The Annals of Applied Probability
2003, Vol. 13, No. 2, 800–815

CRITICAL PRICE NEAR MATURITY FOR AN AMERICAN OPTION ON A DIVIDEND-PAYING STOCK

BY DAMIEN LAMBERTON AND STÉPHANE VILLENEUVE

Université de Marne-la-Vallée and Université d'Evry Val d'Essonne

We study the behavior of the critical price of an American put option near maturity when the underlying stock pays dividends at a continuous rate. The results also apply to foreign currencies American options.

1. Introduction. In the Black–Scholes model, under the risk-neutral probability measure, the stock price process satisfies the following stochastic differential equation:

$$\frac{dS_t}{S_t} = (r - \delta)\, dt + \sigma\, dB_t$$

where the interest rate r, the volatility σ are positive constants, the dividend rate δ is a nonnegative constant and $(B_t)_{0\le t\le T}$ is a standard Brownian motion. We will denote by $(\mathcal{F}_t)_{0\le t\le T}$ the natural filtration of $(B_t)_{0\le t\le T}$. Here we focus on the Black–Scholes model with continuous-dividend payment. However, this model can also be viewed as the classical Garman–Kohlhagen model for foreign currencies, when r stands for the domestic interest rate and δ stands for the foreign interest rate.

In this setting, the price of an American put option with exercise price K and date of maturity T can be written as $P(t, S_t)$ where the function $P(t, x)$ is defined for every $(t, x) \in [0, T] \times \mathbb{R}_+$ by

$$P(t, x) = \sup_{\tau \in \mathcal{T}_{0,T-t}} \mathbb{E}\big[e^{-r\tau}\big(K - xe^{(r-\delta-(\sigma^2/2))\tau+\sigma B_\tau}\big)_+\big],$$

where $\mathcal{T}_{0,T-t}$ is the set of all $(\mathcal{F}_t)$-stopping times with values in $[0, T - t]$. We refer the reader to [8, 12, 13] for basics on American options. For $t \in [0, T)$, define the *critical price* at time t as

$$s(t) = \sup\{x \ge 0 \mid P(t, x) = K - x\}.$$

The graph of $s(t)$ is called the *exercise boundary* or the *free boundary* in the terminology of optimal stopping.

It is well known that the function $t \to s(t)$ is nondecreasing (indeed strictly

Received September 2000; revised October 2001.
AMS 2000 subject classifications. 90A09, 93E20, 60G40.
Key words and phrases. Optimal stopping, free boundary, American options.

increasing, as proved in [16]), C^∞ over $[0, T)$ (see [6, 14]) and that $\lim_{t\to T} s(t) = \min(rK/\delta, K)$ (see [14, 9]). Analogues of the latter result for American options on several assets have been obtained in [15].

In the pioneering work of van Moerbeke [14], the behavior of $s(t)$ as t approaches T was investigated. Although the statements of [14] are given for call options, they can easily be transferred to American puts (see, e.g., Proposition 10.3.4 of [12]). The conclusion seems to lead to a parabolic behavior, without restrictions on the parameters. However, the results of Barles, Burdeau, Romano and Sansoen [2] (see also [11]) show that a parabolic behavior cannot occur in some situations. Indeed, they proved the following estimate for the case $\delta = 0$:

$$\lim_{t\to T} \frac{K - s(t)}{\sigma K \sqrt{(T-t)\log(\frac{1}{T-t})}} = 1.$$

The above estimate remains valid whenever $0 \le \delta < r$, as can be proved by the methods of either [2] (see [1]) or [11] (see [16]).

The purpose of the present paper is to give rigorous results for the cases $r < \delta$ and $r = \delta$. Namely, we will show that the parabolic behavior stated by van Moerbeke does hold in the case $r < \delta$ (see Theorem 2 below). We note that this result is also stated in [17] but with a heuristic proof which does not clarify the role of the condition $r < \delta$. It will appear from our proof that the regularity of the pay-off function near $\lim_{t\to T} s(t)$ is crucial for the parabolic behavior. Our method relies on a rather general expansion of the value function of an optimal stopping problem along parabolas (see Theorem 1).

We will also prove (see Theorem 3) that, if $r = \delta$, the critical price satisfies the following estimate:

$$\lim_{t\to T} \frac{K - s(t)}{\sigma K \sqrt{(T-t)\log(\frac{1}{T-t})}} = \sqrt{2}.$$

This result had been conjectured by Aït-Sahlia in [1], where some partial results were given. Note that when $r = \delta$, the pay-off is not smooth near $\lim_{t\to T} s(t)$ and therefore, Theorem 1 of Section 2 does not apply and we use another approach, similar to that of [11]. The paper is organized as follows. In Section 2, we study the behavior of the value function of an optimal stopping problem along parabolas. In Section 3, we prove the parabolic behavior of the critical price when $r < \delta$ (cf. Theorem 2). In Section 4, we deal with the case $r = \delta$ (cf. Theorem 3).

REMARK 1. After submitting this paper, we were informed by P. Laurence that Evans, Kuske and Keller had obtained similar results (see [5]). However, their methods are quite different and their proofs are somewhat heuristic.

2. The value function of an optimal stopping problem along parabolas.

2.1. *An auxiliary optimal stopping problem.* We first introduce an auxiliary optimal stopping problem, the value function of which will appear naturally in the expansion of the value function of a general stopping problem along parabolas. We consider the real function ϕ defined on $\mathbb{R}$ by

$$\phi(y) = \sup_{\tau \in \mathcal{T}_{0,1}} \mathbb{E} \int_0^\tau (y + B_s)\, ds.$$

Recall that $(B_t)_{t \geq 0}$ is a standard Brownian motion and that $\mathcal{T}_{0,1}$ is the set of all stopping times with values in $[0, 1]$. It is easy to check that ϕ is a continuous, nondecreasing function on $\mathbb{R}$. We will be interested in the largest zero of ϕ.

