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ASYMPTOTIC BEHAVIOR FOR PARTIAL AUTOCORRELATION FUNCTIONS OF FRACTIONAL ARIMA PROCESSES

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We prove a simple asymptotic formula for partial autocorrelation functions of fractional ARIMA processes.

1. Introduction. Let $\{X_n : n \in \mathbb{Z}\}$ be a real, zero-mean, weakly stationary process, which we shall simply call a *stationary process*. We write $\gamma(\cdot)$ for the autocovariance function of $\{X_n\}$:

$$\gamma(n) := E[X_n X_0], \qquad n \in \mathbb{Z}.$$

The *partial autocorrelation* $\alpha(k)$ of $\{X_n\}$ is the correlation coefficient between X_0 and X_k eliminating linear regressions on $X_1, \dots, X_{k-1}$ [see (4.2) for precise definition]. One can calculate the value of $\alpha(k)$ easily, at least numerically, from the values of $\gamma(0), \gamma(1), \dots, \gamma(k)$ via, for example, the Durbin–Levinson algorithm [cf. Brockwell and Davis (1991), Sections 3.4 and 5.2]. The partial autocorrelation function $\alpha(\cdot)$ thus obtained is a real sequence of modulus less than or equal to 1 which is free from restrictions such as nonnegative definiteness [see Ramsey (1974)], unlike the autocovariance function. By virtue of their flexibility, partial autocorrelation functions play a significant role in time series analysis.

The definition of $\alpha(k)$ says that it is a kind of "pure" correlation coefficient between X_0 and X_k. Thus we think that the partial autocorrelation function $\alpha(\cdot)$ closely reflects the dependence structure of $\{X_n\}$. However, in what concrete sense does it do so? More specifically, what does $\alpha(n)$ look like for n large, especially, when $\{X_n\}$ is a *long-memory process* [cf. Brockwell and Davis (1991), Section 13.2]? We dealt with this specific problem in Inoue (2000) and showed that under appropriate conditions there exists a simple asymptotic formula for $\alpha(\cdot)$. However, the main results of Inoue (2000) do not cover an important class of long-memory processes, that is, the *fractional ARIMA* (autoregressive integrated moving-average) *model*. This model was independently introduced by Granger and Joyeux (1980) and Hosking (1981) and has been widely used as a parametric model describing long-memory processes. The purpose of this paper is to extend the asymptotic formula to fractional ARIMA processes.

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We start by recalling the definition of the fractional ARIMA model. Let $\{X_n\}$ be a stationary process with autocovariance function $\gamma(\cdot)$. If there exists an even, nonnegative, and integrable function $\Delta(\cdot)$ on $(-\pi, \pi)$ such that

$$\gamma(n) = \int_{-\pi}^{\pi} e^{in\lambda} \Delta(\lambda)\, d\lambda, \qquad n \in \mathbb{Z},$$

then $\Delta(\cdot)$ is called a *spectral density* of $\{X_n\}$. For $d \in (-1/2, 1/2)$ and $p, q \in \mathbb{N} \cup \{0\}$, $\{X_n\}$ is said to be a fractional ARIMA(p, d, q) process if it has a spectral density $\Delta(\cdot)$ of the form

$$\Delta(\lambda) = \frac{1}{2\pi} \frac{|\theta(e^{i\lambda})|^2}{|\phi(e^{i\lambda})|^2} |1 - e^{i\lambda}|^{-2d}, \qquad -\pi < \lambda < \pi, \tag{1.1}$$

where $\phi(z)$ and $\theta(z)$ are polynomials with real coefficients of degrees p and q, respectively. Throughout the paper we assume that

(A1) $\phi(z)$ and $\theta(z)$ have no common zeros and neither $\phi(z)$ nor $\theta(z)$ has zeros in the closed unit disk $\{z \in \mathbb{C} : |z| \le 1\}$.

We also assume without loss of generality that

$$\theta(0)/\phi(0) > 0. \tag{A2}$$

Note that (A1) and (A2) imply $\theta(1)/\phi(1) > 0$.

For a fractional ARIMA(p, d, q) process $\{X_n\}$ with $d \in (-1/2, 1/2) \setminus \{0\}$, the asymptotic behavior of the autocovariance function $\gamma(\cdot)$ is given by

$$\gamma(n) \sim C n^{2d-1}, \qquad n \to \infty, \tag{1.2}$$

where

$$C = \frac{\Gamma(1-2d)\sin(\pi d)}{\pi} \left\{ \frac{\theta(1)}{\phi(1)} \right\}^2 \tag{1.3}$$

(see Section 4). In particular, if $0 < d < 1/2$, then $\{X_n\}$ is a long-memory process in the sense that $\sum_{n=0}^{\infty} |\gamma(n)| = \infty$ holds. If $d = 0$, then $\{X_n\}$ is also an ARMA(p, q) process [see Brockwell and Davis (1991), Chapter 3], and the sequence $\{\gamma(n)\}_{n=0}^{\infty}$ *decays exponentially*; that is, there exist constants $M > 0$ and $s \in (0, 1)$ such that

$$|\gamma(k)| \le M s^k, \qquad k = 0, 1, \ldots$$

[see Brockwell and Davis (1991), Problem 3.11].

As we stated above, our central concern in this paper is the asymptotic behavior of the partial autocorrelation function $\alpha(\cdot)$ of a fractional ARIMA(p, d, q) process $\{X_n\}$. In this connection, it is instructive to look at the simplest case $(p, q) = (0, 0)$. If $(p, q) = (0, 0)$ and $-1/2 < d < 1/2$, then we have

$$\alpha(n) = \frac{d}{n - d}, \qquad n = 1, 2, \ldots \tag{1.4}$$

[Hosking (1981), Theorem 1; see also Brockwell and Davis (1991), (13.2.10)]. If we further assume $d \neq 0$, then this expression implies the following simple asymptotic behavior for $\alpha(\cdot)$:

$$\alpha(n) \sim \frac{d}{n}, \qquad n \to \infty. \tag{1.5}$$

Notice that the constant d, which is important in a fractional ARIMA process, appears explicitly in (1.5).

