}$$

Now the claim follows by taking into account that

$$\begin{aligned} e^p\big(p^2(p - 1) + q^2\big) &\ge |p^2 - q^2|, \\ e^p pq^2 &\ge 2pq \end{aligned}$$

for all $p \in \mathbb{R}_+$ and $q > \pi$.

A more elegant proof of (6.25) is based on Pólya's criterion (see [44]), which implies that $(1 - |t|)^+ e^{-p|t|}$ is the characteristic function of an absolutely continuous, symmetric probability distribution on $\mathbb{R}$. Now, (6.25) follows immediately by Fourier inversion. □

Now let X be regular affine. Equations (3.7), (3.13) and (3.14) and Proposition 5.2 suggest that $\phi(t,u)$ and $\psi(t,u)$ solve the generalized Riccati equations (6.1) and (6.2).

PROPOSITION 6.4. *There exists a unique continuous extension of $\phi(t,u)$ and $\psi(t,u)$ to $\mathbb{R}_+ \times \mathcal{U}$ such that* (2.2) *holds for all $(t,u) \in \mathbb{R}_+ \times \mathcal{U}$. Moreover, $\phi(\cdot,u)$ and $\psi(\cdot,u)$ solve* (6.1) *and* (6.2) *for all $u \in \mathcal{U}$.*

PROOF. Let $u \in \mathcal{U}^0$ and define $t^* := \sup\{t \mid (t,u) \in \mathcal{O}\}$. From the definition of $\mathcal{O}$ [see (3.1)] we infer $\lim_{t \uparrow t^*} |\phi(t,u)| = \infty$. Every continuous function with continuous right-hand derivatives on $[0,t^*)$ is continuously differentiable on $[0,t^*)$. Therefore and by (3.7), (3.13) and (3.14) and Proposition 6.1, the equalities

$$\phi(t,u) = \Phi(t,u), \qquad \psi(t,u) = \Psi(t,u) \tag{6.27}$$

hold for all $t \in [0,t^*)$. But $|\Phi(t,u)|$ is finite for all finite t. Hence $t^* = \infty$. This way, and since $\mathbb{R}_+ \times \partial\mathcal{U} \subset \mathcal{O}$, we see that $\mathcal{O} = \mathbb{R}_+ \times \mathcal{U}$ and (6.27) holds for all $(t,u) \in \mathbb{R}_+ \times \mathcal{U}^0$. Now $\phi(t,u)$ and $\psi(t,u)$ are jointly continuous on $\mathbb{R}_+ \times \mathcal{U}$. By a limiting argument it follows that $\phi(\cdot,u)$ and $\psi(\cdot,u)$ solve (6.1) and (6.2) also for $u \in \partial\mathcal{U}$. $\square$

Next we recall some well-known conditions for higher regularity of the solution Φ and Ψ of (6.1) and (6.2). These results are strongly connected to the existence of the kth and exponential moments of X_t, as shown in Theorem 2.16. See also Lemma 5.3.

LEMMA 6.5. (i) *Let $k \in \mathbb{N}$. If F and $R_i^{\mathcal{Y}}$ are in $C^k(\mathcal{U})$ for all $i \in \mathcal{I}$, then Φ and $\Psi_i^{\mathcal{Y}}$ are in $C^k(\mathbb{R}_+ \times \mathcal{U})$ for all $i \in \mathcal{I}$.*

(ii) *Suppose that F and $R^{\mathcal{Y}}$ are analytic on some open set U in $\mathbb{C}^d$. Let $T \le \infty$ such that, for every $u \in U$, there exists a U-valued local solution $\Psi(t,u)$ of* (6.1) *for $t \in [0,T)$. Then Φ and Ψ have a unique analytic extension on $(0,T) \times U$.*

PROOF. (i) It follows from [33], Theorem 10.8.2, that Φ and $\Psi_i^{\mathcal{Y}}$ are in $C^k(\mathbb{R}_+ \times \mathcal{U}^0)$. Moreover, for $1 \le l \le m$ and $u \in \mathcal{U}^0$, the mapping $g_u(t) := \partial_{u_l}\Psi^{\mathcal{Y}}(t,u)$ solves the linear equation

$$g_u(t) = \int_0^t D_v R^{\mathcal{Y}}\big(\Psi^{\mathcal{Y}}(s,u), e^{\beta^{\mathcal{Z}} s} w\big) g_u(s)\, ds, \qquad t \in \mathbb{R}_+,$$

where $D_v R^{\mathcal{Y}}(v,w)$ denotes the derivative of the mapping $v \mapsto R^{\mathcal{Y}}(v,w)$, which is continuous on $\mathcal{U}$ by assumption. In view of Proposition 6.4, the mapping $u \mapsto g_u(t)$ can therefore be continuously extended to $\mathcal{U}$. A similar argument works for $m < l \le d$, and induction for higher derivatives yields the assertion for $\Psi^{\mathcal{Y}}$. We conclude by (6.3).

(ii) This follows from a classical approximation argument and a well-known result on convergent sequences of analytic functions; see [33], Theorem 10.8.2. $\square$

7. $\mathcal{C} \times \mathcal{C}^{(m,n)}$-semiflows. The following concepts are the tools for proving the existence of regular affine processes. We denote by $\mathcal{C}$ the convex cone of continuous functions $\phi : \mathcal{U} \to \mathbb{C}_+$ of the form

$$\phi(u) = \langle Aw, w\rangle + \langle B, u\rangle - C + \int_{D\setminus\{0\}} \big(e^{\langle u,\xi\rangle} - 1 - \langle w, \chi_{\mathcal{J}}(\xi)\rangle\big) M(d\xi), \tag{7.1}$$

for $u = (v, w) \in \mathcal{U}$, where $A \in \mathrm{Sem}^n$, $B \in D$, $C \in \mathbb{R}_+$ and $M(d\xi)$ is a non-negative Borel measure on $D \setminus \{0\}$ integrating $\langle \chi_{\mathcal{I}}(\xi), \mathbf{1}\rangle + \|\chi_{\mathcal{J}}(\xi)\|^2$.

The following result is classical.

LEMMA 7.1. *There exists a unique and infinitely divisible substochastic measure μ on D such that*

$$\int_D f_u \, d\mu = e^{\phi(u)} \qquad \forall u \in \mathcal{U}, \tag{7.2}$$

if and only if $\phi \in \mathcal{C}$. Moreover, the representation (7.1) *of $\phi(u)$ by A, B, C and M is unique.*

PROOF. Consider first only $u \in \partial\mathcal{U}$. Then the lemma is essentially the Lévy–Khintchine representation theorem for infinitely divisible distributions on $\mathbb{R}^d$ (see [79], Theorem 8.1). Note that $\mu(\mathbb{R}^d) = \mu(D) = e^{-C}$. The special properties of the parameters follow by [79], Proposition 11.10 and Theorem 24.7. Analytic extension by the Schwarz reflexion principle yields the validity of (7.2) for all $u \in \mathcal{U}$. $\square$

Define the convex cone $\mathcal{C}^{(m,n)} \subset \mathcal{C}^m \times \mathcal{C}^n$ of mappings $\psi : \mathcal{U} \to \mathcal{U}$ by

$$\mathcal{C}^{(m,n)} := \big\{\psi = (\psi^{\mathcal{Y}}, \psi^{\mathcal{Z}}) \mid \psi^{\mathcal{Y}} \in \mathcal{C}^m \text{ and } \psi^{\mathcal{Z}}(v, w) = Bw, \text{ for some } B \in \mathbb{R}^{n\times n}\big\}.$$

Here are some elementary properties of $\mathcal{C}$ and $\mathcal{C}^{(m,n)}$.

PROPOSITION 7.2. *Let $\phi, \phi_k \in \mathcal{C}$ and $\psi, \psi_k \in \mathcal{C}^{(m,n)}$, $k \in \mathbb{N}$.*

(i) *For every $x \in D$ there exists a unique and infinitely divisible substochastic measure $\mu(x, d\xi)$ on D such that*

$$\int_D f_u(\xi)\, \mu(x, d\xi) = e^{\langle \psi(u), x\rangle} \qquad \forall u \in \mathcal{U}.$$

(ii) *The composition $\phi \circ \psi$ is in $\mathcal{C}$.*

(iii) *The composition $\psi_1 \circ \psi$ is in $\mathcal{C}^{(m,n)}$.*

(iv) *If ϕ_k converges pointwise to a continuous function ϕ^* on $\mathcal{U}^0$, then ϕ^* has a continuous extension on $\mathcal{U}$ and $\phi^* \in \mathcal{C}$.*

(v) *If ψ_k converges pointwise to a continuous mapping ψ^* on $\mathcal{U}^0$, then ψ^* has a continuous extension on $\mathcal{U}$ and $\psi^* \in \mathcal{C}^{(m,n)}$.*

PROOF. (i) Since $\langle \psi, x\rangle \in \mathcal{C}$, for all $x \in D$, the claim follows from Lemma 7.1.

(ii) By Lemma 7.1 and part (i) there exist two infinitely divisible substochastic measures m and $\mu(x,\cdot)$ on D specified by

$$\int_D f_u(\xi)\, m(d\xi) = e^{\phi(u)}, \qquad \int_D f_u(\xi)\, \mu(x, d\xi) = e^{\langle \psi(u), x\rangle}.$$

Set $\nu(d\xi) = \int_D m(dx)\mu(x, d\xi)$, which is a substochastic measure on D. For $k \in \mathbb{N}$, denote by $m^{(k)}$ the kth root of m, that is, $m = m^{(k)} * \cdots * m^{(k)}$ (k times). Then we have

$$\int_D f_u(\xi) \int_D m^{(k)}(dx)\, \mu(x, d\xi) = e^{(1/k)\phi(\psi(u))} \qquad \forall u \in \mathcal{U}.$$

Hence ν is infinitely divisible with $\nu^{(k)} = \int_D m^{(k)}(dx)\, \mu(x, d\xi)$. Now the claim follows by Lemma 7.1.

(iii) It is immediate by part (ii).

(iv) Let $\lambda \in \mathbb{R}^m_{++}$ and write

$$\mathcal{U}(-\lambda) := \{(v, w) \in \mathcal{U} \mid \operatorname{Re} v = -\lambda\} \subset \mathcal{U}^0.$$

By (7.2) there corresponds a unique infinitely divisible measure, say μ_k, to ϕ_k. With a slight abuse of notation,

$$\mathcal{U}(-\lambda) \ni u \mapsto e^{\phi_k(u)},$$

is the characteristic function of the measure $e^{-\langle \lambda, \eta\rangle}\mu_k(d\eta, d\zeta)$. Since $\phi_k \to \phi^*$ pointwise on $\mathcal{U}(-\lambda)$, the continuity theorem of Lévy (see, e.g., [44]) implies that $e^{-\langle \lambda, \eta\rangle}\mu_k(d\eta, d\zeta) \to \mu^*(d\eta, d\zeta)$ weakly on D, for some substochastic measure μ^*. On the other hand, there exists a subsequence μ_{k_l} which converges vaguely on D to a substochastic measure μ. By uniqueness of the vague limit we conclude that $\mu^*(d\eta, d\zeta) = e^{-\langle \lambda, \eta\rangle}\mu(d\eta, d\zeta)$ and the entire sequence (μ_k) converges weakly to μ. Hence μ is infinitely divisible and

$$\int_D f_u\, d\mu = e^{\phi^*(u)} \qquad \forall u \in \mathcal{U}^0.$$

Now Lemma 7.1 yields the assertion.

(v) It is immediate by parts (i) and (iv). □

In view of Proposition 7.2(ii) and (iii), the next definition makes sense.

DEFINITION 7.3. A one-parameter family $\{(\phi_t, \psi_t)\}_{t\in\mathbb{R}_+}$ of elements in $\mathcal{C} \times \mathcal{C}^{(m,n)}$ is called a $\mathcal{C} \times \mathcal{C}^{(m,n)}$-*semiflow* if, for all $t, s \in \mathbb{R}_+$ and $u \in \mathcal{U}$,

$$\phi_{t+s}(u) = \phi_t(u) + \phi_s(\psi_t(u)) \quad \text{and} \quad \phi_0 = 0,$$
$$\psi_{t+s}(u) = \psi_t(\psi_s(u)) \quad \text{and} \quad \psi_0(u) = u.$$

It is called a *regular* $\mathcal{C} \times \mathcal{C}^{(m,n)}$-*semiflow* if $\phi_t(u)$ and $\psi_t(u)$ are continuous in $t \in \mathbb{R}_+$, and the right-hand derivatives $\partial_t^+\phi_t(u)|_{t=0}$ and $\partial_t^+\psi_t(u)|_{t=0}$ exist for every $u \in \mathcal{U}$ and are continuous at $u = 0$.

Here is the link to regular affine processes and the generalized Riccati equations.

PROPOSITION 7.4. (i) *Suppose* $\{(\phi_t, \psi_t)\}_{t\in\mathbb{R}_+}$ *is a regular* $\mathcal{C} \times \mathcal{C}^{(m,n)}$-*semiflow. Then there exists a unique regular affine Markov process with state-space* D *and exponents* $\phi(t,u) = \phi_t(u)$ *and* $\psi(t,u) = \psi_t(u)$ [*which in turn solve the corresponding generalized Riccati equations* (6.1) *and* (6.2); *see Proposition* 6.4].

(ii) *Conversely, the solution* Φ *and* Ψ *of* (6.1) *and* (6.2) *uniquely defines a regular* $\mathcal{C} \times \mathcal{C}^{(m,n)}$-*semiflow* $\{(\phi_t, \psi_t)\}_{t\in\mathbb{R}_+}$ *by* $\phi_t = \Phi(t,\cdot)$ *and* $\psi_t = \Psi(t,\cdot)$.

PROOF. By Lemma 7.1 and Proposition 7.2(i) there exists, for every $(t,x) \in \mathbb{R}_+ \times D$, a unique, infinitely divisible substochastic measure $p_t(x,\cdot)$ on D with

$$\int_D f_u(\xi)\, p_t(x, d\xi) = e^{\phi(t,u)+\langle\psi(t,u),x\rangle} \qquad \forall u \in \mathcal{U}.$$

A simple calculation shows that $p_t(x,\cdot)$ satisfies the Chapman–Kolmogorov equation

$$p_{t+s}(x,\cdot) = \int_D p_t(x, d\xi) p_s(\xi,\cdot) \qquad \forall t,s \in \mathbb{R}_+,\ \forall x \in D.$$

Hence $p_t(x,\cdot)$ is the transition function of a Markov process on D, which by construction is regular affine. This proves (i).