LEMMA 1. *There exists* $y^* \in [-1, 0)$ *such that*

$$\forall\, y \leq y^*, \qquad \phi(y) = 0 \quad \textit{and} \quad \forall\, y > y^*, \qquad \phi(y) > 0.$$

PROOF. It follows from [14], Lemma 2 that $\phi(0) > 0$ (see also [15], Lemma 3.1). Therefore, it suffices to prove that $\phi(-1) \leq 0$. First, notice that for all stopping times $\tau \in \mathcal{T}_{0,1}$, $\int_0^\tau B_s\, ds = \tau B_\tau - \int_0^\tau s\, dB_s$. Therefore,

$$\begin{aligned} \mathbb{E} \int_0^\tau B_s\, ds &= \mathbb{E}\tau B_\tau \qquad \text{(since } \tau \text{ is bounded)} \\ &\leq \mathbb{E}\left(\frac{\tau^2 + B_\tau^2}{2}\right). \end{aligned}$$

Now, if $\tau \in \mathcal{T}_{0,1}$, we have $\tau^2 \leq \tau$ and, consequently,

$$\mathbb{E} \int_0^\tau B_s\, ds \leq \mathbb{E}\left(\frac{\tau + B_\tau^2}{2}\right) = \mathbb{E}\tau,$$

which proves that, for all $\tau \in \mathcal{T}_{0,1}$,

$$\mathbb{E} \int_0^\tau (-1 + B_s)\, ds \leq 0. \qquad \square$$

A more precise characterization of the optimal threshold y^* can be given as follows. Define, for $\theta \geq 0$ and $y \in \mathbb{R}$,

$$u(\theta, y) = \sup_{\tau \in \mathcal{T}_{0,\theta}} \mathbb{E}(y + B_\tau)^3.$$

We deduce from Itô's formula that

$$u(\theta, y) = y^3 + 3 \sup_{\tau \in \mathcal{T}_{0,\theta}} \mathbb{E} \int_0^\tau (y + B_s)\, ds.$$

By the scaling property of Brownian motion, we also have

$$u(\theta, y) = \theta^{3/2} u\left(1, \frac{y}{\sqrt{\theta}}\right).$$

Hence

$$u(\theta, y) = y^3 + 3\theta^{3/2}\phi\left(\frac{y}{\sqrt{\theta}}\right).$$

On the other hand, we know from the standard theory of optimal stopping (see [3], Chapter 3, Section 2, for a detailed presentation) that u satisfies the following variational inequality:

$$\max\left(-\frac{\partial u}{\partial \theta} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}, \psi - u\right) = 0,$$
$$u(0, \cdot) = \psi,$$

where $\psi(y) = y^3$. Therefore, $u(\theta, y) \geq y^3$ and, for $y \leq y^*\sqrt{\theta}$, $u(\theta, y) = y^3$. Moreover, u satisfies

$$\frac{\partial u}{\partial \theta}(\theta, y) = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(\theta, y) \qquad \text{for } y > y^*\sqrt{\theta}$$

and for $t \in [0, \theta]$,

$$\left|\frac{\partial u}{\partial y}(t, y)\right| \leq 3(y^2 + \theta),$$

therefore, the process $M_t = \int_0^t \frac{\partial u}{\partial y}(\theta - s, y + B_s)\, dB_s$ is a square integrable martingale. Applying the generalized Ito formula (see [10], Theorem 1, page 122) to $(u(\theta - s, y + B_s))_{0 \leq s \leq \theta}$, we get, after taking expectations,

$$\mathbb{E}\big(u(0, y + B_\theta)\big) = u(\theta, y) + \mathbb{E}\int_0^\theta \left(-\frac{\partial u}{\partial \theta} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\right)(\theta - s, y + B_s)\, ds.$$

Thus,

$$\mathbb{E}(y + B_\theta)^3 = u(\theta, y) + 3\mathbb{E}\int_0^\theta (y + B_s)\mathbb{1}_{\{y + B_s \leq y^*\sqrt{\theta - s}\}}\, ds.$$

We take $y = y^*\sqrt{\theta}$ in the previous equality to obtain

$$(y^*\sqrt{\theta})^3 + 3y^*\theta^{3/2} = (y^*\sqrt{\theta})^3 + 3\mathbb{E}\int_0^\theta (y^*\sqrt{\theta} + B_s)\mathbb{1}_{\{y^*\sqrt{\theta} + B_s \leq y^*\sqrt{\theta - s}\}}\, ds.$$

We set $v = \frac{s}{\theta}$ in the last integral thus

$$\mathbb{E}\int_0^\theta (y^*\sqrt{\theta} + B_s)\mathbb{1}_{\{y^*\sqrt{\theta} + B_s \leq y^*\sqrt{\theta - s}\}}\, ds$$
$$= \theta^{3/2}\mathbb{E}\int_0^1 (y^* + B_v)\mathbb{1}_{\{y^* + B_v \leq y^*\sqrt{1 - v}\}}\, dv.$$

After some rearrangement, we find that $-y^*$ solves the equation $G(x)=0$ where

$$G(x)=\mathbb{E}\int_0^1 (x-\sqrt{v}B_1)\mathbb{1}_{\{B_1\geq x(1-\sqrt{1-v})/\sqrt{v}\}}\,dv.$$

It is proved in the Appendix that G admits a unique root in the interval $(0,1/\sqrt{2}]$ and this can be found numerically to be 0.638748. We note that this numerical value agrees with the one given in [17] for a related problem.

2.2. *Expansion of the value function.* The main result of this section is the following.

THEOREM 1. *Let* $(t_0,x_0)\in\mathbb{R}^+\times\mathbb{R}$, $h>0$ *and* $f:[t_0,t_0+h]\times\mathbb{R}\to\mathbb{R}$ *satisfy*:

(1) f *is continuous on* $[t_0,t_0+h]\times\mathbb{R}$ *and*

$$\forall\,\varepsilon>0,\ \exists\, M_\varepsilon>0,\ \forall\,(t,x)\in[t_0,t_0+h]\times\mathbb{R},\qquad |f(t,x)|\leq M_\varepsilon e^{\varepsilon x^2}.$$

(2) f *admits continuous derivatives* $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x}$ *and* $\frac{\partial^2 f}{\partial x^2}$ *near* (t_0,x_0) *and the function* $\mathcal{D}f=\frac{\partial f}{\partial t}+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}$ *is differentiable at* (t_0,x_0).