If $(p, q) \neq (0, 0)$, then there does not exist such an explicit expression as (1.4). However, numerical calculation [cf. Hosking (1981), page 173] suggests that the asymptotic formula (1.5) might still be valid even if $(p, q) \neq (0, 0)$ and $d \in (-1/2, 1/2) \setminus \{0\}$. The main contribution of this paper is to show that, modulo sign, this is indeed the case when $0 < d < 1/2$.

Here is the main theorem.

THEOREM 1.1. *Let $p, q \in \mathbb{N} \cup \{0\}$ and $0 < d < 1/2$, and let $\{X_n\}$ be a fractional* ARIMA(p, d, q) *process with partial autocorrelation function $\alpha(\cdot)$. Then we have*

$$|\alpha(n)| \sim \frac{d}{n}, \qquad n \to \infty. \tag{1.6}$$

We recall the results of Inoue (2000) that are closely related to Theorem 1.1. Let $-\infty < d < 1/2$ and $\ell(\cdot)$ be a slowly varying function at infinity [cf. Bingham, Goldie and Teugels (1989), Chapter 1]. Then Theorem 2.1 of Inoue (2000) shows that, under certain conditions on the MA(∞) coefficients c_n and the AR(∞) coefficients a_n (see Section 2) of a stationary process $\{X_n\}$,

$$\gamma(n) \sim n^{2d-1}\ell(n), \qquad n \to \infty, \tag{1.7}$$

implies

$$|\alpha(n)| \sim \frac{\gamma(n)}{\sum_{k=-n}^{n} \gamma(k)}, \qquad n \to \infty. \tag{1.8}$$

Now if $0 < d < 1/2$, then (1.2) implies

$$\frac{\gamma(n)}{\sum_{k=-n}^{n} \gamma(k)} \sim \frac{d}{n}, \qquad n \to \infty.$$

Thus (1.6) also falls into (1.8). However, Theorem 2.1 of Inoue (2000) does not include Theorem 1.1 because the MA(∞) coefficients c_n of a fractional ARIMA(p, d, q) process do not generally verify the following rather arbitrary assumption of Inoue [(2000), Theorem 2.1]:

$$c_n \geq 0 \qquad \text{for all } n \geq 0. \tag{C1}$$

The rough line of the proof of Theorem 1.1 is close to that of Inoue [(2000), Theorem 2.1] in two points: first, we deduce the desired asymptotic behavior (1.6) from that of a relevant prediction error; and second, an explicit expression for the prediction error in terms of c_n and a_n plays an important role.

In the present paper, however, there arises an extra complication as we explain now. When we deduce (1.6) from the asymptotic behavior of the prediction error, we use a Tauberian argument. So naturally we need an adequate Tauberian condition. Whereas we can use monotonicity as the necessary Tauberian condition in Inoue (2000), it is difficult to verify it in the present paper because we are lacking (C1). We overcome this trouble by verifying another Tauberian condition (Proposition 4.4) which is weaker than monotonicity but enough for our purpose. The verification, however, is not straightforward. In fact, the most of the proof of Theorem 1.1 is devoted to this task. There, some estimates for sums involving c_n and a_n play an important role. These estimates, in turn, are obtained by using the asymptotic behavior with remainder (Lemma 2.2), for $\{c_n\}$, $\{a_n\}$ and their differences, extending Kokoszka and Taqqu [(1995), Corollary 3.1]. In a sense, we compensate for the lack of (C1) with this type of asymptotics for $\{c_n\}$ and $\{a_n\}$ of a fractional ARIMA process.

The necessary results on the asymptotics for $\{c_n\}$ and $\{a_n\}$ are given in Section 2, followed by key estimates for sums involving c_n and a_n in Section 3. We prove Theorem 1.1 in Section 4, and close with some remarks in Section 5. Throughout this paper, n and k designate nonnegative integers.

2. MA(∞) and AR(∞) coefficients.

Let $d \in (-1/2, 1/2)$, and let $\{X_n\}$ be a fractional ARIMA(p, d, q) process with spectral density $\Delta(\cdot)$ given by (1.1). This section deals with the asymptotics for the MA(∞) coefficients c_n and the AR(∞) coefficients a_n of $\{X_n\}$.

First we recall some basic facts and notation. It is readily checked that

$$\int_{-\pi}^{\pi} |\log \Delta(\lambda)|\, d\lambda < \infty;$$

in other words, $\{X_n\}$ is a purely nondeterministic stationary process [cf. Brockwell and Davis (1991), Section 5.7]. We define the outer function $h(\cdot)$ of $\{X_n\}$ by

$$h(z) := \sqrt{2\pi} \exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{e^{i\lambda}+z}{e^{i\lambda}-z} \log \Delta(\lambda)\, d\lambda\right\}, \qquad z \in \mathbb{C},\ |z| < 1.$$

The function $h(\cdot)$ is actually an outer function in the sense of Rudin [(1987), Definition 17.14]. We have

$$h(z) = \frac{\theta(z)}{\phi(z)}(1-z)^{-d}, \qquad |z| < 1. \tag{2.1}$$

Indeed, the function on the right-hand side of (2.1) is an outer function [cf. Rozanov (1967), Theorem 5.3] with modulus $\sqrt{2\pi\,\Delta(\lambda)}$ for $z = e^{i\lambda}$ and takes a positive value at $z = 0$ since we have assumed $\theta(0)/\phi(0) > 0$; hence it coincides with $h(z)$. Using $h(\cdot)$, we define the MA(∞) coefficients c_n of $\{X_n\}$ by

$$h(z) = \sum_{n=0}^{\infty} c_n z^n, \qquad |z| < 1 \tag{2.2}$$

and the AR(∞) coefficients a_n of $\{X_n\}$ by

$$-\frac{1}{h(z)} = \sum_{n=0}^{\infty} a_n z^n, \qquad |z| < 1. \tag{2.3}$$

See Inoue [(2000), (4.7) and (4.9)] for background.