For part (ii), we first suppose that

$$\int_{D\setminus\{0\}} \chi_i(\xi)\, \mu_i(d\xi) < \infty, \tag{7.3}$$

$$\alpha_{i,ik} = \alpha_{i,ki} = 0 \qquad \forall k \in \mathcal{J}(i), \tag{7.4}$$

for all $i \in \mathcal{I}$. Consequently, $R_i^{\mathcal{Y}}$ can be written in the form

$$R_i^{\mathcal{Y}}(u) = \tilde{R}_i^{\mathcal{Y}}(u) - c_i v_i \qquad \text{with } \tilde{R}_i^{\mathcal{Y}} \in \mathcal{C},\ c_i \in \mathbb{R}_+,\ i \in \mathcal{I}.$$

Then (6.4) is equivalent to the following integral equations:

$$\Psi_i^{\mathcal{Y}}(t,u) = e^{-c_i t} v_i + \int_0^t e^{-c_i(t-s)} \tilde{R}_i^{\mathcal{Y}}(\Psi(s,u))\, ds, \qquad i \in \mathcal{I}.$$

By a classical fixed point argument, the solution $\Psi_i^{\mathcal{Y}}(t,u)$ is the pointwise limit of the sequence $(\Psi_i^{\mathcal{Y},(k)}(t,u))_{k\in\mathbb{N}_0}$, for $(t,u) \in \mathbb{R}_+ \times \mathcal{U}^0$, obtained by the iteration

$$\Psi_i^{\mathcal{Y},(0)}(t,u) = v_i,$$

$$\Psi_i^{\mathcal{Y},(k+1)}(t,u) = e^{-c_i t} v_i + \int_0^t e^{-c_i(t-s)} \tilde{R}_i^{\mathcal{Y}}\big(\Psi^{\mathcal{Y},(k)}(s,u), e^{\beta^{\mathcal{Z}} s} w\big)\, ds.$$

Proposition 7.2(ii) and the convex cone property of $\mathcal{C}$ yield $\Psi_i^{\mathcal{Y},(k)}(t,\cdot) \in \mathcal{C}$ for all $k \in \mathbb{N}_0$. In view of Proposition 7.2(iv) there exists a unique continuous extension

of $\Psi_i^{\mathcal{Y}}$ on $\mathbb{R}_+ \times \mathcal{U}$, and $\Psi_i^{\mathcal{Y}}(t,\cdot) \in \mathcal{C}$. Hence $\Psi(t,\cdot) \in \mathcal{C}^{(m,n)}$. Since $F \in \mathcal{C}$, by Proposition 7.2(ii) also $\Phi(t,\cdot) = \int_0^t F(\Psi(s,\cdot))\,ds \in \mathcal{C}$ and the proposition is proved if (7.3) and (7.4) hold. For the general case it is enough to notice that the solution of (6.4) depends continuously on the right-hand side of (6.4) with respect to uniform convergence on compacts. Now Lemma 7.5 below completes the proof. $\square$

LEMMA 7.5. *Let $i \in \mathcal{I}$. There exists a sequence of functions $(g_k)_{k\in\mathbb{N}}$ which converges uniformly on compacts to $R_i^{\mathcal{Y}}$. Moreover, every g_k is of the form* (2.17) *and satisfies* (7.3) *and* (7.4).

PROOF. We use the same notation as in the proof of Lemma 6.2. Introduce the restricted measures

$$\mu_i^{(k)} := \mu_i|_{\{\xi\in D \mid \|\xi\|>(1/k)\}}, \qquad k \in \mathbb{N},$$

and denote by $\tilde{g}_k$ the corresponding map given by (2.17) with μ_i replaced by $\mu_i^{(k)}$. It is easy to see that every $\mu_i^{(k)}$ satisfies (7.3) and that the sequence of bounded measures $\overline{\mu}_i^{(k)}(d\xi) := d(\xi)\,\mu_i^{(k)}(d\xi)$, $k \in \mathbb{N}$, converges weakly to $\overline{\mu}_i(d\xi) := d(\xi)\,\mu_i(d\xi)$ on $D \setminus \{0\}$. Write

$$f^{(k)}(u) := \int_{D\setminus\{0\}} h_u(\xi)\,\overline{\mu}_i^{(k)}(d\xi), \qquad k \in \mathbb{N}.$$

Let $\varepsilon > 0$ and K a compact subset of D. By (4.12) there exists $\delta > 0$ such that for all $u, u' \in \mathcal{U}$ with $\|u - u'\| < \delta$, we have

$$\sup_{\xi\in K} |h_u(\xi) - h_{u'}(\xi)| \le \varepsilon. \tag{7.5}$$

As in the proof of (3.2) one now can show that

$$f^{(k)}(u) \to f(u) := \int_{D\setminus\{0\}} h_u(\xi)\,\overline{\mu}_i(d\xi) \qquad \text{for } k \to \infty,$$

uniformly in u on compacts. By construction, it follows that $\tilde{g}_k$ converges uniformly on compacts to $R_i^{\mathcal{Y}}$.

It remains to show that, for all $k \in \mathbb{N}$, there exists a sequence $(\tilde{g}_k^{(l)})_{l\in\mathbb{N}}$ of functions that are of the form (2.17) and satisfy (7.3) and (7.4), such that $\tilde{g}_k^{(l)} \to \tilde{g}_k$ uniformly on compacts. The lemma is then proved by choosing an appropriate subsequence $\tilde{g}_k^{(l_k)} =: g_k$, $k \in \mathbb{N}$.

To simplify the notation we suppress the index k in what follows and assume that μ_i already satisfies (7.3). We recapture some of the analysis from Step 2 in the proof of Lemma 4.1 [see (4.12)–(4.18)]. Instead of expanding the integrand in (6.18) we decompose the integral

$$\int_{D\setminus\{0\}} h_u(\xi)\,\overline{\mu}_i(d\xi) = \int_\Lambda \tilde{h}_u\,d\tilde{\mu}_i + \int_{D\setminus Q(0)} h_u(\xi)\,\overline{\mu}_i(d\xi), \tag{7.6}$$

where $\tilde{h}_u := h_u \circ \Gamma^{-1}$, $\tilde{\mu}_i$ is the image of $\overline{\mu}_i$ by Γ, $\Lambda := \Gamma(Q_0(0))$ and $\Gamma(\xi) := \Gamma(0, \xi)$ is given by (4.16). See also (4.18). Recall that $I = \mathcal{I}(i)$, $J = \mathcal{J}(i)$. We can assume that $\alpha_{i,ii} > 0$ since otherwise (7.4) is already satisfied. Hence $\xi^* \in Q_0(0)$, defined by

$$\xi_I^* := 0, \qquad \xi_J^* := \frac{\alpha_{i,iJ}}{\|\alpha_{i,iJ}\|},$$

satisfies

$$d(\xi^*) = 1 \quad \text{and} \quad \xi_i^* \xi_J^* = p^* \alpha_{i,iJ} \qquad \text{for } p^* := \frac{\alpha_{i,ii}}{\|\alpha_{i,iJ}\|^2}.$$

This implies that

$$\alpha_{i,ik} = \frac{1}{p^*} \mathbf{a}_{ik}(\xi^*) \qquad \forall k \in J,$$

where $\mathbf{a}_{kl}(\xi^*) = \mathbf{a}_{kl}(0, \xi^*)$ is defined in (4.14). Since

$$\mathbf{a}_{kl}(r\xi^*) = \mathbf{a}_{kl}(\xi^*) =: A_{kl} \qquad \forall r > 0,\ k, l \in J,$$

we get that

$$\lambda_l := \Gamma\left(\frac{\xi^*}{l}\right) = \left(\frac{\xi^*}{l}, 0, A\right) \quad \longrightarrow \quad \lambda_\infty := (0, 0, A) \qquad \text{in } \overline{\Lambda},\ \text{for } l \to \infty,$$

where in fact $\lambda_\infty \in \overline{\Lambda} \setminus \Lambda$. As seen from (4.17), we therefore can write

$$\tfrac{1}{2}\langle A u_J, u_J \rangle = \tilde{h}_u(\lambda_\infty).$$

On the other hand, by construction, we have

$$\langle \alpha_{i,JJ} u_J, u_J \rangle = \frac{1}{p^*} \langle A u_J, u_J \rangle + \left\langle \left(\alpha_{i,\mathcal{J}\mathcal{J}} - \frac{1}{p^*} A_{\mathcal{J}\mathcal{J}} \right) u_{\mathcal{J}}, u_{\mathcal{J}} \right\rangle, \qquad u \in \mathbb{C}^d.$$

We claim that

$$\tilde{\alpha} := \alpha_{i,\mathcal{J}\mathcal{J}} - \frac{1}{p^*} A_{\mathcal{J}\mathcal{J}} \in \mathrm{Sem}^n. \tag{7.7}$$

Indeed, by construction, we have

$$\tilde{\alpha}_{kl} = \alpha_{i,kl} - \frac{1}{p^*} A_{kl} = \alpha_{i,kl} - \frac{\alpha_{i,ki}}{\alpha_{i,ii}} \alpha_{i,il}, \qquad k, l \in \mathcal{J},$$

which simply is Gaussian elimination with Pivot element $\alpha_{i,ii}$. Now let $q \in \mathbb{R}^n$. Then

$$\begin{aligned}
\langle \tilde{\alpha} q, q \rangle &= \langle \alpha_{i,\mathcal{J}\mathcal{J}} q, q \rangle - \frac{(\sum_{k \in \mathcal{J}} \alpha_{i,ik} q_k)^2}{\alpha_{i,ii}} = \langle \alpha_{i,\mathcal{J}\mathcal{J}} q, q \rangle - \frac{\langle \alpha_{i,JJ} e_1, \tilde{q} \rangle^2}{\langle \alpha_{i,JJ} e_1, e_1 \rangle} \\
&\geq \langle \alpha_{i,\mathcal{J}\mathcal{J}} q, q \rangle - \frac{\langle \alpha_{i,JJ} e_1, e_1 \rangle \langle \alpha_{i,JJ} \tilde{q}, \tilde{q} \rangle}{\langle \alpha_{i,JJ} e_1, e_1 \rangle} = 0,
\end{aligned}$$

by the Cauchy–Schwarz inequality, where $e_1 = (1, 0, \dots, 0)$, $\tilde{q} = (0, q) \in \mathbb{R}^{n+1}$. Whence (7.7).

We now define the measure on $\overline{\Lambda}$,

$$\tilde{\mu}_i^{(\infty)} := \tilde{\mu}_i + \frac{2}{p^*}\delta_{\lambda_\infty} \qquad (\delta_{\lambda_\infty} \text{ denotes the Dirac measure in } \lambda_\infty);$$

then we can write

$$R_i^{\mathcal{Y}}(u) = \int_{\overline{\Lambda}} \tilde{h}_u \, d\tilde{\mu}_i^{(\infty)} + \langle \tilde{\alpha} u_{\mathcal{J}}, u_{\mathcal{J}} \rangle + \int_{D \setminus Q(0)} h_u(\xi) \, \overline{\mu}_i(d\xi) + \langle \beta_i^{\mathcal{Y}}, u \rangle - \gamma_i.$$

That is, the relevant terms $\alpha_{i,iJ}$ are now "absorbed" in the integral over $\overline{\Lambda}$.

On the other hand, the measures

$$\tilde{\mu}_i^{(l)} := \tilde{\mu}_i + \frac{2}{p^*}\delta_{\lambda_l}, \qquad l \in \mathbb{N},$$

satisfy $\tilde{\mu}_i^{(l)}(\overline{\Lambda} \setminus \Lambda) = 0$. Hence, in view of (7.6) and (7.7),

$$\tilde{g}^{(l)}(u) := \langle \tilde{\alpha} u_{\mathcal{J}}, u_{\mathcal{J}} \rangle + \langle \beta_i^{\mathcal{Y}}, u \rangle - \gamma_i + \int_{\overline{\Lambda}} \tilde{h}_u \, d\tilde{\mu}_i^{(l)} + \int_{D \setminus Q(0)} h_u(\xi) \, \overline{\mu}_i(d\xi)$$

is of the form (2.17) and satisfies (7.3) and (7.4). Taking into account

$$\tilde{g}^{(l)}(u) - R_i^{\mathcal{Y}}(u) = \frac{2}{p^*}\big(\tilde{h}_u(\lambda_l) - \tilde{h}_u(\lambda_\infty)\big),$$

(7.5) and the simple fact that

$$\sup_{\lambda \in \overline{\Lambda}} \big|\tilde{h}_u(\lambda) - \tilde{h}_{u'}(\lambda)\big| = \sup_{\xi \in Q_0(0)} |h_u(\xi) - h_{u'}(\xi)|,$$

it is obvious that $\tilde{g}^{(l)}$ converge to $R_i^{\mathcal{Y}}$ uniformly on compacts, and the lemma is proved. $\square$

8. Feller property. Let X be regular affine and $(a, \alpha, b, \beta(\beta^{\mathcal{Y}}, \beta^{\mathcal{Z}}), c, \gamma, m, \mu)$ the corresponding admissible parameters, given by Proposition 5.2. In this section we show that X shares the Feller property, and we provide a strong connection between the admissible parameters and the infinitesimal generator of X.

We start with some preliminary considerations. Let U be an open set or the closure of an open set in $\mathbb{R}^N$, $N \in \mathbb{N}$. For $f \in C^2(U)$ we write

$$\|f\|_{2;U} := \sup_{x \in U} \left\{ |f(x)| + \|\nabla f(x)\| + \sum_{k,l=1}^{N} \left| \frac{\partial^2 f(x)}{\partial x_k \, \partial x_l} \right| \right\}. \tag{8.1}$$

Let U be a closed set in D. For $f \in C^2(D)$, we write

$$\|f\|_{\sharp;U} := \|f\|_{\mathcal{Y};U} + \|f\|_{\mathcal{Z};U},$$

where

$$\|f\|_{\mathcal{Y};U} := \sup_{x=(y,z)\in U}\left\{(1+\|y\|)\left(|f(x)|+\|\nabla f(x)\|+\sum_{k,l=1}^{d}\left|\frac{\partial^2 f(x)}{\partial x_k\,\partial x_l}\right|\right)\right\}$$

$$\|f\|_{\mathcal{Z};U} := \sup_{x=(y,z)\in U}\left|\langle z,\beta^{\mathcal{Z}}\nabla_{\mathcal{J}} f(x)\rangle\right|.$$

The product rule yields

$$\|fg\|_{\mathcal{Y};U} \le K_{\mathcal{Y}}\|f\|_{\mathcal{Y};U}\|g\|_{2;U} \qquad \forall\, f,g\in C^2(D), \tag{8.2}$$

where $K_{\mathcal{Y}} = K_{\mathcal{Y}}(m,n)$ is a universal constant.

For $f\in C^2(D)\cap C_{\mathrm{b}}(D)$ we define $\mathcal{A}^\sharp f(x)$ literally as the right-hand side of (2.12). A straightforward verification shows that $\partial_t P_t f_u(x) = \mathcal{A}^\sharp P_t f_u(x)$ for all $(t,u,x)\in\mathbb{R}_+\times\mathcal{U}\times D$; see (3.10) and (3.11). The connection between $\mathcal{A}^\sharp$ and $\|\cdot\|_{\sharp;D}$ becomes clear with the next lemma. Notice that $f(x+\cdot)\in C^2(D)$ if $f\in C^2(D)$ for any $x\in D$.

LEMMA 8.1. *For any $f\in C^2(D)\cap C_{\mathrm{b}}(D)$, we have*

$$|\mathcal{A}^\sharp f(x)| \le C\left(\|a\|+\|\alpha\|+\|b\|+\|\beta\|+|c|+\|\gamma\|+M+\sum_{i\in\mathcal{I}}\mathcal{M}_i\right)\|f(x+\cdot)\|_{\sharp;D}$$

for all $x\in D$ [see (2.10) *and* (2.11)], *where C only depends on d.*

Hence, if (g_k) is a sequence in $\mathcal{D}(\mathcal{A})$ with $\mathcal{A}g_k = \mathcal{A}^\sharp g_k$ and $\|g_k - g\|_{\sharp;D}\to 0$ for some $g\in C^2(D)\cap C_0(D)$, then $\mathcal{A}^\sharp g_k\to\mathcal{A}^\sharp g$ in $C_0(D)$. Since $\mathcal{A}$ is closed, we conclude $g\in\mathcal{D}(\mathcal{A})$ and $\mathcal{A}g = \mathcal{A}^\sharp g$.

