The Annals of Applied Probability
2003, Vol. 13, No. 3, 890–913

VOLATILITY TIME AND PROPERTIES OF OPTION PRICES

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We use a notion of stochastic time, here called *volatility time*, to show convexity of option prices in the underlying asset if the contract function is convex as well as continuity and monotonicity of the option price in the volatility. The volatility time is obtained as the almost surely unique stopping time solution to a random ordinary differential equation related to volatility. This enables us to write price processes, or processes modeled by local martingales, as Brownian motions with respect to volatility time. The results are shown under very weak assumptions and are of independent interest in the study of stochastic differential equations. Options on several underlying assets are also studied and we prove that if the volatility matrix is independent of time, then the option prices decay with time if the contract function is convex. However, the option prices are no longer necessarily convex in the underlying assets and the option prices do not necessarily decay with time, if a time-dependent volatility is allowed.

1. Introduction. Consider the spot price S of some asset following the risk neutral process

$$dS = S\sigma(S,t)\,dB, \tag{1}$$

with initial condition $S(t) = s$, where B is a Brownian motion and σ is called the volatility of S. We here compute the price with respect to some suitable numeraire process, for instance the price of a zero coupon bond, maturing at some future time T, to avoid the drift in the process for S associated with interest rates. We are interested in general properties of prices of simple contingent claims maturing at T. The arbitrage free price of a simple claim with contract function Φ is given by

$$F(s,t) = E_{s,t}[\Phi(S(T))], \tag{2}$$

according to [2]. In [1] it is shown that the price $F(s,t)$ is a convex function of s if Φ is a convex function. From the Black–Scholes equation,

$$F_t + \tfrac{1}{2}s^2\sigma^2(s,t)F_{ss} = 0, \tag{3}$$

corresponding to (2) through the Feynman–Kac stochastic representation formula, it follows that the convexity of F in s is equivalent to F decaying with time when the price of the underlying asset is constant. In [1] it is also shown that the price F

Received September 2001; revised April 2002.
AMS 2000 subject classifications. 91B28, 60H30, 35K15.
Key words and phrases. Volatility, stochastic time, contingent claim, convexity, time decay.

is monotonic in the volatility. However, the arguments presented in [1] for these results require the volatility to be a differentiable function of the underlying asset price.

In the present paper (see Sections 5 and 7), we generalize the results above to volatilities that are not even continuous in time and only satisfy a local Hölder$(1/2)$ condition in s. We believe that these conditions, especially the lack of continuity assumption in t, are natural in applications as well as mathematically satisfying. To obtain these results, we use the standard fact that a local martingale can be represented as a time-change of a Brownian motion. In our context, this entails defining a notion of stochastic time for risky assets which we refer to as *volatility time*. It is defined in Section 3 and its aim is to reduce the study of price processes modeled by local martingales in the form (1) to the study of Brownian motion. This stochastic time has also been used by Hobson in [7] in the same context. However, in [7], existence and uniqueness of volatility time, which is needed in these applications, is assumed to hold without further discussing conditions for this. Sections 3 and 4 in the present paper deal with finding very general conditions under which the volatility time exists uniquely. The methods used in these sections are also very general in nature, and we believe they can be of independent interest in the study of stochastic differential equations. From these arguments we are also able to establish the continuity (see Section 6) of the option price in the volatility, a result which we have not seen elsewhere. The question of the continuity of option prices under perturbations of the volatility is of obvious interest in applications. A related reference, where the relation between option prices and volatility is treated, especially in the context of hedging, is [3].

We also consider properties of option prices in the case of several underlying assets; see Section 8. Let the assets S_i have risk neutral processes given by

$$dS_i = S_i(t) \sum_{j=1}^{n} \sigma_{ij}\big(S(t), t\big)\, dB_j$$

for $i = 1, \dots, n$, where B_j are independent Brownian motions and $S(t) = (S_1(t), \dots, S_n(t))$. The matrix σ with entries σ_{ij} is called the *volatility matrix*. As above we get rid of the interest rate by using a bond as a numeraire. The pricing function of a contingent claim with the contract $\Phi(S(T))$ is given by

$$F(s,t) = E_{s,t}[\Phi(S(T))].$$

Alternatively, one has that the pricing function is a solution of the partial differential equation

$$F_t + \tfrac{1}{2} \sum_{i,j=1}^{n} s_i s_j F_{s_i s_j} C_{ij} = 0$$

with the boundary condition $F(s,T) = \Phi(s)$, where $C_{ij} = [\sigma\sigma^*]_{ij}$. However, as in the case of one underlying asset, we will only consider the solutions that come from the stochastic representation. Our main result for several underlying assets is that if the volatility matrix is independent of time, and the contract function is convex, then indeed the option prices decay with time. However, many examples show that the case of several underlying assets is essentially different from the case of one underlying asset. For instance, even if the contract function is convex and the solutions decay with time, the option price for a fixed time t need not be a convex function of the underlying assets, in contrast to the one-dimensional case. Furthermore, if a time dependent volatility matrix is allowed, it is easy to find examples where the option prices do not decrease with time. The examples in [1] on, for instance, "bloating" option prices in the case of stochastic volatility have counterparts in our examples on the loss of time decay for options on several underlying assets.

We have mentioned both the partial differential equation approach in (3) as well as the martingale approach, compare (2), to option pricing (see, e.g., [6] for an equivalence theorem under very general conditions). Section 2 provides examples of solutions to the Black–Scholes equation that clearly do not represent arbitrage free option prices and also do not have the desired convexity properties. This will serve as a motivation for us, in the rest of the paper, to use the martingale approach. However, this does of course not prevent a translation of our result to the language of partial differential equations; compare, for instance, related results in [4] and [8].

2. A nontrivial solution to the Black–Scholes equation with zero contract function. It is well known that there are solutions to the Black–Scholes equation that are not given by the stochastic representation formula (2). In this section we show that if one considers such solutions, one can easily construct solutions that do not have any reasonable economical interpretation and also that fail to have the desired convexity properties. After this section we will therefore only consider solutions given by the Feynman–Kac representation formula. In fact, our Theorem 2 of Section 3 is an alternative formulation of the stochastic representation formula.

PROPOSITION 1. *There is a function $F(s,t)$ that solves the Black–Scholes equation* (3) *with contract function identically equal to zero and which is strictly increasing in time. In other words, in this example F is concave as a function of s.*

PROOF. Let $\sigma^2(s,t) = \alpha(T-t)^{-1}$ for some constant α so that the Black–Scholes equation becomes

$$F_t + \frac{\alpha s^2}{2(T-t)} F_{ss} = 0.$$

Then a direct computation shows that

$$F(s,t) = \varphi(T-t)s^2,$$

where $\varphi(y) = -y^\alpha$, for y positive and zero otherwise, satisfies the given Black–Scholes equation with value 0 on expiration but with $F_t > 0$. It is easily seen that this solution has m continuous derivatives if $\alpha > m$. A smooth example is given by the same construction but with $\varphi(y) = -\exp(-1/y)$ for positive y and the volatility $\sigma^2(s,t) = (T-t)^{-2}$. The smooth example can be improved by instead considering the function $\phi(y) = -\exp(-|\log y|^\gamma)$ for $y \le 1/2$, smoothly extended to larger y, for some $\gamma > 1$. In that case, for t close to T,

$$\sigma^2(s,t) = \gamma \frac{|\log(T-t)|^{\gamma-1}}{T-t}.$$

Hence the rate at which $\sigma^2(s,t)$ blows up in the smooth case and in the C^m case only differs by a power of $\log(T-t)$. $\square$

We note that the solutions to the Black–Scholes equation described above can be added, for instance, to a solution with a contract function of a call option. The resulting sum is not convex in s since the price of a call option is bounded by s and the examples given here decay like $-s^2$ for s large.

3. Existence and uniqueness of volatility time. We study the local martingale that describes the value of the underlying asset using the stochastic time change by Dambis, Dubins and Schwarz (cf., e.g., [9], Section V.1), which converts the price process to a Brownian motion. We call this stochastic time *volatility time*, since it is related to the volatility. Its main properties are collected in Theorem 1 below, and as a consequence we have the formula in Theorem 2 yielding an alternative to the Feynman–Kac representation formula.

In option pricing, we are generally only interested in nonnegative values, but we state the result in a more general form and consider the nonnegative case in the next section. We also change notation, writing $\alpha(x,t) = x\sigma(x,t)$.

The proof of Theorem 1 is rather technical and occupies the rest of this section, although the idea is simple. We start with some definitions.