For $\delta \in \mathbb{R}$ and a real sequence $\{\lambda_n\}$, we write $\lambda_n(\delta)$ for the power series coefficients of $(1-z)^{\delta} \sum_{n=0}^{\infty} \lambda_n z^n$:

$$\lambda_n(\delta) = \sum_{k=0}^{n} \lambda_k (-1)^{n-k} \binom{\delta}{n-k}, \qquad n \geq 0,$$

where we used binomial coefficients. It readily follows that

$$\lambda_n(\delta) - \lambda_{n-1}(\delta) = \lambda_n(\delta + 1), \qquad \delta \in \mathbb{R},\ n \geq 1.$$

The next lemma, which is essentially Kokoszka and Taqqu [(1995), Corollary 3.1], plays an important role in this paper.

LEMMA 2.1. *Suppose that* $\delta \in (-1, \infty) \setminus \{0, 1, 2, \ldots\}$ *and that a real sequence* $\{\lambda_n\}$ *decays exponentially. Then we have*

$$\frac{\lambda_n(\delta)}{n^{-\delta-1}} = \frac{\sum_{k=0}^{\infty} \lambda_k}{\Gamma(-\delta)} + O(n^{-1}), \qquad n \to \infty. \tag{2.4}$$

For the proof of this lemma, see Kokoszka and Taqqu [(1995), Section 3].

We define a positive constant K_1 by

$$K_1 := \theta(1)/\phi(1).$$

LEMMA 2.2. *Let* $d \in (-1/2, 1/2) \setminus \{0\}$. *Then we have, as* $n \to \infty$,

$$\frac{c_n}{n^{d-1}} = \frac{K_1}{\Gamma(d)} + O(n^{-1}), \tag{2.5}$$

$$\frac{a_n}{n^{-d-1}} = -\frac{1}{\Gamma(-d) K_1} + O(n^{-1}), \tag{2.6}$$

$$\frac{c_n - c_{n-1}}{n^{d-2}} = \frac{K_1}{\Gamma(d-1)} + O(n^{-1}), \tag{2.7}$$

$$\frac{a_n - a_{n-1}}{n^{-d-2}} = -\frac{1}{\Gamma(-d-1)K_1} + O(n^{-1}), \tag{2.8}$$

$$\frac{(a_n - a_{n-1}) - (a_{n-1} - a_{n-2})}{n^{-d-3}} = -\frac{1}{\Gamma(-d-2)K_1} + O(n^{-1}). \tag{2.9}$$

PROOF. [Note that (2.5) and (2.6) are direct consequences of Kokoszka and Taqqu (1995), Corollary 3.1.] Let λ_n be the power series coefficients of the rational function $-\phi(z)/\theta(z)$:

$$-\frac{\phi(z)}{\theta(z)} = \sum_{n=0}^{\infty} \lambda_n z^n.$$

Since we have assumed that $\theta(z)$ has no zeros in $|z| \le 1$, the sequence $\{\lambda_n\}$ decays exponentially. By (2.1) and (2.3), we have

$$\sum_{n=0}^{\infty} a_n z^n = (1-z)^d \sum_{n=0}^{\infty} \lambda_n z^n.$$

This implies

$$a_n = \lambda_n(d), \qquad n \ge 0,$$

so that

$$\begin{aligned} a_n - a_{n-1} &= \lambda_n(d+1), \qquad n \ge 1, \\ (a_n - a_{n-1}) - (a_{n-1} - a_{n-2}) &= \lambda_n(d+2), \qquad n \ge 2. \end{aligned}$$

Since $\sum_{k=0}^{\infty} \lambda_k = -1/K_1$, Lemma 2.1 yields (2.6), (2.8) and (2.9).

If we define μ_n $(n \ge 0)$ by

$$\frac{\theta(z)}{\phi(z)} = \sum_{n=0}^{\infty} \mu_n z^n,$$

then the sequence $\{\mu_n\}$ also has exponential decay since $\phi(z)$ has no zeros in $|z| \le 1$. Thus, we get (2.5) and (2.7) in a similar fashion. $\square$

We show some consequences of Lemma 2.2. We write $[\cdot]$ for the integer part.

LEMMA 2.3. *Let $d \in (-1/2, 1/2) \setminus \{0\}$ and $r > 1$. Then there exists an $N \in \mathbb{N}$ such that the following inequalities hold for $n \ge N$, $v \ge 0$, $s \ge 0$ and $u \ge 0$:*

$$|c_n| \le \frac{1}{(n+1)^{1-d}} \cdot \frac{rK_1}{|\Gamma(d)|}, \tag{2.10}$$

$$|a_{[nv]+[ns]+[nu]+n+2}| \le \frac{1}{(v+s+u+1)^{1+d}} \cdot \frac{r}{|\Gamma(-d)|K_1 n^{1+d}}, \tag{2.11}$$

$$|a_{[nv]+[ns]+[nu]+n+3}| \le \frac{1}{(v+s+u+1)^{1+d}} \cdot \frac{r}{|\Gamma(-d)|K_1 n^{1+d}}, \tag{2.12}$$

$$\begin{aligned} &|a_{[nv]+[ns]+[nu]+n+2} - a_{[nv]+[ns]+[nu]+n+3}| \\ &\quad \le \frac{1}{(v+s+u+1)^{2+d}} \cdot \frac{r}{|\Gamma(-d-1)|K_1 n^{2+d}}. \end{aligned} \tag{2.13}$$