PROOF. We let $Q(0)$ and $Q_0(0)$ be as in (4.11). Let $i\in\mathcal{I}$ and write $\xi^\perp$ for the projection of $\xi\in D$ onto the subspace of $\mathbb{R}^d$ spanned by $\{e_k \mid k\in\mathcal{J}(i)\}$; that is,

$$\xi_k^\perp := \begin{cases} 0, & \text{if } k\in\mathcal{I}(i),\\ \xi_k, & \text{if } k\in\mathcal{J}(i).\end{cases}$$

Taylor's formula gives

$$\begin{aligned}
&f(x+\xi)-f(x)-\langle\nabla_{\mathcal{J}(i)}f(x),\xi_{\mathcal{J}(i)}\rangle\\
&\quad= f(x+\xi)-f(x+\xi^\perp)+f(x+\xi^\perp)-f(x)-\langle\nabla f(x),\xi^\perp\rangle\\
&\quad=\left\langle\int_0^1\nabla_{\mathcal{I}(i)}f\big(x+s(\xi-\xi^\perp)\big)\,ds,\xi_{\mathcal{I}(i)}\right\rangle\\
&\qquad+\sum_{k,l\in\mathcal{J}(i)}\left(\int_0^1\frac{\partial f(x+s\xi^\perp)}{\partial x_k\,\partial x_l}(1-s)\,ds\right)\xi_k\xi_l
\end{aligned}$$

for $x, \xi \in D$. Notice the simple fact that $|y_i| \le \|y+\eta\|$ for all $y, \eta \in \mathbb{R}^m_+$. Hence

$$\begin{aligned}
&\int_{D\setminus\{0\}} |f(x+\xi) - f(x) - \langle \nabla_{\mathcal{J}} f(x), \chi_{\mathcal{J}}(\xi)\rangle| y_i \, \mu_i(t, d\xi) \\
&\quad \le C_1 \|f(x+\cdot)\|_{\mathcal{Y};D} \int_{Q_0(0)} \big(\|\xi_{\mathcal{I}(i)}\| + \|\xi_{\mathcal{J}(i)}\|^2\big) \mu_i(t, d\xi) \\
&\qquad + \int_{D\setminus Q(0)} \|y+\eta\| |f(x+\xi)| \, \mu_i(d\xi) \\
&\qquad + C_2(|f(x)| + \|\nabla f(x)\|)\|y\| \mu_i\big(D \setminus Q(0)\big) \\
&\quad \le 3(1 + C_1 + C_2)\mathcal{M}_i \|f(x+\cdot)\|_{\mathcal{Y};D},
\end{aligned}$$

where C_1, C_2 only depend on d. A similar inequality can be shown for the integral with respect to $m(d\xi)$. The rest of the assertion is clear. □

PROPOSITION 8.2. *The process X is Feller. Let $\mathcal{A}$ be its infinitesimal generator. Then $C_c^\infty(D)$ is a core of $\mathcal{A}$, $C_c^2(D) \subset \mathcal{D}(\mathcal{A})$ and* (2.12) *holds for $f \in C_c^2(D)$.*

PROOF. Let $\mathcal{S}_n$ denote the Fréchet space of rapidly decreasing C^∞-functions on $\mathbb{R}^n$ (see [78], Chapter 7). Define the set of functions on D,

$$\Theta := \big\{ f(y,z) = e^{\langle v, y\rangle} g(z) \mid v \in \mathbb{C}^m_{--},\ g \in \mathcal{S}_n \big\},$$

and denote its complex linear hull by $\mathcal{L}(\Theta)$.

It is well known (see [78], Chapter 7) that $C_c^\infty(\mathbb{R}^n)$ is dense in $\mathcal{S}_n$, and that the Fourier transform is a linear homeomorphism on $\mathcal{S}_n$. Hence there exists a subset $\Theta_0 \subset \Theta$ such that its complex linear hull $\mathcal{L}(\Theta_0)$ is $\|\cdot\|_{\sharp;D}$-dense in $\mathcal{L}(\Theta)$, and every $f \in \Theta_0$ can be written as

$$\begin{aligned}
f(y,z) &= e^{\langle v, y\rangle} \int_{\mathbb{R}^n} e^{i\langle q, z\rangle} \tilde{g}(q)\, dq \\
&= \int_{\mathbb{R}^n} f_{(v,iq)}(y,z) \tilde{g}(q)\, dq
\end{aligned} \tag{8.3}$$

for some $\tilde{g} \in C_c^\infty(\mathbb{R}^n)$. Let $t \in \mathbb{R}_+$. From Proposition 6.4 and (6.5), we infer

$$\begin{aligned}
P_t f(y,z) &= \int_{\mathbb{R}^n} P_t f_{(v,iq)}(y,z) \tilde{g}(q)\, dq \\
&= \int_{\mathbb{R}^n} e^{i\langle \exp(\beta^{\mathcal{Z}} t) q, z\rangle} e^{\phi(t,v,iq) + \langle \psi^{\mathcal{Y}}(t,v,iq), y\rangle} \tilde{g}(q)\, dq.
\end{aligned} \tag{8.4}$$

Since $\psi(t,u)$ is continuous on $\mathbb{R}_+ \times \mathcal{U}^0$, we obtain, by dominated convergence, that $P_t f \in C^\infty(D)$ and

$$\begin{aligned}\partial_t P_t f(x) &= \int_{\mathbb{R}^n} \partial_t P_t f_{(v,iq)}(x)\tilde{g}(q)\,dq \\ &= \int_{\mathbb{R}^n} \big(F(\psi(t,v,iq)) + \langle R(\psi(t,v,iq)), x\rangle\big) P_t f_{(v,iq)}(x)\tilde{g}(q)\,dq \\ &= \int_{\mathbb{R}^n} \mathcal{A}^\sharp P_t f_{(v,iq)}(x)\tilde{g}(q)\,dq \\ &= \mathcal{A}^\sharp P_t f(x) \qquad \forall\,(t,x) \in \mathbb{R}_+ \times D.\end{aligned} \tag{8.5}$$

In particular,

$$\lim_{t\downarrow 0} P_t f(x) = f(x) \qquad \forall\, x \in D. \tag{8.6}$$

Let $y \in \mathbb{R}^m_+$ and consider the function

$$q \mapsto h(q) := e^{\phi(t,v,iq)+\langle \psi^{\mathcal{Y}}(t,v,iq),y\rangle}\tilde{g}(q) \in C_c(\mathbb{R}^n).$$

Denote by $\tilde{h}$ its Fourier transform,

$$\tilde{h}(z) = \int_{\mathbb{R}^n} e^{i\langle q,z\rangle} h(q)\,dq.$$

By the Riemann–Lebesgue theorem the functions $\tilde{h}$, $\partial_{z_k}\tilde{h}$ and $\partial_{z_k}\partial_{z_l}\tilde{h}$ are in $C_0(\mathbb{R}^n)$. However, it is not clear at this stage whether $\|\tilde{h}\|_{\mathcal{Z};D}$ is finite or not (it *is* finite, as will be shown below), since $h(q)$ might not be differentiable. But from the identity

$$P_t f(y,z) = \tilde{h}\Big(\big(e^{\beta^{\mathcal{Z}}t}\big)^T z\Big)$$

and (8.4), we obtain, by dominated convergence,

$$(1+\|y\|)\left(|P_t f(x)| + \|\nabla P_t f(x)\| + \sum_{k,l=1}^d \left|\frac{\partial^2 P_t f(x)}{\partial x_k\,\partial x_l}\right|\right) \in C_0(D). \tag{8.7}$$

In particular, $P_t f \in C_0(D)$. From Lemma 8.4 below, we now infer that (8.6) holds for every $f \in C_0(D)$ and $P_t C_0(D) \subset C_0(D)$. This implies that X is Feller (see [76], Section II.2).

Obviously $\mathcal{A}^\sharp f \in C_0(D)$ for every $f \in \mathcal{L}(\Theta_0)$. A nice result for Feller processes ([79], Lemma 31.7) now states that the pointwise equality (8.5), for $t = 0$, already implies that $f \in \mathcal{D}(\mathcal{A})$ and $\mathcal{A}f = \mathcal{A}^\sharp f$. Hence $\mathcal{L}(\Theta_0) \subset \mathcal{D}(\mathcal{A})$. By the closedness of $\mathcal{A}$ therefore $\mathcal{L}(\Theta) \subset \mathcal{D}(\mathcal{A})$ and (2.12) holds for $f \in \mathcal{L}(\Theta)$.

Let $h \in C^2_c(D)$ and (h_k) a sequence in $\mathcal{L}(\Theta_0)$ with $\varepsilon_k := \|h_k - h\|_{2;D} \to 0$, given by Lemma 8.4. We modify (h_k) such that we can assert $\|\cdot\|_{\sharp;D}$-convergence.

Therefore, choose a function $\rho \in C_c^\infty(\mathbb{R}_+)$ with

$$\rho(t) = \begin{cases} 1, & \text{if } t \le 1, \\ 0, & \text{if } t > 5, \end{cases}$$

and $0 \le \rho(t) \le 1$, $|\partial_t \rho(t)| \le 1$ and $|\partial_t^2 \rho(t)| \le 1$ for all $t \in \mathbb{R}_+$. Such a function obviously exists. Define

$$g_k(y, z) := e^{-\sqrt{\varepsilon_k}\langle \mathbf{1}, y\rangle} \rho(\varepsilon_k \|z\|^2).$$

Then $g_k h_k \in \mathcal{L}(\Theta)$ and it is easily verified that $\|g_k\|_{\mathcal{Y};D} \le C/\sqrt{\varepsilon_k}$, where C does not depend on k. With regard to (8.2), we derive

$$\begin{aligned} \|g_k h_k - h\|_{\mathcal{Y};D} &\le \|g_k(h_k - h)\|_{\mathcal{Y};D} + \|h(g_k - 1)\|_{\mathcal{Y};D} \\ &\le K_{\mathcal{Y}} \|g_k\|_{\mathcal{Y};D} \|h - h_k\|_{2;D} + \|h(g_k - 1)\|_{\mathcal{Y};\operatorname{supp} h} \\ &\le K_{\mathcal{Y}} C \sqrt{\varepsilon_k} + K_{\mathcal{Y}} \|h\|_{\mathcal{Y};\operatorname{supp} h} \|g_k - 1\|_{2;\operatorname{supp} h} \\ &\to 0 \qquad \text{for } k \to \infty. \end{aligned}$$

On the other hand, for $x = (y, z) \in D$,

$$\begin{aligned} &\|z\| \|\nabla_{\mathcal{J}}(g_k h_k - h)(x)\| \\ &\quad \le \|z\| \big(\|\nabla_{\mathcal{J}} g_k(x) h_k(x)\| + \|g_k(x) \nabla_{\mathcal{J}}(h_k - h)(x)\| + \big\|\nabla_{\mathcal{J}} h(x)\big(g_k(x) - 1\big)\big\| \big) \\ &\quad \le 2\varepsilon_k \|z\|^2 \partial_t \rho(\varepsilon_k \|z\|^2) \big(|h_k(x) - h(x)| + |h(x)| \big) + \frac{\sqrt{5}}{\sqrt{\varepsilon_k}} \|h_k - h\|_{2;D} \\ &\qquad + \sup_{x \in \operatorname{supp} h} \big(\|z\| \|\nabla h(x)\| \big) \|g_k - 1\|_{2;\operatorname{supp} h} \\ &\quad \le 10 \|h_k - h\|_{2;D} + \sqrt{5\varepsilon_k} + C \|g_k - 1\|_{2;\operatorname{supp} h} \end{aligned}$$

for k large enough such that $\{x = (y, z) \mid \varepsilon_k \|z\|^2 \le 1\} \supset \operatorname{supp} h$, since $\partial_t \rho(t) = 0$ for $t \le 1$. Hence $\|g_k h_k - h\|_{\mathcal{Z};D} \to 0$ and thus $\|g_k h_k - h\|_{\sharp;D} \to 0$ for $k \to \infty$. By the closedness of $\mathcal{A}$ therefore $h \in \mathcal{D}(\mathcal{A})$ and $\mathcal{A}h = \mathcal{A}^\sharp h$. Hence $C_c^2(D) \subset \mathcal{D}(\mathcal{A})$ and (2.12) holds for $f \in C_c^2(D)$.

It remains to consider cores. Define

$$\mathcal{D}(\mathcal{A}^\sharp) := \left\{ f \in C^2(D) \;\middle|\; \begin{array}{l} (1 + \|y\|) \\ \quad \times \left(|f(x)| + \|\nabla f(x)\| + \displaystyle\sum_{k,l=1}^{d} \left| \frac{\partial^2 f(x)}{\partial x_k \, \partial x_l} \right| \right) \in C_0(D), \\ \langle z, \beta^{\mathcal{Z}} \nabla_{\mathcal{J}} f(x) \rangle \in C_0(D) \end{array} \right\}.$$

In view of Lemma 8.1, $f \in \mathcal{D}(\mathcal{A}^\sharp)$ implies $\mathcal{A}^\sharp f \in C_0(D)$.

We show that $C_c^2(D)$ is $\|\cdot\|_{\sharp;D}$-dense in $\mathcal{D}(\mathcal{A}^\sharp)$. Let $f \in \mathcal{D}(\mathcal{A}^\sharp)$ and $\rho(t)$ as above. Define $f_k(x) := f(x)\rho(\|x\|^2/k) \in C_c^2(D)$. We have, for $x = (y, z) \in D$,

$$\begin{aligned}
&|\langle z, \beta^{\mathcal{Z}} \nabla_{\mathcal{J}}(f_k - f)(x)\rangle| \\
&\quad \le |\langle z, \beta^{\mathcal{Z}} \nabla_{\mathcal{J}} f(x)\rangle| \left(1 - \rho\left(\frac{\|x\|^2}{k}\right)\right) + 2\|\beta^{\mathcal{Z}}\| |f(x)| \frac{\|x\|^2}{k} \partial_t \rho\left(\frac{\|x\|^2}{k}\right) \\
&\quad \le 10(1 + \|\beta^{\mathcal{Z}}\|) \sup_{\|x\| \ge \sqrt{k}} \{|\langle z, \beta^{\mathcal{Z}} \nabla_{\mathcal{J}} f(x)\rangle| + |f(x)|\} \to 0 \qquad \text{for } k \to \infty.
\end{aligned}$$

Similarly we see that $\|f_k - f\|_{\mathcal{Y};D} \to 0$, and hence $\|f_k - f\|_{\sharp;D} \to 0$ for $k \to \infty$. As a consequence we obtain $\mathcal{D}(\mathcal{A}^\sharp) \subset \mathcal{D}(\mathcal{A})$ and $\mathcal{A} f = \mathcal{A}^\sharp f$ for all $f \in \mathcal{D}(\mathcal{A}^\sharp)$.

Next we claim that $P_t \mathcal{L}(\Theta_0) \subset \mathcal{D}(\mathcal{A}^\sharp)$ for all $t \in \mathbb{R}_+$. Indeed, since $P_t \mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\mathcal{A})$, we know that $P_t \mathcal{L}(\Theta_0) \subset \mathcal{D}(\mathcal{A})$. Let $f \in \Theta_0$ be given by (8.3). With regard to (8.5), we have

$$\begin{aligned}
\mathcal{A} P_t f(x) &= I(x) + \int_{\mathbb{R}^n} \langle \beta^{\mathcal{Z}} e^{\beta^{\mathcal{Z}} t} iq, z\rangle P_t f_{(v,iq)}(x) \tilde{g}(q)\, dq \\
&= I(x) + \langle z, \beta^{\mathcal{Z}} \nabla_{\mathcal{J}} P_t f(x)\rangle,
\end{aligned}$$

where

$$I(x) := \int_{\mathbb{R}^n} \big(F(\psi(t, v, iq)) + \langle R^{\mathcal{Y}}(\psi(t, v, iq)), y\rangle\big) P_t f_{(v,iq)}(x) \tilde{g}(q)\, dq.$$

Using the same arguments as for (8.7), we see that I is in $C_0(D)$, but so is $\mathcal{A} P_t f$, hence

$$\langle z, \beta^{\mathcal{Z}} \nabla_{\mathcal{J}} P_t f(x)\rangle \in C_0(D).$$

Combining this with (8.7) gives $P_t f \in \mathcal{D}(\mathcal{A}^\sharp)$.