DEFINITION 1. A function $\alpha(x,t)$, defined on $\mathbb{R} \times [t_0,\infty)$ for some $t_0 \in \mathbb{R}$, is said to be locally Hölder(1/2) in the x-variable if for every $K > 0$, there exists a constant C_K such that

$$|\alpha(x,t) - \alpha(y,t)| \le C_K |x-y|^{1/2} \qquad \text{when } |x|, |y| \text{ and } |t| \le K. \tag{4}$$

DEFINITION 2. Let X_t be a stochastic process on $[t_0,\infty)$ that satisfies the stochastic differential equation

$$dX_t = \alpha(X_t,t)\, d\tilde{B}_t, \qquad X_{t_0} = x_0,$$

where $\tilde{B}$ is a Brownian motion and $x_0 \in \mathbb{R}$. Then the quadratic variation $\tau(t) = \langle X, X\rangle_t$ is called the volatility time of X corresponding to equation (7). Explicitly, by a standard formula,

$$\tau(t) = \int_{t_0}^{t} \alpha^2(X_s, s)\, ds, \qquad t \geq t_0. \tag{5}$$

THEOREM 1. *Assume that $\alpha(x,t)$ is measurable on $\mathbb{R} \times [t_0, \infty)$ and locally* Hölder(1/2) *in x and that for some constant C and all x and $t \geq t_0$*,

$$\alpha(x,t) \leq C(1 + |x|). \tag{6}$$

Let X_t be a stochastic process on $[t_0, \infty)$ that satisfies the stochastic differential equation,

$$dX_t = \alpha(X_t, t)\, d\tilde{B}_t, \qquad X_{t_0} = x_0, \tag{7}$$

where $\tilde{B}$ is a Brownian motion and $x_0 \in \mathbb{R}$, and let $\tau(t)$ be the corresponding volatility time. Then there exists (possibly on a larger probability space) a Brownian motion B with $B_0 = x_0$ such that

$$X_t = B_{\tau(t)}, \qquad t \geq t_0. \tag{8}$$

This τ satisfies the (random) integral equation

$$\tau(t) = \int_{t_0}^{t} \alpha^2(B_{\tau(s)}, s)\, ds, \qquad t \geq t_0, \tag{9}$$

or, equivalently, τ is locally absolutely continuous with $\tau(t_0) = 0$ and

$$\tau'(t) = \alpha^2(B_{\tau(t)}, t) \qquad a.e.\ t \geq t_0. \tag{10}$$

Moreover, for each $t \geq t_0$, $\tau(t)$ is a stopping time with respect to the completed filtration $(\mathcal{B}_s)$ generated by B.

Conversely, given a Brownian motion B with $B_0 = x_0$, there exists a solution $\tau(t)$ to (9) such that $\tau(t)$ is a stopping time for each t; this solution is almost surely unique and $X_t = B_{\tau(t)}$ is a solution to (7) for some Brownian motion $\tilde{B}_t$.

Our treatment will center around equation (9), or, equivalently, (10). We say that a stochastic process $\tau(t)$ satisfying (9) is a *stopping time solution* if $\tau(t)$ is a stopping time with respect to $(\mathcal{B}_s)$ for every $t \geq t_0$.

THEOREM 2. *Let α be as in Theorem 1. If X_t is a solution to (7) and $\tau(t)$ is a stopping time solution to (9), then for any $T \geq t_0$, X_T and $B_{\tau(T)}$ have the same distribution and thus for any measurable function Φ*,

$$E\Phi(X_T) = E\Phi(B_{\tau(T)}), \tag{11}$$

in the sense that if one of the expectations exists so does the other and they are equal.

REMARK 1. The local Hölder(1/2) condition in (4) is weaker than the more commonly used local Lipschitz condition. Note also that no continuity of α in t is assumed. We also remark that the bound in (6) is used to prevent the solution from exploding.

REMARK 2. If $\alpha > 0$, then τ is strictly increasing and has a continuous increasing inverse A_t, $t \geq 0$, with $A_0 = t_0$. Note that A_t is adapted and that (10) is equivalent to

$$A_t' = \alpha^{-2}(B_t, A_t). \tag{12}$$

Conversely, every adapted solution to (12) is the inverse of a stopping time solution to (10). Hobson [7] uses (12), assuming that α is such that the equation almost surely has a unique solution, without further discussing conditions for this. (Lipschitz continuity in both x and t is enough.) We will instead work with (10) and (9). The main advantage of this approach is that we do not need any continuity of α in t; as a bonus it also allows α to vanish on part of the space. [In other situations, it may be simpler to work with (12); in particular, this is the case when α is time independent.]

PROOF OF THEOREM 1. We may for notational convenience assume that $t_0 = 0$. We first observe that the local Hölder(1/2) condition (4) implies pathwise uniqueness and thus (see [9], Theorem IX.(1.7)) uniqueness in law for the solutions to (7). For a global Hölder(1/2) condition, that is, if C_K does not depend on K, this is stated in [9], Theorem IX.(3.5.ii); the general case follows the same proof or by stopping.

Formula (5) for the quadratic variation of the continuous local martingale X_t is standard, and so is the fact that $B_t = X_{\tau^{-1}(t)}$ is a Brownian motion; see, for instance, [9], Section V.1. Here $\tau^{-1} = \inf\{s : \tau(s) > t\}$ is well defined on $[0, \tau(\infty))$ since τ is continuous and increasing. If τ is constant on some intervals, τ^{-1} will have jumps, but then X_t is constant on the same intervals so $X_{\tau^{-1}(t)}$ is nevertheless continuous. It is possible that $\tau(\infty) < \infty$, in which case we have to enlarge the probability space in order to define B_t, but that makes no significant difference. Clearly, (8) holds by the definition of B_t and (9) follows by (5) and (8). Furthermore, $B_0 = X_{\tau^{-1}(0)} = x_0$. Since α^2 is locally bounded, (9) implies that $\tau(t)$ is locally Lipschitz, and thus locally absolutely continuous. Thus $\tau(t)$ is differentiable almost everywhere and (10) holds.

In order to show the almost sure uniqueness of the solution to (9), we study this equation pathwise as a (nonstochastic) ordinary differential equation with a given Brownian path B. A problem that we now encounter is that even if α is Lipschitz, the Brownian path B is not, and thus the standard theorems on existence and uniqueness of solutions to ordinary differential equations do not apply. Instead, we will construct a solution which is a stopping time, and then use the uniqueness of solutions to (7) to show that the solution is almost surely unique. We prove some lemmas and begin with a nonstochastic existence result for (9) for the case of bounded α; we write $\beta = \alpha^2$.

LEMMA 1. *Let β be a bounded and measurable function on $\mathbb{R} \times [0, \infty)$ that is continuous in x for each fixed t. If $B(t)$ is a continuous function on $[0, \infty)$, then there exists a solution f to*

$$f(t) = \int_0^t \beta\bigl(B(f(s)), s\bigr)\, ds, \qquad t \geq 0. \tag{13}$$

The solution can be chosen so that, as B varies over $C[0, \infty)$, $f(t)$ is, for each $t \geq 0$, a stopping time with respect to the right-continuous filtration $\mathcal{B}^0_{t+} = \bigcap_{u>t} \mathcal{B}^0_u$, where $\mathcal{B}^0_u$ is generated by the coordinate maps $B \mapsto B(s)$, $s \leq u$.

PROOF. We begin by defining approximate solutions f_n, $n = 1, 2, \dots,$ by $f_n(0) = 0$ and, inductively for $k = 0, 1, \dots,$

$$f_n(t) = f_n(k/n) + \int_{k/n}^t \beta\bigl(B\bigl(f_n(k/n)\bigr), s\bigr)\, ds, \qquad \frac{k}{n} < t \leq \frac{k+1}{n}. \tag{14}$$

We can also write this definition as

$$f_n(t) = \int_0^t \beta\bigl(B\bigl(f_n([ns]/n)\bigr), s\bigr)\, ds. \tag{15}$$

We will show that $f(t) = \limsup_{n\to\infty} f_n(t)$ yields a solution to (13). We first verify that our construction yields stopping times. Write the functions defined above as $f_n(t, B)$ and $f(t, B)$ and let $t, u \geq 0$. Note first that (14) implies that $f_n(t, B)$ is a continuous and thus measurable function of $B \in C[0, \infty)$. Hence $B \mapsto f(t, B)$ is measurable. Suppose that $f(t, B) \leq u$ and that $B(s) = B_1(s)$ for $s \in [0, u + \varepsilon)$, where $\varepsilon > 0$. Then, for some n_0, $f_n(t, B) \leq u + \varepsilon$ for $n \geq n_0$. It then follows easily from (14) that $f_n(s, B_1) = f_n(s, B)$ for $0 \leq s \leq t$ and $n \geq n_0$ and thus $f(t, B_1) = f(t, B)$. It follows by Galmarino's test (see [9], Exercise I.(4.21)) that $f(t, B) + \varepsilon$ is a $(\mathcal{B}^0_t)$-stopping time. Letting ε tend to 0, we find that $f(t, B)$ is a $(\mathcal{B}^0_{t+})$-stopping time.