Assume that $\mathcal{D}f(t_0,x_0)=0$ *and* $\frac{\partial\mathcal{D}f}{\partial x}(t_0,x_0)>0$ *and define*, *for* $(\theta,x)\in[0,h]\times\mathbb{R}$,

$$\hat{f}(\theta,x)=\sup_{\tau\in\mathcal{T}_{0,\theta}}\mathbb{E}f(t_0+\tau,x+B_\tau).$$

For any fixed $y\in\mathbb{R}$, *we have*

$$\hat{f}(\theta,x_0+y\sqrt{\theta})=f(t_0,x_0+y\sqrt{\theta})+\left(\frac{\partial\mathcal{D}f}{\partial x}(t_0,x_0)\phi(y)\right)\theta^{3/2}+o(\theta^{3/2}),$$

where ϕ *is defined by Lemma* 1.

Using the fact that, if $y>y^*$, $\phi(y)>0$, we immediately deduce the following corollary from Theorem 1.

COROLLARY 1. *Under the assumptions and with the notation of Theorem* 1, *if* $y>y^*$, *there exists* $\theta_y>0$ *such that*:

$$\forall\,\theta\in(0,\theta_y],\qquad \hat{f}(\theta,x_0+y\sqrt{\theta})>f(t_0,x_0+y\sqrt{\theta}).$$

PROOF OF THEOREM 1. First, we have (scaling property of Brownian motion)

$$\hat{f}(\theta,x)=\sup_{\tau\in\mathcal{T}_{0,1}}\mathbb{E}f(t_0+\theta\tau,x+\sqrt{\theta}B_\tau).$$

We can assume, without loss of generality, f to be a $C^{1,2}$ function on $[t_0, t_0+h] \times [x_0-h, x_0+h]$.

Fix x in $(x_0 - (h/2), x_0 + (h/2))$ and $0 < \theta < h$. Define the function F on $[0, 1] \times \mathbb{R}$ by

$$F(s, z) = f(t_0 + \theta s, x + \sqrt{\theta} z).$$

F is a $C^{1,2}$ function on $[0, 1] \times [-\frac{h}{2\sqrt{\theta}}, \frac{h}{2\sqrt{\theta}}]$ and

$$\left(\frac{\partial F}{\partial s} + \frac{1}{2}\frac{\partial^2 F}{\partial z^2}\right)(s, z) = \theta \mathcal{D} f(t_0 + \theta s, x + \sqrt{\theta} z).$$

Introduce the following stopping time:

$$\rho_\theta = \inf\left\{s \geq 0 \,\Big|\, |B_s| \geq \frac{h}{2\sqrt{\theta}}\right\}.$$

Notice that $\mathbb{P}(\rho_\theta < 1) = o(\theta^n)$ for every $n \in \mathbb{N}$ (see [4], page 172). We then have, for all $\tau \in \mathcal{T}_{0,1}$,

$$\begin{aligned}
&\mathbb{E} f(t_0 + \theta\tau, x + \sqrt{\theta} B_\tau) \\
&\quad = \mathbb{E} F(\tau, B_\tau) = \mathbb{E} F(\tau \wedge \rho_\theta, B_{\tau\wedge\rho_\theta}) + \mathbb{E}\big(F(\tau, B_\tau) - F(\rho_\theta, B_{\rho_\theta})\big)\mathbb{1}_{\{\rho_\theta < \tau\}}.
\end{aligned}$$

Now, assumption (1) of Theorem 1 yields $|F(s, z)| \leq M_\varepsilon e^{2\varepsilon(x^2+\theta z^2)}$, therefore,

$$\begin{aligned}
&\mathbb{E}\big|F(\tau, B_\tau) - F(\rho_\theta, B_{\rho_\theta})\big|\mathbb{1}_{\{\rho_\theta < \tau\}} \\
&\qquad \leq 2M_\varepsilon e^{2\varepsilon x^2}\mathbb{E}\exp\left(2\varepsilon\theta \sup_{0\leq s\leq 1} B_s^2\right)\mathbb{1}_{\{\rho_\theta < \tau\}} \\
&\qquad \leq 2M_\varepsilon e^{2\varepsilon x^2}\sqrt{\mathbb{E}\exp\left(4\varepsilon h \sup_{0\leq s\leq 1} B_s^2\right)}\sqrt{\mathbb{P}(\rho_\theta < 1)}.
\end{aligned}$$

Therefore, $\mathbb{E}|F(\tau, B_\tau) - F(\rho_\theta, B_{\rho_\theta})|\mathbb{1}_{\{\rho_\theta < \tau\}} = o(\theta^n)\ \forall n \in \mathbb{N}$.

Moreover, Ito's formula yields

$$\begin{aligned}
\mathbb{E} F(\tau \wedge \rho_\theta, B_{\tau\wedge\rho_\theta}) &= F(0, 0) + \mathbb{E}\int_0^{\tau\wedge\rho_\theta} \theta \mathcal{D} f(t_0 + \theta s, x + \sqrt{\theta} B_s)\, ds \\
&= f(t_0, x) + \theta\mathbb{E}\int_0^{\tau\wedge\rho_\theta} \mathcal{D} f(t_0 + \theta s, x + \sqrt{\theta} B_s)\, ds.
\end{aligned}$$

Choose $x = x_0 + y\sqrt{\theta}$ with θ small enough to ensure that $x_0 - \frac{h}{2} < x_0 + y\sqrt{\theta} < x_0 + \frac{h}{2}$. We obtain, for $\tau \in \mathcal{T}_{0,1}$,

$$\begin{aligned}
&\mathbb{E} F(\tau \wedge \rho_\theta, B_{\tau\wedge\rho_\theta}) \\
&\quad = f(t_0, x_0 + y\sqrt{\theta}) + \theta\mathbb{E}\int_0^{\tau\wedge\rho_\theta} \mathcal{D} f\big(t_0 + \theta s, x_0 + \sqrt{\theta}(y + B_s)\big)\, ds.
\end{aligned}$$