PROOF. We restrict attention to (2.13); we can handle (2.10)–(2.12) in like manner. By (2.8), we may choose $N \in \mathbb{N}$ such that the following inequality holds for $n \ge N$:

$$|a_{n+2} - a_{n+3}| \le (n+3)^{-d-2} \frac{r}{|\Gamma(-d-1)|K_1}.$$

Since

$$[nv] + [ns] + [nu] + n + 3 > nv + ns + nu + n,$$

(2.13) follows. □

3. Estimates. The purpose of this section is to derive some estimates needed in the proof of Theorem 1.1. Let d be a constant in $(0, 1/2)$ and let $\{X_n\}$ be a fractional ARIMA(p, d, q) process with spectral density $\Delta(\cdot)$ given by (1.1). As in Section 2, we write c_n and a_n for the MA(∞) and AR(∞) coefficients of $\{X_n\}$, respectively. Throughout this section, we fix a constant $r \in (1, \infty)$.

By Lemmas 2.2 and 2.3, we may take $N_1 \in \mathbb{N}$ such that (2.10) as well as $c_n \ge 0$ holds for $n \ge N_1$. We define

$$c_n^0 := \begin{cases} 0, & \text{if } 0 \le n \le N_1 - 1, \\ c_n, & \text{if } n \ge N_1, \end{cases}$$

and

$$c_n^1 := \begin{cases} c_n, & \text{if } 0 \le n \le N_1 - 1, \\ 0, & \text{if } n \ge N_1. \end{cases}$$

Recall K_1 from Section 2. We define $K_2 = K_2(r)$ by

$$K_2 := \frac{\Gamma(d+1)N_1}{rK_1} \max_{0 \le j \le N_1 - 1} |c_j|.$$

LEMMA 3.1. *For $n \ge 1$, $x \ge 1$ and $i = 0, 1$, the following inequality holds*:

$$\int_0^\infty |c^i_{[nv]}| \frac{1}{(v+x)^{1+d}}\, dv \le \left(\frac{K_2}{n^d}\right)^i \frac{rK_1}{\Gamma(d+1)n^{1-d}x}. \tag{3.1}$$

PROOF. By (2.10), we have

$$0 \le c^0_{[nv]} \le \frac{1}{(nv)^{1-d}} \cdot \frac{rK_1}{\Gamma(d)}, \qquad v > 0,\ n \ge 1. \tag{3.2}$$

This and the identity

$$\int_0^\infty \frac{dv}{v^{1-d}(v+y)^{1+d}} = \frac{1}{yd}, \qquad y > 0, \tag{3.3}$$

show that if $i = 0$, then the integral on the left-hand side of (3.1) is at most

$$\frac{rK_1}{\Gamma(d)n^{1-d}} \int_0^\infty \frac{dv}{v^{1-d}(v+x)^{1+d}} = \frac{rK_1}{\Gamma(d+1)n^{1-d}x}.$$

This proves (3.1) for $i = 0$.

If $i = 1$, then, using

$$(v+x)^{-1-d} \le x^{-1}, \qquad x \ge 1,\ v > 0,$$

we see that the integral on the left-hand side of (3.1) is at most

$$x^{-1} \max_{0 \le j \le N_1 - 1} |c_j| \int_0^{N_1/n} dv = \frac{N_1}{nx} \max_{0 \le j \le N_1 - 1} |c_j|.$$

This proves (3.1) for $i = 1$. □

We introduce some notation. For $u > 0$ and $k \ge 1$, we define $f_k(u)$ by

$$f_1(u) := \frac{1}{\pi(1+u)},$$

$$f_2(u) := \frac{1}{\pi^2} \int_0^\infty \frac{ds_1}{(s_1+1+u)(s_1+1)},$$

$$f_k(u) := \frac{1}{\pi^k} \int_0^\infty ds_{k-1} \cdots \int_0^\infty ds_1 \frac{1}{(s_{k-1}+1+u)} \times \left\{ \prod_{m=1}^{k-2} \frac{1}{(s_{m+1}+s_m+1)} \right\} \frac{1}{(s_1+1)}, \qquad k \ge 3.$$

As in Inoue [(2000), Section 6], we set

$$A_k := \int_0^\infty f_k(u)^2\, du, \qquad k \ge 1.$$

Then we know [Inoue (2000), Lemma 6.5] that

$$\sum_{k=1}^\infty A_k x^{2k} = \pi^{-2} \arcsin^2 x, \qquad |x| < 1 \tag{3.4}$$

or

$$A_k = \frac{1}{\pi^2} \cdot \frac{(2k-2)!!}{(2k-1)!!k}, \qquad k \geq 1.$$

For $I = (i_1, \dots, i_k) \in \{0, 1\}^k$, we write $|I|$ for the sum $i_1 + \cdots + i_k$.

We choose $N_2 = N_2(r) \in \mathbb{N}$ such that the inequalities (2.11)–(2.13) hold for $n \geq N_2$, $v \geq 0$, $s \geq 0$ and $u \geq 0$. For $n \geq N_2$ and $p \in \mathbb{N} \cup \{0\}$, we define

$$d_1(n, p; I) := \sum_{v_1=0}^{\infty} c_{v_1}^{i} a_{v_1+n+2+p}, \qquad I = i \in \{0, 1\}, \tag{3.5}$$

$$\begin{aligned} d_2(n, p; I) := \sum_{v_2=0}^{\infty} c_{v_2}^{i_2} \sum_{v_1=0}^{\infty} c_{v_1}^{i_1} \sum_{m=0}^{\infty} a_{v_2+m+n+2+p} a_{v_1+m+n+2}, \\ I = (i_1, i_2) \in \{0, 1\}^2. \end{aligned} \tag{3.6}$$