To show that f actually is a solution, let M be a fixed number with $M \geq \sup\beta$ and say that a function $g : [0, \infty) \to [0, \infty)$ is a ϕ-approximate solution to (13), where $\phi : [0, \infty) \to [0, \infty)$ is a measurable function with $\phi \leq M$, provided

$$g(0) = 0, \tag{16}$$

$$|g(s) - g(t)| \leq M|s - t| \qquad \text{for all } s, t \geq 0, \tag{17}$$

$$\bigl|g'(t) - \beta\bigl(B(g(t)), t\bigr)\bigr| \leq \phi(t) \qquad \text{for a.e. } t \geq 0. \tag{18}$$

Note that (17) implies that g is locally absolutely continuous, and thus $g'(t)$ exists almost everywhere and $g(t) = \int_0^t g'(s)\, ds$. It follows that, given (17), (18) is equivalent to

$$\left|g(u) - g(t) - \int_t^u \beta\bigl(B(g(s)), s\bigr)\,ds\right| \le \int_t^u \phi(s)\,ds \tag{19}$$

for all t, u with $0 \le t \le u$.

LEMMA 2. *If f_1 and f_2 are ϕ-approximate solutions to* (13), *then so are $f_1 \vee f_2$ and $f_1 \wedge f_2$.*

PROOF. We consider $f = f_1 \vee f_2$; the case $f = f_1 \wedge f_2$ being similar. Clearly (16) and (17) hold for f by the corresponding properties for f_1 and f_2. It follows from (17) that $f'(t)$, $f_1'(t)$ and $f_2'(t)$ exist almost everywhere. Now, let t be such that these derivatives exist and further that (18) holds for f_1 and f_2. If $f_1(t) > f_2(t)$, then $f(s) = f_1(s)$ for all s in a neighborhood of t, and thus $f'(t) = f_1'(t)$ and

$$\bigl|f'(t) - \beta\bigl(B(f(t)), t\bigr)\bigr| = \bigl|f_1'(t) - \beta\bigl(B(f_1(t)), t\bigr)\bigr| \le \phi(t).$$

The case $f_2(t) > f_1(t)$ is, of course, analogous. Finally, assume $f_1(t) = f_2(t)$, and let $b = \beta(B(f(t)), f) = \beta(B(f_1(t)), t) = \beta(B(f_2(t)), t)$. Then for $i = 1, 2$, we have $|f_i'(t) - b| \le \phi(t)$, so if $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f_i(t + h) - f(t) - hb| \le |h|\bigl(\phi(t) + \varepsilon\bigr)$$

for $|h| < \delta$. In particular, for such h,

$$|f(t + h) - f(t) - hb| \le |h|\bigl(\phi(t) + \varepsilon\bigr)$$

and since we have assumed that $f'(t)$ exists, we conclude that $|f'(t) - b| \le \phi(t) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we find that $|f'(t) - b| \le \phi(t)$, as required. $\square$

LEMMA 3. *Suppose that f_n is a ϕ_n-approximate solution to* (13), $n \ge 1$. *Then* $\limsup_{n\to\infty} f_n$ *and* $\liminf_{n\to\infty} f_n$ *are ϕ-approximate solutions with* $\phi = \limsup_{n\to\infty} \phi_n$.

PROOF. Consider first the special case when $f(t) = \lim_{n\to\infty} f_n(t)$ and $\phi(t) = \lim_{n\to\infty} \phi_n(t)$ exist for all $t \ge 0$. Clearly, (16) and (17) hold for f since they hold for each f_n. Moreover, (19) holds with f_n and ϕ_n for each n and by dominated convergence on both sides (19) holds for f too. [Recall that β and ϕ_n are bounded by M and that $x \to \beta(B(x), s)$ is continuous.] Thus f is a ϕ-approximate solution.

In general, it follows from Lemma 2 that if $n, m \ge 1$, then $\max_{n\le k\le n+m} f_k$ is a $(\sup_{k\ge n} \phi_k)$-approximate solution. Letting first $m \to \infty$ and then $n \to \infty$ and twice applying the special case just proved, we see that $\sup_{k\ge n} f_k$ is a $(\sup_{k\ge n} \phi_k)$-approximate solution and that $\limsup f_n$ is a $(\limsup \phi_n)$-approximate solution. The case of $\liminf$ is similar. $\square$

CONCLUSION OF THE PROOF OF LEMMA 1. For almost every t, by (15),

$$f_n'(t) - \beta\big(B(f_n(t)), t\big) = \beta\big(B\big(f_n([nt]/n)\big), t\big) - \beta\big(B(f_n(t)), t\big).$$

Moreover, since $\sup \beta \leq M$,

$$0 \leq f_n(t) - f_n([nt]/n) \leq M(t - [nt]/n) \leq M/n$$

and $f_n(t) \leq Mt$. Consequently, if

$$\phi_n(t) = \sup_{0 \leq y \leq x \leq Mt, |x-y| \leq 1/n} \big|\beta\big(B(x), t\big) - \beta\big(B(y), t\big)\big|,$$

then

$$\big|f_n'(t) - \beta\big(B(f_n(t)), f\big)\big| \leq \phi_n(t)$$

and, since (16) and (17) evidently hold, f_n is a ϕ_n-approximate solution. Moreover, since $x \mapsto \beta(B(x), t)$ is continuous for every t, and thus uniformly continuous on compact sets, $\phi_n(t) \to 0$ as $n \to \infty$. Lemma 3 now shows that $\limsup_n f_n$ is a null-approximate solution, that is, a solution to (13), thus completing the proof of Lemma 1. $\square$

We now wish to extend Lemma 1 to unbounded α and β. In that case we can no longer guarantee the absence of explosions for all continuous functions B, but we prove that this is true for almost every Brownian path B. We thus have to switch to stochastic arguments including switching to the notation of Theorem 1. We begin with a standard estimate. Note that by a solution to (9) we mean a stochastic process τ such that almost surely (9) holds for all $t \geq t_0$.

LEMMA 4. *If B_t is a Brownian motion with $B_0 = x_0$ and $\tau(t)$, $t \geq 0$, is a stopping time solution to* (9), *where α is measurable on $\mathbb{R} \times [t_0, \infty)$ and satisfies* (6), *then*

$$E\tau(t) \leq (1 + x_0^2)e^{2C^2(t-t_0)}, \qquad t \geq t_0.$$

PROOF. Let $N > 0$ be an integer. It follows from (9) that

$$\tau(t) \wedge N = \int_{t_0}^t \alpha^2(B_{\tau(s)}, s)\mathbf{1}_{[\tau(s)<N]}\, ds \leq \int_{t_0}^t \alpha^2(B_{\tau(s)\wedge N}, s)\, ds.$$

Moreover, (6) implies $\alpha^2(x, s) \leq C^2(1 + |x|)^2 \leq 2C^2(1 + x^2)$. Hence, using the fact that $\tau(s) \wedge N$ is a bounded stopping time,

$$\begin{aligned} E\big(\tau(t) \wedge N\big) &\leq \int_{t_0}^t E\big(2C^2(1 + B_{\tau(s)\wedge N}^2)\big)\, ds \\ &= 2C^2 \int_{t_0}^t (1 + EB_{\tau(s)\wedge N}^2)\, ds \\ &= 2C^2 \int_{t_0}^t \big(1 + x_0^2 + E\big(\tau(s) \wedge N\big)\big)\, ds. \end{aligned}$$

Gronwall's lemma (see, e.g., Appendix Section 1 in [9]) shows that

$$1 + x_0^2 + E\big(\tau(t) \wedge N\big) \le (1 + x_0^2)e^{2C^2(t-t_0)},$$

and the result follows by letting $N \to \infty$. $\square$

REMARK 3. Lemma 4 can be improved to

$$E\tau(t)^p < \infty$$

for all $t \ge t_0$ and $p < \infty$. Indeed, if $p \ge 1$, by Hölder's inequality and the Burkholder–Davis–Gundy inequalities (see, e.g., [9], IV.4)

$$\begin{aligned} E(\tau(t) \wedge N)^p &\le (t - t_0)^{p-1} E \int_{t_0}^{t} \alpha^{2p}(B_{\tau(s)\wedge N}, s)\, ds \\ &\le C_p (t - t_0)^{p-1} E \int_{t_0}^{t} (1 + B_{\tau(s)\wedge N}^{2p})\, ds \\ &\le C'_p (t - t_0)^{p-1} \int_{t_0}^{t} \big(1 + x_0^{2p} + E\big(\tau(s) \wedge N\big)^p\big)\, ds \end{aligned}$$

and the result follows by Gronwall's lemma as above.

LEMMA 5. *If B_t is a Brownian motion with $B_0 = x_0$ and α is measurable on* $\mathbb{R} \times [t_0, \infty)$, *continuous in x and satisfies* (6), *then there exists a stopping time solution τ to* (9).

PROOF. Assume, as we may, that $t_0 = 0$. For $N = 1, 2, \dots,$ define $\beta_N(x, t) = \alpha^2(x, t) \wedge N^2$, and let $\tau_N(t)$ be the solution to

$$\tau'_N(t) = \beta_N(B_{\tau_N(t)}, t)$$

constructed in Lemma 1. Define $\tau(t) = \liminf_{N\to\infty} \tau_N(t)$; this is a $(\mathscr{B}^0_{t+})$-stopping time. Moreover, Lemma 4 shows that $\sup_N E\tau_N(t) < \infty$, so by Fatou's lemma,

$$E\tau(t) \le \liminf_{N\to\infty} E\tau_N(t) < \infty$$

and $\tau(t)$ is almost surely finite.

