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# A COMPLETE EXPLICIT SOLUTION TO THE LOG-OPTIMAL PORTFOLIO PROBLEM

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D. Kramkov and W. Schachermayer [*Ann. Appl. Probab.* **9** (1999) 904–950] proved the existence of log-optimal portfolios under weak assumptions in a very general setting. For many—but not all—cases, T. Goll and J. Kallsen [*Stochastic Process. Appl.* **89** (2000) 31–48] obtained the optimal solution explicitly in terms of the semimartingale characteristics of the price process. By extending this result, this paper provides a complete explicit characterization of log-optimal portfolios without constraints.

Moreover, the results of Goll and Kallsen are generalized here in two further respects: First, we allow for random convex trading constraints. Second, the remaining consumption time—or more generally the consumption clock—may be random, which corresponds to a life-insurance problem.

Finally, we consider neutral derivative pricing in incomplete markets.

**1. Introduction.** A classical problem in mathematical finance is how to choose an optimal investment strategy in a securities market, or more precisely, how to maximize the expected utility from consumption or terminal wealth (often called *Merton's problem*). We focus on logarithmic utility in this paper. On an intuitive level, this utility function is supported by the so-called *Weber–Fechner law*, which says that stimuli are often perceived on a logarithmic rather than linear scale. From a mathematical point of view, logarithmic utility distinguishes itself by a desirable feature: in contrast to any other utility function, the optimal solution can be calculated quite explicitly in general dynamic models—even in the presence of complex dependencies. Third, the log-optimal strategy also maximizes the long-term growth rate in an almost-sure sense. For an historical account, details and further references on expected utility maximization we refer the reader to Karatzas and Shreve (1998), Goll and Kallsen (2000) (henceforth GK) and the overview article Schachermayer (2001).

Kramkov and Schachermayer (1999) (henceforth KS) proved the existence of optimal portfolios for terminal wealth in a general semimartingale framework and for a large class of utility functions. Semimartingales are in some sense the largest class of processes that allows for the definition of a stochastic integral or, in financial terms, a gains process. Therefore, KS cover essentially the most general case in which Merton's problem can be formulated.

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Explicit solutions to this problem in terms of the semimartingale characteristics of the underlying price process are provided in GK for logarithmic utility. They generalize well-known earlier results, for example, for discrete-time or Itô processes. However, GK do not achieve the same degree of generality as KS: in some cases optimal solutions are known to exist but they do not meet the sufficient condition (3.2) in GK, Theorem 3.1.

The main goal of this paper is to fill this gap by providing a more general sufficient condition which turns out to be necessary as well. Moreover, we extend the earlier results in two respects: first, we allow for random convex constraints similar to Cvitanić and Karatzas (1992); second, the consumption clock may be stochastic as well. As a particularly interesting example consider the case of a random remaining lifetime [cf. Richard (1975)]. This can be interpreted as a life-insurance or, more precisely, an old-age pension problem. It turns out that investment and consumption can no longer be treated separately in this case.

In Section 2 we state the problem and some preparatory results. The explicit solution of Merton's problem with random consumption clock and constraints can be found in Section 3. With the help of KS, it is shown in Section 4 that our sufficient condition is actually necessary in the absence of constraints. Some further properties and illuminating examples concerning the log-optimal portfolio are discussed in Section 5.

Another important issue in mathematical finance is derivative pricing. If one leaves the small set of complete market models, unique arbitrage-free contingent claim values do not exist any more. A way out is to consider *neutral* derivative prices. These are the only derivative prices such that the optimal expected utility is not increased by trading in contingent claims. For motivation and more background on neutral pricing see Kallsen (2001). Existence, uniqueness and computation of neutral derivative prices in the context of logarithmic utility is treated in Section 6. The Appendix contains results from stochastic calculus that are needed in the preceding sections.

Throughout, we use the notation of Jacod and Shiryaev (1987) (henceforth JS) and Jacod (1979, 1980). For any $\mathbb{R}^d$-valued semimartingale $S$, we denote by $L(S)$ the class of $\mathbb{R}^d$-valued predictable processes that are integrable with respect to $S$ in the sense of Jacod (1980). The transpose of a vector or matrix $x$ is denoted as $x^\top$ and its components by superscripts. Increasing processes are identified with their corresponding Lebesgue–Stieltjes measure.

**2. Optimal portfolios and supermartingales.** Our mathematical framework for a frictionless market model is as follows: we work with a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in\mathbb{R}_+}, P)$ in the sense of JS, I.1.2. We consider traded *securities* $0, \dots, d$ whose price processes are expressed in terms of multiples of the *numeraire* security 0. Put differently, these securities are modelled by their discounted *price processes* $S^0, \dots, S^d$, where $S^0 = 1$. We assume that $S := (S^1, \dots, S^d)$ is an $\mathbb{R}^d$-valued semimartingale.

We consider an investor (hereafter called "you") who disposes of an *initial endowment* $\varepsilon \in (0,\infty)$. *Trading strategies* are modelled by $\mathbb{R}^d$-valued, predictable stochastic processes $\varphi = (\varphi^1,\dots,\varphi^d)$, where $\varphi^i_t$ denotes the number of shares of security $i$ in your portfolio at time $t$. A trading strategy $\varphi \in L(S)$ with $\varphi_0 = 0$ belongs to the set $\mathfrak{S}$ of all *admissible* strategies if its discounted *gains process* $\varphi^\top \cdot S$ is bounded from below by $-\varepsilon$ (no debts allowed). Note that we have not considered the number $\varphi^0$ of shares of the numeraire in the portfolio. However, this number is uniquely specified as $\varphi^0 := \varphi^\top \cdot S - \varphi^\top S = \varphi^\top \cdot S_- - \varphi^\top S_-$ if we want $(\varphi^0,\dots,\varphi^d)$ to satisfy the *self-financeability* condition $(\varphi^0,\dots,\varphi^d)^\top (S^0,\dots,S^d) = (\varphi^0,\dots,\varphi^d)^\top \cdot (S^0,\dots,S^d)$.

We assume that your discounted *consumption* up to time $t$ is of the form $\kappa \cdot K_t$, where $\kappa$ denotes your discounted *consumption rate* according to the *consumption clock* $K$. We suppose that $K \in \mathscr{A}^+$; that is, $K$ is an adapted increasing process with $K_0 = 0$ and $E(K_\infty) < \infty$. Typical choices are $K_t := \mathbb{1}_{[\![T,\infty[\![}$ (consumption only at time $T$), $K_t := 1 - e^{-\lambda t}$ (consumption with impatience rate $\lambda$), $K_t := \sum_{s\le t} \mathbb{1}_{\{1,\dots,N\}}(s)$ for some $N \in \mathbb{N}$ (consumption only at integer times less than or equal to $N$), $K_t := t \wedge T$ for some stopping time $T$ (consumption uniformly in time during your lifetime $[\![0,T]\!]$). $\kappa$ is supposed to be an element of the set $\mathfrak{K}$ of all nonnegative, optional processes such that $\kappa \cdot K$ is finite on $\mathbb{R}_+$. Your discounted *wealth* at time $t$ is given by $V_t(\varphi,\kappa) := \varepsilon + \varphi^\top \cdot S_t - \kappa \cdot K_t$. A pair $(\varphi,\kappa) \in \mathfrak{S} \times \mathfrak{K}$ belongs to the set $\mathfrak{P}$ of *admissible portfolio–consumption pairs* if the discounted wealth process $V(\varphi,\kappa)$ is nonnegative.

To handle the stochastic clock we define a martingale $M$ by $M_t := E(K_\infty|\mathscr{F}_t)$. Moreover, we set $D := \{(\omega,t) \in \Omega \times \mathbb{R}_+ : t = 0 \text{ or } M_{t-}(\omega) - K_{t-}(\omega) > 0\} \in \mathscr{P}$ and $T := \inf\{t \in (0,\infty) : M_{t-} - K_{t-} = 0\}$. Since $M_t - K_t = 0$ implies that $M_s - K_s = 0$ for any $s \ge t$, we have that $T = \sup_{n\in\mathbb{N}} T_n$ and $[\![0,T[\![ \subset D = \bigcup_{n\in\mathbb{N}} [\![0,T_n]\!] \subset [\![0,T]\!]$, where the sequence of stopping times $(T_n)_{n\in\mathbb{N}}$ is defined by $T_n := \inf\{t \in \mathbb{R}_+ : M_t - K_t < \frac{1}{n}\}$. Moreover, we have $K = \mathbb{1}_D \cdot K$.

*Trading constraints* are given in terms of subsets of the set of all trading strategies. More specifically, we consider a process $\Gamma$ whose values are convex subsets of $\mathbb{R}^d$. The *constrained set* of trading strategies $\mathfrak{S}(\Gamma)$ and portfolio–consumption pairs $\mathfrak{P}(\Gamma)$ are defined as above but with the additional requirement that $\varphi_t \in V_{t-}(\varphi,\kappa)\Gamma_t$ pointwise on $\Omega \times (0,\infty)$, that is, $\Gamma_t$ restricts the portfolio relative to one unit of wealth. Important examples are $\Gamma := \mathbb{R}^d$ (no constraints) and $\Gamma := (\mathbb{R}_+)^d$ (no short sales).

The aim of this paper is to determine how you can make the best out of your money in the following sense:

DEFINITION 2.1. We say that $(\varphi,\kappa) \in \mathfrak{P}(\Gamma)$ is an *optimal portfolio–consumption pair for the constraints* $\Gamma$ if it maximizes $(\widetilde{\varphi},\widetilde{\kappa}) \mapsto E(\log(\widetilde{\kappa}) \cdot K_\infty)$ over all $(\widetilde{\varphi},\widetilde{\kappa}) \in \mathfrak{P}(\Gamma)$. [By convention, we set $E(\log(\widetilde{\kappa}) \cdot K_\infty) := -\infty$ if $E((-\log(\widetilde{\kappa}) \vee 0) \cdot K_\infty) = \infty$.]

REMARKS. (i) Observe that maximization of expected utility from terminal wealth is recovered as a special case of the previous definition if we choose $K := \mathbb{1}_{[\![T,\infty[\![}$ and $\kappa_T := \varepsilon + \varphi^\top \cdot S_T$, where $T \in \mathbb{R}_+$ denotes the terminal time (cf. GK).

(ii) Let us briefly touch on the subject of discounting and numeraire changes. To this end let $\overline{S}^0$ denote the undiscounted price process of the numeraire. Suppose that $\overline{S}^0$ is a semimartingale, $\overline{S}^0_0$ is deterministic and $E(\log(\overline{S}^0) \cdot K_\infty)$ is finite. The *undiscounted consumption rate* is obtained by multiplying $\kappa$ with the current undiscounted price $\overline{S}^0$ of the numeraire [cf. Goll and Kallsen (2001) for details]. Since $E(\log(\kappa\overline{S}^0) \cdot K_\infty) = E(\log(\kappa) \cdot K_\infty) + E(\log(\overline{S}^0) \cdot K_\infty)$, it follows that an optimal portfolio–consumption pair maximizes the expected logarithm of *un*discounted consumption $(\varphi, \kappa) \mapsto E(\log(\kappa\overline{S}^0) \cdot K_\infty)$ as well. Note also that the notion of admissibility does not depend on the chosen numeraire because it means that the wealth does not fall below 0, which is a numeraire-independent statement. Therefore, we have that the optimal solution to Merton's problem does not depend on the chosen numeraire. However, one has to be careful about interpreting this fact: the consumption rate $\kappa$ is always expressed in terms of multiples of the numeraire. Therefore, the discounted rate $\kappa$ of one and the same optimal solution does in fact depend on the numeraire. The situation is even more subtle for the optimal strategy $\varphi$: it only depends on the numeraire because the remaining endowment is implicitly invested in the numeraire without appearing in $\varphi$. Therefore, the same optimal investment is noted slightly differently if the numeraire is changed (cf. Example 5.1).