We also define, for $k \geq 3$, $n \geq N_2$, $p \in \mathbb{N} \cup \{0\}$ and $I = (i_1, \dots, i_k) \in \{0, 1\}^k$,

$$\begin{aligned} d_k(n, p; I) := \sum_{v_k=0}^{\infty} c_{v_k}^{i_k} \cdots \sum_{v_1=0}^{\infty} c_{v_1}^{i_1} \sum_{m_{k-1}=0}^{\infty} a_{v_k+m_{k-1}+n+2+p} \\ \times \sum_{m_{k-2}=0}^{\infty} a_{v_{k-1}+m_{k-1}+m_{k-2}+n+2} \cdots \sum_{m_2=0}^{\infty} a_{v_3+m_3+m_2+n+2} \\ \times \sum_{m_1=0}^{\infty} a_{v_2+m_2+m_1+n+2} a_{v_1+m_1+n+2}. \end{aligned} \tag{3.7}$$

By the next lemma, we see that these sums converge absolutely, so that $d_k(n, p; I)$ are well defined.

LEMMA 3.2. *Let $k \geq 1$, $p \in \mathbb{N} \cup \{0\}$ and $I \in \{0, 1\}^k$. Then for $n \geq N_2$ all the sums on the right-hand sides of* (3.5)–(3.7) *converge absolutely. Moreover, for $n \geq N_2$, $u > 0$, and $m = n, n+1$, the following inequality holds*:

$$\left| d_k(m, [nu]; I) \right| \leq n^{-1} \{r^2 \sin(d\pi)\}^k \left(\frac{K_2}{n^d} \right)^{|I|} f_k(u). \tag{3.8}$$

PROOF. We prove only (3.8), assuming the assertion on absolute convergence; the proof of the latter is similar. We also restrict attention to the case $k \geq 3$.

Let $I = (i_1, \ldots, i_k) \in \{0,1\}^k$. Expressing sums using integrals and applying change of variable, we get

$$
\begin{aligned}
d_k(n,[nu];I) :=\, & n^{2k-1} \int_0^\infty dv_k\, c^{i_k}_{[nv_k]} \cdots \int_0^\infty dv_1 c^{i_1}_{[nv_1]} \\
& \times \int_0^\infty ds_{k-1}\, a_{[nv_k]+[ns_{k-1}]+n+2+[nu]} \\
& \times \int_0^\infty ds_{k-2}\, a_{[nv_{k-1}]+[ns_{k-1}]+[ns_{k-2}]+n+2} \\
& \times \cdots \times \int_0^\infty ds_2\, a_{[nv_3]+[ns_3]+[ns_2]+n+2} \\
& \times \int_0^\infty a_{[nv_2]+[ns_2]+[ns_1]+n+2} a_{[nv_1]+[ns_1]+n+2}\, ds_1.
\end{aligned}
\tag{3.9}
$$

Therefore, by (2.11), $|d_k(n,[nu];I)|$ is at most

$$
\begin{aligned}
& n^{2k-1}\left(\frac{r}{|\Gamma(-d)|K_1 n^{1+d}}\right)^k \int_0^\infty dv_k |c^{i_k}_{[nv_k]}| \cdots \int_0^\infty dv_1 |c^{i_1}_{[nv_1]}| \\
& \times \int_0^\infty ds_{k-1} \frac{1}{(v_k + s_{k-1} + 1 + u)^{1+d}} \\
& \times \int_0^\infty ds_{k-2} \frac{1}{(v_{k-1} + s_{k-1} + s_{k-2} + 1)^{1+d}} \\
& \times \cdots \times \int_0^\infty ds_2 \frac{1}{(v_3 + s_3 + s_2 + 1)^{1+d}} \\
& \times \int_0^\infty \frac{1}{(v_2 + s_2 + s_1 + 1)^{1+d}(v_1 + s_1 + 1)^{1+d}}\, ds_1,
\end{aligned}
$$

which, by Lemma 3.1, is at most

$$
\begin{aligned}
& n^{2k-1}\left(\frac{r}{|\Gamma(-d)|K_1 n^{1+d}}\right)^k \left(\frac{rK_1}{\Gamma(d+1)n^{1-d}}\right)^k \left(\frac{K_2}{n^d}\right)^{|I|} \pi^k f_k(u) \\
& = n^{-1}\{r^2 \sin(d\pi)\}^k \left(\frac{K_2}{n^d}\right)^{|I|} f_k(u).
\end{aligned}
$$

This proves (3.8) for $m = n$. The result for $m = n+1$ follows in the same way if we use (2.12) instead of (2.11). $\square$

LEMMA 3.3. *Let $k \geq 1$, $n \geq N_2$, $u > 0$ and $I \in \{0,1\}^k$. Then the following inequality holds*:

$$
\begin{aligned}
&|d_k(n,[nu];I) - d_k(n+1,[nu];I)| \\
&\qquad \leq n^{-2}(d+1)k\{r^2\sin(d\pi)\}^k\left(\frac{K_2}{n^d}\right)^{|I|} f_k(u).
\end{aligned}
\tag{3.10}
$$

PROOF. For simplicity, we restrict attention to the case $k = 4$ but the method of proof also applies to the general case.