Next, let $t \ge 0$. Consider a point ω in the probability space Ω and assume $\tau(t) < \infty$ at ω. Take $A_1 = A_1(\omega) > \tau(t)$ and $A_2 = A_2(\omega) = \max_{x \le A_1} |B(x)|$. For infinitely many N, $\tau_N(t) < A_1$; fix one such N with $N > C(1 + A_2)$. In the proof of Lemma 1, the solution τ_N was constructed as $\limsup_{n\to\infty} \tau_{Nn}$. For sufficiently large n, we thus have $\tau_{Nn}(t) < A_1$. For these n, the definition (14) uses, on the interval $[0, t]$, only $\beta_N(B(x), s)$ with $x \le \tau_{Nn}(t) < A_1$ and thus $|B(x)| \le A_2$ and $|\alpha(B(x), s)| \le C(1 + A_2)$. It follows that $\beta_N(B(x), s) = \alpha^2(B(x), s) = \beta_M(B(x), s)$ for all such x and s and all $M \ge N$. Hence $\tau_{Mn}(s) = \tau_{Nn}(s)$ for $s \le t$ and $M \ge N$ and these n, and thus $\tau_M(s) = \tau_N(s)$ for $s \le t$ and

$M \geq N$. Consequently, $\tau(s) = \tau_N(s)$ for $s \leq t$, and τ satisfies (9) on $[0, t]$. Hence τ is a solution to (9). $\square$

CONCLUSION OF THE PROOF OF THEOREM 1. By assumption, X_t is a local martingale with respect to some complete right-continuous filtration $(\mathcal{F}_t)$. Then $\tau^{-1}(t)$ is a right-continuous family of $(\mathcal{F}_t)$-stopping times, B_t is a Brownian motion with respect to $\hat{\mathcal{F}}_t = \mathcal{F}_{\tau^{-1}(t)}$ and $\tau(t)$ is a $(\hat{\mathcal{F}}_s)$-stopping time. Let $(\mathcal{B}_t^0)$ be the filtration generated by B_t, and $\mathcal{B}_t$ the completion of $\mathcal{B}_{t+}^0$. We have $\mathcal{B}_t \subseteq \hat{\mathcal{F}}_t$, but we have not yet shown that $\tau(t)$ is a $(\mathcal{B}_t)$-stopping time.

Conversely, if B_t is a Brownian motion with respect to a complete right-continuous filtration $(\mathcal{F}_t)$, and $\tau(t)$ are $(\mathcal{F}_t)$-stopping times satisfying (9), then $X_t = B_{\tau(t)}$ defines a local martingale (with respect to the σ-fields $\mathcal{F}_{\tau(t)}$, see Proposition V.1.5 in [9]), with the quadratic variation, by (9),

$$\langle X, X\rangle_t = \tau(t) = \int_0^t \alpha^2(X_s, s)\, ds.$$

Hence, if we define $\tilde{B}$ by

$$\tilde{B}_t = \int_0^t \alpha(X_s, s)^{-1} \mathbf{1}_{[\alpha(X_s,s)\neq 0]}\, dX_s + \int_0^t \mathbf{1}_{[\alpha(X_s,s)=0)]}\, d\overline{B}_s,$$

where $\overline{B}$ is another Brownian motion, independent of everything else, then $\tilde{B}_t$ is a local martingale with quadratic variation $\langle \tilde{B}, \tilde{B}\rangle_t = t$; that is, another Brownian motion by Lévy's theorem (see, e.g., [9], Theorem IV, (3.6)), and (7) holds.

Now, suppose that $\overline{\tau}(t)$ is another family of $(\mathcal{F}_t)$-stopping times satisfying (9). Then $\overline{X}_t = B_{\overline{\tau}}(t)$ is another solution to (7) (for another $\tilde{B}_t$), but as is pointed out in the beginning of the proof, (4) implies uniqueness in law for (7). Hence, the processes X_t and $\overline{X}_t$ have the same distribution, and (5) implies that

$$E\tau(t) = \int_0^t E\alpha^2(X_s, s)\, ds = E\overline{\tau}(t),$$

for each t. Moreover, $\tau(t) \vee \overline{\tau}(t)$ is another solution to (9), by applying Lemma 2 with $\phi = 0$ locally; thus the argument just given shows

$$E\tau(t) = E\big(\tau(t) \vee \overline{\tau}(t)\big) = E\overline{\tau}(t),$$

which is possible only if $\tau(t) = \overline{\tau}(t)$ almost surely, since $E\tau(t) < \infty$ by Lemma 4. In other words, two solutions to (9) that are $(\mathcal{F}_t)$-stopping times are almost surely identical. Since Lemma 5 provides a solution which is a family of $(\mathcal{B}_t)$-stopping times, and $\mathcal{B}_t \subseteq \mathcal{F}_t$, every solution consists of $(\mathcal{B}_t)$-stopping times.

Similarly, if B is any Brownian motion and $\tau(t)$ and $\overline{\tau}(t)$ are two $(\mathcal{B}_t)$-stopping time solutions to (9), the same argument shows that $\tau = \overline{\tau}$ almost surely. This completes the proof of Theorem 1. $\square$

PROOF OF THEOREM 2. By the argument above, $B_{\tau(t)}$ is another solution to (7), for some other Brownian motion $\tilde{B}$, and by uniqueness in law as discussed above, X_t has the same distribution as $B_{\tau(t)}$. $\square$

REMARK 4. We have shown in Theorem 1 that there is only one stopping time solution to (9) (up to almost sure equivalence). We do not know whether there are other solutions that are not stopping times. Note that there is no uniqueness in Lemma 1 under our weak assumptions; a simple example is given by $\beta(x,t) = \min(2|x|, 1)$ and $B(t) = \sqrt{t}$, when $f(t) = \min(H(t-a)(t-a)^2, |t - \frac{1}{2} - a| + \frac{1}{4})$ is a solution for every $a \geq 0$, with H denoting the Heaviside function. Thus there exist Brownian paths where the solution to (9) is nonunique, although it is possible that the set of such paths has probability 0.

4. Existence and uniqueness for volatility time for absorbed processes. In applications to option pricing, we are only interested in nonnegative values of the underlying asset. We thus consider a stochastic process $X_t \geq 0$ that satisfies (7) when $X_t > 0$. We also assume that $X_{t_0} = x_0 > 0$. We allow the possibility that X_t reaches 0 in finite time, and then assume that X_t remains 0 forever, that is, 0 is an absorbing state.

In this setting we need α to be defined on $(0,\infty) \times [t_0, \infty)$ only; we extend α to $\mathbb{R} \times [t_0,\infty)$ by defining $\alpha(x,t) = 0$ for $x \leq 0$. We still assume that α is measurable and that the bound (6) holds, but we require the Hölder(1/2) condition only locally on $(0,\infty) \times [t_0,\infty)$; that is, (4) holds for $x, y \in [K^{-1}, K]$ and $|t| \leq K$. Note that we allow a discontinuity of α at $x = 0$. In this context it is customary to use the volatility $\sigma \geq 0$ as in (1) instead of α; they are related by $\alpha(x,t) = x\sigma(x,t)$. Note that α is locally Hölder(1/2) on $(0,\infty) \times [t_0,\infty)$ if and only if σ is, and that (6) holds if and only if

$$\sigma(s,t) \leq C(1 + s^{-1}).$$

Then the following version of the results in Section 3 holds.