From Lemma 8.3 below, with $\mathcal{D}_0 = \mathcal{L}(\Theta_0)$ and $\mathcal{D}_1 = \mathcal{D}(\mathcal{A}^\sharp)$, we now infer that $\mathcal{D}(\mathcal{A}^\sharp)$ is a core of $\mathcal{A}$. But $C_c^\infty(D)$ is $\|\cdot\|_{\sharp;D}$-dense in $C_c^2(D)$, which is $\|\cdot\|_{\sharp;D}$-dense in $\mathcal{D}(\mathcal{A}^\sharp)$. This yields the assertion. $\square$

LEMMA 8.3. *Let $\mathcal{D}_0$ and $\mathcal{D}_1$ be dense linear subspaces of $C_0(D)$ such that*

$$P_t \mathcal{D}_0 \subset \mathcal{D}_1 \subset \mathcal{D}(\mathcal{A}) \qquad \forall t \ge 0,$$

then $\mathcal{D}_1$ is a core of $\mathcal{A}$.

For the proof, see [79], Lemma 31.6.

LEMMA 8.4. *For every $h \in C_c^2(D)$ there exists a sequence (h_k) in $\mathcal{L}(\Theta_0)$ with*

$$\lim_{k \to \infty} \|h_k - h\|_{2;D} = 0. \tag{8.8}$$

Consequently, $\mathcal{L}(\Theta_0)$ is dense in $C_0(D)$.

PROOF. Let $h \in C_c^2(D)$ and define h^* on $D^* := (0,1)^m \times (-1,1)^n$ by

$$h^*(\eta,\theta) := h\big(\log(\eta_1^{-1}),\dots,\log(\eta_m^{-1}),\tanh^{-1}(\theta_1),\dots,\tanh^{-1}(\theta_n)\big).$$

By construction, $h^* \in C^2(\overline{D^*})$ and $h^*(\eta,\theta) = 0$ if θ is a boundary point of $[-1,1]^n$. Introduce the coordinates

$$\upsilon_j = \rho_j \cos(\pi\theta_j), \qquad \zeta_j = \rho_j \sin(\pi\theta_j),$$
$$\rho_j \in \mathbb{R}_+,\ \theta_j \in [-1,1],\ j = 1,\dots,n.$$

The inverse transforms $\rho_j = \rho_j(\upsilon_j,\zeta_j)$ and $\theta_j = \theta_j(\upsilon_j,\zeta_j)$ are smooth on $\mathbb{R}^2$ and $\mathbb{R}^2 \setminus \{(r,0) \mid r \le 0\}$, respectively. By smooth extension, the function

$$h^{**}(\eta,\upsilon,\zeta) := \rho_1 \cdots \rho_n\, h^*(\eta,\theta)$$

is well defined on $D^{**} := (0,1)^m \times (-2,2)^{2n}$ and satisfies $h^{**} \in C^2(\overline{D^{**}})$. A version of the Stone–Weierstrass approximation theorem yields a sequence of polynomials (p_k) with

$$\lim_{k\to\infty} \|p_k - h^{**}\|_{2;D^{**}} = 0;$$

see [29], Section II.4. This convergence holds in particular on the subset $[0,1]^m \times \{(\upsilon,\zeta) \mid \rho_1 = \cdots = \rho_n = 1\}$, which can be identified with D_Δ by the preceding transforms. Indeed, for $z \in \mathbb{R}^n$, write shorthand

$$c(z) := \big(\cos(\pi\tanh(z_1)),\dots,\cos(\pi\tanh(z_n))\big),$$
$$s(z) := \big(\sin(\pi\tanh(z_1)),\dots,\sin(\pi\tanh(z_n))\big).$$

Then we have $h(y,z) = h^{**}(e^{-y_1},\dots,e^{-y_m},c(z),s(z))$. Hence the functions

$$\tilde{h}_k(y,z) := p_k\big(e^{-y_1},\dots,e^{-y_m},c(z),s(z)\big), \qquad k \in \mathbb{N},$$

satisfy $\lim_{k\to\infty} \|\tilde{h}_k - h\|_{2;D} = 0$. Now let K be a compact subset in $\mathbb{R}^n$ such that $\operatorname{supp} h \subset \mathbb{R}_+^m \times K$ and let $\sigma \in C_c^\infty(\mathbb{R}^n)$ with $\sigma = 1$ on K. Each $\tilde{h}_k$ is a complex linear combination of terms of the form

$$e^{-\langle \upsilon, y\rangle} \prod_{j=1}^n \cos^{\kappa_j}(\pi\tanh(z_j)) \sin^{\lambda_j}(\pi\tanh(z_j)), \qquad \upsilon \in \mathbb{N}_0^m,\ \kappa_j,\lambda_j \in \mathbb{N}_0.$$

Hence $h_k(y,z) := e^{-2^{-k}\langle \mathbf{1}, y\rangle}\sigma(z)\tilde{h}_k(y,z) \in \mathcal{L}(\Theta)$ for every $k \in \mathbb{N}$, and (8.8) holds.

Since $\mathcal{L}(\Theta_0)$ is $\|\cdot\|_{2;D}$-dense in $\mathcal{L}(\Theta)$, and $C_c^2(D)$ is dense in $C_0(D)$, the lemma is proved. $\square$

9. Conservative regular affine processes. Let X be regular affine and $(a, \alpha, b, \beta, c, \gamma, m, \mu)$ the corresponding admissible parameters, given by Proposition 5.2. In this section we investigate under which conditions X is conservative.

PROPOSITION 9.1. *If $c = 0$, $\gamma = 0$ and $g = 0$ is the only $\mathbb{R}^m_-$-valued solution of*

$$\begin{aligned} \partial_t g(t) &= \operatorname{Re} R^{\mathcal{Y}}\big(g(t), 0\big), \\ g(0) &= 0, \end{aligned} \tag{9.1}$$

then X is conservative.

On the other hand, if X is conservative then $c = 0$, $\gamma = 0$ and there exists no solution g of (9.1) *with $g(t) \in \mathbb{R}^m_{--}$ for all $t > 0$.*

PROOF. Observe that X is conservative if and only if $\phi(t, 0) = 0$ and $\psi(t, 0) = 0$, for all $t \in \mathbb{R}_+$. In view of Proposition 6.4 and since

$$\operatorname{Im} \phi\big(t, (v, 0)\big) = 0, \qquad \operatorname{Im} \psi\big(t, (v, 0)\big) = 0 \qquad \forall\, v \in \mathbb{R}^m_-,\ \forall\, t \in \mathbb{R}_+,$$

X is conservative if and only if

$$\begin{aligned} &\lim_{v \to 0,\ v \in \mathbb{R}^m_-} \operatorname{Re} \phi\big(t, (v, 0)\big) = 0, \\ &\lim_{v \to 0,\ v \in \mathbb{R}^m_-} \operatorname{Re} \psi\big(t, (v, 0)\big) = 0 \qquad \forall\, t \in \mathbb{R}_+. \end{aligned} \tag{9.2}$$

Suppose $c = 0$ and $\gamma = 0$. Then $\operatorname{Re} \psi(\cdot, (v, 0))$ converges to an $\mathbb{R}^m_-$-valued solution of (9.1) for $v \to 0$. Under the stated assumptions, we conclude that (9.2) holds and X is conservative.

Now suppose X is conservative. From (2.16) and conditions (2.4) and (2.8), we see that $\operatorname{Re} F(v, 0) \le -c$ and thus

$$\operatorname{Re} \phi\big(t, (v, 0)\big) \le -ct \qquad \forall\, v \in \mathbb{R}^m_-.$$

Together with (6.9) this implies that (9.2) only can hold if $c = 0$ and $\gamma = 0$. Now let g be a solution of (9.1) with $g(t) \in \mathbb{R}^m_{--}$ for all $t > 0$. By the uniqueness result from Proposition 6.1 we have $g(t + s) = \operatorname{Re} \psi(s, (g(t), 0))$ for all $s, t > 0$. But $g(t) \to 0$ and thus $\operatorname{Re} \psi(s, (g(t), 0)) \to \operatorname{Re} \psi(s, 0) = g(s)$ for $t \to 0$, which contradicts (9.2). $\square$

We can give sufficient conditions for conservativity in terms of the parameters of X directly.

LEMMA 9.2. *Suppose*

$$\int_{D \setminus \{0\}} (\|\eta\| \wedge \|\eta\|^2)\, \mu_i(d\xi) < \infty \qquad \forall\, i \in \mathcal{I}. \tag{9.3}$$

Then $R^{\mathcal{Y}}(v, w)$ is locally Lipschitz continuous in $v \in \mathbb{C}^m_-$, uniformly in w on compact sets in $\mathbb{R}^n$.

Consequently, if in addition $c = 0$ and $\gamma = 0$ then X is conservative.

PROOF. As in Lemma 5.3 we derive that $\nabla_{\mathcal{I}} R_i^{\mathcal{Y}} \in C(\mathcal{U})$ for all $i \in \mathcal{I}$. Thus the first part of the lemma follows. Consequently, if $\gamma = 0$, the unique $\mathbb{R}_-^m$-valued solution of (9.1) is $g = 0$. Now Proposition 9.1 yields the assertion. $\square$

We now give an example of a nonconservative regular affine Markov process.

EXAMPLE 9.3. Let $(m, n) = (1, 0)$, that is, $D = \mathbb{R}_+$. We suppress the index i and set

$$\mu(d\xi) = \frac{1}{2\sqrt{\pi}} \frac{d\xi}{\xi^{3/2}}.$$

Then condition (9.3) is not satisfied. A calculation shows

$$\int_{D\setminus\{0\}} \big(e^{v\xi} - 1 - v\chi(\xi)\big)\,\mu(d\xi) = -\sqrt{-v} - \frac{2}{\sqrt{\pi}} v, \qquad v \in \mathbb{R}_-.$$

Choose $F = 0$ and $\alpha = 0$, $\beta = 2/\sqrt{\pi}$, $\gamma = 0$. Hence $R(v) = -\sqrt{-v}$ (where $\sqrt{\cdot}$ is the unique holomorphic square root function on $\mathbb{C}_{++}$ with $\sqrt{1} = 1$), which is not Lipschitz continuous at $v = 0$. The solution to (6.1) is $\psi(t, v) = -(2\sqrt{-v} + t)^2/4$ for $\operatorname{Re} v < 0$. The limit for $v \to 0$ is $\psi(t, 0) = -t^2/4$. Hence

$$P_t \mathbb{1}(x) = e^{-(t^2/4)x}$$

and X is not conservative. Its infinitesimal generator is

$$\mathcal{A} f(x) = \int_{D\setminus\{0\}} \big(f(x + \xi) - f(x)\big) \frac{x}{2\sqrt{\pi}} \frac{d\xi}{\xi^{3/2}}.$$

Thus X is a pure jump process with increasing (in time along every path) jump intensity such that explosion occurs in finite time for $X_0 = x > 0$.

10. Proof of the main results.

10.1. *Proof of Theorem* 2.7. The first part of Theorem 2.7 is a summary of Propositions 5.2, 6.4 and 8.2. The second part follows from Propositions 6.1 and 7.4.

10.2. *Proof of Theorem* 2.12. We first prove that X is a semimartingale. Write $\tilde{X}_t = (\tilde{Y}_t, \tilde{Z}_t) := X_t \mathbb{1}_{\{t<\tau_X\}}$ and let $x \in D$. For every $u \in \mathcal{U}$ and $T > 0$, we see that

$$e^{\phi(T-t,u) + \langle \psi(T-t,u), X_t\rangle} \mathbb{1}_{\{t<\tau_X\}} = \mathbb{E}_x[f_u(X_T) | \mathcal{F}_t], \qquad t \in [0, T],$$

is a $\mathbb{P}_x$-martingale. Therefore,

$$\exp\Big(\big\langle \psi^{\mathcal{Y}}(T - t, (v, w)), \tilde{Y}_t\big\rangle + \big\langle e^{\beta^{\mathcal{Z}}(T-t)} w, \tilde{Z}_t\big\rangle\Big) \quad \text{and} \quad \big\langle \psi^{\mathcal{Y}}(T - t, (v, 0)), \tilde{Y}_t\big\rangle, \qquad t \in [0, T],$$

are $\mathbb{P}_x$-semimartingales for all $(v, w) \in \mathcal{U}$. There exists $T^* > 0$ such that

$$\psi^{\mathcal{Y}}(t, (e_1, 0)), \dots, \psi^{\mathcal{Y}}(t, (e_m, 0))$$

are linearly independent vectors in $\mathbb{R}^m$ for every $t \in [0, T^*)$. By induction, we obtain that $(\tilde{Y}_t \mathbb{1}_{\{t<kT^*\}})$ is a $\mathbb{P}_x$-semimartingale for all $k \in \mathbb{N}$. Since being a semimartingale is a local property (see [57], Proposition I.4.25), $\tilde{Y}$ is a $\mathbb{P}_x$-semimartingale. Now let $T > 0$ and write $\hat{Z}_t := e^{-\beta^{\mathcal{Z}}(T-t)}\tilde{Z}_t$. From the above we obtain that $(\sin(q_j \hat{Z}_t^j))_{t\in[0,T]}$ are $\mathbb{P}_x$-semimartingales for all $q_j \in \mathbb{R}$, $j \in \mathcal{J}$. Define the stopping time $T_k := \inf\{t \mid |\hat{Z}_t^j| > k\} \wedge \tau_X$. Then $\hat{Z}^j$ coincides with the $\mathbb{P}_x$-semimartingale $k \arcsin(\sin(\hat{Z}^j/k))$ on the stochastic interval $[0, T_k)$. But $T_k \uparrow \tau_x$, and the above localization argument ([57], Proposition I.4.25) implies that $\hat{Z}$ and hence $\tilde{Z}$ are $\mathbb{P}_x$-semimartingales. Therefore X is a semimartingale.

Now let X be conservative. Theorem 2.7 tells us that

$$f(X_t) - f(x) - \int_0^t \mathcal{A} f(X_s)\, ds, \qquad t \in \mathbb{R}_+, \tag{10.1}$$

is a $\mathbb{P}_x$-martingale for every $f \in C_c^2(D)$ and $x \in D$. Conversely, by Itô's formula, we see that

$$f(X'_t) - f(x) - \int_0^t \mathcal{A} f(X'_s)\, ds, \qquad t \in \mathbb{R}_+, \tag{10.2}$$

is a $\mathbb{P}'$-martingale for every $f \in C_c^2(D)$. Theorem 2.12 now follows from Lemmas 10.1 and 10.2.

LEMMA 10.1. *Suppose X is conservative. Then the following two conditions are equivalent*:

(i) *X admits the characteristics* (2.20)–(2.22) *on* $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}_x)$.
(ii) *For every $f \in C_c^2(D)$, the process* (10.1) *is a $\mathbb{P}_x$-martingale.*

PROOF. The implication (i)$\Rightarrow$(ii) is a consequence of Itô's formula. Now suppose (ii) holds. For every $u \in \mathcal{U}$, there exists a sequence (g_k) in $C_c^2(D)$ such that

$$\lim_{k\to\infty} \|g_k - f_u\|_{2;K} = 0$$

[see (8.1)] for every compact set $K \subset D$. Hence the set $C_c^2(D)$ completely (up to modification) determines the characteristics of a semimartingale, as is shown in [57], Lemma II.2.44. Since X is conservative, we have $c = 0$ and $\gamma = 0$ by Proposition 9.1, and (i) now follows as in the proof of [57], Theorem II.2.42(a). $\square$

Condition (i) of Lemma 10.1 defines X as a solution of the *martingale problem for* $(\mathcal{A}, \mathbb{P}_x)$; see [42], Section 4.3. We recall a basic uniqueness result for Markov processes.