For $I = (i_1,\dots,i_4) \in \{0,1\}^4$, we have, as in the previous proof,

$$
d_4(n,[nu];I) - d_4(n+1,[nu];I) = \sum_{j=1}^{4} D_4^j(n,u;I),
$$

where

$$
\begin{aligned}
D_4^1(n,u;I) := n^{2\cdot 4-1}&\int_0^\infty dv_4\, c_{[nv_4]}^{i_4}\cdots\int_0^\infty dv_1\, c_{[nv_1]}^{i_1}\\
&\times\int_0^\infty ds_3(a_{[nv_4]+[ns_3]+n+2+[nu]} - a_{[nv_4]+[ns_3]+n+3+[nu]})\\
&\times\int_0^\infty ds_2\, a_{[nv_3]+[ns_3]+[ns_2]+n+2}\\
&\times\int_0^\infty a_{[nv_2]+[ns_2]+[ns_1]+n+2} a_{[nv_1]+[ns_1]+n+2}\, ds_1,
\end{aligned}
$$

$$
\begin{aligned}
D_4^2(n,u;I) := n^{2\cdot 4-1}&\int_0^\infty dv_4\, c_{[nv_4]}^{i_4}\cdots\int_0^\infty dv_1\, c_{[nv_1]}^{i_1}\\
&\times\int_0^\infty ds_3\, a_{[nv_4]+[ns_3]+n+3+[nu]}\\
&\times\int_0^\infty ds_2(a_{[nv_3]+[ns_3]+[ns_2]+n+2} - a_{[nv_3]+[ns_3]+[ns_2]+n+3})\\
&\times\int_0^\infty a_{[nv_2]+[ns_2]+[ns_1]+n+2} a_{[nv_1]+[ns_1]+n+2}\, ds_1,
\end{aligned}
$$

$$
\begin{aligned}
D_4^3(n,u;I) := n^{2\cdot 4-1}&\int_0^\infty dv_4\, c_{[nv_4]}^{i_4}\cdots\int_0^\infty dv_1\, c_{[nv_1]}^{i_1}\\
&\times\int_0^\infty ds_3\, a_{[nv_4]+[ns_3]+n+3+[nu]}\int_0^\infty ds_2\, a_{[nv_3]+[ns_3]+[ns_2]+n+3}\\
&\times\int_0^\infty (a_{[nv_2]+[ns_2]+[ns_1]+n+2} - a_{[nv_2]+[ns_2]+[ns_1]+n+3})\\
&\qquad\times a_{[nv_1]+[ns_1]+n+2}\, ds_1,
\end{aligned}
$$

$$
\begin{aligned}
D_4^4(n,u;I) := n^{2\cdot 4-1} \int_0^\infty dv_4\, c_{[nv_4]}^{i_4} \cdots \int_0^\infty dv_1\, c_{[nv_1]}^{i_1} \\
\times \int_0^\infty ds_3\, a_{[nv_4]+[ns_3]+n+3+[nu]} \int_0^\infty ds_2\, a_{[nv_3]+[ns_3]+[ns_2]+n+3} \\
\times \int_0^\infty a_{[nv_2]+[ns_2]+[ns_1]+n+3} \\
\times (a_{[nv_1]+[ns_1]+n+2} - a_{[nv_1]+[ns_1]+n+3})\, ds_1.
\end{aligned}
$$

We observe that $\Gamma(-d-1) = -\Gamma(-d)/(d+1)$ and that

$$
(x+1)^{-d-2} \le (x+1)^{-d-1}, \qquad x > 0.
$$

Then it follows from (2.11)–(2.13) and Lemma 3.1 that $|D_4^1(n,u;I)|$ is at most

$$
\begin{aligned}
& n^{2\cdot 4-2}(d+1)\left(\frac{r}{|\Gamma(-d)|K_1 n^{1+d}}\right)^4 \\
& \quad \times \int_0^\infty dv_4 |c_{[nv_4]}^{i_4}| \cdots \int_0^\infty dv_1 |c_{[nv_1]}^{i_1}| \int_0^\infty ds_3 \frac{1}{(v_4+s_3+1+u)^{1+d}} \\
& \quad \times \int_0^\infty ds_2 \frac{1}{(v_3+s_3+s_2+1)^{1+d}} \\
& \quad \times \int_0^\infty \frac{1}{(v_2+s_2+s_1+1)^{1+d}(v_1+s_1+1)^{1+d}}\, ds_1 \\
& \qquad \le n^{-2}(d+1)\{r^2 \sin(d\pi)\}^4 \left(\frac{K_2}{n^d}\right)^{|I|} f_4(u).
\end{aligned}
$$

Similarly, we have

$$
|D_4^j(n,u;I)| \le n^{-2}(d+1)\{r^2 \sin(d\pi)\}^4 \left(\frac{K_2}{n^d}\right)^{|I|} f_4(u), \qquad j = 2,3,4.
$$

In summary,

$$
|d_4(n,[nu];I) - d_4(n+1,[nu];I)| \le n^{-2} 4(d+1)\{r^2 \sin(d\pi)\}^4 \left(\frac{K_2}{n^d}\right)^{|I|} f_4(u).
$$

This proves (3.10) for $k=4$. □

For $k \ge 1$, $n \ge N_2$, and $p \in \mathbb{N} \cup \{0\}$, we set

$$
\begin{aligned}
g_k(n,p) &:= d_k(n,p;I) \qquad \text{with } I = (0,\dots,0), \\
e_k(n,p) &:= \sum\nolimits_I' d_k(n,p;I),
\end{aligned}
$$

where $\sum_I'$ stands for the sum

$$
\sum_{I \in \{0,1\}^k \setminus \{(0,\dots,0)\}} .
$$

For $n \geq N_2$ and $p \in \mathbb{N} \cup \{0\}$, we define

$$d_1(n,p) := \sum_{v_1=0}^{\infty} c_{v_1} a_{v_1+n+2+p}, \tag{3.11}$$

$$d_2(n,p) := \sum_{v_2=0}^{\infty} c_{v_2} \sum_{v_1=0}^{\infty} c_{v_1} \sum_{m=0}^{\infty} a_{v_2+m+n+2+p} a_{v_1+m+n+2} \tag{3.12}$$

and, for $k \geq 3$,

$$\begin{aligned} d_k(n,p) := {} & \sum_{v_k=0}^{\infty} c_{v_k} \cdots \sum_{v_1=0}^{\infty} c_{v_1} \sum_{m_{k-1}=0}^{\infty} a_{v_k+m_{k-1}+n+2+p} \\ & \times \sum_{m_{k-2}=0}^{\infty} a_{v_{k-1}+m_{k-1}+m_{k-2}+n+2} \cdots \sum_{m_2=0}^{\infty} a_{v_3+m_3+m_2+n+2} \\ & \times \sum_{m_1=0}^{\infty} a_{v_2+m_2+m_1+n+2} a_{v_1+m_1+n+2}. \end{aligned} \tag{3.13}$$