Due to the strict concavity of the logarithm, optimal portfolio–consumption pairs are essentially unique:

LEMMA 2.2 (Uniqueness). *Let $(\varphi, \kappa)$ and $(\widetilde{\varphi}, \widetilde{\kappa})$ be optimal portfolio–consumption pairs for the constraints $\Gamma$ with finite expected utility $E(\log(\kappa) \cdot K_\infty)$. Then $\kappa = \widetilde{\kappa}$ holds $(P \otimes K)$-almost everywhere. Moreover, we have $\varphi^\top \cdot S = \widetilde{\varphi}^\top \cdot S$ and hence $V(\varphi, \kappa) = V(\widetilde{\varphi}, \widetilde{\kappa})$ on $D$.*

PROOF. *Step* 1. Define $\widehat{\varphi} := \frac{1}{2}(\varphi + \widetilde{\varphi})$, $\widehat{\kappa} := \frac{1}{2}(\kappa + \widetilde{\kappa})$. Obviously, $V(\widehat{\varphi}, \widehat{\kappa}) = \frac{1}{2}(V(\varphi, \kappa) + V(\widetilde{\varphi}, \widetilde{\kappa}))$. From the convexity of $\Gamma$ it follows that $\frac{1}{2}(\varphi + \widetilde{\varphi}) \in \frac{1}{2}(V_-(\varphi, \kappa) + V_-(\widetilde{\varphi}, \widetilde{\kappa}))\Gamma$ and hence $(\widehat{\varphi}, \widehat{\kappa}) \in \mathfrak{P}(\Gamma)$. By optimality of $(\varphi, \kappa)$ and $(\widetilde{\varphi}, \widetilde{\kappa})$, we have $\int (\log(\widehat{\kappa}_t) - \frac{1}{2}(\log(\kappa_t) + \log(\widetilde{\kappa}_t)))\, d(P \otimes K) = \int \log(\widehat{\kappa}_t)\, d(P \otimes K) - \int \log(\kappa_t)\, d(P \otimes K) \leq 0$. Since the logarithm is concave, the integrand $\log(\widehat{\kappa}_t) - \frac{1}{2}(\log(\kappa_t) + \log(\widetilde{\kappa}_t))$ is nonnegative, which implies that it is 0 $(P \otimes K)$-almost everywhere. Therefore $\widetilde{\kappa} = \kappa$ $(P \otimes K)$-almost everywhere because the logarithm is strictly concave.

*Step* 2. Let $t_0 \in \mathbb{R}_+$ with $P(\{\omega \in \Omega : (\omega, t_0) \in D\}) \neq 0$. Moreover, define

$$A := \{\varphi^\top \cdot S_{t_0} < \widetilde{\varphi}^\top \cdot S_{t_0}\} \in \mathscr{F}_{t_0} \quad \text{and} \quad R := \frac{\varepsilon + \widetilde{\varphi}^\top \cdot S_{t_0} - \widetilde{\kappa} \cdot K_{t_0-}}{\varepsilon + \varphi^\top \cdot S_{t_0} - \kappa \cdot K_{t_0-}}.$$

Note that $\varepsilon + \varphi^\top \cdot S_{t_0} - \kappa \cdot K_{t_0-} \geq V_{t_0}(\varphi, \kappa) \geq 0$. For the denominator to be nonzero, we may assume w.l.o.g. that $\varepsilon + \varphi^\top \cdot S_{t_0} - \kappa \cdot K_{t_0-} > 0$ on $A$. Otherwise, replace $\varphi$ with $\widehat{\varphi}$ from Step 1, which satisfies $\varphi^\top \cdot S_{t_0} < \widehat{\varphi}^\top \cdot S_{t_0} < \widetilde{\varphi}^\top \cdot S_{t_0}$ on $A$. Define a new portfolio–consumption pair $(\overline{\varphi}, \overline{\kappa})$ by

$$\overline{\varphi}_t(\omega) := \begin{cases} \widetilde{\varphi}_t(\omega), & \text{if } t \leq t_0 \text{ or } \omega \in A^C, \\ R\varphi_t(\omega), & \text{if } t > t_0 \text{ and } \omega \in A, \end{cases}$$

$$\overline{\kappa}_t := \begin{cases} \kappa_t, & \text{for } t < t_0 \text{ or } \omega \in A^C, \\ R\kappa_t, & \text{for } t \geq t_0 \text{ and } \omega \in A. \end{cases}$$

For $t \geq t_0$ and $\omega \in A$ we have $V_t(\overline{\varphi}, \overline{\kappa}) = (\varepsilon + \widetilde{\varphi}^\top \cdot S_{t_0} - \widetilde{\kappa} \cdot K_{t_0-}) + R((\varphi 1\!\!1_{]\!]t_0,\infty[\![})^\top \cdot S_t - (\kappa 1\!\!1_{[\![t_0,\infty[\![}) \cdot K_t) = RV_t(\varphi, \kappa) \geq V_t(\varphi, \kappa) \geq 0$. Hence $\overline{\varphi} \in V_-(\overline{\varphi}, \overline{\kappa})\Gamma$, which implies that $(\overline{\varphi}, \overline{\kappa}) \in \mathfrak{P}(\Gamma)$. Obviously, $\overline{\kappa} > \kappa$ on $(A \times [t_0, \infty)) \cap D$. In view of the first step, this is only possible if $P(A \cap \{\omega \in \Omega : (\omega, t_0) \in D\}) = 0$. $\square$

It is well known that the optimal solution to Merton's problem is *myopic*; that is, it depends only on the local behavior of the price process. This local behavior of semimartingales is described by its *characteristics* in the sense of JS, II.2.6. Fix a truncation function $h : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$, that is, a bounded function with compact support that satisfies $h(x) = x$ in a neighborhood of 0. We assume that the characteristics $(B, C, \nu)$ of the $\mathbb{R}^{d+1}$-valued semimartingale $(S^1, \dots, S^d, M)$ relative to $h$ are given in the form

$$B = b \cdot A, \qquad C = c \cdot A, \qquad \nu = A \otimes F, \tag{2.1}$$

where $A \in \mathscr{A}^+_{\mathrm{loc}}$ is a predictable process, $b$ is a predictable $\mathbb{R}^{d+1}$-valued process, $c$ is a predictable $\mathbb{R}^{(d+1)\times(d+1)}$-valued process whose values are nonnegative, symmetric matrices, and $F$ is a transition kernel from $(\Omega \times \mathbb{R}_+, \mathscr{P})$ into $(\mathbb{R}^{d+1}, \mathscr{B}^{d+1})$. By JS, II.2.9, such a representation always exists. Typical choices for $A$ are $A_t := t$ (e.g., for Lévy processes, diffusions, Itô processes etc.) and $A_t := \sum_{s \leq t} 1\!\!1_{\mathbb{N}\setminus\{0\}}(s)$ (discrete-time processes). Especially for $A_t = t$, one can interpret $b_t$ or rather $b_t + \int (x - h(x)) F_t(dx)$ as a drift rate, $c_t$ as a diffusion coefficient and $F_t$ as a local jump measure. It is straightforward to obtain the semimartingale characteristics from other local desriptions of $(S, M)$, for example, in terms of stochastic differential equations or one-step transition densities in the discrete-time case (cf. GK, Section 4).

Even in the unconstrained case, the trading strategy $\varphi$ cannot be freely chosen because the wealth process is not allowed to jump to negative values. Moreover, it should not jump to 0 either because this prevents future consumption unless the market allows arbitrage. It turns out to be useful to express this fact in terms of a constraint set $\Gamma^0$ defined by

$$\Gamma^0_t := \{\psi \in \mathbb{R}^d : (\psi, 0)^\top x > -1 \text{ for } F_t\text{-almost any } x \in \mathbb{R}^{d+1}\}. \tag{2.2}$$

We call $\Gamma^0$ the *neutral constraints* and set $\dot{\Gamma} := \Gamma \cap \Gamma^0$.

The following lemma relates optimal portfolio–consumption pairs with supermartingales. It will serve as an important tool to prove our main results in the subsequent sections. Moreover, it is of interest on its own and we will discuss it more thoroughly in Section 4.

LEMMA 2.3. *Suppose that a $\dot{\Gamma}$-valued process $H \in L(S)$ exists and let $(\varphi, \kappa) \in \mathfrak{P}(\Gamma)$ be a portfolio–consumption pair. Assume that there is a nonnegative process $Z$ with the following properties*:

(i) $Z_0 = \frac{E(K_\infty|\mathscr{F}_0)}{\varepsilon}$ *and* $Z = \frac{1}{\kappa}$ $(P \otimes K)$*-almost everywhere.*

(ii) $Z^{T_n}$ *is a semimartingale and* $(Z\mathscr{E}(\psi^\top \cdot S))^{T_n}$ *is a supermartingale for any* $n \in \mathbb{N}$ *and any* $\dot{\Gamma}$*-valued* $\psi \in L(S)$.

*Then* $(\varphi, \kappa)$ *is an optimal portfolio–consumption pair for the constraints* $\Gamma$.

PROOF. Since $H$ is $\Gamma^0$-valued, we have $E(\sum_{t\in\mathbb{R}_+} \mathbb{1}_{(-\infty,-1]}(\Delta(H^\top \cdot S)_t)) = E(\mathbb{1}_{(-\infty,-1]}((H,0)^\top x) * \mu_\infty^{(S,M)}) = E(\mathbb{1}_{(-\infty,-1]}((H,0)^\top x) * \nu_\infty) = 0$. Therefore $P(\text{Ex. } t \in \mathbb{R}_+ \text{ with } \Delta(H^\top \cdot S)_t \leq -1) = 0$, which implies that $V := \varepsilon\mathscr{E}(H^\top \cdot S)$ and $V_-$ are positive processes (cf. JS, I.4.61c). Define $(\varphi^H, \kappa^H) \in \mathfrak{P}(\Gamma)$ by $\varphi_t^H := V_{t-}H_t$ for $t \in (0, \infty)$ and $\kappa^H := 0$.

Now, let $(\widetilde{\varphi}, \widetilde{\kappa}) \in \mathfrak{P}(\Gamma)$ with $E(\log(\widetilde{\kappa}) \cdot K_\infty) > -\infty$. Fix $\delta \in (0,1)$. Define a portfolio–consumption pair $(\overline{\varphi}, \overline{\kappa})$ by $\overline{\varphi} := (1-\delta)\widetilde{\varphi} + \delta\varphi^H$, $\overline{\kappa} := (1-\delta)\widetilde{\kappa} + \delta\kappa^H$. Obviously, $\overline{V} := V(\overline{\varphi}, \overline{\kappa}) = (1-\delta)V(\widetilde{\varphi}, \widetilde{\kappa}) + \delta V$. From the convexity of $\Gamma$ it follows that $\overline{\varphi} \in \overline{V}_-\Gamma$ and hence $(\overline{\varphi}, \overline{\kappa}) \in \mathfrak{P}(\Gamma)$. The positivity of $V$ and $V_-$ implies that $\overline{V}$ and $\overline{V}_-$ are positive as well and hence $P(\text{Ex. } t \in \mathbb{R}_+ \text{ with } \Delta(\psi^\top \cdot S)_t \leq -1) = 0$ for the $\Gamma$-valued process $\psi := \frac{\overline{\varphi}}{\overline{V}_-} \in L(S)$. Arguing conversely as for $H$ above, we conclude that $\psi$ is $\dot{\Gamma}$-valued $(P \otimes A)$-almost everywhere. Without loss of generality we may assume that $\psi$ is $\dot{\Gamma}$-valued: otherwise replace $\psi$ with $H$ on the set $\{\psi \notin \Gamma^0\}$, which is predictable because the mapping $(\omega, t, y) \mapsto \int \mathbb{1}_{(-\infty,-1]}((y,0)^\top x) F((\omega,t), dx)$ is $(\mathscr{P} \otimes \mathscr{B}^d)$-measurable.