THEOREM 3. *Suppose that α is measurable on $(0,\infty) \times [t_0,\infty)$ for some real t_0, and that (6) holds and also that α is locally Hölder(1/2) on $(0,\infty) \times [t_0,\infty)$. Let further $\alpha(x,t) = 0$ for $x \leq 0$. Then, for every $x_0 > 0$, and a Brownian motion $\tilde{B}$, there exists an almost surely unique stochastic process X_t on $[t_0,\infty)$ such that (7) holds and 0 is an absorbing state for X_t. Moreover, for a Brownian motion B_t with $B_0 = x_0$, there exists an almost surely unique stopping time solution $\tau(t)$ to (9). Finally, for any such X_t and B, X_T and $B_{\tau(T)}$ have the same distribution and thus, for any measurable function Φ,*

$$E\Phi(X_T) = E\Phi(B_{\tau(T)}).$$

PROOF. We may again assume that $t_0 = 0$. We first consider the process X_t. Define for $N \geq 1/x_0$,

$$\alpha_N(x,t) = \alpha\big(x \vee (1/N), t\big).$$

Then α_N satisfies (6) and (4), and it follows from Theorem 1 that there exists a unique solution $X^{(N)}$ to

$$dX_t^{(N)} = \alpha_N\big(X_t^{(N)}, t\big)\, d\tilde{B}_t, \qquad X_0^{(N)} = x_0. \tag{20}$$

By stopping $X^{(N)}$ at $T_N = \inf\{t : X_t^{(N)} = 1/N\}$, we obtain a solution to (7) and (20) on $[0, T_N]$. The same holds if $M > N$ and we stop $X^{(M)}$ when it reaches $1/N$. Since the latter solution may be restarted to give a solution to (20) defined for all $t \geq 0$, the uniqueness of $X^{(N)}$ implies that $X^{(M)} = X^{(N)}$ on $[0, T_N]$ for $M > N$. It follows that $T_M > T_N$ when $M > N$; that is, the sequence $(T_N)_N$ of stopping times is strictly increasing. Define $T_\infty = \lim_{N\to\infty} T_N = \sup_N T_N$ and

$$X_t = \begin{cases} \liminf\limits_{N\to\infty} X_t^{(N)}, & \text{if } t < T_\infty, \\ 0, & \text{if } t \geq T_\infty. \end{cases}$$

Then $X_t = X_t^{(N)}$ when $0 \leq t \leq T_N$; thus X is a solution to (7) on $[0, T_N]$ for each N, and thus on $[0, T_\infty)$. In particular, X_t is a continuous local martingale on $[0, T_\infty)$. Since a continuous local martingale is a time change of a Brownian motion, it almost surely either converges or oscillates between $-\infty$ and $+\infty$; by construction $X_t \geq 0$ for $t < T_\infty$, so the latter alternative is impossible and $\lim_{t\uparrow T_\infty} X_t$ exists almost surely. Moreover, since $X_{T_N} = X_{T_N}^{(N)} = 1/N$, this limit is 0, so $t \mapsto X_t$ is continuous at T_∞ too, and thus for all $t \geq 0$.

Conversely, if $\tilde{X}_t$ is any solution to (7) with 0 absorbing and $\tilde{X}_0 = x_0$ we obtain by stopping at the time $\tilde{T}_N$ when $\tilde{X}_t$ reaches $1/N$ a solution to (20) on $[0, \tilde{T}_N]$. As in the first part of the proof it follows that $\tilde{T}_N = T_N$ and that $\tilde{X} = X^{(N)} = X$ on $[0, T_N]$. Hence, $\tilde{X} = X$ on $[0, T_\infty)$, and since $\tilde{X}_{T_\infty} = X_{T_\infty} = 0$ and 0 is an absorbing state for both processes,

$$\tilde{X} = X.$$

It follows further that the distribution of $(X_t)_t$ does not depend on the choice of Brownian motion $\tilde{B}_t$.

We now consider the volatility time τ. We argue as in the proof of Lemma 5, this time defining

$$\beta_N(x,t) = \begin{cases} \alpha^2(x,t) \wedge N^2, & \text{if } x \geq 1/N, \\ \alpha^2(1/N, t) \wedge N^2, & \text{if } x < 1/N. \end{cases} \tag{21}$$

We again let $\tau_N(t)$ be the stopping time solutions to $\tau_N'(t) = \beta_N(B_{\tau_N(t)}, t)$ constructed in Lemma 1, and let $\overline{\tau}(t) = \liminf_{N\to\infty} \tau_N(t)$; this is, by Fatou's lemma and Lemma 4 an almost surely finite stopping time. Consider a point $\omega \in \Omega$. Let $T_0 = \inf\{t : B_t = 0\}$ and suppose that $t \geq 0$ and $\overline{\tau}(t) < T_0$. Take $A_1 = A_1(\omega)$ with $\overline{\tau}(t) < A_1 < T_0$ and let $A_2 = A_2(\omega) = \max_{x \leq A_1}(|B(x)| + |B(x)|^{-1})$. It follows as in the proof of Lemma 5, that $\overline{\tau}(s) = \tau_N(s)$ for all $s \leq t$ and $N \geq N_0(\omega) = (1 \vee C)(1 + A_2)$. Consequently, $\overline{\tau}$ satisfies (9) for all t such that $\overline{\tau}(t) < T_0$. Note also that $\overline{\tau}(s) \leq \overline{\tau}(t)$ when $s \leq t$. Now, suppose that $\overline{\tau}(t) \geq T_0$ for some finite t, and let $t_0 = \inf\{t : \overline{\tau}(t) \geq T_0\}$. We claim that $\overline{\tau}(t) \uparrow T_0$ as $t \uparrow t_0$. Indeed, let $\overline{\tau}(t_0-) = \sup_{s<t_0} \overline{\tau}(s)$ and assume $\overline{\tau}(t_0-) < T_0$. Taking $A_1 \in (\overline{\tau}(t_0-), T_0)$ and A_2 as above we find as before that $\overline{\tau}(s) = \tau_N(s)$ for all $s < t_0$ and $N \geq N_0$. In particular, since τ_{N_0} is continuous, $\tau_{N_0}(t_0) = \overline{\tau}(t_0-) < A_1$ and thus $\tau_{N_0} < A_1$ for $t \leq t_0 + \varepsilon$, for some $\varepsilon > 0$. The same argument now shows that $\tau_N(t) = \tau_{N_0}(t)$ for all $N \geq N_0$ and $t \leq t_0 + \varepsilon$, and thus $\overline{\tau}(t) = \tau_{N_0}(t) < A_1 < T_0$ for $t \leq t_0 + \varepsilon$, which contradicts the definition of t_0. Hence $\overline{\tau}(t_0-) = T_0$ as claimed. It follows that

$$\tau(t) = \overline{\tau}(t) \wedge T_0 = \begin{cases} \overline{\tau}(t), & \text{if } t < t_0; \\ T_0, & \text{if } t \geq t_0 \end{cases}$$

is continuous, and that it satisfies (9) for all $t \leq t_0$, and thus for all t since $\alpha(B_{T_0}, t) = 0$. Moreover, each $\tau(t)$ is a stopping time. This proves the existence of a stopping time solution τ.

Let τ be any stopping time solution to (9). It follows as in the proof of Theorem 1 that $X_t = B_{\tau(t)}$ is a solution to (7) for some Brownian motion $\tilde{B}_t$. We next show that this solution is absorbed at 0.

Consider again an $\omega \in \Omega$. If $B_{\tau(t)} < 0$ for some $t = t_1$, say, then by continuity, this holds in an open interval $t_1 - \varepsilon < t < t_1 + \varepsilon$. In this interval we thus have $\alpha(B_{\tau(t)}, t) = 0$, and hence $\tau'(t) = 0$. In other words, $\tau(t)$ and thus $B_{\tau(t)}$ are constant on $t_1 - \varepsilon < t < t_1 + \varepsilon$. Consequently, for each $y < 0$, the set $E_y = \{t : B_{\tau(t)} = y\}$ is a both closed and open subset of $[0, \infty)$, and since $B_{\tau(0)} = x_0 > 0$, $E_y = \varnothing$ for every $y < 0$. Consequently, $B_{\tau(t)} \geq 0$ for all t.

Next, almost surely, every interval $(T_0, T_0 + \varepsilon)$ contains points t where $B_t < 0$. Hence, if $\tau(t) > T_0$, there exists an $s \leq t$ such that $B_{\tau(s)} < 0$, which is a contradiction. Consequently, $\tau(t) \leq T_0$ almost surely. It follows that if $B_{\tau(t)} = 0$ for some t, then $\tau(t) = T_0$. Hence, for $u \geq t$ we have $\tau(t) \leq \tau(u) \leq T_0$, so $\tau(u) = T_0$ and $B_{\tau(u)} = 0$. In other words, $B_{\tau(t)}$ is absorbed at 0. It now follows, arguing as in the proof of Theorem 1, that if τ_1 and τ_2 are two stopping time solutions to (9), then so are $\tau_1 \wedge \tau_2$ and $\tau_1 \vee \tau_2$, and hence, from the uniqueness assertion in the first part of the proof, $\tau_1 = \tau_2$ almost surely. The proof is complete. □

5. Time decay of option prices. In this and the following two sections, the general results above on volatility time will be applied to several problems concerning option pricing, using arguments similar to [7]. (See also [5] for related arguments.) The following simple lemma provides a link between monotonicity properties of option prices and monotonicity properties of stopping times.

LEMMA 6. *Suppose that B_t is a Brownian motion with $B_0 = x_0 \in \mathbb{R}$ and that τ_1 and τ_2 are two stopping times such that $\tau_1 \leq \tau_2$ almost surely and $E\tau_2 < \infty$. If Φ is any convex function, then*

$$E\Phi(B_{\tau_1}) \leq E\Phi(B_{\tau_2}),$$

where the expectations are finite or $+\infty$.