LEMMA 10.2. *Let X' be a D-valued, right-continuous, adapted process defined on some filtered probability space $(\Omega', \mathcal{F}', (\mathcal{F}_t'), \mathbb{P}')$ with $\mathbb{P}'[X_0' = x] = 1$ such that* (10.2) *is a martingale for every $f \in C_c^2(D)$. Then $\mathbb{P}' \circ X'^{-1} = \mathbb{P}_x$.*

For the proof, combine Theorem 2.7 and [42], Theorem 4.1, Chapter 4.

10.3. *Proof of Theorem* 2.15. We proceed as in [85].

LEMMA 10.3. *Let $(\mathbb{P}_x^{(i)})_{x\in D} \in \mathcal{P}_{\mathrm{RM}}$ for $i = 1, 2, 3$. Then*

$$\mathbb{P}_x^{(1)} * \mathbb{P}_\xi^{(2)} = \mathbb{P}_{x+\xi}^{(3)} \qquad \forall x, \xi \in D, \tag{10.3}$$

if and only if, for all $\mathbf{t} = (t_0, \dots, t_N) \in \mathbb{R}_+^{N+1}$ and $\mathbf{u} = (u^{(0)}, \dots, u^{(N)}) \in \mathcal{U}^{N+1}$, $N \in \mathbb{N}_0$, there exist $\rho^{(i)}(\mathbf{t}, \mathbf{u}) \in \mathbb{C}$ and $\psi(\mathbf{t}, \mathbf{u}) \in \mathbb{C}^d$ such that $\rho^{(1)}(\mathbf{t}, \mathbf{u})\rho^{(2)}(\mathbf{t}, \mathbf{u}) = \rho^{(3)}(\mathbf{t}, \mathbf{u})$ and

$$\mathbb{E}_x^{(i)}\left[\exp\left(\sum_{k=0}^{N} \langle u^{(k)}, X_{t_k}\rangle\right)\right] = \rho^{(i)}(\mathbf{t}, \mathbf{u}) e^{\langle \psi(\mathbf{t},\mathbf{u}), x\rangle} \qquad \forall x \in D,\ i = 1, 2, 3. \tag{10.4}$$

PROOF. Notice that (10.3) holds if and only if, for all $\mathbf{t}$, $\mathbf{u}$ and $N \in \mathbb{N}_0$ as above, we have

$$g^{(1)}(x) g^{(2)}(\xi) = g^{(3)}(x + \xi) \qquad \forall x, \xi \in D, \tag{10.5}$$

where $g^{(i)}(x) := \mathbb{E}_x^{(i)}[\exp(\sum_{k=0}^N \langle u^{(k)}, X_{t_k}\rangle)]$. From (10.5) it follows that $g^{(1)}(x)g^{(2)}(0) = g^{(1)}(0)g^{(2)}(x) = g^{(3)}(x)$. Hence $g^{(i)}(0) = 0$ for some i if and only if $g^{(i)} = 0$ for all i, and (10.4) holds for $\rho^{(i)}(\mathbf{t}, \mathbf{u}) = 0$. Now suppose $g^{(i)}(0) > 0$ for all i. Then $g^{(i)}(x)/g^{(i)}(0) = g^{(3)}(x)/(g^{(1)}(0)g^{(2)}(0)) =: g(x)$ for $i = 1, 2$. The function g is measurable and satisfies the functional equation $g(x)g(\xi) = g(x + \xi)$. Hence there exists $\rho' \in \mathbb{C} \setminus \{0\}$ and $\psi(\mathbf{t}, \mathbf{u}) \in \mathbb{C}^d$ such that $g(x) = \rho' e^{\langle \psi(\mathbf{t},\mathbf{u}), x\rangle}$. Define $\rho^{(i)}(\mathbf{t}, \mathbf{u}) := g^{(i)}(0)\rho'$, $i = 1, 2$, and $\rho^{(3)}(\mathbf{t}, \mathbf{u}) := g^{(1)}(0)g^{(2)}(0)\rho'$, then (10.4) follows. Since conversely (10.4) implies (10.5), the lemma is proved. □

Let $(X, (\mathbb{P}_x)_{x\in D})$ be regular affine with parameters $(a, \alpha, b, \beta, c, \gamma, m, \mu)$ and exponents $\phi(t, u)$ and $\psi(t, u)$. Let $k \in \mathbb{N}$. Then $(a/k, \alpha, b/k, \beta, c/k, \gamma, m/k, \mu)$ are admissible parameters, which induce a regular affine Markov process $(X, (\mathbb{P}_x^{(k)})_{x\in D})$ with exponents $\phi'(t, u) = \phi(t, u)/k$ and $\psi'(t, u) = \psi(t, u)$; see Theorem 2.7. Now let $\mathbf{t}$, $\mathbf{u}$ and N as in Lemma 10.3. Without loss of generality we may assume $\Delta t_l := t_l - t_{l-1} > 0$ for all $l = 1, \dots, N$. By the Markov property,

we have

$$\begin{aligned}
&\mathbb{E}_x\left[\exp\left(\sum_{l=0}^{N}\langle u^{(l)}, X_{t_l}\rangle\right)\right] \\
&\quad = e^{\phi(\Delta t_N, u^{(N)})}\mathbb{E}_x\left[\exp\left(\sum_{l=0}^{N-1}\langle u^{(l)}, X_{t_l}\rangle\right)e^{\langle\psi^{(1)}(\mathbf{t},\mathbf{u}), X_{t_{N-1}}\rangle}\right] \\
&\quad = \cdots = e^{-\phi(\mathbf{t},\mathbf{u})+\langle\psi^{(N)}(\mathbf{t},\mathbf{u}),x\rangle},
\end{aligned}$$

where we have defined inductively

$$\begin{aligned}
\psi^{(0)}(\mathbf{t},\mathbf{u}) &:= u_N, \\
\psi^{(l+1)}(\mathbf{t},\mathbf{u}) &:= \psi\big(\Delta t_{N-l}, \psi^{(l)}(\mathbf{t},\mathbf{u})\big) + u_{N-l}, \qquad l = 0, \dots, N-1, \\
\phi(\mathbf{t},\mathbf{u}) &:= \sum_{l=1}^{N}\phi\big(\Delta t_l, \psi^{(l)}(\mathbf{t},\mathbf{u})\big).
\end{aligned}$$

Similarly, we obtain $\mathbb{E}_x^{(k)}[\exp(\sum_{l=0}^{N}\langle u^{(l)}, X_{t_l}\rangle)] = e^{\phi'(\mathbf{t},\mathbf{u})+\langle\psi^{(N)}(\mathbf{t},\mathbf{u}),x\rangle}$, with $\phi'(\mathbf{t},\mathbf{u}) = \phi(\mathbf{t},\mathbf{u})/k$. Now Lemma 10.3 yields (2.25).

Conversely, let $(\mathbb{P}_x)_{x\in D}$ be infinitely decomposable. By definition, we have $(\mathbb{P}_x)_{x\in D} \in \mathcal{P}_{\mathrm{RM}}$, and $\mathbb{P}_x \circ X_t^{-1}$ is an infinitely divisible (substochastic) distribution on D for all $(t,x) \in \mathbb{R}_+ \times D$. It is well known that the characteristic function of an infinitely divisible distribution on $\mathbb{R}^d$ has no (real) zeros, (see, e.g., [10], Chapter 29). Hence $P_t f_u(x) \neq 0$ for all $(t,u,x) \in \mathbb{R}_+ \times \partial\mathcal{U} \times D$. Set $N = 0$. From (10.4), we now obtain that $(X, (\mathbb{P}_x)_{x\in D})$ is affine, and Theorem 2.15 is proved.

As an immediate consequence of Lemma 10.3 and the preceding proof we obtain the announced additivity property of regular affine processes.

COROLLARY 10.4. *Let $(X, (\mathbb{P}_x^{(i)})_{x\in D})$ be regular affine Markov processes with parameters $(a^{(i)}, \alpha, b^{(i)}, \beta, c^{(i)}, \gamma, m^{(i)}, \mu)$ for $i = 1, 2$. Then*

$$\big(a^{(1)} + a^{(2)}, \alpha, b^{(1)} + b^{(2)}, \beta, c^{(1)} + c^{(2)}, \gamma, m^{(1)} + m^{(2)}, \mu\big)$$

are admissible parameters generating a regular affine Markov process $(X, (\mathbb{P}_x^{(3)})_{x\in D})$ such that (10.3) *holds.*

11. Discounting. For the entire section we suppose that X is conservative regular affine with parameters $(a, \alpha, b, \beta, 0, 0, m, \mu)$ (we recall that Proposition 9.1 yields $c = 0$ and $\gamma = 0$). We shall investigate the behavior of X under a particular transformation that is inevitable for financal applications: discounting. Let $\ell \in \mathbb{R}$, $\lambda = (\lambda^{\mathcal{Y}}, \lambda^{\mathcal{Z}}) \in \mathbb{R}^m \times \mathbb{R}^n$ and define the affine function $L(x) := \ell + \langle\lambda, x\rangle$ on $\mathbb{R}^d$.

In many applications $L(X)$ is a model for the short rates. The price of a claim of the form $f(X_t)$, where $f \in bD$, is given by the expectation

$$Q_t f(x) := \mathbb{E}_x\Big[\exp\Big(-\int_0^t L(X_s)\,ds\Big) f(X_t)\Big]. \tag{11.1}$$

Suppose that, for fixed $x \in D$ and $t \in \mathbb{R}_+$, we have $\mathbb{E}_x[\exp(-\int_0^t L(X_s)\,ds)] < \infty$. Then $\mathbb{R}^d \ni q \mapsto Q_t f_{iq}(x)$ is the characteristic function of X_t with respect to the bounded measure $\exp(-\int_0^t L(X_s)\,ds)\,\mathbb{P}_x$. Hence if one knows $Q_t f_u(x)$ for all $u \in \partial\mathcal{U}$, the integral (11.1) can be (explicitly or numerically) calculated via Fourier inversion.

Depending on whether L is non-negative on D or not there are two approaches for the calculation of $Q_t f_u(x)$. The first one is based on an analytic version of the Feynman–Kac formula. The second one uses martingale methods and is more general. But it requires an enlargement of the state space and an analytic extension of the exponents $\phi'(t,\cdot)$ and $\psi'(t,\cdot)$ of some $(d+1)$-dimensional regular affine process X'.

11.1. *The Feynman–Kac formula.* Suppose $L(x) \geq 0$ for all $x \in D$. This is equivalent to $\ell \in \mathbb{R}_+$ and $\lambda = (\lambda^{\mathcal{Y}}, 0) \in \mathbb{R}_+^m \times \{0\}$. The function L induces a closed linear operator $\mathcal{L}$ on bD with $\mathcal{D}(\mathcal{L}) = \{f \in bD \mid Lf \in bD\}$ by $\mathcal{L}f(x) := L(x)f(x)$.

PROPOSITION 11.1. *The family $(Q_t)_{t\in\mathbb{R}_+}$ forms a regular affine semigroup with infinitesimal generator*

$$\mathcal{B}f = \mathcal{A}f - \mathcal{L}f, \qquad f \in C_c^2(D).$$

The corresponding admissible parameters are $(a, \alpha, b, \beta, \ell, \lambda, m, \mu)$.

PROOF. Define the shift operators $\theta_t : \Omega \to \Omega$ by $\theta_t(\omega)(s) = \omega(t+s)$. Then by the Markov property of X,

$$\begin{aligned}
Q_{t+s} f(x) &= \mathbb{E}_x\Big[\exp\Big(-\int_0^{t+s} L(X_r)\,dr\Big) f(X_{t+s})\Big] \\
&= \mathbb{E}_x\Big[\exp\Big(-\int_0^t L(X_r)\,dr\Big) \\
&\qquad \times \mathbb{E}_x\Big[\Big(\exp\Big(-\int_0^s L(X_r)\,dr\Big) f(X_s)\Big)\circ\theta_t \Big| \mathcal{F}_t\Big]\Big] \\
&= \mathbb{E}_x\Big[\exp\Big(-\int_0^t L(X_r)\,dr\Big) Q_s f(X_t)\Big] = Q_t Q_s f(x) \qquad \forall f \in bD.
\end{aligned}$$

Since $Q_0 f = f$ and $0 \leq Q_t 1 \leq 1$, we conclude that (Q_t) is a positive contraction semigroup on bD.

By the right-continuity of X it follows from (11.1) that $(t,x) \mapsto Q_t f(x)$ is measurable on $\mathbb{R}_+ \times D$. Hence, for every $q > 0$, the resolvent of (Q_t),

$$R_q^{\mathcal{L}} g(x) := \int_{\mathbb{R}_+} e^{-qt} Q_t g(x)\, dt,$$

is a well-defined bounded operator from bD into bD. Denote by B_0 the set of elements $h \in bD$ with $\lim_{t\to 0} Q_t h = h$ in bD. It is well known that $R_q^{\mathcal{L}} bD \subset B_0$, and $R_q^{\mathcal{L}} : B_0 \to \mathcal{D}(\mathcal{B})$ is one-to-one with $(R_q^{\mathcal{L}})^{-1} = qI - \mathcal{B}$, where we consider $\mathcal{B}$ a priori as the infinitesimal generator of the semigroup (Q_t) acting on bD. We denote the resolvent of the Feller semigroup (P_t) by R_q. We claim that

$$R_q g = R_q^{\mathcal{L}}\big(g + \mathcal{L}(R_q g)\big) \tag{11.2}$$

for all $g \in C_0(D)$ with $R_q g \in \mathcal{D}(\mathcal{L})$. Indeed, since R_q is a positive operator, we may assume $g \geq 0$. Using Tonelli's theorem and following [77], Section III.19, we calculate

$$\begin{aligned}
&R_q g(x) - R_q^{\mathcal{L}} g(x) \\
&\quad = \mathbb{E}_x\bigg[\int_{\mathbb{R}_+} e^{-qt} g(X_t)\bigg(1 - \exp\bigg(-\int_0^t L(X_r)\, dr\bigg)\bigg)\, dt\bigg] \\
&\quad = \mathbb{E}_x\bigg[\int_{\mathbb{R}_+} e^{-qt} g(X_t)\bigg(\int_0^t L(X_s) \exp\bigg(-\int_0^s L(X_r)\, dr\bigg)\, ds\bigg)\, dt\bigg] \\
&\quad = \mathbb{E}_x\bigg[\int_{\mathbb{R}_+} \exp\bigg(-\int_0^s L(X_r)\, dr\bigg) L(X_s)\bigg(\int_{\mathbb{R}_+} e^{-q(t+s)} g(X_{t+s})\, dt\bigg)\, ds\bigg] \\
&\quad = \mathbb{E}_x\bigg[\int_{\mathbb{R}_+} e^{-qs} \exp\bigg(-\int_0^s L(X_r)\, dr\bigg) L(X_s)\bigg(\int_{\mathbb{R}_+} e^{-qt} P_t g(X_s)\, dt\bigg)\, ds\bigg] \\
&\quad = \mathbb{E}_x\bigg[\int_{\mathbb{R}_+} e^{-qs} \exp\bigg(-\int_0^s L(X_r)\, dr\bigg) \mathcal{L}(R_q g)(X_s)\, ds\bigg] = R_q^{\mathcal{L}} \mathcal{L}(R_q q)(x),
\end{aligned}$$

whence (11.2).