Fix $n \in \mathbb{N}$. If we write $C := \overline{\kappa} \cdot K$, we have $(Z\overline{\kappa}) \cdot K = Z \cdot C = Z_- \cdot C + [Z, C] = (Z_-\overline{V}_-\psi)^\top \cdot S - Z_- \cdot \overline{V} + \overline{V}_- \cdot [Z, \psi^\top \cdot S] - [Z, \overline{V}]$ on $[\![0, T_n]\!]$ by JS, I.4.49, and the definition of $V(\overline{\varphi}, \overline{\kappa})$. Since $(Z\mathscr{E}(\psi^\top \cdot S))^{T_n}$ is a supermartingale and hence locally of class (D) [cf. Jacod (1979), (2.18) and its proof], it follows that

$$\frac{\overline{V}_-}{\mathscr{E}(\psi^\top \cdot S)_-} \cdot \big(Z\mathscr{E}(\psi^\top \cdot S)\big)^{T_n} = \overline{V}_- \cdot \big((Z_-\psi)^\top \cdot S + Z + [Z, \psi^\top \cdot S]\big)^{T_n}$$

is a local supermartingale and can be written as $N - A$ for some $N \in \mathscr{M}_{\mathrm{loc}}$, $A \in \mathscr{A}_{\mathrm{loc}}^+$ (cf. JS, I.3.38, I.4.34). This implies $Z \cdot C^{T_n} = N - A - \overline{V}_- \cdot Z^{T_n} - Z_- \cdot \overline{V}^{T_n} - [Z, \overline{V}^{T_n}]$, which equals $N - A - (Z\overline{V})^{T_n} + Z_0\overline{V}_0$ by partial integration. If $(U_m)_{m\in\mathbb{N}}$ denotes a localizing sequence of stopping times for both $N$ and $A$, we have that $E(Z \cdot C_{U_m \wedge T_n}) = E(N_{U_m\wedge T_n} - A_{U_m\wedge T_n} - (Z\overline{V})_{U_m\wedge T_n} + Z_0\overline{V}_0) \leq E(Z_0\overline{V}_0) = E(K_\infty)$ for any $m \in \mathbb{N}$. By monotone convergence, this implies

$E(Z\cdot C_{T_n})\leq E(K_\infty)$. Another application of monotone convergence yields $E((Z\overline{\kappa})\cdot K_\infty)\leq E(K_\infty)$ because $Z\cdot C_{T_n}\uparrow Z\cdot C_\infty$ for $n\to\infty$. Since the logarithm is concave, it follows that

$$\begin{aligned}\log(1-\delta)&E(K_\infty)+E\big(\log(\widetilde{\kappa})\cdot K_\infty\big)\\&=E\big(\log(\overline{\kappa})\cdot K_\infty\big)\\&\leq E\big(\big(\log(\kappa)+Z(\overline{\kappa}-\kappa)\big)\cdot K_\infty\big)\\&=E\big(\log(\kappa)\cdot K_\infty\big)+E\big((Z\overline{\kappa})\cdot K_\infty\big)-E(K_\infty)\\&\leq E\big(\log(\kappa)\cdot K_\infty\big).\end{aligned}$$

Letting $\delta\to 0$, we have $E(\log(\widetilde{\kappa})\cdot K_\infty)\leq E(\log(\kappa)\cdot K_\infty)$, which proves the claim. $\square$

**3. Explicit solution in terms of characteristics.** We place ourselves in the setup of the previous section. The following theorem provides a sufficient condition for optimality of a portfolio–consumption pair in terms of the characteristics of the price process. In Section 4 we will show that this condition is also necessary in the absence of constraints.

THEOREM 3.1. *Let $H\in L(S)$ be a $\dot{\Gamma}$-valued process and define $\Lambda:\Omega\times\mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}\cup\{\infty\}$ on $D\times\mathbb{R}^d$ by setting*

$$\begin{aligned}\Lambda(\psi)&:=(\psi,0)^\top b+(\psi,0)^\top c\left(-H,\frac{1}{(M-K)_-}\right)\\&\quad+\int\left(\frac{(\psi,0)^\top x}{1+(H,0)^\top x}\left(1+\frac{x^{d+1}}{(M-K)_-}\right)-(\psi,0)^\top h(x)\right)F(dx)\end{aligned}\tag{3.1}$$

*for $\psi\in\mathbb{R}^d$ if $\int|\frac{(\psi,0)^\top x}{1+(H,0)^\top x}(1+\frac{x^{d+1}}{(M-K)_-})-(\psi,0)^\top h(x)|F(dx)<\infty$ and $\Lambda(\psi):=\infty$ otherwise. Outside $D\times\mathbb{R}^d$ we set $\Lambda(\psi):=0$. Suppose that*

$$\sup\{\Lambda(\psi-H):\psi\in\dot{\Gamma}\}=0\qquad (P\otimes A)\text{-almost everywhere on } D.\tag{3.2}$$

*Let*

$$\kappa:=\frac{\varepsilon\mathscr{E}(H^\top\cdot S)}{E(K_\infty|\mathscr{F}_0)\mathscr{E}((1/(M-K)_-)\cdot M)}\times\mathbb{1}_{D\cap\{E(K_\infty|\mathscr{F}_0)\mathscr{E}((1/(M-K)_-)\cdot M)>0\}},$$

$$V_t:=\begin{cases}\kappa_t(M_t-K_t), & \text{if } t<T,\\ V_{T-}\big(1+H_T^\top\Delta S_T\big), & \text{if } t=T,\ \Delta K_T=0,\\ 0, & \text{if } t=T,\ \Delta K_T\neq 0,\\ V_T\mathscr{E}\big((H\mathbb{1}_{]\!]T,\infty[\![})^\top\cdot S\big), & \text{if } t>T,\end{cases}\tag{3.3}$$

$$\varphi:=V_-H,$$

*where we set $V_{0-} := 0$. Then $(\varphi, \kappa) \in \mathfrak{P}(\Gamma)$ is an optimal portfolio–consumption pair for the constraints $\Gamma$ with wealth process $V$.*

We have to say a few words about the definition of $\kappa$. Since $\frac{1}{(M-K)_-}$ is bounded on $[\![0, T_n]\!]$, we have that $\frac{1}{(M-K)_-}\mathbb{1}_{[\![0,T_n]\!]} \in L(M)$ for any $n \in \mathbb{N}$. Therefore, it makes sense to define

$$\kappa_t := \begin{cases} \kappa_t^{T_n}, & \text{if } t \leq T_n \text{ for some } n \in \mathbb{N}, \\ 0, & \text{otherwise,} \end{cases}$$

where the process $\kappa^{T_n}$ is defined by stopping the right-hand side of (3.3) at $T_n$. Note that these processes $\kappa^{T_n}$ are optional. Since $\kappa = \lim_{n\to\infty}(\kappa^{T_n}\mathbb{1}_D)$, it follows that $\kappa$ is optional as well.

PROOF OF THEOREM 3.1. *Step* 1. As in the proof of Lemma 2.3 it follows that $\mathscr{E}(H^\top \cdot S)$ and $\mathscr{E}(H^\top \cdot S)_-$ are positive on $\mathbb{R}_+$. Define

$$Z := \frac{1}{\kappa}\mathbb{1}_{\{\kappa>0\}} = \frac{E(K_\infty|\mathscr{F}_0)\mathscr{E}((1/(M-K)_-)\cdot M)}{\varepsilon\mathscr{E}(H^\top \cdot S)}\mathbb{1}_D.$$

Fix $n \in \mathbb{N}$. All processes in this and the next step are supposed to be stopped at $T_n$, that is, equalities and so on refer to the stochastic interval $[\![0, T_n]\!]$. Let

$$\begin{aligned} N :=& \left(-H, \frac{1}{(M-K)_-}\right)^\top \cdot (S, M) + \left((-H, 0)^\top c\left(-H, \frac{1}{(M-K)_-}\right)\right) \cdot A \\ & - (H, 0)^\top x\left(\frac{1}{1+(H,0)^\top x}\left(1 + \frac{x^{d+1}}{(M-K)_-}\right) - 1\right) * \mu^{(S,M)}. \end{aligned}$$

We will show that $N$ is well defined and $Z = Z_0\mathscr{E}(N)$.

By Itô's formula [cf., e.g., Goll and Kallsen (2000), Lemma A.5], we have

$$\begin{aligned} \frac{1}{\mathscr{E}(H^\top \cdot S)} =& \mathscr{E}\Big(-H^\top \cdot S + \big((H,0)^\top c(H,0)\big) \cdot A \\ & + \left(\frac{1}{1+(H,0)^\top x} - 1 + (H,0)^\top x\right) * \mu^{(S,M)}\Big). \end{aligned}$$

Since $\mathscr{E}(X)\mathscr{E}(Y) = \mathscr{E}(X + Y + [X, Y])$ for any two semimartingales $X, Y$ [cf. Jacod (1979), (6.4)], we have that $Z = Z_0\mathscr{E}(\widetilde{N})$ with

$$\begin{aligned} \widetilde{N} :=& \left(-H, \frac{1}{(M-K)_-}\right)^\top \cdot (S, M) + \big((H,0)^\top c(H,0)\big) \cdot A \\ & + \left(\frac{1}{1+(H,0)^\top x} - 1 + (H,0)^\top x\right) * \mu^{(S,M)} \\ & + \Big[-H^\top \cdot S + \big((H,0)^\top c(H,0)\big) \cdot A \\ & \quad + \left(\frac{1}{1+(H,0)^\top x} - 1 + (H,0)^\top x\right) * \mu^{(S,M)}, \frac{1}{(M-K)_-} \cdot M\Big]. \end{aligned}$$

By JS, I.4.52, the quadratic covariation term equals

$$\begin{aligned}
&\Big\langle -H^\top \cdot S^c, \frac{1}{(M-K)_-}\cdot M^c\Big\rangle + \sum_{s\le\cdot}\Big(\frac{1}{1+H_s^\top \Delta S_s}-1\Big)\frac{\Delta M_s}{(M-K)_{s-}}\\
&\quad = \Big((-H,0)^\top c\Big(0,\frac{1}{(M-K)_-}\Big)\Big)\cdot A\\
&\qquad + \Big(\frac{1}{1+(H,0)^\top x}-1\Big)\frac{x^{d+1}}{(M-K)_-} * \mu^{(S,M)}.
\end{aligned}$$

It follows that $\widetilde{N}=N$.

*Step* 2. Let $\psi \in L(S)$ be a $\dot\Gamma$-valued process. By (3.2) we have that

$$\int \left|\frac{(\psi-H,0)^\top x}{1+(H,0)^\top x}\Big(1+\frac{x^{d+1}}{(M-K)_-}\Big) - (\psi-H,0)^\top h(x)\right| F(dx) < \infty$$

$(P\otimes A)$-almost everywhere.

Define

$$\begin{aligned}
D_n := \Big\{&(\omega,t)\in\Omega\times\mathbb{R}_+ : |\psi_t - H_t|(\omega)\le n \text{ and}\\
&\int\left|\frac{(\psi_t-H_t,0)^\top x}{1+(H_t,0)^\top x}\Big(1+\frac{x^{d+1}}{(M-K)_{t-}}\Big)-(\psi_t-H_t,0)^\top h(x)\right|\\
&\qquad\qquad F_t(dx)(\omega)\in[0,n]\cup\{\infty\}\Big\}
\end{aligned}$$

for $n\in\mathbb{N}$. Since $\psi$ and $H$ are predictable, we have that $(D_n)_{n\in\mathbb{N}}$ is an increasing sequence of predictable sets with $D_n \uparrow \Omega\times\mathbb{R}_+$. Fix $n\in\mathbb{N}$. For any semimartingale $X$, we write $X^{D_n} := X_0 \mathbb{1}_{D_n}(0) + \mathbb{1}_{D_n}\cdot X$. Partial integration in the sense of JS, I.4.45, yields that $Z\mathscr{E}(\psi^\top\cdot S) = Z_0 + (Z_-\mathscr{E}(\psi^\top\cdot S)_-)\cdot(\psi^\top\cdot S + N + [N,\psi^\top\cdot S])$ and hence $(Z\mathscr{E}(\psi^\top\cdot S))^{D_n} = Z_0\mathbb{1}_{D_n}(0) + (Z_-\mathscr{E}(\psi^\top\cdot S)_-)\cdot X$ with $X := (\mathbb{1}_{D_n}\psi)^\top\cdot S + \mathbb{1}_{D_n}\cdot N + \mathbb{1}_{D_n}\cdot[N,\psi^\top\cdot S]$. From

$$\begin{aligned}
[N,\psi^\top\cdot S] &= \Big\langle \frac{1}{(M-K)_-}\cdot M^c - H^\top\cdot S^c, \psi^\top\cdot S^c\Big\rangle\\
&\quad + \sum_{s\le\cdot}\Big(\frac{\Delta M_s}{(M-K)_{s-}} + \Big(\frac{1}{1+H_s^\top\Delta S_s}-1\Big)\\
&\qquad\qquad \times\Big(1+\frac{\Delta M_s}{(M-K)_{s-}}\Big)\Big)\psi_s^\top \Delta S_s\\
&= \Big((\psi,0)^\top c\Big(-H,\frac{1}{(M-K)_-}\Big)\Big)\cdot A\\
&\quad + (\psi,0)^\top x\Big(\frac{1}{1+(H,0)^\top x}\Big(1+\frac{x^{d+1}}{(M-K)_-}\Big)-1\Big) * \mu^{(S,M)}
\end{aligned}$$

it follows that

$$
\begin{aligned}
X = {} & \biggl(\Bigl(\psi - H, \frac{1}{(M-K)_-}\Bigr)\mathbb{1}_{D_n}\biggr)^{\top} \cdot (S, M) \\
& + \biggl((\psi - H, 0)^{\top} c\Bigl(-H, \frac{1}{(M-K)_-}\Bigr)\mathbb{1}_{D_n}\biggr) \cdot A \\
& + (\psi - H, 0)^{\top} x \biggl(\frac{1}{1 + (H,0)^{\top} x}\Bigl(1 + \frac{x^{d+1}}{(M-K)_-}\Bigr) - 1\biggr)\mathbb{1}_{D_n} * \mu^{(S,M)}.
\end{aligned}
$$