PROOF. As is well known, the stopped Brownian motion $B_{\tau_2 \wedge t}$ is a square integrable martingale, and

$$B_{\tau_1} = E(B_{\tau_2} | \mathcal{F}_{\tau_1}).$$

If $|\Phi(x)| = \mathcal{O}(1 + |x|)$, then $E\Phi(B_{\tau_1})$ and $E\Phi(B_{\tau_2})$ are finite and by the conditional version of Jensen's inequality,

$$\Phi(B_{\tau_1}) \leq E\Phi(B_{\tau_2} | \mathcal{F}_{\tau_1})$$

almost surely and thus

$$E\Phi(B_{\tau_1}) \leq E\Phi(B_{\tau_2}).$$

In general, Φ is the limit of an increasing sequence of convex functions Φ_n such that $\Phi_n = \Phi$ on $[-n, n]$ and Φ_n is linear on $(-\infty, -n]$ and on $[n, \infty)$. The result follows by the preceding case and monotone convergence. □

In our first application we compare option prices at two different times. We begin with a simple monotonicity result for the corresponding volatility times.

LEMMA 7. *Let α be as in Theorem* 1 *or* 3 *and let $t_0 \leq t_1 \leq t_2$. If B_t is a Brownian motion and τ_1 and τ_2 are the stopping time solutions to*

$$\tau_j'(t) = \alpha^2(B_{\tau_j(t)}, t), \qquad t \geq t_j,$$

with $\tau_j(t_j) = 0$, $j = 1, 2$, then

$$\tau_1(t) \geq \tau_2(t)$$

almost surely for every $t \geq t_2$.

PROOF. Define $\tau(t) = \tau_1(t) \wedge \tau_2(t)$. As in the proof of Lemma 2,

$$\tau'(t) = \alpha^2(B_{\tau(t)}, t)$$

for almost every $t \geq t_2$. Moreover, $\tau_1(t_2) \geq 0$ and thus $\tau(t_2) = \tau_2(t_2) = 0$. By the uniqueness assertions in Theorems 1 and 3, for any $t \geq t_2$, $\tau(t) = \tau_2(t)$ almost surely, and thus $\tau_1(t) \geq \tau_2(t)$. □

REMARK 5. We conjecture that actually $\tau_1(t) > \tau_2(t)$ almost surely except in trivial cases, but we have not been able to show this without stricter requirements on α.

THEOREM 4. *Let α be as in Theorem* 1 *or* 3 *and let $t_0 \le t_1 \le t_2 \le T$. Let $X_t^{(1)}$ and $X_t^{(2)}$ be solutions to*

$$dX_t = \alpha(X_t, t)\, d\tilde{B}_t,$$

where $\tilde{B}$ is a Brownian motion, with $X_{t_1}^{(1)} = x_0 = X_{t_2}^{(2)}$. In the setting of Theorem 3, *assume further that* 0 *is an absorbing state for $X_t^{(j)}$, $j = 1, 2$. Finally, let Φ be a convex function. Then*

$$E\Phi(X_T^{(1)}) \ge E\Phi(X_T^{(2)}).$$

Before proving this theorem we note that interpreting X as a price process with volatility given by α we have, in view of (2) that option prices with convex contract functions decay with time, or in view of (3), that the price is convex as a function of the price of underlying asset. This has earlier been proved by [1] and [7] under somewhat different conditions.

PROOF OF THEOREM 4. Let τ_1 and τ_2 be as in Lemma 7. Thus

$$\tau_2(T) \le \tau_1(T)$$

almost surely. By Lemma 4, $E\tau_1(T) < \infty$, and Lemma 6 yields, together with Theorem 2 or 3,

$$E\Phi(X_T^{(2)}) = E\Phi(B_{\tau_2(T)}) \le E\Phi(B_{\tau_1(T)}) = E\Phi(X_T^{(1)}). \qquad \square$$

REMARK 6. A basic idea in our approach is to compare different price processes (or, as in Theorem 4, the same price processes started at different times), by generating them from the *same* Brownian motion, using different time changes as defined by the respective volatility times. This idea, an example of *coupling*, is also used by Hobson [7].

We know that the prices tend to the contract function as time t approaches the time of expiration T and that the prices remain convex under the conditions of Theorem 4. But what happens if we let t tend to minus infinity? We have the following result.

THEOREM 5. *Let $\alpha(x, t) = x\sigma(x, t)$ be as in Theorem* 3 *with $t_0 = -\infty$ and assume further that $\sigma(x, t) \ge \gamma(x)\delta(t)$ for some positive functions γ and δ such that $\inf_{a \le x \le b} \gamma(x) > 0$ for any $0 < a < b < \infty$ and $\int_{-\infty}^{0} \delta^2(t)\, dt = \infty$.*

Then the price of a call option with any strike price K at a fixed time T increases to s, the price of the underlying asset, as time tends to minus infinity.

PROOF. Let $X_u^{(t)}$, $u \geq t$, denote the solution to $dX_u = \alpha(X_u, u)\, d\tilde{B}_u$ starting at $X_t = x_0 = s$. Recall that the price of the option at time t is according to (2) given by $E\Phi(X_T^{(t)})$ with the contract function $\Phi(s) = (s-K)_+$.

Consider a decreasing sequence $t_n \downarrow -\infty$ of times less than T, and let $\tau_n(t)$ denote the solution to (9) starting at $t_0 = t_n$. By Lemma 7, $\tau_n(T)$ is an increasing sequence; denote its limit by $\tau(T)$. Let as before $T_0 = \inf\{t : B_t = 0\}$. We claim that $\tau(T) = T_0$ almost surely. Indeed, we know that each $\tau_n(T) \leq T_0$, so $\tau(T) \leq T_0$. Suppose that, at some $\omega \in \Omega$, $\tau(T) < T_0$. Then $a = \inf_{0 \leq u \leq \tau(T)} B_u > 0$ and $b = \sup_{0 \leq u \leq \tau(T)} B_u < \infty$, and thus, by the assumptions, $\alpha(B_u, t) \geq c\delta(t)$ for some $c > 0$ and all $u \leq \tau(T)$. Then, for each n,

$$\tau(T) \geq \tau_n(T) = \int_{t_n}^{T} \alpha^2\bigl(B_{\tau_n(u)}, u\bigr)\, du \geq \int_{t_n}^{T} c^2 \delta(u)^2\, du,$$

which tends to $\int_{-\infty}^{T} c^2\delta(u)^2\, du = \infty$ as $n \to \infty$, a contradiction. Hence $\tau_n(T) \to T_0$ almost surely, and thus $B_{\tau_n(T)} \to 0$.

Since $X_T^{(t_n)}$ and $B_{\tau_n(T)}$ have the same distribution, by Theorem 3, this shows that $X_T^{(t_n)} \to 0$ in probability as $n \to \infty$. Since the sequence t_n is arbitrary, this shows that $X_T^{(t)} \to 0$ in probability as $t \to -\infty$. It follows by dominated convergence that $E(X_T^{(t)} \wedge K) \to 0$. Moreover, $X_u^{(t)}$ is a martingale and thus $EX_T^{(t)} = x_0$, and consequently,

$$E\bigl(X_T^{(t)} - K\bigr)_+ = EX_T^{(t)} - E\bigl(X_T^{(t)} \wedge K\bigr) \to x_0 = s. \qquad \square$$

REMARK 7. Note that this result is *not* the consequence of the declining value of currency units since we in this article express assets in constant units.

Instead, the proof shows that $X_T^{(t)} \to 0$ in probability as $t \to -\infty$. In other words, in the long run the asset is with large probability almost worthless (relative to bonds), but with a small probability it has a very large value, making the expectation constant; thus making the strike price irrelevant.

6. Continuity of option prices. In this section we consider continuity of option prices under perturbations of the volatility. We begin with a weak lemma on (pathwise) monotonicity; it will be improved in Lemma 10.

LEMMA 8. *Suppose that $\beta(x,t)$ and $\tilde{\beta}(x,t)$ are locally* Hölder(1/2) *in x in the sense of Definition* 1 *with $t_0 = 0$ and suppose further that*

$$\beta(x,t) + \varepsilon \leq \tilde{\beta}(x,t)$$

for all x, t and some $\varepsilon > 0$. If $B(t)$ is a continuous function on $[0,\infty)$, *and*

$$\tau'(t) = \beta\bigl(B(\tau(t)), t\bigr), \qquad \tilde{\tau}'(t) = \tilde{\beta}\bigl(B(\tilde{\tau}(t)), t\bigr),$$

then $\tau(t) < \tilde{\tau}(t)$ for all $t > 0$.