Let $f \in C_c^2(D)$. There exists a unique $g \in C_0(D)$ with $R_q g = f$, in fact, $g = (qI - \mathcal{A}) f$. From (11.2), we obtain

$$f = R_q^{\mathcal{L}}(g + \mathcal{L} f). \tag{11.3}$$

Hence $C_c^2(D) \subset B_0$, but $C_c^2(D)$ is dense in $C_0(D)$ and B_0 closed in bD, hence $C_0(D) \subset B_0$. Since g and $\mathcal{L} f$ are in $C_0(D)$, we infer from (11.3) that $C_c^2(D) \subset R_q^{\mathcal{L}} C_0(D) \subset \mathcal{D}(\mathcal{B})$. Moreover, by (11.3) again, $(qI - \mathcal{B}) f = g + \mathcal{L} f = (qI - \mathcal{A}) f + \mathcal{L} f$. Thus

$$\mathcal{B} = \mathcal{A} - \mathcal{L} \qquad \text{on } C_c^2(D), \tag{11.4}$$

and Proposition 8.2 yields the assertion. $\square$

11.2. *Enlargement of the state space.* The preceding approach requires nonnegativity of L. But there is a large literature on affine term structures for which the short rate is not necessarily non-negative. See, for example, [90] and [31]. We shall provide a different approach, using the martingale argument from Theorem 2.12. Let L be as at the beginning of Section 11. For $r \in \mathbb{R}$, write

$$R_t^r := r + \int_0^t L(X_s)\, ds.$$

It can be shown that (X, R^r) is a Markov process on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}_x)$ for every $x \in D$ and $r \in \mathbb{R}$. In fact, we enlarge the state space $D \rightsquigarrow D \times \mathbb{R}$ and $\mathcal{U} \rightsquigarrow \mathcal{U} \times i\mathbb{R}$, and write accordingly $(x, r) = (y, z, r)$, $(\xi, \rho) = (\eta, \zeta, \rho) \in D \times \mathbb{R}$ and $(u, q) = (v, w, q) \in \mathcal{U} \times i\mathbb{R}$. Let $X' = (Y', Z', R')$ be the regular affine process with state space $D \times \mathbb{R}$ given by the admissible parameters

$$\begin{aligned} a' &= \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, & \alpha_i' &= \begin{pmatrix} \alpha_i & 0 \\ 0 & 0 \end{pmatrix}, & i &\in \mathcal{I}, \\ b' &= (b, \ell), & \beta' &= \begin{pmatrix} \beta & 0 \\ \lambda & 0 \end{pmatrix}, \\ c' &= c = 0, & \gamma' &= \gamma = 0 \end{aligned}$$

and

$$\begin{aligned} m'(d\xi, d\rho) &= m(d\xi) \times \delta_0(d\rho), \\ \mu_i'(d\xi, d\rho) &= \mu_i(d\xi) \times \delta_0(d\rho), \qquad i \in \mathcal{I}. \end{aligned}$$

The existence of X' is guaranteed by Theorem 2.7. We let X' be defined on the canonical space $(\Omega', \mathcal{F}', (\mathcal{F}_t'), (\mathbb{P}'_{(x,r)})_{(x,r)\in D\times\mathbb{R}})$, as described in Section 1. The corresponding mappings F' and $R' = (R'^{\mathcal{Y}}, R'^{\mathcal{Z}}, R'^{\mathcal{R}})$ satisfy

$$\begin{aligned} F'(u, q) &= F(u) + \ell q, \\ R'^{\mathcal{Y}}(u, q) &= R^{\mathcal{Y}}(u) + \lambda^{\mathcal{Y}} q, \\ R'^{\mathcal{Z}}(u, q) &= R^{\mathcal{Z}}(u) + \lambda^{\mathcal{Z}} q, \\ R'^{\mathcal{R}}(u, q) &= 0; \end{aligned}$$

see (2.16), (2.17) and (5.3). Let ϕ' and $\psi' = (\psi'^{\mathcal{Y}}, \psi'^{\mathcal{Z}}, \psi'^{\mathcal{R}})$ be the solution of the corresponding generalized Riccati equations (6.1) and (6.2). The variation of constants formula yields

$$\psi'^{\mathcal{Z}}(t, u, q) = e^{\beta^{\mathcal{Z}} t} w + q \int_0^t e^{\beta^{\mathcal{Z}}(t-s)} \lambda^{\mathcal{Z}}\, ds, \tag{11.5}$$

and clearly $\psi'^{\mathcal{R}}(t, u, q) = q$. The dependence of $\phi'(t, u, q)$ and $\psi'^{\mathcal{Y}}(t, u, q)$ on q

is more implicit. In fact,

$$\phi'(t,u,q) = \int_0^t F\big(\psi'^{\mathcal{Y}}(s,u,q), \psi'^{\mathcal{Z}}(s,u,q)\big)\,ds + t\ell q, \tag{11.6}$$

$$\psi'^{\mathcal{Y}}(t,u,q) = \int_0^t R^{\mathcal{Y}}\big(\psi'^{\mathcal{Y}}(s,u,q), \psi'^{\mathcal{Z}}(s,u,q)\big)\,ds + t\lambda^{\mathcal{Y}} q. \tag{11.7}$$

PROPOSITION 11.2. *Let $(x,r) \in D \times \mathbb{R}$. We have $\mathbb{P}_x \circ (X, R^r)^{-1} = \mathbb{P}'_{(x,r)}$, and in particular,*

$$\begin{aligned} \mathbb{E}_x\big[e^{qR_t^r} f_u(X_t)\big] &= \mathbb{E}'_{(x,r)}\big[e^{qR'_t} f_u(Y'_t, Z'_t)\big] \\ &= e^{\phi'(t,u,q) + \langle \psi'^{\mathcal{Y}}(t,u,q), y\rangle + \langle \psi'^{\mathcal{Z}}(t,u,q), z\rangle + qr} \end{aligned} \tag{11.8}$$

for all $(t,u,q) \in \mathbb{R}_+ \times \mathcal{U} \times i\mathbb{R}$.

PROOF. First notice that the restriction $\mathcal{A}'^{\sharp}$ of the infinitesimal generator $\mathcal{A}'$ of X' on $C_c^2(D \times \mathbb{R})$ is $\mathcal{A}'^{\sharp} f(x,r) = \mathcal{A}^{\sharp} f(x,r) + L(x)\partial_r f(x,r)$; see (2.12). On the other hand, the process $X'' = (X, R^r)$ is a $D \times \mathbb{R}$-valued semimartingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}_x)$ with characteristics (B'', C'', ν'') given by

$$B''_t = (B_t, R_t),$$

$$C''_t = \begin{pmatrix} C_t & 0 \\ 0 & 0 \end{pmatrix},$$

$$\nu''(dt, d\xi, d\rho) = \nu(dt, d\xi) \times \delta_0(d\rho).$$

Now Theorem 2.12 yields the assertion. □

In view of (11.1), we have to ask whether (11.8) has a meaning for $q = -1$. Theorem 2.16(ii) provides conditions which justify an extension of equality (11.8) for $q = -1$. These conditions have to be checked from case to case, using Lemmas 5.3 and 6.5. We do not intend to give further general results in this direction but refer to a forthcoming paper in which we proceed by means of explicit examples.

We end this section by giving (implicit) conditions which are equivalent to the existence of a continuous extension of $\phi'(t,\cdot)$ and $\psi'(t,\cdot)$ in (11.8). If

$$\mathbb{E}'_{(x,0)}\big[e^{-R'_t}\big] < \infty \qquad \forall x \in D, \tag{11.9}$$

then $Q_t f_u(x) = \mathbb{E}'_{(x,0)}[e^{-R'_t} f_u(Y'_t, Z'_t)]$ is well defined. Condition (11.9) holds true if, for instance, $\lambda^{\mathcal{Y}} \in \mathbb{R}_+^m$ and $\lambda^{\mathcal{Z}} = 0$. Indeed, if in addition $\ell \in \mathbb{R}_+$, we are back to the situation from Section 11.1. It is easy to see that then $\phi'(t,u,-1)$ and $(\psi'^{\mathcal{Y}}(t,u,-1), \psi'^{\mathcal{Z}}(t,u,-1))$, well-defined solutions of (11.5)–(11.7) for $q = -1$, are the exponents which correspond to the regular affine semigroup (Q_t) from Proposition 11.1.

In general, Lemma 3.2 tells us the following. Let $t \in \mathbb{R}_+$. Suppose (11.9) is satisfied and

$$\mathbb{E}'_{(0,0)}\big[e^{-sR'_t} f_u(Y'_t, Z'_t)\big] \neq 0 \qquad \forall\,(u,s) \in \mathcal{U} \times [0,1].$$

Then there exists a unique continuous extension of $\phi'(t,\cdot)$ and $\psi'(t,\cdot)$ on $U' := \mathcal{U} \times (i\mathbb{R} \cup [-1,0])$ such that (11.8) holds for all $(x,r,u,q) \in D \times \mathbb{R} \times U'$, in particular, for $q = -1$ (notice that $\mathbb{E}'_{(x,r)}[e^{-R'_t}] = e^{-r}\mathbb{E}'_{(x,0)}[e^{-R'_t}]$).

For the price of a bond [$f \equiv 1$ in (11.1), see Section 13.1], we then have

$$Q_t \mathbb{1}(x) = e^{A(t)+\langle B(t),x\rangle}, \tag{11.10}$$

where $A(t) := \phi'(t,0,-1)$, $B(t) := (\psi'^{\mathcal{Y}}(t,0,-1), \psi'^{\mathcal{Z}}(t,0,-1))$, and we have set $r = 0$; see (11.8).

12. The choice of the state space. We have, throughout, assumed that the state space D is $\mathbb{R}^m_+ \times \mathbb{R}^n$. Among other reasons for this choice, it allows for a unified treatment of CBI and OU-type processes. We shall see in Section 13 that this state space covers essentially all applications appearing in the finance literature. Indeed, in modeling the term structure of interest rates with affine processes that are diffusions, Dai and Singleton [31] propose $\mathbb{R}^m_+ \times \mathbb{R}^n$ as *the* natural state space to consider.

In this section, we briefly analyze whether this state space for an affine process is actually canonical. We therefore denote by D an a priori arbitrary subset of $\mathbb{R}^d$, and start by extending Definition 2.1 of an affine process to this setting.

DEFINITION 12.1. Let D be an arbitrary subset of $\mathbb{R}^d$ and $(X, (\mathbb{P}_x)_{x\in D})$ the canonical realization of a time-homogeneous Markov process with state space D. We call this process *affine* if, for each $t \geq 0$ and $u \in \mathbb{R}^d$, there is an affine function $x \mapsto h(t,u)(x)$, defined for $x \in D$, such that

$$\mathbb{E}_x\big[e^{i\langle u, X_t\rangle}\big] = e^{h(t,u)(x)}. \tag{12.1}$$

For the case of $d = m + n$ and $D = \mathbb{R}^m_+ \times \mathbb{R}^n$, the above definition is merely a reformulation of Definition 2.1.

Slightly more generally, let $T : \mathbb{R}^d \to \mathbb{R}^d$ be an affine isomorphism, that is, $T(x) = x_0 + S(x)$, where $x_0 \in \mathbb{R}^d$ and S is a linear isomorphism of $\mathbb{R}^d$ onto itself, and let $D = T(\mathbb{R}^m_+ \times \mathbb{R}^n)$. It is rather obvious that a time-homogeneous Markov process X on D is affine, if and only if $T^{-1}(X)$ is affine on the state space $\mathbb{R}^m_+ \times \mathbb{R}^n$. Hence the above analysis carries over verbatim to images $D = T(\mathbb{R}^m_+ \times \mathbb{R}^n)$ of the canonical state space $\mathbb{R}^m_+ \times \mathbb{R}^n$ under an affine isomorphism T. By a convenient rotation, for instance, we obtain an affine process with nondiagonal diffusion matrix for the CBI part; see (2.27). Or, by a translation, we can construct an affine term structure model with negative short rates which are bounded from below, etc.

12.1. *Degenerate examples.* Now, we pass to state spaces of a different form than the image under an affine isomorphism of $\mathbb{R}^m_+ \times \mathbb{R}^n$. We begin with some relatively easy and degenerate examples.

EXAMPLE 12.2. Let $(X, (\mathbb{P}_x)_{x\in\mathbb{R}^d})$ be the deterministic process

$$p_t(x, d\xi) = \delta_{(e^{-t\|x\|}x)} \qquad \text{for } x \in \mathbb{R}^d,\ t \geq 0,$$

which simply describes a homogeneous radial flow towards 0 on $\mathbb{R}^d$. The infinitesimal generator $\mathcal{A}$ of the process X is given by

$$\mathcal{A}f = -\sum_{i=1}^{d} x_i \frac{\partial f}{\partial x_i}.$$

This process is affine on $\mathbb{R}^d$, as we have

$$\mathbb{E}_x\big[e^{i\langle u, X_t\rangle}\big] = e^{i\langle u, e^{-t\|x\|}x\rangle} = e^{h(t,u)(x)}.$$

In this—rather trivial—example, the state space $D = \mathbb{R}^d$ is, of course, not unique. In fact, each $D \subseteq \mathbb{R}^d$ that is star-shaped around the origin is invariant under X, and therefore X induces a Markov process on D. Clearly, for each such set D, the induced process is still affine on D. What is slightly less obvious is that, if D affinely spans $\mathbb{R}^d$, the process $(X, (\mathbb{P}_x)_{x\in\mathbb{R}^d})$ is determined already (as an affine Markov process) by its restriction to D. This fact is isolated in the subsequent lemma.

LEMMA 12.3. *Let D_1, D_2 be subsets of $\mathbb{R}^d$, $D_1 \subseteq D_2$, and suppose that D_1 affinely spans $\mathbb{R}^d$. Let $(X, (\mathbb{P}_x)_{x\in D_2})$ be an affine time-homogeneous Markov process on D_2, so that D_1 is invariant under X. Then the affine Markov process $(X, (\mathbb{P}_x)_{x\in D_2})$ is already determined by its restriction to D_1, that is, by the family $(\mathbb{P}_x)_{x\in D_1}$.*

PROOF. If suffices to remark that an affine function $x \mapsto h(t,u)(x)$ defined on $\mathbb{R}^d$, is already specified by its values on a subset D_1 that affinely spans $\mathbb{R}^d$. This determines the values $\mathbb{E}_x[e^{i\langle u, X_t\rangle}]$ for $x \in D_2$, $t \geq 0$ and $u \in \mathbb{R}^d$, which in turn determines the time-homogeneous Markov process $(X, (\mathbb{P}_x)_{x\in D_2})$. $\square$

We now pass to an example of a Markov process X whose "maximal domain" $D \subseteq \mathbb{R}^3$ is not of the form $\mathbb{R}^m_+ \times \mathbb{R}^n$.

EXAMPLE 12.4. Let $d = 3$, $D = [0,1] \times \mathbb{R}^2$, and define the infinitesimal generator $\mathcal{A}$ by

$$\mathcal{A}f(x_1, x_2, x_3) = -x_1 \frac{\partial f}{\partial x_1} + x_1 \frac{\partial^2 f}{\partial x_2^2} + (1 - x_1)\frac{\partial^2 f}{\partial x_3^2},$$

defined for $f \in C_c^\infty(D)$.

The corresponding time-homogeneous Markov process X is most easily described in prose as follows. In the first coordinate x_1, this process is simply a deterministic flow towards the origin, as in Example 12.2. In the second and third coordinate, the process is a diffusion with zero drift, and with volatility equal to x_1 and $(1 - x_1)$, respectively.

It is straightforward to check that $\mathcal{A}$ well defines an affine Markov process X on D. The point of the above example is that the set D is the maximal domain on which X may be defined. Indeed, the volatility of the second and third coordinate of X must clearly be non-negative, which forces x_1 to be in $[0, 1]$. This can also be verified by calculating explicitly the characteristic functions of the process X, a task left to the energetic reader (compare the calculation in the subsequent example).