By JS, II.2.34 and II.1.30, we have

$$
\begin{aligned}
\bigl((\psi - H)\mathbb{1}_{D_n}\bigr)^{\top} \cdot S = {} & \bigl((\psi - H)\mathbb{1}_{D_n}\bigr)^{\top} \cdot S^c + (\psi - H, 0)^{\top} h(x)\mathbb{1}_{D_n} * (\mu^{(S,M)} - \nu) \\
& + (\psi - H, 0)^{\top}\bigl(x - h(x)\bigr)\mathbb{1}_{D_n} * \mu^{(S,M)} \\
& + \bigl((\psi - H, 0)^{\top} b \mathbb{1}_{D_n}\bigr) \cdot A.
\end{aligned}
$$

Using JS, II.1.28, we obtain

$$
\begin{aligned}
X = {} & \bigl((\psi - H)\mathbb{1}_{D_n}\bigr)^{\top} \cdot S^c + \frac{\mathbb{1}_{D_n}}{(M-K)_-} \cdot M \\
& + \frac{(\psi - H, 0)^{\top} x}{1 + (H,0)^{\top} x}\Bigl(1 + \frac{x^{d+1}}{(M-K)_-}\Bigr)\mathbb{1}_{D_n} * (\mu^{(S,M)} - \nu) \\
& + \bigl(\Lambda(\psi - H)\mathbb{1}_{D_n}\bigr) \cdot A.
\end{aligned}
$$

Since $Z_- \mathscr{E}(\psi^{\top} \cdot S)_- \Lambda(\psi - H) \leq 0$, it follows that $(Z\mathscr{E}(\psi^{\top} \cdot S))^{D_n}$ is a local supermartingale [cf. JS, I.4.34, I.4.23], which in turn implies that $Z\mathscr{E}(\psi^{\top} \cdot S)$ is a $\sigma$-supermartingale [cf. Kallsen (2002), Lemma 2.3]. By Kallsen [(2002), Proposition 3.5] it is even a supermartingale.

*Step* 3. Recall that $K = \mathbb{1}_D \cdot K$. On $D$, the process $\kappa$ attains the value 0 only if $K = 0$ or if $\mathscr{E}(\frac{1}{(M-K)_-} \cdot M)$ jumps to 0. It is easy to see that the latter can only happen if $t = T$ and $\Delta K_t = 0$. Together, it follows that $\kappa > 0$ and hence $Z = \frac{1}{\kappa}$ $(P \otimes K)$-almost everywhere. In view of Lemma 2.3, it remains to be shown that $(\varphi, \kappa) \in \mathfrak{P}(\Gamma)$ is an admissible portfolio–consumption pair with wealth process $V$.

Fix $n \in \mathbb{N}$. Since $\mathscr{E}(X)\mathscr{E}(Y) = \mathscr{E}(X + Y + [X, Y])$ for any two semimartingales $X, Y$, straightforward calculations yield that

$$
1 = \mathscr{E}\Bigl(\frac{1}{(M-K)_-} \cdot M\Bigr)\mathscr{E}\Bigl(-\frac{1}{(M-K)_-} \cdot M + \frac{1}{(M-K)_-(M-K_-)} \cdot [M, M]\Bigr)
$$

on $[\![0, T_n[\![$, which implies that

$$\kappa = \frac{\varepsilon}{E(K_\infty|\mathscr{F}_0)}\mathscr{E}\biggl(H^\top \cdot S - \frac{1}{(M-K)_-} \cdot M + \frac{1}{(M-K)_-(M-K_-)} \cdot [M,M] + \biggl[H^\top \cdot S, -\frac{1}{(M-K)_-} \cdot M + \frac{1}{(M-K)_-(M-K_-)} \cdot [M,M]\biggr]\biggr)$$

on $[\![0, T_n[\![$. Another straightforward calculation yields $(M-K)_- \cdot \kappa + [\kappa, M] = -\kappa_- \cdot M + ((M-K)_-\kappa_- H)^\top \cdot S$ on $[\![0, T_n[\![$. Since $V = \varepsilon + \kappa_- \cdot (M-K) + (M-K)_- \cdot \kappa + [\kappa, M-K]$ by partial integration and $\kappa_- \cdot K + [\kappa, K] = \kappa \cdot K$ by JS, I.4.49a, we conclude that $V = \varepsilon + \varphi \cdot S - \kappa \cdot K$ on $[\![0, T_n[\![$.

So far all equalities have referred to the stochastic interval $[\![0, T_n[\![$ for given $n \in \mathbb{N}$. Letting $n \to \infty$, it follows that $V = \varepsilon + \varphi \cdot S - \kappa \cdot K = \varepsilon + V_- \cdot (H^\top \cdot S) - \kappa \cdot K$ holds on $[\![0, T[\![$. Since $\kappa \cdot K$ is nondecreasing, we have that $0 \leq V \leq \varepsilon\mathscr{E}(H^\top \cdot S)$ on $[\![0, T_n[\![$ for any $n$ by Proposition A.1. Therefore $1_D V_-$ is locally bounded and the limit $(1_D V_-) \cdot (H^\top \cdot S)_{T-} = \varphi^\top \cdot S_{T-}$ exists. Since $\kappa \cdot K$ is nondecreasing and bounded from above by $\varepsilon + \varphi \cdot S$, the limits $\kappa \cdot K_{T-}$ and $V_{T-}$ exist as well. Hence $V_T$ and $\varphi_T$ are well defined.

Suppose that $\Delta K_T = 0$. Then $V_T = V_{T-}(1 + H_T^\top \Delta S_T) = V_{T-} + \varphi_T^\top \Delta S_T - \kappa_T \Delta K_T$ and hence $V = \varepsilon + \varphi \cdot S - \kappa \cdot K$ holds on $[\![0, T]\!]$.

Alternatively, suppose that $\Delta K_T \neq 0$. A straightforward calculation yields that

$$1 + \frac{1}{(M-K)_{T-}}\Delta M_T = \frac{\Delta K_T}{(M-K)_{T-}}$$

and hence

$$\kappa_T = \kappa_{T-}(1 + H_T^\top \Delta S_T)\frac{(M-K)_{T-}}{\Delta K_T} = \frac{1}{\Delta K_T}(V_{T-} + \varphi_t^\top \Delta S_T).$$

Hence $V = \varepsilon + \varphi^\top \cdot S - \kappa \cdot K$ holds on $[\![0, T]\!]$ in this case as well.

The extension to $\mathbb{R}_+$ is straightforward. Together, it follows that $(\varphi, \kappa)$ is indeed an admissible portfolio–consumption pair with wealth process $V$. $\square$

The following corollary is not as general, but the condition on $H$ is more transparent.

COROLLARY 3.2. *Suppose that* $\Gamma = \mathbb{R}^d$ (*i.e., there are no constraints*). *Let* $H \in L(S)$ *be an* $\mathbb{R}^d$*-valued process with the following properties*:

(i) $1 + (H,0)_t^\top x > 0$ *for* $(P \otimes A \otimes F)$*-almost all* $(\omega, t, x) \in D \times \mathbb{R}^{d+1}$;

(ii) $\int |\frac{x^i}{1+(H,0)^\top x}(1 + \frac{x^{d+1}}{(M-K)_-}) - h^i(x)|F(dx) < \infty$ $(P \otimes A)$*-almost everywhere on* $D$ *for* $i = 1, \dots, d$;

(iii)

$$
\begin{aligned}
0 &= b^i + c^i\Big(-H, \frac{1}{(M-K)_-}\Big) \\
&\quad + \int \Big(\frac{x^i}{1+(H,0)^\top x}\Big(1 + \frac{x^{d+1}}{(M-K)_-}\Big) - h^i(x)\Big) F(dx)
\end{aligned}
\tag{3.4}
$$

*$(P \otimes A)$-almost everywhere on $D$ for $i = 1, \dots, d$.*

*Define $\kappa$, $V$ and $\varphi$ as in Theorem* 3.1. *Then $(\varphi, \kappa) \in \mathfrak{P}$ is an optimal portfolio–consumption pair with wealth process $V$.*

PROOF. Note that $\Lambda(\psi) = 0$ for any $\psi \in \mathbb{R}^d$ in Theorem 3.1. $\square$

In which sense does Theorem 3.1 provide an explicit solution to Merton's problem? The crucial part of both Theorem 3.1 and Corollary 3.2 is the condition on $H$. Let us start with Corollary 3.2. Here, all we have to do is to solve (3.4) pointwise for any $(\omega, t)$. At least from a numerical point of view, this is relatively easy because the characteristics of the price process are typically known and we only need to find a solution to $d$ equations in $d$ unknowns. Various concrete examples are given in GK.

Formally, the right-hand side of (3.4) can be interpreted as the derivative of some concave function of $H$ that is to be maximized (cf. GK, Remark 4 following Theorem 3.1). However, the derivative at a maximal point need not be 0 in the presence of convex constraints. Instead, it suffices that the directional derivative is nonpositive for those directions that point inside the constrained set. In our setting, this directional derivative is represented by $\Lambda(\psi - H)$, where $\Lambda$ is defined in (3.1), $H$ is the reference point and $\psi - H$ denotes the direction of interest. The corresponding nonpositivity statement is to be found in condition (3.2). Note that, regardless of its more complex form, this condition on $H$ in Theorem 3.1 is still a pointwise one.

Even in the unconstrained case, the optimal solution is sometimes not of the form in Corollary 3.2 (cf. Example 5.2 below and Example 5.1bis in KS). The way out is to treat this case artificially as a constrained one by introducing the neutral constraints $\Gamma^0$, which has been done in Theorem 3.1. This leads to a necessary condition as is shown in Section 4.

Another interesting issue is the role of the consumption clock in the utility maximization problem. Let us start with the simple case where $K$ is deterministic, which implies that $M = K_\infty$ is constant. In this case, (3.1) can be rewritten as

$$
\Lambda(\psi) = \psi^\top \overline{b} - \psi^\top \overline{c} H + \int \Big(\frac{\psi^\top x}{1 + H^\top x} - \psi^\top \overline{h}(x)\Big) \overline{F}(dx),
$$

where $\overline{h}: \mathbb{R}^d \to \mathbb{R}^d$ denotes a truncation function and $(\overline{b}, \overline{c}, \overline{F})$ is defined in the same way as $(b, c, F)$ but for the $\mathbb{R}^d$-valued process $S$ instead of $(S, M)$. Since the specific form of $K$ does not affect $H$, it follows that the portfolio and the consumption problem can be separated. Although the optimal portfolio $\varphi$ depends on $V$ and hence on $K$, the *relative portfolio* $H = \frac{1}{V_-}(\varphi^1, \dots, \varphi^d)$, that is, the number of securities relative to one unit of wealth, does not. On the other hand, a simple calculation shows that

$$\kappa_t = \frac{V_t}{K_\infty - K_t} = \frac{V_{t-} + \varphi_t^\top \Delta S_t}{K_\infty - K_{t-}}$$

for the optimal portfolio–consumption pair. The numerator of the second fraction is the wealth at time $t$ before consumption has taken place. The denominator stands for the remaining consumption time in the interval $[t, \infty)$. Therefore, the optimal strategy tries to spread consumption of current wealth uniformly over the remaining lifetime as it is measured by the consumption clock $K$.