Clearly we have, for $k \geq 1$, $n \geq N_2$ and $p \in \mathbb{N} \cup \{0\}$,

$$d_k(n,p) = \sum\nolimits_I d_k(n,p;I) = g_k(n,p) + e_k(n,p), \tag{3.14}$$

where $\sum_I$ stands for the sum $\sum_{I \in \{0,1\}^k}$. In the sequel, we shall show that we may regard $g_k(n,p)$ as the main part [hence $e_k(n,p)$ as the negligible one] of $d_k(n,p)$ in an adequate sense.

We choose $N_3 = N_3(r) \in \mathbb{N}$ such that

$$N_3 \geq \max\left\{N_2, \left(\frac{K_2}{r-1}\right)^{1/d}\right\}.$$

Notice that $1 + (K_2/n^d) \leq r$ for $n \geq N_3$.

PROPOSITION 3.4. *For $k \geq 1$, $n \geq N_3$, $u > 0$ and $m = n, n+1$, the following inequalities hold*:

$$|g_k(m,[nu])| \leq n^{-1}\{r^2 \sin(d\pi)\}^k f_k(u), \tag{3.15}$$

$$|e_k(m,[nu])| \leq n^{-1-d} k K_2 \{r^3 \sin(d\pi)\}^k f_k(u), \tag{3.16}$$

$$|d_k(m,[nu])| \leq n^{-1}\{r^3 \sin(d\pi)\}^k f_k(u). \tag{3.17}$$

PROOF. Inequality (3.15) immediately follows if we put $I = (0,\ldots,0)$ in (3.8). Using (3.8) and

$$(1+x)^k - 1 \leq kx(1+x)^k, \qquad x \geq 0,$$

we get

$$
\begin{aligned}
|e_k(m,[nu])| &\le {\sum_I}' |d_k(m,[nu];I)| \\
&\le n^{-1}\{r^2\sin(d\pi)\}^k f_k(u) {\sum_I}' (K_2/n^d)^{|I|} \\
&= n^{-1}\{r^2\sin(d\pi)\}^k f_k(u)\big[\{1+(K_2/n^d)\}^k-1\big] \\
&\le n^{-1-d} k K_2 \{r^2\sin(d\pi)\}^k \{1+(K_2/n^d)\}^k f_k(u) \\
&\le n^{-1-d} k K_2 \{r^3\sin(d\pi)\}^k f_k(u).
\end{aligned}
$$

This proves (3.16).

Similarly,

$$
\begin{aligned}
|d_k(m,[nu])| &\le \sum_I |d_k(m,[nu];I)| \\
&\le n^{-1}\{r^2\sin(d\pi)\}^k f_k(u) \sum_I (K_2/n^d)^{|I|} \\
&= n^{-1}\{r^2\sin(d\pi)\}^k f_k(u)\{1+(K_2/n^d)\}^k \\
&\le n^{-1}\{r^3\sin(d\pi)\}^k f_k(u),
\end{aligned}
$$

whence (3.17). □

PROPOSITION 3.5. *For $k \ge 1$, $n \ge N_3$ and $u > 0$, the following inequalities hold*:

$$
|g_k(n,[nu]) - g_k(n+1,[nu])| \le n^{-2}(d+1)k\{r^2\sin(d\pi)\}^k f_k(u), \tag{3.18}
$$

$$
|e_k(n,[nu]) - e_k(n+1,[nu])| \le n^{-2-d}(d+1)K_2k^2\{r^3\sin(d\pi)\}^k f_k(u). \tag{3.19}
$$

PROOF. Inequality (3.18) is nothing but (3.10) with $I = (0,\dots,0)$.

A further application of (3.10) shows that

$$
\begin{aligned}
&\big|e_k(n,[nu]) - e_k(n+1,[nu])\big| \\
&\quad\le {\sum_I}' \big|d_k(n,[nu];I) - d_k(n+1,[nu];I)\big| \\
&\quad\le n^{-2}(d+1)k\{r^2\sin(d\pi)\}^k f_k(u) {\sum_I}' (K_2/n^d)^{|I|} \\
&\quad= n^{-2}(d+1)k\{r^2\sin(d\pi)\}^k f_k(u)\big[\{1+(K_2/n^d)\}^k-1\big] \\
&\quad\le n^{-2-d}(d+1)K_2k^2\{r^3\sin(d\pi)\}^k f_k(u).
\end{aligned}
$$

Thus (3.19) follows. □

4. Proof of Theorem 1.1. Let d, $\{X_n\}$, c_n and a_n be as in Section 3. In this section, r is a fixed constant such that

$$1 < r < \{\sin(\pi d)\}^{-1/3}. \tag{4.1}$$

Notice that $0 < r^{5/2}\sin(d\pi) < r^3\sin(d\pi) < 1$. We shall continue to use the notation of Section 3.