PROOF. Suppose that $\tau(t) = \tilde{\tau}(t)$ for some $t \geq 0$. Let $x = \tau(t)$ and $y = B(x)$. By the local Hölder assumption, for $|s - t| \leq 1$ and some finite C,

$$|\tau'(s) - \beta(y, s)| = \big|\beta\big(B(\tau(s)), s\big) - \beta(y, s)\big| \leq C|B(\tau(s)) - y|^{1/2}$$
$$= C|B(\tau(s)) - B(\tau(t))|^{1/2},$$

which by the continuity of B and τ is less than $\varepsilon/3$ if $|s - t|$ is small enough. Similarly,

$$|\tilde{\tau}'(s) - \tilde{\beta}(y, s)| < \varepsilon/3$$

if $|s - t|$ is small enough. Consequently, for some $\delta > 0$ and $|s - t| < \delta$,

$$\tilde{\tau}'(s) - \tau'(s) > \tilde{\beta}(y, s) - \beta(y, s) - 2\varepsilon/3 \geq \varepsilon/3.$$

Hence $\tilde{\tau}(s) - \tau(s)$ is 0 for $s = t$, negative for $s \in (t - \delta, t)$ and positive for $s \in (t, t + \delta)$. In particular, taking $t = 0$, $\tilde{\tau}(s) - \tau(s)$ is positive for small positive s. If $\tilde{\tau}(s) \leq \tau(s)$ for some $s > 0$, we let t be the infimum of all such s and obtain $t > 0$ and $\tilde{\tau}(t) = \tau(t)$. As just proved, then $\tilde{\tau}(s) < \tau(s)$ for some $s < t$, which is a contradiction, thus completing the proof. □

LEMMA 9. *Suppose that α and $\alpha_1, \alpha_2, \ldots$ satisfy the conditions of Theorem* 1 *or* 3 *uniformly*, *that is*, *with the same constants C and C_K in* (6) *and* (4) *for all α_n. Suppose further that $\alpha_n(x, t) \to \alpha(x, t)$ as n tends to infinity for all x and t. If B_t is a Brownian motion and $\tau, \tau_1, \tau_2, \ldots$ are stopping time solutions to*

$$\tau' = \alpha^2(B_{\tau(t)}, t), \qquad \tau_n' = \alpha_n^2\big(B_{\tau_n(t)}, t\big)$$

with $\tau(t_0) = \tau_n(t_0) = 0$, then

$$\tau_n(t) \to \tau(t)$$

almost surely as n tends to infinity for every $t \geq t_0$.

PROOF. Assume again that $t_0 = 0$ and let $\beta = \alpha^2$ and $\beta_n = \alpha_n^2$. Suppose first that $\alpha_1, \alpha_2, \ldots$ are uniformly bounded, say $|\alpha_n| \leq C$, and satisfy (4) with some constant C_K not depending on n. Consider a fixed Brownian path B_t and let

$$B_t^* = \sup_{0 \leq s \leq t} |B_s|.$$

Let $M = C^2$, and note that $\tau_n(t) \leq Mt$ for all n and t. Then, for almost every $t \geq 0$,

$$\big|\tau_n'(t) - \beta\big(B_{\tau_n(t)}, t\big)\big| = \big|\beta_n\big(B_{\tau_n(t)}, t\big) - \beta\big(B_{\tau_n(t)}, t\big)\big|$$
$$\leq \phi_n(t),$$

where

$$\phi_n(t) = \sup\big\{|\beta_n(x, t) - \beta(x, t)| : |x| \leq B_{Mt}^*\big\}$$
$$\leq 2C \sup\big\{|\alpha_n(x, t) - \alpha(x, t)| : |x| \leq B_{Mt}^*\big\}.$$

Hence, τ_n is a ϕ_n-approximate solution to (13).

For fixed t, the functions $\alpha_n(x,t) - \alpha(x,t)$ are uniformly continuous on the compact set $K = \{x : |x| \le B^*_{Mt}\}$ and converge pointwise to 0; hence they converge uniformly to 0 on K, and thus $\phi_n(t) \to 0$ as $n \to \infty$. Lemma 3 now shows that $\liminf_{n\to\infty} \tau_n$ and $\limsup_{n\to\infty} \tau_n$ both are solutions to (13).

Thus $\liminf_{n\to\infty} \tau_n$ and $\limsup_{n\to\infty} \tau_n$ are two stopping time solutions to (9), so by the uniqueness in Theorem 1,

$$\liminf_{n\to\infty} \tau_n = \limsup_{n\to\infty} \tau_n = \tau$$

almost surely, which proves the result in this case.

Assume next that the α_n satisfy the conditions of Theorem 1, uniformly in n. Let $\tilde{\alpha}(x,t) = 1 + C(1+|x|)$ and $\tilde{\beta}(x,t) = \tilde{\alpha}^2(x,t)$. Then $\tilde{\alpha}$ also satisfies the conditions of Theorem 1, so there exists a stopping time solution $\tilde{\tau}$ to

$$\tilde{\tau}'(t) = \tilde{\alpha}^2(B_{\tilde{\tau}(t)}, t).$$

Moreover, for every n, $\tilde{\alpha}^2(x,t) > 1 + \alpha_n^2(x,t)$, and thus $\tau_n(t) \le \tilde{\tau}(t)$ by Lemma 8.

For $1 \le n \le \infty$ and $N \ge 1$, define $\beta_{nN} = \alpha_n^2 \wedge N^2$, where $\alpha_\infty = \alpha$, and let τ_{nN} be the stopping time solution to

$$\tau'_{nN} = \beta_{nN}(B_{\tau_{nN}(t)}, t).$$

Suppose that $t \ge 0$ and that $\tilde{\tau}(t) < \infty$. By the proof of Lemma 5 and the uniqueness in Theorem 1, $\tau_n = \liminf_{N\to\infty} \tau_{nN}$ almost surely for each $n \le \infty$. Moreover, taking $A_1 > \tilde{\tau}(t)$ in the proof of Lemma 5 so that $A_1 > \tau_n(t)$ for all n, there exists $N_0(\omega) < \infty$, independent of n, such that $\tau_n(t) = \tau_{nN}(t)$ for all $n \le \infty$ and $N \ge N_0$. Since $\tau_{nN}(t) \to \tau_{\infty N}(t)$ almost surely for each N by the first part of the proof, we find that $\tau_n(t) \to \tau_\infty(t) = \tau(t)$ almost surely, as asserted.

Finally, assume that α and the α_n satisfy the conditions of Theorem 3, uniformly in n, and that $B_0 = x_0 > 0$. Consider an $\omega \in \Omega$ and let $T_0 = \inf\{t : B_t = 0\}$. Then, almost surely, $0 < T_0 < \infty$, and thus $\sup\{|B_t| : t \le T_0\} < \infty$. By (6), there thus exists a constant $C(\omega)$ such that $|\alpha_n(B_s,t)| \le C$ for $s \le T_0$ and $t \ge 0$ and thus $|\tau_n'(t)| \le C^2$ for almost every $t \ge 0$. Hence, the functions $t \mapsto \tau_n(t)$ are Lipschitz, uniformly in n, and thus $\liminf_{n\to\infty} \tau_n(t)$ and $\limsup_{n\to\infty} \tau_n(t)$ are Lipschitz and hence continuous.

If t is such that $\tau(t) < T_0$ and $\limsup_{n\to\infty} \tau_n(t) < T_0$, we take $A_1(\omega)$ with $\tau(t) < A_1 < T_0$ and $\limsup_{n\to\infty} \tau_n(t) < A_1$, and thus $\tau_n(t) < A_1$ for all large n, say $n \ge n_0(\omega)$. Let β_N be defined by (21) and let τ_{nN} be the stopping time solution to $\tau'_{nN}(t) = \beta_N(B_{\tau_{nN}(t)}, t)$. It follows from the proof of Theorem 3, arguing as in the preceding case of the present proof, that $\tau_\infty(t) = \tau_{\infty N}(t)$ and $\tau_n(t) = \tau_{nN}(t)$ for $N \ge N_0(\omega)$ and $n \ge n_0(\omega)$. The first part of the proof yields that $\tau_{nN}(t) \to \tau_{\infty N}(t)$ for each N, and it follows that $\lim_{n\to\infty} \tau_n(t) = \tau(t)$.

It follows, by first considering rational t, that the three continuous functions $\limsup_{n\to\infty} \tau_n(t)$, $\liminf_{n\to\infty} \tau_n(t)$ and $\tau(t)$ almost surely coincide for all t

such that all three functions are strictly less than T_0. Since all three functions are continuous, nondecreasing and bounded by T_0, it follows that they hit T_0 simultaneously (if at all) and then remain constant. Hence, almost surely, the three functions coincide for all t; that is, $\lim_{n\to\infty} \tau_n(t) = \tau(t)$. $\square$

THEOREM 6. *Suppose that α and $\alpha_1, \alpha_2, \ldots$ satisfy the conditions of Theorem 1 or 3 uniformly and suppose that*

$$\alpha_n(x,t) \to \alpha(x,t),$$

as $n \to \infty$ for all x and t. Let X_t and $X_t^{(n)}$ be solutions to

$$dX_t = \alpha(X_t, t)\, d\tilde{B}_t, \qquad dX_t^{(n)} = \alpha_n(X_t^{(n)}, t)\, d\tilde{B}_t$$

with $X_{t_0} = X_{t_0}^{(n)} = x_0$. In the setting of Theorem 3, assume further that 0 is an absorbing state for X_t and $X_t^{(n)}$. Let $T \geq t_0$. Then $X_T^{(n)}$ converges in distribution to X_T as n tends to infinity. Further, if Φ is a continuous function with $|\Phi(x)| \leq C_1(1+|x|)^k$ for some C_1 and $k < \infty$, then

$$E\Phi(X_T^{(n)}) \to E\Phi(X_T).$$

PROOF. Let τ and τ_n be as in Lemma 9. Then $\tau_n(T) \to \tau(T)$ almost surely as n tends to infinity, and thus $B_{\tau_n(T)} \to B_{\tau(T)}$ almost surely and thus in distribution. Since $B_{\tau_n(T)}$ and $X_T^{(n)}$ as well as $B_{\tau(T)}$ and X_T agree in distribution, by Theorem 1 or 3, the first assertion follows.