If the consumption clock is random, the two aspects can no longer be separated. It turns out that the optimal relative portfolio is affected by the consumption clock $K$ if the tradable securities $S^1, \dots, S^d$ and the martingale $M$ are not independent. The intuitive reason is that your uncertain remaining lifetime creates a risk that you want to hedge partially by trading securities. Put differently, you invest in a portfolio that insures you against the expenses of old age—even if this portfolio has a negative drift and is hence unprofitable. The consumption strategy, on the other hand, remains essentially the same. Since

$$\kappa_t = \frac{V_t}{M_t - K_t} = \frac{V_{t-} + \varphi_t^\top \Delta S_t}{E(K_\infty - K_{t-} | \mathscr{F}_t)},$$

you still try to spread your wealth over the remaining lifetime $K_\infty - K_{t-}$. However, because the latter is unknown, it is replaced with its conditional expectation.

**4. Necessity and existence in the absence of constraints.** So far, we have not addressed the question whether an optimal portfolio–consumption pair exists and if it is of the form in Theorem 3.1. In this section we show that this is indeed the case—at least in the absence of constraints. The proof will be based on Theorem 2.2 in KS, which states that optimal portfolios exist in the terminal wealth case and which characterizes them in terms of a dual minimization problem. Interestingly, this deep result will allow us to prove the existence of a solution even for some random consumption clocks.

The general setting is as before. In addition to the assumptions in the previous sections we suppose that $\Gamma = \mathbb{R}^d$ (i.e., there are no constraints) and that "Condition NFLVR" (no free lunch with vanishing risk) in the sense of Delbaen and Schachermayer (1998) holds. Moreover, we assume that

$$\sup\{E(\log(\varepsilon + \varphi^\top \cdot S_{T\wedge n})) : \varphi \in \mathfrak{S}\} < \infty \qquad \text{for any } n \in \mathbb{N}. \tag{4.1}$$

Finally, we suppose that $\frac{1}{(M-K)_-} \in L^1_{\mathrm{loc}}(M)$ and $\mathscr{E}(\frac{1}{(M-K)_-} \cdot M)$ is a positive, locally bounded process, which holds trivially, for example, if the consumption clock is deterministic.

THEOREM 4.1. *There exists an optimal portfolio–consumption pair* $(\varphi, \kappa) \in \mathfrak{P}$ *which meets the conditions in Theorem* 3.1 *for* $\Gamma = \mathbb{R}^d$. *If* $S$ *is continuous, it also meets the conditions in Corollary* 3.2. *By Lemma* 2.2, *there is essentially no other optimal portfolio–consumption pair*.

PROOF. *Step* 1. Suppose that $M$ is constant and $T < \infty$. In Step 5 we treat the general case. All processes in Steps 1–4 of this proof are supposed to be stopped at $T$; that is, equalities and so on refer to the interval $[\![0, T]\!]$. Condition NFLVR implies that there exists a weak local martingale measure in the sense of the remark following Definition 5.2 in Kallsen (2002). By KS, Theorem 2.2(ii), there exists a strategy $\varphi \in \mathfrak{S}$ such that $Z(\varepsilon + \psi^\top \cdot S)$ is a supermartingale for any $\psi \in \mathfrak{S}$, where $Z := 1/(\varepsilon + \varphi^\top \cdot S)$. In particular, $Z$ is a semimartingale. Let $H := \varphi/(\varepsilon + \varphi^\top \cdot S_-)$. Since $\varepsilon \mathscr{E}(H^\top \cdot S) = \frac{1}{Z}$ is a positive process, we have that $H^\top \Delta S > -1$ up to an evanescent set. In particular, we have $H \in \Gamma^0$ $(P \otimes A)$-almost everywhere because

$$
\begin{aligned}
0 &= E\left(\sum_{s \in [\![0,T]\!]} 1\!\!1_{(-\infty,-1]}(H_t^\top \Delta S_t)\right) \\
&= E\big(1\!\!1_{(-\infty,-1]}\big((H,0)^\top x\big) * \mu_T^{(S,M)}\big) \\
&= E\big(1\!\!1_{(-\infty,-1]}\big((H,0)^\top x\big) * \nu_T\big) \\
&= \int_{[\![0,T]\!]} F\big(\{x \in \mathbb{R}^{d+1} : (H,0)^\top x \le -1\}\big)\, d(P \otimes A).
\end{aligned}
$$

Note that the set $G := \{H \notin \Gamma^0\}$ is predictable because the mapping $(\omega, t, y) \mapsto \int 1\!\!1_{(-\infty,-1]}\ ((y,0)^\top x) F((\omega,t), dx)$ is $(\mathscr{P} \otimes \mathscr{B}^d)$-measurable. Therefore, $\widetilde{H} := H 1\!\!1_{G^C} \in L(S)$ is a predictable $\Gamma^0$-valued process. Moreover, we have $H^\top \cdot S = \widetilde{H}^\top \cdot S$ because $(H 1\!\!1_G)^\top \cdot S = 0$ [cf. Kallsen and Shiryaev (2001), Lemma 2.5, and the fact that a semimartingale with vanishing characteristics is constant, cf. JS, II.4.19]. Hence we may assume without loss of generality that $H$ is $\Gamma^0$-valued.

*Step* 2. By Itô's formula (e.g., as in GK, Lemma A.5), we have that $Z = Z_0 \mathscr{E}(N)$ with

$$
\begin{aligned}
N := &-H^\top \cdot S + \big((H,0)^\top c(H,0)\big) \cdot A \\
&+ \left(\frac{1}{1 + (H,0)^\top x} - 1 + (H,0)^\top x\right) * \mu^{(S,M)}.
\end{aligned}
$$

Define $\Delta := \{(\omega, t, x) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}^{d+1} : |x| > 1 \text{ or } |(H,0)_t^\top(\omega) x| > 1\} \in \mathscr{P} \otimes \mathscr{B}^{d+1}$. By GK, Propositions A.2 and A.3, we have

$$
\begin{aligned}
N = &-H^\top \cdot S^c - (H,0)^\top x 1\!\!1_{\Delta^C}(x) * (\mu^{(S,M)} - \nu) - (H,0)^\top \cdot \widetilde{B} \\
&+ \big((H,0)^\top c(H,0)\big) \cdot A + \left(\frac{1}{1 + (H,0)^\top x} - 1 + (H,0)^\top x 1\!\!1_{\Delta^C}(x)\right) * \mu^{(S,M)},
\end{aligned}
$$

where $\widetilde{B} := (b - \int(h(x) - x\mathbb{1}_{\Delta^C}(x))F(dx)) \cdot A$ is a predictable process whose components are in $\mathscr{V}$. Since $Z$ is a supermartingale, we have that $N := \frac{1}{Z_-} \cdot Z$ is a special semimartingale (cf. JS, I.4.26, I.4.34). By JS, I.4.23, this implies that

$$\left(\frac{1}{1+(H,0)^\top x} - 1 + (H,0)^\top x\mathbb{1}_{\Delta^C}(x)\right) * \mu^{(S,M)} \in \mathscr{A}_{\mathrm{loc}}$$

and hence, by JS, II.1.28,

$$\begin{aligned}
N &= -H^\top \cdot S^c + \left(\frac{1}{1+(H,0)^\top x} - 1\right) * (\mu^{(S,M)} - \nu) \\
&\quad - (H,0)^\top \cdot \widetilde{B} + \big((H,0)^\top c(H,0)\big) \cdot A \\
&\quad - \left(\frac{(H,0)^\top x}{1+(H,0)^\top x} - (H,0)^\top x\mathbb{1}_{\Delta^C}(x)\right) * \nu \\
&= -H^\top \cdot S^c - \frac{(H,0)^\top x}{1+(H,0)^\top x} * (\mu^{(S,M)} - \nu) - \Lambda(H) \cdot A,
\end{aligned} \tag{4.2}$$

where $\Lambda$ is defined as in (3.1).

*Step* 3. Let $\psi \in L(S)$ be a $\Gamma^0$-valued process, which implies that $\mathscr{E}(\psi^\top \cdot S)$ and $\mathscr{E}(\psi^\top \cdot S)_-$ are positive processes (cf. the proof of Lemma 2.3). By Yor's formula [cf. Jacod (1979), (6.4)], we have that

$$\begin{aligned}
\frac{1}{Z_-\mathscr{E}(\psi^\top \cdot S)_-} \cdot \big(Z\mathscr{E}(\psi^\top \cdot S)\big) &= N + \psi^\top \cdot S + [N, \psi^\top \cdot S] \\
&= \psi^\top \cdot (S + [N,S]) + N.
\end{aligned}$$

Since $Z\varepsilon\mathscr{E}(\psi^\top \cdot S)$ is a supermartingale (cf. Step 1) and $1/(Z_-\mathscr{E}(\psi^\top \cdot S)_-)$ is positive and locally bounded, it follows that $\psi^\top \cdot (S + [N,S]) + N$ is a local supermartingale and in particular a special semimartingale (cf. JS, I.3.38, I.4.34). Note that

$$\begin{aligned}
[N, S^i] &= \langle -H^\top \cdot S^c, S^{i,c}\rangle + \sum_{t \le \cdot} \Delta N_t \Delta S^i_t \\
&= -\big(c^{i\cdot}(H,0)\big) \cdot A - x^i \frac{(H,0)^\top x}{1+(H,0)^\top x} * \mu^{(S,M)} \qquad \text{for } i = 1, \dots, d.
\end{aligned}$$

Using the same arguments as for $N$ in the previous step, we conclude that

$$\psi^\top \cdot (S + [N,S]) = \psi^\top \cdot S^c + \frac{(\psi,0)^\top x}{1+(H,0)^\top x} * (\mu^{(S,M)} - \nu) + \Lambda(\psi) \cdot A.$$

By (4.2), it follows that the local supermartingale $\psi^\top \cdot (S + [N,S]) + N$ equals $\Lambda(\psi - H) \cdot A$ up to a local martingale. Since $\Lambda(\psi - H) \cdot A$ is predictable, it must be nonincreasing, which implies that $\Lambda(\psi - H) \le 0$ $(P \otimes A)$-almost everywhere.

*Step* 4. Define the set $G := \{(\omega, t) \in [\![0, T]\!] : \text{There exists } y \in \Gamma^0_t(\omega) \text{ with } \Lambda_t(y - H_t)(\omega) > 0\}$. Then $G = \{(\omega, t) \in [\![0, T]\!] : \text{There exists } y \in \mathbb{R}^d \text{ with } (\omega, t, y) \in g^{-1}(\{1\} \times (0, \infty])\}$, where $g : [\![0, T]\!] \times \mathbb{R}^d \to \mathbb{R} \times (\mathbb{R} \cup \{\infty\})$, $(\omega, t, y) \mapsto (\mathbb{1}_{\Gamma^0_t(\omega)}(y), \Lambda_t(y - H_t)(\omega))$. Since $g$ is $(\mathscr{P} \otimes \mathscr{B}^d)$-measurable, Sainte-Beuve [(1974), Theorem 4] yields that $G$ is $\mathscr{P}^{P\otimes A}$-measurable, where $\mathscr{P}^{P\otimes A}$ denotes the $(P \otimes A)$-completion of the $\sigma$-field $\mathscr{P}$ [cf. Halmos (1974), Theorem 13.C, in this context]. Hence, the set $\widetilde{G} := (G^C \times \{0\}) \cup g^{-1}(\{1\} \times (0, \infty])$ is $(\mathscr{P}^{P\otimes A} \otimes \mathscr{B}^d)$-measurable. By the measurable selection theorem [cf. Sainte-Beuve (1974), Theorem 3], there exists a $\mathscr{P}^{P\otimes A}$-measurable mapping $\psi : [\![0, T]\!] \to \mathbb{R}^d$ with $(\omega, t, \psi_t(\omega)) \in \widetilde{G}$ for any $(\omega, t) \in [\![0, T]\!]$. Outside some $(P \otimes A)$-null set, $\psi$ coincides with some $\Gamma^0$-valued predictable process, which we denote again by $\psi$. Fix $n \in \mathbb{N}$ and let $\widetilde{\psi} := \psi \mathbb{1}_{\{|\psi| \leq n\}}$. One easily verifies that $\widetilde{\psi} \in L(S)$ is a $\Gamma^0$-valued process and $\Lambda(\widetilde{\psi} - H) > 0$ $(P \otimes A)$-almost everywhere on $G \cap \{|\psi| \leq n\}$. From the previous step it follows that $G \cap \{|\psi| \leq n\}$ is a $(P \otimes A)$-null set. Since $n$ was chosen arbitrarily, we have that $G$ is a $(P \otimes A)$-null set, which implies that $\sup\{\Lambda(\psi - H) : \psi \in \Gamma^0\} = 0$ $(P \otimes A)$-almost everywhere on $[\![0, T]\!]$.