Since $\mathcal{D}f$ is differentiable at (t_0, x_0) and satisfies $\mathcal{D}f(t_0, x_0) = 0$, we have

$$\mathcal{D}f\big(t_0+\theta s, x_0+\sqrt{\theta}(y+B_s)\big) = \frac{\partial \mathcal{D}f}{\partial t}(t_0,x_0)\theta s + \frac{\partial \mathcal{D}f}{\partial x}(t_0,x_0)\sqrt{\theta}(y+B_s) + \big\|\big(\theta s, \sqrt{\theta}(y+B_s)\big)\big\| R\big(\theta s, \sqrt{\theta}(y+B_s)\big)$$

where R is a bounded function satisfying $\lim_{(u,v)\to(0,0)} R(u,v) = 0$. Thus,

$$\mathbb{E}F\big(\tau\wedge\rho_\theta, B_{\tau\wedge\rho_\theta}\big) = f(t_0, x_0 + y\sqrt{\theta}) + \theta^{3/2}\mathbb{E}\int_0^\tau (y+B_s)\,ds\,\frac{\partial \mathcal{D}f}{\partial x}(t_0,x_0) + R_1(\theta,\tau),$$

with

$$|R_1(\theta,\tau)| \le I_1(\theta,\tau) + \frac{\theta^2}{2}\left|\frac{\partial \mathcal{D}f}{\partial t}(t_0,x_0)\right| + I_2(\theta,\tau),$$

where we have sets

$$I_1(\theta,\tau) = \theta^{3/2}\mathbb{E}\int_{\tau\wedge\rho_\theta}^{\tau} |y+B_s|\,ds\left|\frac{\partial \mathcal{D}f}{\partial x}(t_0,x_0)\right|$$

and

$$I_2(\theta,\tau) = \theta\mathbb{E}\int_0^{\tau\wedge\rho_\theta} \big\|\big(\theta s, \sqrt{\theta}(y+B_s)\big)\big\|\big|R\big(\theta s, \sqrt{\theta}(y+B_s)\big)\big|\,ds.$$

We shall prove that $\sup_{\tau\in\mathcal{T}_{0,1}} |R_1(\theta,\tau)| = o(\theta^{3/2})$. For I_1, we have

$$\mathbb{E}\int_{\tau\wedge\rho_\theta}^{\tau} |y+B_s|\,ds \le \mathbb{E}\int_0^1 |y+B_s|\,ds\mathbb{1}_{\{\rho_\theta<1\}} = o(\theta^n) \qquad \text{for all } n\in\mathbb{N}.$$

For I_2, we have

$$I_2(\theta,\tau) \le \theta^{3/2}\mathbb{E}\int_0^{1\wedge\rho_\theta} (1+|y|+|B_s|)\big|R\big(\theta s, \sqrt{\theta}(y+B_s)\big)\big|\,ds.$$

The integral on the right-hand side goes to zero as θ goes to 0 by the dominated convergence theorem.

Finally, we conclude that, for $\tau\in\mathcal{T}_{0,1}$, we have

$$\begin{aligned}(1)\qquad &\mathbb{E}f\big(t_0+\theta\tau, x_0+\sqrt{\theta}(y+B_\tau)\big)\\ &\quad = f(t_0, x_0+y\sqrt{\theta}) + \theta^{3/2}\frac{\partial \mathcal{D}f}{\partial x}(t_0,x_0)\mathbb{E}\int_0^\tau (y+B_s)\,ds + \bar{R}(\theta,\tau),\end{aligned}$$

where $\sup_{\tau\in\mathcal{T}_{0,1}} |\bar{R}(\theta,\tau)| = o(\theta^{3/2})$ and the theorem follows easily. □

REMARK 2. If $\frac{\partial \mathcal{D}f}{\partial x}(t_0,x_0) \le 0$, we deduce from (1) that

$$\hat{f}(\theta, x_0+y\sqrt{\theta}) = f(t_0, x_0+y\sqrt{\theta}) + \left|\frac{\partial \mathcal{D}f}{\partial x}(t_0,x_0)\right|\phi(-y)\theta^{3/2} + o(\theta^{3/2}).$$

3. The case $r < \delta$. Throughout this section, we assume $r < \delta$. We will prove the following theorem:

THEOREM 2. *If $r < \delta$, as t approaches T, the critical stock-price satisfies the following*:

$$\lim_{t \to T} \frac{s(T) - s(t)}{\sigma s(T)\sqrt{T-t}} = -y^*,$$

where $s(T) = \frac{rK}{\delta}$ and y^* is given by Lemma 1.

PROOF. We will apply Theorem 1 to the following situation. Set $t_0 = 0$, $x_0 = \frac{1}{\sigma} \log \frac{rK}{\delta}$ and

$$f(t,x) = e^{-rt}\bigl(K - e^{(r-\delta-(\sigma^2/2))t+\sigma x}\bigr)_+.$$

Notice that $f(0,x) = (K - e^{\sigma x})_+$.

Observe that f satisfies conditions (1) and (2) of Theorem 1. Near $(0, x_0)$, we have

$$\mathcal{D} f(t,x) = -rKe^{-rt} + \delta e^{-(\delta+(\sigma^2/2))t+\sigma x}$$

and thus $\mathcal{D} f(0, x_0) = 0$ and $\frac{\partial \mathcal{D} f}{\partial x}(t_0, x_0) = rK\sigma > 0$.