12.2. *Nondegenerate example.* The above examples are somewhat artificial, as they have, at least in one coordinate, purely deterministic behavior. The next example does not share this feature.

Let $(W_t)_{t\geq 0}$ denote standard Brownian motion, and define the $\mathbb{R}^2$-valued process $(X_t^{(0,0)})_{t\geq 0} = (Y_t^{(0,0)}, Z_t^{(0,0)})_{t\geq 0}$, starting at $X_0^{(0,0)} = (0, 0)$, by

$$X_t^{(0,0)} = (W_t^2, W_t), \qquad t \geq 0.$$

This process satisfies the stochastic differential equation

$$\begin{aligned} dY_t &= 2Z_t\, dW_t + dt, \\ dZ_t &= dW_t \end{aligned}$$

and takes its values in the parabola $P = \{(z^2, z) : z \in \mathbb{R}\}$.

Obviously $(X_t^{(0,0)})_{t\geq 0}$ is a time-homogeneous Markov process, and we may calculate the associated characteristic function

$$\begin{aligned} \mathbb{E}_{(0,0)}\big[e^{i\langle u, X_t\rangle}\big] &= \mathbb{E}\big[e^{i(vW_t^2 + wW_t)}\big] \\ &= \frac{1}{\sqrt{1 - 2ivt}} e^{-tw^2/(2(1-2ivt))}, \qquad u = (v, w) \in \mathbb{R}^2. \end{aligned}$$

More generally, fixing a starting point $(z^2, z) \in P$, we obtain for

$$\big(X_t^{(z^2,z)}\big)_{t\geq 0} = \big((W_t + z)^2, (W_t + z)\big)_{t\geq 0}$$

the characteristic function

$$\begin{aligned} \mathbb{E}_{(z^2,z)}\big[e^{i\langle u, X_t\rangle}\big] &= \mathbb{E}\big[e^{i(v(W_t+z)^2 + w(W_t+z))}\big] \\ &= \frac{1}{\sqrt{1 - 2ivt}} e^{(-tw^2 + 2ivz^2 + 2iwz)/(2(1-2ivt))}. \end{aligned} \tag{12.2}$$

Now, one makes the crucial observation that the last expression is exponential-affine in the variables $(y, z) = (z^2, z)$, as (y, z) ranges through P. In other words, the process $(X, (\mathbb{P}_x)_{x \in P})$ is affine.

What is the maximal domain $D \subseteq \mathbb{R}^2$, to which this affine Markov process can be extended?

We know from Lemma 12.3 that, if such an extension to a set $D \supseteq P$ is possible, then it is uniquely determined, and, for $x = (y, z) \in D$, we necessarily have

$$\begin{aligned} \mathbb{E}_{(y,z)}\big[e^{i\langle u, X_t\rangle}\big] &= \frac{1}{\sqrt{1-2ivt}} e^{(-tw^2+2ivy+2iwz)/(2(1-2ivt))} \\ &=: e^{h(t,u)(y,z)}. \end{aligned} \tag{12.3}$$

If $y < z^2$, then one verifies that—at least for $t > 0$ sufficiently small—the expression on the right-hand side of (12.3) is not the characteristic function of a probability distribution on $\mathbb{R}^2$. Indeed, letting $u = (\frac{1}{2}, -z)$ we may estimate

$$\begin{aligned} \big|e^{h(t,u)(y,z)}\big|^2 &= \frac{1}{\sqrt{1+t^2}} \exp\Big[-\frac{t}{1+t^2}(y - z^2)\Big] \\ &= 1 - (y - z^2)t + O(t^2), \end{aligned}$$

which is strictly bigger than 1, for $t > 0$ small enough. Hence, the process X cannot have a state space extending below the parabola P. The maximal remaining candidate for an extension is $D = \{(y, z) : y \geq z^2\}$, that is, the epigraph of the parabola P.

We shall now show that X may indeed be extended to D. In order to do so, let $(B_t)_{t \geq 0}$ denote another standard Brownian motion, independent of $(W_t)_{t \geq 0}$, and define the $\mathbb{R}_+$-valued Feller diffusion process $(F_t)_{t \geq 0}$ (which is just the Bessel-squared process of dimension zero in the terminology of [76]) by

$$dF_t = 2\sqrt{F_t}\, dB_t.$$

The characteristic function of F_t, starting at $F_0 = f \geq 0$, is given by

$$\mathbb{E}_f\big[e^{ivF_t}\big] = e^{ivf/(1-2ivt)}, \qquad v \in \mathbb{R}. \tag{12.4}$$

After this preparation, we define the D-valued process $X^{(y,z)}$ starting at $X_0^{(y,z)} = (y, z) \in D$ in the following way. The point $(y, z) \in D$ may be uniquely written as $(y, z) = (z^2 + f, z)$, where $f \geq 0$. We define $(X_t^{(y,z)})_{t \geq 0} = (Y_t^{(y,z)}, Z_t^{(y,z)})_{t \geq 0}$ by

$$\begin{aligned} Y_t^{(y,z)} &= (W_t + z)^2 + F_t^{(f)}, \\ Z_t^{(y,z)} &= (W_t + z). \end{aligned} \tag{12.5}$$

Calculating the characteristic function of $X_t^{(y,z)}$, we obtain from (12.4), (12.2) and the independence of $(F_t)_{t \geq 0}$ and $(W_t)_{t \geq 0}$, that

$$\mathbb{E}_{(y,z)}[e^{i\langle u, X_t\rangle}] = \frac{1}{\sqrt{1-2ivt}} e^{(-tw^2+2ivz^2+2iwz+2ivf)/(2(1-2ivt))},$$

which is the desired expression (12.3).

Hence, (12.5) defines an affine Markov process with maximal state space $D = \{(y, z) : y \geq z^2\}$, extending the process (12.2) from P to D. This affine process obeys the stochastic differential equation

$$\begin{aligned} dY_t &= 2Z_t\, dW_t + 2\sqrt{Y_t - Z_t^2}\, dB_t + dt, \\ dZ_t &= dW_t. \end{aligned} \tag{12.6}$$

The coefficient matrix of the diffusion term

$$\sigma(y, z) = \begin{pmatrix} 2z & 2\sqrt{y - z^2} \\ 1 & 0 \end{pmatrix}$$

satisfies

$$\begin{aligned} \sigma(y, z)\sigma^T(y, z) &= \begin{pmatrix} 2z & 2\sqrt{y - z^2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2z & 1 \\ 2\sqrt{y - z^2} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 4y & 2z \\ 2z & 1 \end{pmatrix}, \end{aligned}$$

so that $\sigma(y, z)\sigma^T(y, z)$ indeed is an affine function in $(y, z) \in \mathbb{R}^2$. One also notes that $\sigma(y, z)\sigma^T(y, z)$ is non-negative definite if and only if $y \geq z^2$; that is, $(y, z) \in D$.

Summing up, we have exhibited a diffusion process, driven by a two-dimensional Brownian motion via (12.6), which is an affine Markov process with maximal state space $D = \{(y, z) : y \geq z^2\}$. This shows that there are nontrivial situations in which the natural domain of an affine process is not of the form $\mathbb{R}_+^m \times \mathbb{R}^n$. A more systematic investigation of this phenomenon is left to future research.

We end this section by formulating a rather bold conjecture. For diffusion processes in $\mathbb{R}^2$ (or maybe even in $\mathbb{R}^d$, for $d \geq 3$), this last example is "essentially" the only situation of an affine Markov process whose maximal state space is not (up to the image under an affine isomorphism) of the form $\mathbb{R}_+^m \times \mathbb{R}^n$. We are deliberately vague about the meaning of the word "essentially." What we have in mind is excluding counterexamples of a trivial kind, as considered in Examples 12.2 and 12.4.

13. Financial applications. Several strands of financial modeling have made extensive use of the special properties of affine processes, both for their tractability and for their flexibility in capturing certain stochastic properties that are apparent in many financial markets, such as jumps and stochastic volatility in various forms. In addition to applications summarized below regarding the valuation of financial assets in settings of affine processes, recent progress [17, 22, 66, 67, 80–82, 93] in the modeling of optimal dynamic portfolio and consumption choice has exploited the special structure of controlled affine state-process models.

We fix a conservative regular affine process X with semigroup (P_t), as in Section 2, and a "discounting" semigroup $(Q_t)_{t\in\mathbb{R}_+}$ based, as in Section 11, on a short-rate process $L(X)$. We shall view $Q_t f(X_s)$ as the price at time s of a financial asset paying the amount $f(X_{s+t})$ at time $s+t$. This implies a particular "risk-neutral" interpretation [52, 32] of the semigroup (P_t) that we shall not detail here. We emphasize, however, that statistical analysis of the time series of X, or measurement of the risk of changes in market values of financial assets, would be based on the distribution of X under an "actual" probability measure that is normally distinct from that associated with the "risk-neutral" semigroup (P_t).

13.1. *The term structure of interest rates.* A central object of study in finance is the term structure $t \mapsto Q_t\mathbb{1}$ of prices of "bonds," assets that pay one unit of account at a given maturity t. (From these, the prices of bonds that make payments at multiple dates, and other "fixed-income" securities, can be built up.)

Early prominent models of interest-rate behavior were based on such simple models of the short rate $L(X)$ as the Vasicek (Gaussian Ornstein–Uhlenbeck) process [90], or the Cox–Ingersoll–Ross process [30], which is the continuous branching diffusion of Feller [43]. Both of these short-rate processes are of course themselves affine $[L(x) = x]$, as are many variants [20, 23, 30, 49, 58, 59, 72, 90, 74].

In general, because $\mathbb{1} = e^{\langle 0,x\rangle}$, the bond price

$$Q_t\mathbb{1}(x) = e^{A(t)+\langle B(t),x\rangle} \tag{13.1}$$

is easily calculated from the generalized Riccati equations for a broad range of affine processes [see (11.10)]. Indeed, given the desire to model interest rates with ever increasing realism, various higher-dimensional ($d > 1$) variants have appeared [5, 6, 11, 14, 23, 30, 31, 54, 64, 69], and efforts [13, 18, 31, 39, 46, 45], including this paper, have been directed to the classification and unification of affine term-structure models. Beyond the scope of our analysis here, there are in fact "infinite-dimensional affine term-structure models" [51, 28, 27].

Empirical analyses of interest-rate behavior based on the properties of affine models include [19, 25, 31, 34, 36, 50, 55, 65, 74, 75, 91], with a related analysis of foreign-currency forwards in [2]. Statistical methods developed specifically for the analysis of time-series data from affine models have been based on approximation of the likelihood function [68, 40], on generalized method of moments [48] or on spectral properties, making use of the easily calculated complex moments of affine processes [61, 21, 86].

13.2. *Default risk.* In order to model the timing of default of financial contracts, we suppose that N is a nonexplosive counting process [16] (defined on an enlarged probability space) that is doubly stochastic driven by X, with intensity $\{\Lambda(X_{t-}) : t \geq 0\}$, where $x \mapsto \Lambda(x) \geq 0$ is affine. That is, conditional on X, N is Poisson with time-varying intensity $\{\Lambda(X_{t-}) : t \geq 0\}$. In fact, (X, N) is an affine

process, and one can enlarge the augmented filtration $(\mathcal{F}_t)$ of X to that of (X, N). The default time τ is defined as the first jump time of N.

From the doubly stochastic property of N, the survival probability is

$$\mathbb{P}_x(\tau > t) = \mathbb{E}_x\left[\exp\left(-\int_0^t \Lambda(X_s)\,ds\right)\right],$$

which is of the same form as the bond-price calculation (13.1), although with a different effective discount rate. The popularity of affine models of interest rates has thus led to the common application of affine processes to default modeling, as in [35, 41] and [63].

A defaultable bond with maturity t is a financial asset paying $\mathbb{1}_{\{\tau>t\}}$ at t. Applying the doubly stochastic property, Lando [63] showed that the defaultable bond has a price of

$$\mathbb{E}_x\left[\exp\left(-\int_0^t L(X_s)\,ds\right)\mathbb{1}_{\{\tau>t\}}\right] = \mathbb{E}_x\left[\exp\left(-\int_0^t \big(L(X_s)+\Lambda(X_s)\big)\,ds\right)\right].$$

Because $x \mapsto L(x) + \Lambda(x)$ is affine, the defaultable bond price is again of the tractable form of the default-free bond price (13.1), with new coefficients. Various approaches [63, 60, 41, 71] to modeling nonzero recovery at default have been adopted.

For a model of the default times $\tau_1, \dots, \tau_k$ of $k > 1$ different financial contracts, an approach is to suppose that τ_i is the first jump time of a nonexplosive counting process N_i with intensity $\{\Lambda_i(X_{t-}) : t \geq 0\}$, for affine $x \mapsto \Lambda_i(x) \geq 0$, where $N_1, \dots, N_k$ are doubly stochastic driven by X, and moreover are independent conditional on X. [Again, $(X, N_1, \dots, N_k)$ may be viewed as an affine process.] The doubly stochastic property implies that, for any sequence of times $t_1, t_2, \dots, t_k$ in $\mathbb{R}_+$ that is (without loss of generality) increasing,

$$\mathbb{P}_x(\tau_1 \geq t_1, \dots, \tau_k \geq t_k) = \mathbb{E}_x\left[\exp\left(-\int_0^{t_k} \Lambda(X_s, s)\,ds\right)\right],$$

where

$$\Lambda(x, s) = \sum_{\{i\,:\,s \leq t_i\}} \Lambda_i(x).$$

By the law of iterated expectations, the joint distribution of the default times is given by

$$\mathbb{P}_x(\tau_1 \geq t_1, \dots, \tau_k \geq t_k) = e^{\phi_0 + \langle \psi_0, x\rangle}, \tag{13.2}$$

where ϕ_i and ψ_i are defined inductively by $\phi_k = 0$, $\psi_k = 0$, and

$$e^{\phi_i + \langle \psi_i, x\rangle} = \mathbb{E}_x\left[\exp\left(-\int_0^{t_{i+1}-t_i} \Lambda(X_t, t_i + t)\,dt\right) e^{\phi_{i+1} + \langle \psi_{i+1}, X_{t_{i+1}-t_i}\rangle}\right],$$

taking $t_0 = 0$, and using the fact that $x \mapsto \Lambda(x, s) \geq 0$ is affine with constant coefficients for s between t_i and t_{i+1}. Because the coefficients ϕ_i and ψ_i are easily

calculated recursively from the associated generalized Riccati equations, one can implement (13.2) for the calculation of the probability distribution of the total default losses on a portfolio of financial contracts, as in [38].

Similarly, the "first default time" $\tau^* = \inf\{\tau_1, \dots, \tau_k\}$, which is the basis of certain financial contracts [37], satisfies

$$\mathbb{P}_x(\tau^* > t) = \mathbb{E}_x\left[\exp\left(-\int_0^t \Lambda^*(X_s)\,ds\right)\right],$$

where $\Lambda^*(x) = \sum_{i=1}^k \Lambda_i(x)$, again of the form of (13.1).

13.3. *Option pricing.* We consider an option that conveys the opportunity, but not the obligation, to sell an underlying asset at time t for some fixed price of $K > 0$ in $\mathbb{R}$. This is known as a "put" option; the corresponding "call" option to buy the asset may be treated similarly.