*Step* 5. Now, we consider the general case. Without loss of generality we may assume $S = S^T$ because the condition on $H$ in Theorem 3.1 pertains only to $D \subset [\![0, T]\!]$ or, put differently, trading after $T$ does not affect the expected utility of consumption. Let $(R_n)_{n\in\mathbb{N}}$ be a sequence of stopping times with $R_n \uparrow \infty$ almost surely and $\mathscr{E}(\frac{1}{(M-K)_-} \cdot M)^{R_n} \leq n$ for any $n \in \mathbb{N}$. Without loss of generality let $R_n \leq n$ for any $n \in \mathbb{N}$.

Fix $n \in \mathbb{N}$. Define a probability measure $P^\star \sim P$ by its Radon–Nikodym density $\frac{dP^\star}{dP} := \mathscr{E}(\frac{1}{(M-K)_-} \cdot M)_{R_n}$. Since $\frac{dP^\star}{dP}$ is bounded, condition (4.1) implies that $\sup\{E_{P^\star}(\log(\varepsilon + \varphi^\top \cdot S_t)) : \varphi \in \mathfrak{S}\} < \infty$ for any $t \in \mathbb{R}_+$. Moreover, NFLVR holds relative to $P^\star$. By Kallsen [(2002), Lemma 5.1] and straightforward calculations, the $P^\star$-characteristics of $(S^1, \dots, S^d, M)$ are of the form (2.1), but with $b^\star, c^\star, F^\star$ instead of $b, c, F$, where

$$\begin{aligned}
b^\star &= b + c\left(0, \frac{1}{(M-K)_-}\right) + \int h(x) \frac{x^{d+1}}{(M-K)_-} F(dx),\\
c^\star &= c,\\
F^\star(G) &= \int_G \left(1 + \frac{x^{d+1}}{(M-K)_-}\right) F(dx) \qquad \text{for } G \in \mathscr{B}^{d+1}
\end{aligned}$$

on $[\![0, R_n]\!]$. It follows that

$$\begin{aligned}
\Lambda(\psi) &= (\psi, 0)^\top b^\star - (\psi, 0)^\top c^\star (H, 0)\\
&\quad + \int \left(\frac{(\psi, 0)^\top x}{1 + (H, 0)^\top x} - (\psi, 0)^\top h(x)\right) F^\star(dx) \qquad \text{on } [\![0, R_n]\!],
\end{aligned}$$

where $\Lambda(\psi)$ is defined as in (3.1). Now we can apply Steps 1–4 of this proof to $P^\star$ instead of $P$, $\mathbb{1}_{[\![n,\infty[\![}$ instead of $K$ and hence $n$ instead of $T$. This yields the existence of a $\Gamma^0$-valued process $H^{(n)} \in L(S)$ with $\sup\{\Lambda(\psi - H^{(n)}) : \psi \in \Gamma^0\} = 0$ $(P \otimes A)$-almost everywhere on $[\![0, R_n]\!]$. Letting $H := \sum_{n\in\mathbb{N}\setminus\{0\}} H^{(n)} \mathbb{1}_{]\!]R_{n-1},R_n]\!]}$, the claim follows. □

Along with other recent articles on the subject, this paper is based on a key insight which relates utility maximization and equivalent martingale measures: very roughly speaking, a portfolio–consumption pair $(\varphi, \kappa)$ is optimal if and only if $u'(\kappa)$ is up to a normalizing constant the density process of an equivalent martingale measure (EMM). Here, $u$ denotes the utility function under consideration, that is, the logarithm in our case. Similarly, an admissible strategy $\varphi$ maximizes the expected utility from terminal wealth at time $T$ iff $u'(\varepsilon + \varphi^\top \cdot S_T)$ is proportionate to the density of an EMM. This relationship has been termed the *fundamental theorem of utility maximization* (FTUM) in Kallsen (2001) because of its similarity with the fundamental theorem of asset pricing (FTAP), which relates the absence of arbitrage with EMMs. For bibliography, we refer the reader to Kallsen [(2001), Section 2.2] and in particular to Foldes (1990), who stated a version of Corollary 4.2 in a quite general semimartingale setting.

Similarly to the FTAP, the FTUM holds literally true only for markets of a simple structure, for example, in finite probability spaces. In general, the process $u'(\kappa)$ may no longer be the density process of a probability measure, let alone an EMM. The following result states a general version of the relationship between log-optimal portfolio–consumption pairs and martingales. Here, $Z$ plays the role of the density process of an EMM. However, Example 5.1bis in KS shows that in general $Z$ may fail to be even a local martingale.

COROLLARY 4.2 (Fundamental theorem of utility maximization). *Suppose that the consumption clock $K$ is deterministic. For any $(\varphi, \kappa) \in \mathfrak{P}$ the following statements are equivalent*:

(i) *$(\varphi, \kappa)$ is an optimal portfolio–consumption pair without constraints.*

(ii) *There exists a nonnegative semimartingale $Z$ with $Z_0 = \frac{K_\infty}{\varepsilon}$ such that the following hold*:

(a) $Z = \frac{1}{\kappa}$ *$(P \otimes K)$-almost everywhere*;

(b) $(Z(\varepsilon + \psi^\top \cdot S))^{T_n}$ *is a supermartingale for any $n \in \mathbb{N}$ and any $\psi \in \mathfrak{S}$.*

PROOF. [(ii) ⇒ (i)] This implication follows from Lemma 2.3: choose $H := 0$ and recall from the proof of Lemma 2.3 that $\mathscr{E}(\psi^\top \cdot S)$ is nonnegative for any $\dot{\Gamma}$-valued $\psi \in L(S)$.

[(i) ⇒ (ii)] *Step* 1. Fix $n \in \mathbb{N}$. As noted in Step 1 of the proof of Theorem 4.1, there exists a strategy $\varphi^{(n)} \in \mathfrak{S}$ such that $(Z^{(n)}(\varepsilon + \psi^\top \cdot S))^{T\wedge n}$ is a supermartingale for any $\psi \in \mathfrak{S}$, where $Z^{(n)} := K_\infty/(\varepsilon + (\varphi^{(n)})^\top \cdot S^{T\wedge n})$. Moreover, it was shown that $H^{(n)} := (\varphi^{(n)}/[\varepsilon + (\varphi^{(n)})^\top \cdot S])1_{[\![0,T\wedge n]\!]}$ meets the conditions in Theorem 3.1, for example, for the consumption clock $K^{(n)} := 1_{[\![T\wedge n,\infty[\![}$.

*Step* 2. Let $m, n \in \mathbb{N}$ with $m > n$. Obviously, $H^{(m)}$ meets the conditions in Theorem 3.1 for the consumption clock $K^{(n)}$ as well. Moreover, $\varphi^{(n)}$ coincides on $[\![0, T \wedge n]\!]$ with the optimal strategy defined in Theorem 3.1 constructed from $H^{(n)}$ and $K^{(n)}$. The same is true for $\varphi^{(m)}$, $H^{(m)}$ and $K^{(n)}$. By Lemma 2.2 we conclude that $(\varphi^{(m)})^\top \cdot S^{T\wedge n} = (\varphi^{(n)})^\top \cdot S^{T\wedge n}$, which implies that $Z^{(m)} = Z^{(n)}$ and $(H^{(m)})^\top \cdot S = (H^{(n)})^\top \cdot S$ on $[\![0, T \wedge n]\!]$.

*Step* 3. Now, define $H := \sum_{n\in\mathbb{N}\setminus\{0\}} H^{(n)} 1_{]\!]T\wedge n-1, T\wedge n]\!]} \in L(S)$ and $Z := K_\infty/(\varepsilon\mathscr{E}(H^\top \cdot S))$, which equals $Z^{(n)}$ on $[\![0, T \wedge n]\!]$ for any $n \in \mathbb{N}$. Note that $H$ meets the conditions in Theorem 3.1 (for the originally given consumption clock $K$), which implies that $\widetilde{\kappa} := (\varepsilon\mathscr{E}(H^\top \cdot S)/K_\infty)1_D = \frac{1}{Z}1_D$ is the consumption rate of some optimal portfolio–consumption pair $(\widetilde{\varphi}, \widetilde{\kappa})$. By Lemma 2.2 and since $D^C$ is a $(P \otimes K)$-null set, we have $Z = \frac{1}{\kappa}$ $(P \otimes K)$-almost everywhere. Statement (b) follows from Step 1. $\square$

If we allow for constraints, we may consider Lemma 2.3 as a sufficient part of the FTUM. The question whether it is also necessary is left to future research.

**5. Special cases and related problems.** First of all, we want to take a closer look at the general assumptions which we made in the previous sections. In Lemma 2.2 we assumed that the maximal expected utility is finite. It is easy to see that unique optimal portfolio–consumption pairs do not exist in cases where the maximal expected utility equals $-\infty$ or $\infty$: if an optimal strategy is given, then cutting trades and consumption in half and investing the remaining endowment in the numeraire yields an optimal portfolio–consumption pair as well. Condition (4.1) is closely related to the natural assumption $\sup\{E(\log(\kappa) \cdot K_\infty) < \infty : (\varphi, \kappa) \in \mathfrak{P}\} \in \mathbb{R}$ of finite maximal expected utility. However, one can show that the latter condition alone does not suffice to conclude existence of an optimal strategy [cf. Goll and Kallsen (2001)].

In Section 4 we also imposed the standard condition of no free lunch with vanishing risk. It is well known that it can be expressed in terms of $\sigma$-martingales or local martingales [cf. Delbaen and Schachermayer (1998)]. One may wonder whether the absence of free lunches does not already follow from the condition of finite maximal expected utility. This is not the case, as will be shown in Example 5.1: even in a market with arbitrage there may still exist an optimal portfolio–consumption pair with finite expected utility. The reason is that the notion of admissibility for the utility maximization problem implies that the wealth process is bounded from below by 0 and not just by an arbitrary constant. Conversely, the absence of free lunches does not imply that the maximal expected utility is finite: one simply has to consider a one-period market with one risky asset $S^1$ satisfying $E(\log(\Delta S^1_1)) = \infty$ in the sense that $E(0 \vee (-\log(\Delta S^1_1))) < \infty$ and $E(0 \vee \log(\Delta S^1_1)) = \infty$.

EXAMPLE 5.1. In this example we consider the same market relative to two different numeraires. Let $S^0 := 1$ and $S^1 := 1 + (\rho 1\!\!1_{[\![0,\tau]\!]}) \cdot W$, where $W$ denotes a standard Wiener process, $\rho_t := 1/\sqrt{1-t}$, and the stopping time $\tau := \inf\{t \in [0,1] : \rho \cdot W_t < -\frac{1}{2}\}$ is bounded $P$-almost surely by 1. Obviously, $S^1$ is a $[\frac{1}{2}, \infty)$-valued local martingale with $S^1_1 = \frac{1}{2}$. Let the initial endowment and the consumption clock be given by $\varepsilon := 1$ and $K := 1\!\!1_{[\![1,\infty[\![}$.