For the second we define, as in the proof of Lemma 9, $\tilde{\alpha}(x,t) = 1 + C(1+|x|)$ and let $\tilde{\tau}$ be a stopping time solution to $\tilde{\tau}'(t) = \tilde{\alpha}^2(B_{\tilde{\tau}(t)}, t)$. By Lemma 8, $\tau_n(T) \leq \tilde{\tau}(T)$ and thus

$$\Phi(B_{\tau_n(T)}) \leq C_1\bigl(1 + |B_{\tau_n(T)}|\bigr)^k \leq C_1\bigl(1 + B^*_{\tilde{\tau}(T)}\bigr)^k.$$

Moreover, by the Burkholder–Davis–Gundy inequalities and Remark 3,

$$E\bigl(B^*_{\tilde{\tau}(T)}\bigr)^k \leq C_k E\bigl(\tilde{\tau}(T)^{k/2}\bigr) < \infty.$$

Hence dominated convergence yields $E\Phi(B_{\tau_n(T)}) \to E\Phi(B_{\tau(T)})$, which yields the second assertion by Theorem 1 or 3. $\square$

7. Monotonicity in volatility. Using the continuity established above we can now sharpen Lemma 8. This will enable us to derive the monotonicity of option prices in the volatility.

LEMMA 10. *Suppose that α and $\tilde{\alpha}$ satisfy the conditions of Theorem 1 or 3 and that*

$$|\alpha(x,t)| \leq |\tilde{\alpha}(x,t)|$$

for all x and t. If B_t is a Brownian motion and $\tau(t)$ and $\tilde{\tau}(t)$ are stopping time solutions to

$$\tau'(t) = \alpha^2(B_{\tau(t)}, t), \qquad \tilde{\tau}'(t) = \tilde{\alpha}^2(B_{\tilde{\tau}(t)}, t),$$

with $\tau(0) = 0 = \tilde{\tau}(0)$, then

$$\tau(t) \le \tilde{\tau}(t),$$

almost surely for every t.

PROOF. We may replace α and $\tilde{\alpha}$ by their absolute values and thus assume that they are both nonnegative. Let $\tilde{\alpha}_n(x,t) = \tilde{\alpha}(x,t) + 1/n$ and let $\tilde{\tau}_n$ be the corresponding solution to (9). By Lemma 8, $\tilde{\tau}_n(t) \ge \tau(t)$ almost surely for each n, and by Lemma 9 $\tilde{\tau}_n(t) \to \tilde{\tau}(t)$ almost surely as n tends to infinity. □

THEOREM 7. *Suppose that α and $\tilde{\alpha}$ are as in Lemma* 10. *Let X_t and $\tilde{X}_t$ be solutions to*

$$dX_t = \alpha(X_t, t)\, d\tilde{B}_t, \qquad d\tilde{X}_t = \tilde{\alpha}(\tilde{X}_t, t)\, d\tilde{B}_t$$

with $X_0 = \tilde{X}_0 = x_0$. If $T \ge t_0$ and Φ is a convex function, then

$$E\Phi(X_T) \le E\Phi(\tilde{X}_T).$$

The proof is immediate by Theorem 1 or 3 and Lemmas 6, 10 and 4.

8. Several underlying assets. We now consider some properties of option prices in the case of several underlying assets. Let the assets S_i have risk neutral processes given by

$$dS_i = S_i(t) \sum_{j=1}^{n} \sigma_{ij}\big(S(t), t\big)\, dB_j$$

for $i = 1, \dots, n$, where B_j are independent Brownian motions and $S(t) = (S_1(t), \dots, S_n(t))$. The matrix σ with entries σ_{ij} is called the *volatility matrix*, where we assume that the each of the entries satisfies the assumptions on volatility of the previous sections. As before we get rid of the interest rate by using a bond as a numeraire. The pricing function of a contingent claim with the contract $\Phi(S(T))$ is given by

$$F(s,t) = E_{s,t}[\Phi(S(T))].$$

Alternatively, one has that the pricing function is a solution of the partial differential equation

$$F_t + \tfrac{1}{2} \sum_{i,j=1} s_i s_j F_{s_i s_j} C_{ij} = 0$$

with the boundary condition $F(s,T) = \Phi(s)$, where $C_{ij} = [\sigma\sigma^*]_{ij}$. The following is our main result on several underlying assets.

THEOREM 8. *If the volatility matrix is independent of time, and the contract function is convex, then the option price* $F(s,t)$ *given by*

$$F(s,t) = E_{s,t}\big(\Phi(S(T))\big)$$

decreases with time.

PROOF. Since

$$F(s,t) = E_{s,t}\big(\Phi(S(T))\big),$$

we know that if $F(s,T) = \Phi(S(T)) \geq 0$, then $F(s,t) \geq 0$ for all $t \leq T$. We also note that if the contract function is an affine function then $F(t,s) = F(T,s)$ for all $t \leq T$. First, we will show that if the contract function is convex then $F(s,t) \geq F(s,T)$ for all $t \leq T$ and all s. To show this for some particular s_0 we compare the solution F with a solution U having a supporting hyperplane at s_0 as contract function. We then have $F(s,T) - U(s,T) \geq 0$, because F is convex, and thus $F(s,t) - U(s,t) \geq 0$ for every $t \leq T$. Moreover, $F(s_0,T) = U(s_0,T)$ and $U(s_0,t) = U(s_0,T)$ for all $t \leq T$ because U is affine. Hence

$$F(s_0,t) \geq U(s_0,t) = U(s_0,T) = F(s_0,T).$$

Thus we have for arbitrary s and $t_1 \geq 0$ that

$$F(s, T - t_1) \geq F(s,T).$$

Now, let us consider both sides of this inequality as contract functions and consider the corresponding solutions at some time $T - t_2$ where $t_2 \geq 0$. The corresponding solutions satisfy the same inequality by the argument above. However, the time independence of the equation yield that these solutions are simply given by translates in time of $F(s,t)$ and we obtain

$$F(s, T - t_1 - t_2) \geq F(s, T - t_2)$$

which is the desired monotonicity in t. $\square$

REMARK (Convexity and time decay of option prices and monotonicity in volatility). Consider a market with two underlying assets S_1 and S_2. Let this market have a diffusion matrix which is independent of time in accordance with the theorem above and with a convex contract function. Then the theorem yields that the option price decays with time. However, let the contract function be that of a call option, with strike price K on one of the assets, say S_1, but let the volatility of S_1 depend on S_2 in such a way that the volatility has a strict local maximum for some value $s_{2,0}$ of S_2. It is then easy to see that the solution to the pricing equation is not convex near the point $(K, s_{2,0})$ in the $(S_1$–$S_2)$-plane. Thus convexity is lost but not time decay of the prices. Of course, if the price remains a convex function of the asset values we see directly from the differential equation that the prices will decay with time. Another property which is lost with the convexity is the monotonicity in volatility. In the present example we see that if the volatility in S_2 is made larger then the value of the option at $(K, s_{2,0})$ decreases.

In the time-dependent case there are no general results corresponding to the theorem above. Instead, different conditions on the contract functions can be combined with various classes of volatility matrices to guarantee time decay of the solutions. The technique with volatility time is harder to use, since each asset, in general, has its own volatility time. However, if these agree, one can show results corresponding to those in the case of one underlying asset. One has that if the volatility matrix σ is a diagonal matrix with

$$S_i \sigma_{ii} = v\bigl(t, S(t)\bigr),$$

where $v(t, S(t))$ satisfies the conditions of α in Theorem 1, then the option price $F(s,t)$ decays with time provided that the contract function Φ is subharmonic. This case is rather special and we leave the proof to the interested reader.

REMARK (Options as "bloating" assets). One can easily modify the example above by letting the volatility for the second asset be time dependent such that it is very large at some time and then decreases to a very small value until the time of expiration. Then the option price at $(K, s_{2,0})$ will increase with time during some interval. This corresponds directly to examples in [1] of bloating option prices when the volatility is stochastic.

We finally state one observation on independent processes of option prices.

PROPOSITION 2. *If the volatility of S_i, for every i, only depends on time and S_i, then the option prices decay with time if the contract function is a finite sum of products of positive convex functions of one of the underlying assets. Again we have to assume the regularity conditions of the previous theorems.*

The proof follows directly from the risk-neutral valuation formula and Theorem 3.

Acknowledgment. We thank Christer Kiselman who gave us a lot of assistance concerning questions of convexity and constructed the examples of Section 2.

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