With the notation of Section 1, we have, for $0 \le \theta \le T$, $P(T-\theta, e^{\sigma x}) = \hat{f}(\theta, x)$. Theorem 1 yields

$$P\bigl(T-\theta, e^{\sigma(x_0+y\sqrt{\theta})}\bigr) = \bigl(K - e^{\sigma(x_0+y\sqrt{\theta})}\bigr)_+ + \theta^{3/2} rK\sigma\phi(y) + o(\theta^{3/2}). \tag{2}$$

We deduce from Corollary 1 that if $y > y^*$, we have, for θ small enough,

$$P\bigl(T-\theta, e^{\sigma(x_0+y\sqrt{\theta})}\bigr) > \bigl(K - e^{\sigma(x_0+y\sqrt{\theta})}\bigr)_+.$$

Therefore, $e^{\sigma(x_0+y\sqrt{\theta})} > s(T-\theta)$. Since $x_0 = \frac{1}{\sigma}\log\frac{rK}{\delta}$, we obtain

$$\log \frac{rK}{\delta} - \log s(T-\theta) > -y\sigma\sqrt{\theta}.$$

Thus, for $y > y^*$,

$$\liminf_{\theta \to 0} \frac{s(T) - s(T-\theta)}{\sigma s(T)\sqrt{\theta}} \ge -y$$

and letting y go to y^*,

$$\liminf_{\theta \to 0} \frac{s(T) - s(T-\theta)}{\sigma s(T)\sqrt{\theta}} \ge -y^*.$$

Now, take $y \le y^*$. Theorem 1 gives

$$\hat{f}(\theta, x_0 + y\sqrt{\theta}) = f(0, x_0 + y\sqrt{\theta}) + o(\theta^{3/2}).$$

Thus,

$$P\big(T-\theta, e^{\sigma(x_0+y\sqrt{\theta})}\big) = \big(K - e^{\sigma(x_0+y\sqrt{\theta})}\big)_+ + g(\theta)$$

with $\lim_{\theta\to 0} \frac{g(\theta)}{\theta^{3/2}} = 0$. Moreover, if $s(t) < x < (rK/\delta)$, we have, using Taylor's formula and the *smooth fit* property (see, e.g., [12], Corollary 10.3.10),

$$P(t,x) - (K-x) = \frac{(x-s(t))^2}{2}\frac{\partial^2 P}{\partial x^2}(t,\zeta) \qquad \text{for some } \zeta \in \big(s(t), x\big).$$

As $\zeta > s(t)$, we get

$$\begin{aligned}
&\frac{\sigma^2\zeta^2}{2}\frac{\partial^2 P}{\partial x^2}(t,\zeta)\\
&\quad = -\frac{\partial P}{\partial t}(t,\zeta) - (r-\delta)\zeta\frac{\partial P}{\partial x}(t,\zeta) + rP(t,\zeta)\\
&\quad \geq (\delta - r)\zeta\frac{\partial P}{\partial x}(t,\zeta) + rP(t,\zeta) \qquad \text{[since } P(\cdot, x) \text{ is nondecreasing]}\\
&\quad \geq (\delta-r)\zeta(-1) + r(K-\zeta) \qquad \text{[since } P(t,\cdot) \text{ is convex]}\\
&\quad = rK - \delta\zeta\\
&\quad \geq rK - \delta x \qquad (\text{since } \zeta < x).
\end{aligned}$$

Whence, for $x < \frac{rK}{\delta}$,

$$\big((x-s(t))_+\big)^2 \leq \frac{\sigma^2 x^2}{rK - \delta x}\big(P(t,x) - (K-x)\big).$$

We apply the last inequality with $x = e^{\sigma(x_0+y\sqrt{\theta})}$ to obtain

$$\begin{aligned}
\big(e^{\sigma(x_0+y\sqrt{\theta})} - s(T-\theta)\big)_+^2 &\leq \frac{\sigma^2 e^{2\sigma(x_0+y\sqrt{\theta})}}{rK - \delta e^{\sigma(x_0+y\sqrt{\theta})}} g(\theta) = \frac{\sigma^2 e^{2\sigma(x_0+y\sqrt{\theta})}}{rK(1-e^{\sigma y\sqrt{\theta}})} g(\theta)\\
&\leq \frac{C}{rK\sigma y}\frac{g(\theta)}{\sqrt{\theta}}.
\end{aligned}$$

Thus,

$$\big(s(T)e^{\sigma y\sqrt{\theta}} - s(T-\theta)\big)_+ \leq o(\sqrt{\theta}).$$

Hence,

$$s(T)(1+\sigma y\sqrt{\theta}) - s(T-\theta) \leq o(\sqrt{\theta})$$

or equivalently,

$$\frac{s(T) - s(T-\theta)}{\sigma s(T)\sqrt{\theta}} \leq -y + o(1),$$

from which we deduce

$$\limsup_{\theta\to 0}\frac{s(T)-s(T-\theta)}{\sigma s(T)\sqrt{\theta}}\le -y^*,$$

which completes the proof. $\square$

4. The case $r=\delta$. This section is devoted to the proof of the following estimate:

THEOREM 3. *If $r=\delta$, the critical price satisfies the following*:

$$\lim_{t\to T}\frac{K-s(t)}{\sigma K\sqrt{(T-t)\log(\frac{1}{T-t})}}=\sqrt{2}.$$

Observe that if $r=\delta$, we have $s(T):=\lim_{t\to T}s(t)=K$, so that the payoff function is not smooth near $s(T)$, and we cannot apply Theorem 1. Our method will follow some ideas of [11]. We introduce the European put price function. The price of a European put option with exercise price K and date of maturity T can be written as $P_e(t,S_t)$ where the function $P_e(t,x)$ is defined for every $(t,x)\in[0,T]\times\mathbb{R}_+$ by

$$P_e(t,x)=\mathbb{E}\big[e^{-rt}\big(K-xe^{(r-\delta-(\sigma^2/2))(T-t)+\sigma B_{(T-t)}}\big)_+\big].$$

It is straightforward to check that the equation

$$P_e(t,x)=(K-x)$$

admits a unique solution in the interval $(0,K)$, if $t<T$ (see [11]). This solution will be denoted by $s_e(t)$ in the sequel. Since the function P_e can be given in closed form, the behavior of $s_e(t)$ is fairly easy to study and we have the following lemma.

LEMMA 2. *If $r=\delta$, the function $t\mapsto s_e(t)$ is nondecreasing near T and satisfies*

$$\lim_{t\to T}\frac{K-s_e(t)}{\sigma K\sqrt{(T-t)\log(\frac{1}{T-t})}}=\sqrt{2}. \tag{3}$$

Theorem 3 will follow immediately from (3) and the next lemma.