Firstly, we consider $S^0$ as the numeraire. Since $\frac{S^1}{S^0} = S^1$ is a $P$-local martingale, the market meets condition NFLVR. A simple application of Corollary 3.2 yields that it is optimal not to trade in security 1 and to consume the initial endowment at time 1; that is, we have $\varphi^1 = 0$ and $\kappa_1 = 1$ for the optimal portfolio–consumption pair $(\varphi^1, \kappa)$.

Alternatively, we treat $S^1$ as numeraire. Then the discounted prices are given by $\widehat{S}^1 := \frac{S^1}{S^1} = 1$ and $\widehat{S}^0 := \frac{S^0}{S^1}$. Note that $\widehat{S}^0$ is $[0,2]$-valued with $\widehat{S}^0_0 = 1$ and $\widehat{S}^0_1 = 2$. Hence, buying this security at time 0 and selling it at time 1 is an arbitrage in this market, which implies that condition NFLVR does not hold. For a thorough account of arbitrage and numeraire changes cf. Delbaen and Schachermayer (1995). Using Itô's formula, we conclude that the characteristics $(b, c, F)$ of $\widehat{S}^0$ relative to $A_t = t$ are given by

$$b_t = (S^1_t)^{-3} \frac{1\!\!1_{[\![0,\tau]\!]}(t)}{1-t} = (\widehat{S}^0_t)^3 \frac{1\!\!1_{[\![0,\tau]\!]}(t)}{1-t},$$

$$c_t = (S^1_t)^{-4} \frac{1\!\!1_{[\![0,\tau]\!]}(t)}{1-t} = (\widehat{S}^0_t)^4 \frac{1\!\!1_{[\![0,\tau]\!]}(t)}{1-t}$$

and $F = 0$, which implies that $H := 1/\widehat{S}^0$ leads to an optimal strategy in the application of Corollary 3.2. Consequently, the optimal portfolio–consumption pair $(\widehat{\varphi}^0, \widehat{\kappa})$ is given by $\widehat{\varphi}^0 = H\mathscr{E}(H \cdot \widehat{S}^0) = (1/\widehat{S}^0)\widehat{S}^0 = 1$. Moreover, $\widehat{\kappa}_1 = \mathscr{E}(H \cdot \widehat{S}^0)_1 = \widehat{S}^0_1 = 2$. Note that the role of the numeraire and the risky asset are now exchanged for the application of Corollary 3.2.

Although $(\widehat{\varphi}^0, \widehat{\kappa})$ looks quite different from $(\varphi^1, \kappa)$ above, it corresponds to the same investment strategy. Since the remaining endowment is implicitly invested in the numeraire, $\varphi^1 = 0$ means investment in zero shares of security 1 and one share of security 0. The same holds true for $\widehat{\varphi}^0 = 1$. Similarly, the undiscounted consumption is calculated from $\kappa$ resp. $\widehat{\kappa}$ by multiplication with the nominal value of the corresponding numeraire. In both cases it equals $\kappa_1 S^0_1 = 1 = \widehat{\kappa}_1 S^1_1$.

We have noted already in Section 3 that the complex sufficient condition in Theorem 3.1 can often be replaced with the simpler one in Corollary 3.2. Now, we want to take a closer look at what (3.4) means. From Lemma 2.3 and Corollary 4.2 we know that the process $Z = \frac{1}{\kappa}$ plays a crucial role if $(\varphi, \kappa) \in \mathfrak{P}$ denotes an optimal portfolio–consumption pair. Let us assume for the time being that $Z^T$ is a positive uniformly integrable martingale and hence up to a normalizing constant the density process of some probability measure $P^\star \sim P$. With this notion, condition (3.4) means that $S^T$ is a $P^\star$-$\sigma$-martingale, that is, $P^\star$ is a $\sigma$-martingale measure for the stopped process $S^T$ [cf. Step 1 in the proof of Theorem 3.1, Kallsen (2002), Lemma 5.4, and some straightforward calculations]. Put differently, it means that $(SZ)^T$ is a $P$-$\sigma$-martingale [cf. Kallsen (2002), Proposition 5.3], which makes sense even if $Z^T$ is only a local martingale and $P^\star$ is not defined. Sometimes, however, $Z^T$ is not even a local martingale, for example, in Example 5.1bis of KS, where (3.4) does not have a solution. It may also happen that $Z^T$ is a uniformly integrable martingale but the corresponding measure $P^\star$ is not a $\sigma$-martingale measure, in which case one cannot apply Corollary 3.2 either:

EXAMPLE 5.2. Let $X$ be a random variable whose law is exponential with parameter 2. Define a simple one-period market as follows: $\mathscr{F}_t := \{\varnothing, \Omega\}$ for $t \in [0, 1)$, $\mathscr{F}_t := \sigma(X)$ for $t \geq 1$, $S^1_t := 1$ for $t \in [0, 1)$, $S^1_t := X$ for $t \geq 1$. Note that $E(S^1_1) = \frac{1}{2} < S^1_0$. If we consider the terminal wealth problem with initial endowment 1 (i.e., $K := \mathbb{1}_{[\![1,\infty[\![}$ and $\varepsilon := 1$), a straightforward application of Theorem 3.1 or Corollary 4.2 shows that $\varphi^1 = 0$ and $\kappa = 1$ for the optimal portfolio–consumption pair $(\varphi, \kappa) \in \mathfrak{P}$. In particular, the corresponding process $Z = \frac{1}{\kappa} = 1$ is the density process of $P^\star := P$, but this measure is not a $\sigma$-martingale measure, not even a weak local martingale measure in the sense of the remark following Definition 5.2 in Kallsen (2002).

At this point let us take the opportunity to correct an inaccuracy in GK: in Theorem 3.1 one has to impose slightly stronger integrability conditions for the proof to work. However, as can be seen from Theorem 3.1 and Corollary 3.2 in this paper, the statements remain valid under the original integrability condition but the proof requires more refined reasoning by $\sigma$-localization.

*Numeraire portfolio.* Sometimes, the *numeraire portfolio* is proposed as an alternative to measure changes in mathematical finance [cf., e.g., Long (1990) and Becherer (2001)]. The idea is to choose some tradable portfolio as a numeraire such that the corresponding discounted securities price processes are martingales or at least $\sigma$-martingales under the real-world probability measure $P$. If a numeraire portfolio exists, then it is essentially unique and it coincides with the log-optimal portfolio for terminal wealth. [Note that the optimal trading strategy $\varphi$ in the terminal wealth problem (i.e., for $K := \mathbb{1}_{[\![T,\infty[\![}$ with some $T \in \mathbb{R}_+$) does not depend on $T$ as long as it is in the future.] To obtain existence and uniqueness of numeraire portfolios under the weak general assumptions in Section 4, Becherer (2001) extended this notion to strategies $\varphi$ such that $(1+\vartheta^\top \cdot S)/(1+\varphi^\top \cdot S)$ is a supermartingale for any $\vartheta \in L(S)$ with $1+\vartheta^\top \cdot S > 0$. By Corollary 4.8 in Becherer (2001), numeraire portfolios in this sense coincide precisely with log-optimal portfolios for terminal wealth and initial endowment 1. Therefore, Theorem 3.1 can be interpreted as a general explicit characterization of the numeraire portfolio if we choose $\varepsilon := 1$, $\Gamma := \mathbb{R}^d$ and $K := \mathbb{1}_{[\![T,\infty[\![}$ for some remote $T \in \mathbb{R}_+$. If you prefer the narrower definition in terms of $\sigma$-martingales, you should turn instead to Corollary 3.2: (3.4) means that the corresponding discounted securities $S^i/(1+\varphi^\top \cdot S) = S^i/\kappa$ are $\sigma$-martingales for $i = 0, \ldots, d$. This follows from a straightforward but tedious calculation.

*Growth rate of wealth.* Finally, we turn to the *growth rate of wealth* which is discussed, for example, in Karatzas and Shreve (1998), Section 3.10. Suppose that NFLVR and condition (4.1) hold for any $T \in \mathbb{R}_+$. By Theorem 4.1 the optimal portfolio for terminal wealth does not depend on the terminal date $T$. Therefore, there is a strategy $\varphi \in \mathfrak{S}$ that maximizes $\widetilde{\varphi} \mapsto \frac{1}{T} E(\log(\varepsilon + \widetilde{\varphi}^\top \cdot S_T))$ for any $T \in \mathbb{R}_+$ and hence also the *expected growth rate of wealth* $\limsup_{t\to\infty} \frac{1}{t} E(\log(\varepsilon + \widetilde{\varphi}^\top \cdot S_t))$. Interestingly, this property can be strengthened in an almost-sure sense. The following lemma extends Theorem 3.10.1 in Karatzas and Shreve (1998) to the general semimartingale case, but the proof remains essentially the same. For references on the growth rate of wealth, cf. Karatzas and Shreve (1998), Section 3.11.

LEMMA 5.3. *By $\varphi \in \mathfrak{S}$ denote the optimal portfolio for terminal wealth as explained above. Then we have*

$$\limsup_{t\to\infty} \frac{1}{t} \log(\varepsilon + \widetilde{\varphi} \cdot S_t) \le \limsup_{t\to\infty} \frac{1}{t} \log(\varepsilon + \varphi \cdot S_t)$$

*$P$-almost surely for any $\widetilde{\varphi} \in \mathfrak{S}$.*

PROOF. Let $\widetilde{\varphi} \in \mathfrak{S}$ and $\delta \in (0,1)$ and define $Z = 1/(\varepsilon + \varphi^\top \cdot S)$. It follows from KS, Theorem 2.2(ii), that $Z$ is a well-defined positive supermartingale and that $Z(\varepsilon + \widetilde{\varphi}^\top \cdot S)$ is a supermartingale. In particular, we have $e^{\delta n} P(\sup_{t\in[n,\infty)} Z_t(\varepsilon + \widetilde{\varphi}^\top \cdot S_t) > e^{\delta n}) \le E(Z_0(\varepsilon + \widetilde{\varphi}^\top \cdot S_0)) = 1$ for any $n \in \mathbb{N}$ by Doob's maximal inequality [cf., e.g., Elliott (1982), Corollary 4.8 and Theorem 4.2]. This implies $\sum_{n=1}^\infty P(\sup_{t\in[n,\infty)} \frac{1}{n} \log(Z_t(\varepsilon + \widetilde{\varphi}^\top \cdot S_t)) > \delta) \le \sum_{n=1}^\infty e^{-\delta n} < \infty$. From the Borel–Cantelli lemma it follows that $P$-almost surely there exists some (random) $n_0 \in \mathbb{N}$ such that $\sup_{t\in[n,\infty)} \frac{1}{n} \log(Z_t(\varepsilon + \widetilde{\varphi}^\top \cdot S_t)) \le \delta$ for any $n \ge n_0$. Since $Z = 1/(\varepsilon + \varphi^\top \cdot S)$, we have that $\limsup_{t\to\infty} \frac{1}{t} \log(\varepsilon + \widetilde{\varphi}^\top \cdot S_t) \le \limsup_{t\to\infty} \frac{1}{t} \log(\varepsilon + \varphi^\top \cdot S_t) + \delta$ $P$-almost surely. $\square$

**6. Neutral derivative pricing.** Contingent claim valuation in incomplete markets cannot be based solely on arbitrage arguments. Additional assumptions have to be made if one wants to obtain unique prices. *Neutral derivative pricing* tries to mimic and generalize the economic reasoning in complete markets by substituting utility maximizers for arbitrage traders. A derivative price is called *neutral* if an investor cannot raise his or her expected utility by trading the claim. For motivation, references and connections to other approaches in the literature we refer the reader to Kallsen (2001).

Of course, neutral prices generally depend on the profile of the representative investor, that is, on his or her utility function, initial endowment, time horizon etc. Logarithmic utility offers a number of advantages in this context. We have noted already that it is supported by the Weber–Fechner law on an intuitive level. Moreover, it turns out that neutral prices for logarithmic utility do not depend on the numeraire, the initial endowment or the time horizon of the investor. Also, opposed to other utility functions, the density process of the pricing measure can be computed explicitly for a great number of semimartingale models for the underlyings.