Suppose the price of the underlying asset at time t is of the form $f(X_t)$, for some non-negative $f \in C(D)$. The option is rationally exercised if and only if $f(X_t) \le K$, and so has the payoff $g(X_t) = \max(K - f(X_t), 0)$, and the initial price

$$\begin{aligned} Q_t g(x) &= \mathbb{E}_x\left[\exp\left(-\int_0^t L(X_s)\,ds\right) g(X_t)\right] \\ &= K\mathbb{E}_x\left[\exp\left(-\int_0^t L(X_s)\,ds\right)\mathbb{1}_{\{f(X_t)\le K\}}\right] \\ &\quad - \mathbb{E}_x\left[\exp\left(-\int_0^t L(X_s)\,ds\right) f(X_t)\mathbb{1}_{\{f(X_t)\le K\}}\right]. \end{aligned}$$

One can exploit the affine modeling approach to computational advantage provided $f(x) = ke^{\langle b,x\rangle}$, for some constant $k > 0$ in $\mathbb{R}$ and some b in $\mathbb{R}^d$, an example of which is the bond price $f(x) = e^{A(T-t)+\langle B(T-t),x\rangle}$ of (13.1), as of time t, for a maturity date $T > t$. In this case, both terms in the calculation above of $Q_t g(x)$ are of the form

$$G_{a,b}(q) = \mathbb{E}_x\left[\exp\left(-\int_0^t L(X_s)\,ds\right) e^{\langle a,X_t\rangle}\mathbb{1}_{\{\langle b,X_t\rangle\le q\}}\right],$$

for some $(a, b, q) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$. (Here, $q = \log K - \log k$.) Because $G_{a,b}(\cdot)$ is the distribution function of $\langle b, X_t\rangle$ with respect to the measure

$$\exp\left(-\int_0^t L(X_s)\,ds\right) e^{\langle a,X_t\rangle}\mathbb{P}_x,$$

it is enough to be able to compute the transform

$$\mathcal{G}_{a,b}(z) = \int_{-\infty}^{+\infty} e^{izq}\, G_{a,b}(dq),$$

for then well-known Fourier-inversion methods can be used to compute $G_{a,b}(q)$. One can see, however, that

$$\begin{aligned}\mathcal{G}_{a,b}(z) &= \mathbb{E}_x\left[\exp\left(-\int_0^t L(X_s)\,ds\right)e^{\langle a,X_t\rangle}e^{iz\langle b,X_t\rangle}\right]\\ &= \mathbb{E}_x\left[\exp\left(-\int_0^t L(X_s)\,ds\right)f_u(X_t)\right],\end{aligned}$$

where $u = a + izb$, and the generalized Riccati equations give the solution under nonnegativity of $L(X)$, or under conditions described at the end of Section 11.

This is the Heston [56] approach to option pricing, building on earlier work of Stein and Stein [87] that did not exploit the properties of affine processes. Heston's objective was to extend the Black–Scholes model [15], for which the underlying price process is a geometric Brownian motion, to allow "stochastic volatility." In [56], the underlying asset price is e^{Z_t}, where (Y, Z) is the affine process ($m = n = 1$) defined by

$$\begin{aligned}dY_t &= (b_1 - \beta Y_t)\,dt + \sigma\sqrt{Y_t}\,dW_t^{(1)},\\ dZ_t &= b_2\,dt + \sqrt{Y_t}\big(\rho\,dW_t^{(1)} + \sqrt{1-\rho^2}\,dW_t^{(2)}\big)\end{aligned}$$

for real constants $\rho \in (-1, 1)$, $b_1, \sigma \geq 0$ and b_2, β, and where $(W^{(1)}, W^{(2)})$ is a standard Brownian motion in $\mathbb{R}^2$. The "stochastic volatility" process Y is constant in the special Black–Scholes case of a geometric Brownian price process e^Z. For Heston's model, the Fourier transform $\mathcal{G}_{a,b}(\cdot)$ is computed explicitly in [56].

A defaultable option may be likewise priced by replacing $L(X_t)$ with $L(X_t) + \Lambda(X_t)$, where $\{\Lambda(X_{t-}) : t \geq 0\}$ determines the default intensity, as for defaultable bond pricing.

Numerous affine generalizations [3, 4, 7–9, 24, 26, 40, 83, 84] of the Heston model have been directed toward more realistic stochastic volatility and jump behavior. Pan [73] conducted a time-series analysis of the S&P 500-index data, both the underlying returns as well as option prices, based on an affine jump-diffusion model of returns. Special numerical methods for more general derivative-asset pricing with affine processes have been based [89] on the Fourier inversion of their characteristic functions.

APPENDIX: ON THE REGULARITY OF CHARACTERISTIC FUNCTIONS

Let $N \in \mathbb{N}$ and ν be a bounded measure on $\mathbb{R}^N$. Denote by

$$g(y) = \int_{\mathbb{R}^N} e^{i\langle y,x\rangle}\,\nu(dx), \qquad y \in \mathbb{R}^N,$$

its characteristic function. To avoid unnecessary notational complications, we introduce the function

$$h(z) = \int_{\mathbb{R}^N} e^{\langle z,x\rangle}\,\nu(dx),$$

which is well defined if $\operatorname{Re} z \in V$ where

$$V := \left\{ y \in \mathbb{R}^N \;\middle|\; \int_{\mathbb{R}^N} e^{\langle y,x\rangle} \nu(dx) < \infty \right\}. \tag{A.1}$$

Clearly, we have $0 \in V$ and $g(y) = h(iy)$ on $\mathbb{R}^N$. We shall investigate the interplay between the (marginal) moments of ν and the corresponding (partial) regularity of g and h, respectively.

LEMMA A.1. *Let $k \in \mathbb{N}$ and $1 \le i \le N$. If $(\partial_{y^i})^{2k} g(0)$ exists then*

$$\int_{\mathbb{R}^N} (x_i)^{2k} \nu(dx) < \infty. \tag{A.2}$$

On the other hand, if $\int_{\mathbb{R}^N} \|x\|^k \nu(dx) < \infty$ then $g \in C^k(\mathbb{R}^N)$ and

$$\partial_{y_{i_1}} \cdots \partial_{y_{i_l}} g(y) = i^l \int_{\mathbb{R}^N} x_{i_1} \cdots x_{i_l} e^{i\langle y,x\rangle} \nu(dx)$$

for all $y \in \mathbb{R}^N$, $1 \le i_1, \dots, i_l \le N$ and $1 \le l \le k$.

PROOF. Let e_i denote the ith basis vector in $\mathbb{R}^N$. Observe that $s \mapsto g(se_i)$ is the characteristic function of the marginal measure $\nu_i(dt)$ on $\mathbb{R}$ given by $\nu_i(U) = \int_{\mathbb{R}^N} \mathbb{1}_U(x_i)\, \nu(dx)$, $U \subset \mathbb{R}$ measurable. Now $(\partial_s)^{2k} g(se_i)|_{s=0} = (\partial_{y^i})^{2k} g(0)$, and the assertion follows from [70], Theorem 2.3.1.

The second part of the lemma follows by dominated convergence. □

LEMMA A.2. *The set V, given in* (A.1), *is convex. Moreover, let V_0 be an open set in $\mathbb{R}^N$ such that $V_0 \subset V$. Then h is analytic on the open strip*

$$S := \{ z \in \mathbb{C}^N \mid \operatorname{Re} z \in V_0 \}. \tag{A.3}$$

PROOF. Let $a, b \in V$. First, we show that $sa \in V$ for all $s \in [0,1]$. Denote by $\nu_a(dt)$ the image measure of ν on $\mathbb{R}$ by the mapping $x \mapsto \langle a, x\rangle$. Then $h(sa) = \int_{\mathbb{R}} e^{st} \nu_a(dt) < \infty$ for $s = 0, 1$, and hence for all $s \in [0,1]$, which is seen by decomposition of the integral $\int_{\mathbb{R}} = \int_{(-\infty,0)} + \int_{[0,\infty)}$.

In general, we write $e^{\langle a,x\rangle} = e^{\langle a-b,x\rangle} e^{\langle b,x\rangle}$ and notice that $\nu'(dx) = e^{\langle b,x\rangle} \nu(dx)$ is a finite measure on $\mathbb{R}^N$. Hence it follows by the above argument that $s(a-b) + b \in V$ for all $s \in [0,1]$, whence V is convex.

Recall the fact that h, being continuous on S, is analytic on S if and only if, for every $1 \le i \le N$ and $z \in S$, the function

$$h_{i,z}(\zeta) := h(z_1, \dots, z_{i-1}, \zeta, z_{i+1}, \dots, z_N)$$

is analytic on $S_i(z) := \{\zeta \in \mathbb{C} \mid \operatorname{Re}(z_1, \dots, z_{i-1}, \zeta, z_{i+1}, \dots, z_N) \in V_0\}$. This follows from the Cauchy formula; see [33], Section IX.9. By the definition of V, we can write

$$h_{i,z}(\zeta) = \int_{\mathcal{R}_{i,-}} e^{\zeta x_i} e^{\langle z_{I(i)}, x_{I(i)}\rangle} \nu(dx) + \int_{\mathcal{R}_{i,+}} e^{\zeta x_i} e^{\langle z_{I(i)}, x_{I(i)}\rangle} \nu(dx), \tag{A.4}$$

where $\mathcal{R}_{i,-} := \{x \in \mathbb{R}^N \mid x_i < 0\}$, $\mathcal{R}_{i,+} := \{x \in \mathbb{R}^N \mid x_i \geq 0\}$ and $I(i) := \{1, \dots, N\} \setminus \{i\}$. Define $\rho_{i,-}(z) := \inf\{t \in \mathbb{R} \mid \operatorname{Re}(z_1, \dots, z_{i-1}, t, z_{i+1}, \dots, z_N) \in V\}$ and $\rho_{i,+}(z) := \sup\{t \in \mathbb{R} \mid \operatorname{Re}(z_1, \dots, z_{i-1}, t, z_{i+1}, \dots, z_N) \in V\}$. By assumption we have $-\infty \leq \rho_{i,-} < \rho_{i,+} \leq \infty$. By dominated convergence we obtain that the two integrals in (A.4) are analytic on the half-planes $\{\zeta \in \mathbb{C} \mid \operatorname{Re}\zeta > \rho_{i,-}(z)\}$ and $\{\zeta \in \mathbb{C} \mid \operatorname{Re}\zeta < \rho_{i,+}(z)\}$, respectively. Hence $h_{i,z}$ is analytic on S_i, and the assertion follows. $\square$

In general, V does not contain an open set V_0 in $\mathbb{R}^N$. The next two lemmas provide sufficient conditions for the existence of such a V_0. Let $\rho = (\rho_1, \dots, \rho_N) \in \mathbb{R}^N_{++}$ and define the open polydisc in $\mathbb{C}^N$ with center 0,

$$P_\rho := \{z \in \mathbb{C}^N \mid |z_i| < \rho_i,\ i = 1, \dots, N\}.$$

LEMMA A.3. *Suppose $g(y) = G(iy)$ for all $y \in P_\rho \cap \mathbb{R}^N$, where G is an analytic function on P_ρ. Then $P_\rho \cap \mathbb{R}^N \subset V$ and $h = G$ on P_ρ.*

PROOF. Let $t \in (0, 1)$. By assumption we have that g is analytic on $P_{t\rho} \cap \mathbb{R}^N$ and, by the Cauchy formula,

$$g(y) = \sum_{i_1, \dots, i_N \in \mathbb{N}_0} c_{i_1, \dots, i_N} y_1^{i_1} \cdots y_N^{i_N} \qquad \forall\, y \in P_{t\rho} \cap \mathbb{R}^N,$$

where $\sum_{i_1, \dots, i_N \in \mathbb{N}_0} c_{i_1, \dots, i_N} z_1^{i_1} \cdots z_N^{i_N} = G(iz)$ on $P_{t\rho}$. This power series is absolutely convergent on $P_{t\rho}$. By Lemma A.1, we have

$$c_{i_1, \dots, i_N} = \frac{i^{i_1 + \cdots + i_N}}{i_1! \cdots i_N!} \int_{\mathbb{R}^N} x_1^{i_1} \cdots x_N^{i_N}\, \nu(dx).$$

From the inequality ${x_l}^{2k-1} \leq ({x_l}^{2k} + {x_l}^{2k-2})/2$ we see that

$$T(y) := \sum_{i_1, \dots, i_N \in \mathbb{N}_0} d_{i_1, \dots, i_N} |y_1^{i_1} \cdots y_N^{i_N}| < \infty \qquad \forall\, y \in P_{t\rho},$$

where

$$d_{i_1, \dots, i_N} := \frac{1}{i_1! \cdots i_N!} \int_{\mathbb{R}^N} |x_1^{i_1} \cdots x_N^{i_N}|\, \nu(dx).$$

However,

$$\begin{aligned} T(y) &\geq \sum_{i_1, \dots, i_N \in \mathbb{N}_0} \frac{|y_1^{i_1} \cdots y_N^{i_N}|}{i_1! \cdots i_N!} \int_K |x_1^{i_1} \cdots x_N^{i_N}|\, \nu(dx) \\ &= \int_K \exp\left(\sum_{i=1}^N |y_i||x_i| \right) \nu(dx) \end{aligned}$$

for every compact $K \subset \mathbb{R}^N$ and $y \in P_{t\rho} \cap \mathbb{R}^N$. Therefore the integral

$$\int_{\mathbb{R}^N} \exp\left(\sum_{i=1}^{N} |y_i||x_i|\right) \nu(dx)$$

is finite for all $y \in P_{t\rho} \cap \mathbb{R}^N$. Hence $P_{t\rho} \cap \mathbb{R}^N \subset V$, and since $t \in (0,1)$ was arbitrary, $P_\rho \cap \mathbb{R}^N \subset V$. We conclude by Lemma A.2. $\square$

An extension of the preceding considerations yields the following useful result.

LEMMA A.4. *Let U be an open convex neighborhood of* 0 *in $\mathbb{C}^N$ and G an analytic function on U. Suppose that $g(y) = G(iy)$ for all $iy \in U \cap i\mathbb{R}^N$. Then $U \cap \mathbb{R}^N \subset V$.*

PROOF. There exists an open polydisc P_ρ in $\mathbb{C}^N$ with center 0 such that $P_\rho \subset U$. By Lemma A.3, $h = G$ on $P_\rho \cap \mathbb{R}^N \subset V$. Now let $a \in U \cap \mathbb{R}^N$. Since U is open convex, there exists $s_0 > 1$ such that $sa \in U \cap \mathbb{R}^N$ for all $s \in [0, s_0]$, and the function $G_a(s) := G(sa)$ is analytic on $(0, s_0)$. As in the proof of Lemma A.2 let $\nu_a(dt)$ be the image measure of ν by the mapping $x \mapsto \langle a, x\rangle$. Then there exists $s_1 > 0$ such that $h_{a,+}(s) := \int_{[0,\infty)} e^{st} \nu_a(dt) < \infty$ for all $s \in [0, s_1]$, and $h_{a,+}$ is analytic on $(0, s_1)$. Now obviously $h_{a,-}(s) := \int_{(-\infty,0)} e^{st} \nu_a(dt)$ is finite and analytic on $\mathbb{R}_+$. We thus have $h_{a,+} = G_a - h_{a,-}$ on $(0, s_0 \wedge s_1)$. By monotone convergence we conclude that $h_{a,+}(s)$ must be finite for all $s \in [0, s_0]$, and thus $a \in V$. $\square$

Convexity of U is in fact a too strong assumption for Lemma A.4. Its proof only requires that U is open in $\mathbb{C}^N$ and that $U \cap \mathbb{R}^N$ is star-shaped with respect to 0 ($a \in U \cap \mathbb{R}^N$ implies $sa \in U \cap \mathbb{R}^N$ for all $s \in [0, 1]$). However, Lemma A.2 then immediately yields that the convex hull, say V_0, of $U \cap \mathbb{R}^N$ lies in V, and G has an analytic extension on the strip S [see (A.3)], which is a convex open neighborhood of 0 in $\mathbb{C}^N$.

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