LEMMA 3. *There exists a constant $C>0$ such that, for $t<T$*,

$$0\le s_e(t)-s(t)\le C\sqrt{T-t}.$$

PROOF OF LEMMA 2. The estimate (3) has been observed by Aït-Sahlia in [1] and by Barles (private communication) (see also [16]). We sketch the proof for completeness. Recall that $s_e(t)$ is the unique solution in $(0, K)$ of the equation

$$P_e(t, x) = (K - x).$$

The inequality $s(t) \leq s_e(t)$ follows from $P_e \leq P$ and yields $\lim_{t\to T} s_e(t) = K$. For notational simplicity, we set $\theta = T - t$ and

$$\alpha = \alpha(\theta) = \frac{\log(K/s_e(t)) + (\sigma^2/2)\theta}{\sigma\sqrt{\theta}}.$$

We have $\lim_{\theta\to 0} \alpha(\theta) = 0$ and (with the same argument as in [11], Lemma 2.2) $\lim_{\theta\to 0} \sqrt{\theta}\alpha(\theta) = +\infty$. From

$$\begin{aligned} K - s_e(t) &= e^{-r\theta}\mathbb{E}\big(K - s_e(t)e^{\sigma B_\theta - (\sigma^2/2)\theta}\big)_+ \\ &= e^{-r\theta}\big(K - s_e(t)\big) + e^{-r\theta}\mathbb{E}\big(s_e(t)e^{\sigma B_\theta - (\sigma^2/2)\theta} - K\big)_+, \end{aligned}$$

we derive

$$(1 - e^{-r\theta})\big(e^{\sigma\sqrt{\theta}\alpha(\theta) - (\sigma^2/2)\theta} - 1\big) = e^{-r\theta}\mathbb{E}\big(e^{\sigma B_\theta - (\sigma^2/2)\theta} - e^{\sigma\sqrt{\theta}\alpha(\theta) - (\sigma^2/2)\theta}\big)_+.$$

Now, with the notation $f_1(\theta) \sim f_2(\theta)$ for $\lim_{\theta\to 0} \frac{f_1(\theta)}{f_2(\theta)} = 1$, we have, following the lines of the proof of Proposition 2.1 of [11],

$$\begin{aligned} r\sigma\theta^{3/2}\alpha(\theta) &\sim \mathbb{E}(e^{\sigma B_\theta} - e^{\sigma\sqrt{\theta}\alpha(\theta)})_+ \\ &\sim \sigma\sqrt{\theta}\mathbb{E}(B_1 - \alpha)_+ \\ &\sim \sigma\sqrt{\theta}\frac{1}{\sqrt{2\pi}\alpha^2 e^{\alpha^2/2}}. \end{aligned}$$

Hence,

$$\alpha^3 e^{\alpha^2/2} \sim \frac{1}{r\sqrt{2\pi}\theta}. \tag{4}$$

This yields $\alpha(\theta) \sim \sqrt{2\log(1/\theta)}$ and (3) follows easily.

We now prove that s_e is nondecreasing near T. Note that this property does not follow as easily as in the case of $s(t)$ because we do not have $\frac{\partial P_e}{\partial t} \leq 0$. We have $F(t, s_e(t)) = 0$, with

$$F(t, x) = P_e(t, x) - (K - x).$$

The function F is of class C^∞ on $(0, T) \times (0, K)$ and satisfies

$$\frac{\partial F}{\partial x}(t, x) = \frac{\partial P_e}{\partial x}(t, x) + 1 > 0.$$

We deduce from the implicit function theorem that s_e is differentiable on $(0, T)$ and that

$$\frac{\partial P_e}{\partial t}\big(t, s_e(t)\big) + \left(\frac{\partial P_e}{\partial x}\big(t, s_e(t)\big) + 1\right)s_e'(t) = 0.$$

It follows from this equality that $s_e'(t)$ and $-\frac{\partial P_e}{\partial t}(t, s_e(t))$ have the same sign. Moreover, we know that P_e satisfies the following PDE:

$$\frac{\partial P_e}{\partial t}(t, x) = r P_e(t, x) - \frac{\sigma^2 x^2}{2}\frac{\partial^2 P_e}{\partial x^2}(t, x).$$

Thus, using the definition of s_e,

$$\frac{\partial P_e}{\partial t}\big(t, s_e(t)\big) = r\big(K - s_e(t)\big) - \frac{\sigma^2 s_e^2(t)}{2}\frac{\partial^2 P_e}{\partial x^2}\big(t, s_e(t)\big). \tag{5}$$

With the same notation as above, we can derive from the Black–Scholes formula

$$\frac{\partial^2 P_e}{\partial x^2}\big(t, s_e(t)\big) = \frac{e^{-r\theta}}{s_e(t)\sigma\sqrt{\theta}\sqrt{2\pi}}\exp\left(-\frac{1}{2}(\alpha - \sigma\sqrt{\theta})^2\right).$$

Using (4), we have

$$\begin{aligned}\frac{\partial^2 P_e}{\partial x^2}\big(t, s_e(t)\big) &\sim \frac{1}{\sigma K\sqrt{2\pi\theta}}\exp\left(-\frac{\alpha^2}{2}\right)\\ &\sim \frac{1}{\sigma K\sqrt{2\pi\theta}}\alpha^3 r\sqrt{2\pi}\theta\\ &\sim \frac{r}{K\sigma}\alpha^3\sqrt{\theta}\\ &\sim \frac{r}{K\sigma}2^{3/2}\big(\log(1/\theta)\big)^{3/2}\sqrt{\theta}.\end{aligned}$$

Hence,

$$\frac{\sigma^2 s_e^2(t)}{2}\frac{\partial^2 P_e}{\partial x^2}\big(t, s_e(t)\big) \sim \sigma r K\sqrt{2\theta}\big(\log(1/\theta)\big)^{3/2}.$$