In this section, we work with a finite time horizon $\tau \in \mathbb{R}_+$. As representative investor we consider an unconstrained log-utility maximizer with deterministic consumption clock. More precisely, we assume that the general setting is as in Section 4, that $S = S^\tau$ and that $K_\infty - K_{\tau-} > 0$, where $K$ is deterministic. Suppose that there exists a process $H$ as in Corollary 3.2 with

$$\int \left| \frac{x}{1+(H,0)^\top x} - h(x) \right| F(dx) \in L(A).$$

Then

$$L := S^c + \frac{x}{1+H^\top x} * (\mu^S - \nu^S)$$

is a well-defined local martingale, where $\nu^S$ denotes the compensator of the measure of jumps of $S$. Moreover, we have the following.

PROPOSITION 6.1. *$H \in L(L)$ and $\mathscr{E}(-H^\top \cdot L) = 1/(\mathscr{E}(H^\top \cdot S))$.*

PROOF. Recall from Step 1 in the proof of Theorem 3.1 that $\mathscr{E}(H^\top \cdot S)$ and $\mathscr{E}(H^\top \cdot S)_-$ are positive on $\mathbb{R}_+$. An application of Itô's formula, for example, as in GK, Lemma A.5, yields that

$$\frac{1}{\mathscr{E}(H^\top \cdot S)} = \mathscr{E}\left(-H^\top \cdot S + \langle H^\top \cdot S^c, H^\top \cdot S^c \rangle - \left(\frac{H^\top x}{1+H^\top x} - H^\top x\right) * \mu^S\right).$$

Observe that $|x/(1+H^\top x) - x| * \mu^S \in \mathscr{V}$ because $|x/(1+H^\top x) - h(x)| * \nu^S \in \mathscr{V}$ and $|x - h(x)| * \mu^S \in \mathscr{V}$. By Proposition A.2 it follows that

$$\left(\frac{H^\top x}{1+H^\top x} - H^\top x\right) * \mu^S = H^\top \cdot \left(\left(\frac{x}{1+H^\top x} - x\right) * \mu^S\right)$$

and hence $1/(\mathscr{E}(H^\top \cdot S)) = \mathscr{E}(-H^\top \cdot \widetilde{L})$ with

$$\widetilde{L} := S - S^0 - \langle S^c, H^\top \cdot S^c\rangle + \left(\frac{x}{1+H^\top x} - x\right) * \mu^S.$$

The canonical decomposition of $S$ and (3.4) yield that

$$\begin{aligned}
S^i &= S_0^i + S^{i,c} + h^i(x) * (\mu^S - \nu^S) + (x^i - h^i(x)) * \mu^S + B^i \\
&= S_0^i + S^{i,c} + h^i(x) * (\mu^S - \nu^S) + (x^i - h^i(x)) * \mu^S \\
&\quad + \langle S^{i,c}, H^\top \cdot S^c\rangle - \left(\frac{x^i}{1+H^\top x} - h^i(x)\right) * \nu^S
\end{aligned}$$

for $i = 1, \dots, d$, which implies that

$$\widetilde{L}^i = S^{i,c} + \frac{x^i}{1+H^\top x} * (\mu^S - \nu^S) = L^i$$

as desired. □

We define $Z := \mathscr{E}(-H^\top \cdot L) = 1/(\mathscr{E}(H^\top \cdot S)) = \exp(X)$, where $X := -H^\top \cdot S + \frac{1}{2}((H,0)^\top c\ (H,0)) \cdot A - (\log(1 + H^\top x) - H^\top x) * \mu^S$ and the last equality follows from Kallsen and Shiryaev (2002), Lemma 2.6. From now on, we assume that $Z$ is a martingale [for sufficient conditions cf., e.g., Kallsen and Shiryaev (2002), Section 3]. Then $Z$ is the density process of some probability measure $P^\star \sim P$. We call $P^\star$ the *dual measure* or *neutral pricing measure* for logarithmic utility. The first name is motivated by the fact that it solves some dual optimization problem (cf. KS), whereas the second terminology will become clear below. Note that $P^\star$ depends neither on $K$ nor on $\tau$ (as long as $S = S^\tau$). Recall from the paragraph following Example 5.1 that $P^\star$ is an equivalent $\sigma$-martingale measure (E$\sigma$MM) [cf. Kallsen (2002), Definition 5.2]. It minimizes the reverse relative entropy $Q \mapsto -E(\log(\frac{dQ}{dP}))$ among all E$\sigma$MM's $Q$ [cf. Becherer (2001), Corollary 4.8, and Goll and Rüschendorf (2001), Corollary 6.2, for a similar statement on equivalent local martingale measures].

The framework of our contingent claim valuation problem is as follows: in addition to underlying securities $1, \dots, d$, we assume that derivatives $d+1, \dots, d+n$ are given in terms of their discounted terminal payoffs $R^{d+1}, \dots, R^{d+n}$, which are supposed to be $\mathscr{F}_\tau$-measurable random variables. We call semimartingales $S^{d+1}, \dots, S^{d+n}$ *derivative price processes* if $S^{d+i} \in [\operatorname{ess\,inf} R^{d+i}, \operatorname{ess\,sup} R^{d+i}]$ and $S_t^{d+i} = R^{d+i}$ for $t \geq \tau$ and $i = 1, \dots, n$. As noted above, we are interested in contingent claim values that have a neutral effect on the derivative market in the sense that they do not cause supply of or demand for derivatives by the representative log-utility maximizer:

DEFINITION 6.2. We call derivative price processes $S^{d+1}, \dots, S^{d+n}$ *neutral* if there exists an optimal portfolio–consumption pair $(\overline{\varphi}, \overline{\kappa})$ in the extended market $S^1, \dots, S^{d+n}$ which satisfies $\overline{\varphi}^{d+1} = \dots = \overline{\varphi}^{d+n} = 0$.

The following result treats existence and uniqueness of neutral derivative price processes. Moreover, it shows that they are obtained via conditional expectation relative to the neutral pricing measure $P^\star$.

THEOREM 6.3. *Suppose that $R^{d+1}, \dots, R^{d+n}$ are bounded. Then there exist unique neutral derivative price processes. These are given by $S_t^{d+i} = E_{P^\star}(R^{d+i}|\mathscr{F}_t)$ for $t \in \mathbb{R}_+$, $i = 1, \dots, n$. Moreover, the extended market $S^1$, $\dots, S^{d+n}$ satisfies the condition NFLVR.*

PROOF. *Existence*. Set $S_t^{d+1} := E_{P^\star}(R^{d+i}|\mathscr{F}_t)$ for $t \in \mathbb{R}_+$, $i = 1, \dots, n$. Define $(\overline{\varphi}, \overline{\kappa})$ by $\overline{\varphi} := (\varphi^1, \dots, \varphi^d, 0, \dots, 0)$ and $\overline{\kappa} = \kappa$, where $(\varphi, \kappa)$ is the optimal portfolio–consumption pair from Corollary 3.2. The definition of $\kappa$ implies that $\frac{1}{\overline{\kappa}} = \frac{1}{\kappa} = \frac{K_\infty}{\varepsilon} Z$ $(P \otimes K)$-almost everywhere. Obviously, $(\overline{\varphi}, \overline{\kappa})$ is an admissible portfolio–consumption pair in the extended market $\overline{S} := (S^1, \dots, S^{d+n})$. Now, let $\psi \in L(\overline{S})$ with $\varepsilon + \psi^\top \cdot \overline{S} \geq 0$. Since $\overline{S}$ is a $\mathbb{R}^{d+n}$-valued $P^\star$-$\sigma$-martingale, we have that $Z(\varepsilon + \psi^\top \cdot \overline{S})$ is a $P$-$\sigma$-martingale [cf. Kallsen (2002), Lemma 3.6 and Proposition 5.3] and hence a $P$-supermartingale [cf. Kallsen (2002), Proposition 3.5]. In view of Corollary 4.2, $(\overline{\varphi}, \overline{\kappa})$ is an optimal portfolio–consumption pair in the extended market. Since $\overline{S}$ is a $\sigma$-martingale with respect to $P^\star \sim P$, we have that NFLVR holds [cf. Delbaen and Schachermayer (1998), Theorem 1.1].

*Uniqueness*. Obviously, one may choose the portfolio–consumption pair $(\overline{\varphi}, \overline{\kappa})$ in Definition 6.2 as in the existence part of this proof. Let $\overline{Z}$ be the supermartingale in Statement 2 of Corollary 4.2 corresponding to the extended market. Since $\overline{Z} = \frac{1}{\overline{\kappa}} = \frac{1}{\kappa} = \frac{K_\infty}{\varepsilon} Z$ $(P \otimes K)$-almost everywhere, one easily concludes that $\overline{Z} = \frac{K_\infty}{\varepsilon} Z$ on $[\![0, \tau]\!]$ up to indistinguishability [e.g., by Jacod (1979), (7.10), and the fact that $Z$ is a martingale].

Fix $i \in \{1, \dots, n\}$. Define $\psi \in L(\overline{S})$ by $\psi^j := 0$ for $j \neq d + i$ and $\psi^{d+i} := \varepsilon / |\operatorname{ess\,sup} R^{d+i} - \operatorname{ess\,inf} R^{d+i}|$. Then $\varepsilon + \psi^\top \cdot \overline{S} = \varepsilon + \psi^{d+i}(S^{d+i} - S_0^{d+i}) \geq 0$. By Corollary 4.2, $Z(\varepsilon + \psi^\top \cdot \overline{S})$ and hence $ZS^{d+i}$ is a supermartingale. Replacing $\psi^{d+i}$ with $-\psi^{d+i}$ yields that $-ZS^{d+i}$ is a supermartingale as well. Together, we have that $S^{d+i}$ is a $P^\star$-martingale (cf. JS, III.3.8), which implies the uniqueness. $\square$

REMARK. If $R^{d+1}, \dots, R^{d+n}$ are $P^\star$-integrable instead of bounded, the above proof still yields the existence of neutral derivative price processes and the NFLVR property of the extended market.

Note that the neutral derivative prices depend neither on the initial capital $\varepsilon$ nor on the specific deterministic consumption clock $K$ nor on $\tau$ because the same holds for $P^\star$. Recall from Section 2 that optimal portfolio–consumption pairs do not depend on the chosen numeraire. Consequently, the property of being neutral is independent of the numeraire as well, which is a very desirable feature.

## APPENDIX

This Appendix contains two simple propositions which are needed in the proofs of Theorem 3.1 and Proposition 6.1.

PROPOSITION A.1. *Let $X$ be a semimartingale with $\Delta X > -1$, let $A \in \mathscr{V}^+$ and let $Y$ be a semimartingale with $Y = 1 + Y_- \cdot X - A$. Then we have $Y \leq \mathscr{E}(X)$.*

PROOF. By Protter [(1992), Theorem V.7], the equation $Y = 1 + Y_- \cdot X - A$ has an up to indistinguishability unique solution $Y$. Partial integration in the sense of JS, I.4.49a, shows that it is given by $Y = \mathscr{E}(X)(1 - \frac{1}{\mathscr{E}(X)} \cdot A)$ [cf. also Jacod (1979), (6.8)]. Since $\frac{1}{\mathscr{E}(X)} \cdot A$ is nonnegative, the assertion follows. $\square$

PROPOSITION A.2. *Let $\mu$ be an integer-valued random measure and let $H, K$ be $\mathbb{R}^d$-valued predictable processes with $K^i * \mu \in \mathscr{V}$ for $i = 1, \dots, d$. Then $(H^\top K) * \mu \in \mathscr{V}$ if and only if $H \in L_s(K * \mu)$. In this case $(H^\top K) * \mu = H^\top \cdot (K * \mu)$.*

PROOF. There exists an $\mathbb{R}^d$-valued predictable process $a$ with $K^i = a^i |K|$ for $i = 1, \dots, d$. Set $A := |K| * \mu$. Then $K^i * \mu = a^i \cdot A$ for $i = 1, \dots, d$. Note that $|H^\top K| * \mu = (|H^\top a|\,|K|) * \mu = |H^\top a| \cdot A$. Since $|H^\top K| * \mu \in \mathscr{V}$ if and only if $(H^\top K) * \mu \in \mathscr{V}$, and $|H^\top a| \cdot A \in \mathscr{V}$ if and only if $H \in L_s(K * \mu)$, the first claim follows. The second statement follows from a similar calculation. $\square$

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