Therefore, for t close to T,

$$r\big(K - s_e(t)\big) < \frac{\sigma^2 s_e^2(t)}{2}\frac{\partial^2 P_e}{\partial x^2}\big(t, s_e(t)\big),$$

and we have from equality (5),

$$\frac{\partial P_e}{\partial t}\big(t, s_e(t)\big) < 0$$

which completes the proof of Lemma 2. □

PROOF OF LEMMA 3. Recall the early exercise premium formula (cf. [13], Corollary 3.1) and use $r = \delta$ to have

$$P(t,x) - P_e(t,x) = r\mathbb{E}\int_0^{T-t} e^{-ru}(K - S_u^x)\mathbb{1}_{\{S_u^x \le s(t+u)\}}\,du$$

where $S_u^x = xe^{\sigma B_u - (\sigma^2/2)u}$. Therefore,

$$\begin{aligned} P(t,x) - P_e(t,x) &\le r\int_0^{T-t} \mathbb{E}[e^{-ru}(K - S_u^x)_+]\,du \\ &= r\int_0^{T-t} P_e(T-u,x)\,du \\ &= r\int_0^{T-t} P_e(t+s,x)\,ds. \end{aligned}$$

In particular,

$$P\big(t,s_e(t)\big) - P_e\big(t,s_e(t)\big) \le r\int_0^{T-t} P_e\big(t+s,s_e(t)\big)\,ds.$$

Using Lemma 2, we have for t close to T, $s_e(t) \le s_e(t+s)$ and thus

$$P_e\big(t+s,s_e(t)\big) \le K - s_e(t),$$

whence,

$$P\big(t,s_e(t)\big) - P_e\big(t,s_e(t)\big) \le r(T-t)\big(K - s_e(t)\big). \tag{6}$$

Using Taylor's formula and the smooth fit property, we have

$$P\big(t,s_e(t)\big) - P_e\big(t,s_e(t)\big) = P\big(t,s_e(t)\big) - \big(K - s_e(t)\big) = \frac{(s_e(t) - s(t))^2}{2}\frac{\partial^2 P}{\partial x^2}(t,\zeta)$$

for some ζ in the interval $[s(t), s_e(t)]$. Now, since $\zeta > s(t)$, we have

$$\begin{aligned} \frac{\sigma^2\zeta^2}{2}\frac{\partial^2 P}{\partial x^2}(t,\zeta) &= -\frac{\partial P}{\partial t}(t,\zeta) + rP(t,\zeta) \\ &\ge rP(t,\zeta) \\ &\ge r(K-\zeta) \\ &\ge r\big(K - s_e(t)\big) \qquad [\text{because } \zeta \le s_e(t)]. \end{aligned}$$

We then deduce that

$$\begin{aligned} &\frac{(s_e(t) - s(t))^2}{2}\big(K - s_e(t)\big) \\ &\qquad \le \frac{2r}{\sigma^2\zeta^2}\Big(P\big(t,s_e(t)\big) - P_e\big(t,s_e(t)\big)\Big) \end{aligned}$$

$$\leq \frac{2r^2}{\sigma^2\zeta^2}(T-t)\big(K-s_e(t)\big) \qquad \text{[using equality (6)]}$$
$$\leq C(T-t)\big(K-s_e(t)\big)$$

which completes the proof of Lemma 3. □

APPENDIX

Let us denote by g a standard normal random variable. We shall prove that the function

$$G(x) = \mathbb{E}\int_0^1 (x-\sqrt{v}g)\mathbb{1}_{\{g\geq x(1-\sqrt{1-v})/\sqrt{v}\}}\,dv$$

admits a unique root in $(0, 1/\sqrt{2}]$. First, a straightforward calculus leads to $G(0) = -\frac{2}{3}\frac{1}{\sqrt{2\pi}}$. Set $\phi(v) = \frac{1-\sqrt{1-v}}{\sqrt{v}} = \frac{\sqrt{v}}{1+\sqrt{1-v}}$ and notice that ϕ is a nondecreasing function satisfying $\phi(0) = 0, \phi(1) = 1$ and $v\phi'(v) = \frac{\phi(v)}{2\sqrt{1-v}}$. We shall proceed in two steps.

Step 1. For every $x \geq \frac{1}{\sqrt{2}}$, we have $G(x) \geq 0$. We write

$$G(x) = x\int_0^1 \mathbb{P}\big(g \geq x\phi(v)\big)\,dv - \int_0^1 \sqrt{v}\mathbb{E}(g\mathbb{1}_{\{g\geq x\phi(v)\}})\,dv.$$

Integrating by parts the first integral above, we obtain

$$G(x) = x\mathbb{P}(g\geq x) + x^2\int_0^1 v\phi'(v)e^{-(x^2/2)\phi^2(v)}\frac{dv}{\sqrt{2\pi}} - \int_0^1 \sqrt{v}e^{-(x^2/2)\phi^2(v)}\frac{dv}{\sqrt{2\pi}}.$$

But, integrating by parts again gives

$$\int_0^1 \sqrt{v}e^{-(x^2/2)\phi^2(v)}\frac{dv}{\sqrt{2\pi}} = \frac{2}{3}\left(\frac{1}{\sqrt{2\pi}}e^{-(x^2/2)} + x^2\int_0^1 v\phi'(v)(1-\sqrt{1-v})e^{-(x^2/2)\phi^2(v)}\frac{dv}{\sqrt{2\pi}}\right).$$

Finally,

$$G(x) = x\mathbb{P}(g\geq x) - \frac{2}{3}\frac{1}{\sqrt{2\pi}}e^{-(x^2/2)} + \frac{x^2}{3}\int_0^1 h(v)e^{-(x^2/2)\phi^2(v)}\frac{dv}{\sqrt{2\pi}}, \tag{7}$$

where $h(v) = (1 + 2\sqrt{1-v})v\phi^{'}(v)$ is a nonnegative function on [0, 1] satisfying $\int_0^1 h(v)\,dv = 1$. Since $e^{-(x^2/2)\phi^2(v)} \geq e^{-x^2/2}$, we get

$$G(x) \geq x\mathbb{P}(g \geq x) + \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\left(\frac{x^2}{3} - \frac{2}{3}\right).$$

At this stage, we need to recall Komatsu's lemma (see [7]): we have, for $x > 0$,

$$\mathbb{P}(g \geq x) \geq \frac{2}{x + \sqrt{x^2+4}} \frac{e^{-x^2/2}}{\sqrt{2\pi}}.$$

Thus,

$$G(x) \geq p(x)\frac{e^{-x^2/2}}{\sqrt{2\pi}},$$

with

$$p(x) = \left(\frac{x^2}{3} - \frac{2}{3} + \frac{2x}{x + \sqrt{x^2+4}}\right).$$

We close the first step since p is a nondecreasing function on $[0, +\infty[$ satisfying $p(1/\sqrt{2}) = 0$.

Step 2. For every $x \in]0, 1/\sqrt{2}]$, $G^{'}(x) > 0$. Using equality (7), we have

$$\begin{aligned} G^{'}(x) &= \mathbb{P}(g \geq x) - \frac{x}{3}\frac{1}{\sqrt{2\pi}} e^{-x^2/2} \\ &\quad + \frac{2x}{3}\int_0^1 (1 + 2\sqrt{1-v})v\phi^{'}(v) e^{-(x^2/2)\phi^2(v)} \frac{dv}{\sqrt{2\pi}} \\ &\quad + \frac{x}{3}\int_0^1 k(v)(-x^2)\phi^{'}(v)\phi(v) e^{-(x^2/2)\phi^2(v)} \frac{dv}{\sqrt{2\pi}}, \end{aligned}$$

where $k(v) = v\phi(v)(1 + 2\sqrt{1-v})$. Integrating by parts the last integral and using equality $v\phi^{'}(v) = \frac{\phi(v)}{2\sqrt{1-v}}$, we get

$$\begin{aligned} G^{'}(x) &= \mathbb{P}(g \geq x) + x\int_0^1 v(2v-1)\phi^{'}(v) e^{-(x^2/2)\phi^2(v)} \frac{dv}{\sqrt{2\pi}} \\ &\geq \mathbb{P}(g \geq x) - \frac{x}{8}\int_0^1 \phi^{'}(v) e^{-(x^2/2)\phi^2(v)} \frac{dv}{\sqrt{2\pi}} \\ &= \mathbb{P}(g \geq x) - \frac{1}{8}\int_0^x e^{-t^2/2}\frac{dt}{\sqrt{2\pi}} \qquad [\text{we set } t = x\phi(v)] \\ &= \frac{9}{8}\mathbb{P}(g \geq x) - \frac{1}{16}. \end{aligned}$$

We close this second step by noting that the function $f(x) = \frac{9}{8}\mathbb{P}(g \geq x) - \frac{1}{16}$ is nonincreasing on $]0, +\infty[$ and satisfies $f(\frac{1}{\sqrt{2}}) > 0$.

REFERENCES

[1] AÏT-SAHLIA, F. (1995). Optimal stopping and weak convergence methods for some problems in financial economics. Ph.D. dissertation, Stanford Univ.

[2] BARLES, G., BURDEAU, J., ROMANO, M. and SANSOEN, N. (1995). Critical stock price near expiration. *Math. Finance* **5** 77–95.

[3] BENSOUSSAN, A. and LIONS, J. L. (1982). *Applications of Variational Inequalities in Stochastic Control*. North-Holland, Amsterdam.

[4] BORODIN, A. N. and SALMINEN, P. (1996). *Handbook of Brownian Motion—Facts and Formulae.* Birkhäuser, Boston.

[5] EVANS, J. D., KUSKE, R. and KELLER, J. B. (2002). American options with dividends near expiry. *Math. Finance* **12** 219–237.

[6] FRIEDMAN, A. (1975). Parabolic variational inequalities in one space dimension and smoothness of the free boundary. *J. Funct. Anal.* **18** 151–176.

[7] ITO, K. and MCKEAN, H. P. (1965). *Diffusion Processes and Their Sample Paths.* Springer, Berlin.

[8] KARATZAS, I. (1988). On the pricing of American options. *Appl. Math. Optim.* **17** 37–60.

[9] KIM, I. J. (1990). The analytic valuation of American options. *Rev. Financial Stud.* **3** 547–572.

[10] KRYLOV, N. V. (1980). *Controlled Diffusion Processes.* Springer.

[11] LAMBERTON, D. (1995). Critical price for an American option near maturity. In *Seminar on Stochastic Analysis* (E. Bolthausen, M. Dozzi and F. Russo, eds.). Birkhäuser, Boston.

[12] LAMBERTON, D. (1998). American options. In *Statistics and Finance* (D. Hand and S. Jacka, eds.). Arnold, London.

[13] MYNENI, R. (1992). The pricing of the American option. *Ann. Appl. Probab.* **2** 1–23.

[14] VAN MOERBEKE, P. (1976). On optimal stopping and free boundary problems. *Arch. Ration. Mech. Anal.* **20** 101–148.

[15] VILLENEUVE, S. (1999). Exercise region of American options on several assets. *Finance Stoch.* **3** 295–322.

[16] VILLENEUVE, S. (1999). Options américaines dans un modèle de Black–Scholes multidimensionnel. Doctoral dissertation, Univ. Marne-la-Vallée.

[17] WILMOTT, P., DEWYNNE, J. and HOWISON, S. *Option Pricing*: *Mathematical Models and Computation.* Cambridge Univ. Press.

LABORATOIRE D'ANALYSE
ET DE MATHÉMATIQUES APPLIQUÉES
PROJET MATHFI
UNIVERSITÉ DE MARNE-LA-VALLÉE
5 BOULEVARD DESCARTES
77 454 MARNE-LA-VALLÉE CEDEX 2
FRANCE
E-MAIL: dlamb@math.univ-mlv.fr

GREMAQ UMR CNRS 5604
UNIVERSITÉ DES
SCIENCES SOCIALES (TOULOUSE 1)
MANUFACTURE DES TABACS
21 ALLÉE DE BRIENNE
31000 TOULOUSE
FRANCE
E-MAIL: villeneuve@gremaq.univ-tlse1.fr