We write H for the real Hilbert space spanned by $\{X_k : k \in \mathbb{Z}\}$ in $L^2(\Omega, \mathcal{F}, P)$, with inner product

$$(Y_1, Y_2) := E[Y_1 Y_2]$$

and norm

$$\|Y\| := (Y, Y)^{1/2}.$$

For $I \subset \mathbb{Z}$, denote by H_I the closed real linear hull of $\{X_k : k \in I\}$ in H. In particular, for $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$ with $m \le n$, we write $H_{(-\infty,m]}$ and $H_{[m,n]}$ for H_I with $I = \{k \in \mathbb{Z} : -\infty < k \le m\}$ and $\{k \in \mathbb{Z} : m \le k \le n\}$, respectively. For $I \subset \mathbb{Z}$, we denote by P_I the orthogonal projection operator of H onto H_I. We write $P_I^{\perp} := I_H - P_I$, where I_H is the identity map of H. So $P_I^{\perp}$ is the orthogonal projection operator of H onto $H_I^{\perp}$. For $Y \in H$, we may think of $P_I Y$ as the best linear predictor of Y on the observations $\{X_k : k \in I\}$, whence $P_I Y = Y - P_I Y$ as its prediction error.

The partial autocorrelation function $\alpha(\cdot)$ of $\{X_n\}$ is defined by

$$\alpha(n) := \frac{E[Z_n^+ Z_n^-]}{E[(Z_n^+)^2]^{1/2} \cdot E[(Z_n^-)^2]^{1/2}}, \qquad n \ge 2, \tag{4.2}$$

where

$$Z_n^+ := X_n - P_{[1,n-1]} X_n, \qquad Z_n^- := X_0 - P_{[1,n-1]} X_0. \tag{4.3}$$

Furthermore, $\alpha(1)$ is defined by $\alpha(1) := \gamma(1)/\gamma(0)$. See Brockwell and Davis [(1991), Section 3.4].

As in Inoue (2000), we set

$$\varepsilon(n) := \frac{\|P_{[-n,0]}^{\perp} X_1\|^2 - \|P_{(-\infty,0]}^{\perp} X_1\|^2}{\|P_{(-\infty,0]}^{\perp} X_1\|^2}, \qquad n = 0, 1, \dots.$$

Recall N_2 and $d_k(n, p)$ from Section 3. Here is the expression of $\varepsilon(\cdot)$ in terms of c_n and a_n [cf. Inoue (2000), Theorems 4.5 and 4.6].

THEOREM 4.1. *For $n \ge N_2$, we have*

$$\varepsilon(n) = \sum_{k=1}^{\infty} \sum_{p=0}^{\infty} d_k(n, p)^2. \tag{4.4}$$

PROOF. We define, for $n \geq 1$ and $p \in \mathbb{N} \cup \{0\}$,

$$D_1(n, p) := d_1(n, p),$$

$$D_k(n, p) := \sum_{m_1=1}^{\infty} a_{n+1+m_1} \sum_{m_2=1}^{\infty} b_{n+m_2}^{m_1} \cdots \sum_{m_{k-1}=1}^{\infty} b_{n+m_{k-1}}^{m_{k-2}} \sum_{m_k=1}^{\infty} b_{n+p+m_k}^{m_{k-1}} c_{m_k-1}, \qquad k \geq 2,$$

where

$$b_j^m := \sum_{k=1}^{m} c_{m-k} a_{k+j}, \qquad m \geq 1, \ j \geq 0.$$

Then, since (2.6) implies $\sum_{k=0}^{\infty} |a_k| < \infty$, it follows from Inoue [(2000), Theorem 4.5] that

$$\varepsilon(n) = \sum_{k=1}^{\infty} \sum_{p=0}^{\infty} D_k(n, p)^2, \qquad n \geq 1.$$

Now Lemma 3.2 allows us to apply Fubini's theorem to exchange the order of sums [cf. the proof of Inoue (2000), Theorem 4.6] to obtain

$$D_k(n, p) = d_k(n, p), \qquad k \geq 1, \ n \geq N_2, \ p \in \mathbb{N} \cup \{0\}.$$

Thus (4.4) follows. $\square$

We need the next lemma to derive the asymptotic behavior of $\varepsilon(\cdot)$.

LEMMA 4.2. *For $k \geq 1$ and $u > 0$, we have*

$$d_k(n, [nu]) \sim n^{-1} \sin^k(d\pi) f_k(u), \qquad n \to \infty. \tag{4.5}$$

PROOF. We restrict attention to the case $k \geq 3$; the proofs of the cases $k = 1, 2$ are similar. By (3.14) and (3.16), it suffices to show that

$$\lim_{n \to \infty} n g_k(n, [nu]) = \sin^k(d\pi) f_k(u), \qquad n \to \infty. \tag{4.6}$$

Using (3.9) with $I = (0, \ldots, 0)$, we see that $n g_k(n, [nu])$ is equal to

$$\int_0^{\infty} dv_k \cdots \int_0^{\infty} dv_1 \int_0^{\infty} ds_{k-1} \cdots \int_0^{\infty} ds_1 \, B_k(n, u; v_1, \ldots, v_k; s_1, \ldots, s_{k-1}),$$

where

$$\begin{aligned} &B_k(n, u; v_1, \ldots, v_k; s_1, \ldots, s_{k-1}) \\ &\quad := \left\{ \prod_{m=1}^{k} n^{1-d} c_{[nv_m]}^0 \right\} \times n^{1+d} a_{[nv_k]+[ns_{k-1}]+n+2+[nu]} \\ &\qquad \times \left\{ \prod_{m=1}^{k-2} n^{1+d} a_{[nv_{m+1}]+[ns_{m+1}]+[ns_m]+n+2} \right\} \times n^{1+d} a_{[nv_1]+[ns_1]+n+2}. \end{aligned}$$

Now (2.5) and (2.6) imply

$$c_n \sim n^{d-1} \frac{K_1}{\Gamma(d)}, \qquad n \to \infty \tag{4.7}$$

and

$$a_n \sim n^{-(d+1)} \frac{\Gamma(d)}{K_1} \cdot \frac{d \sin(d\pi)}{\pi}, \qquad n \to \infty, \tag{4.8}$$

respectively, so that

$$\begin{aligned} &\lim_{n\to\infty} B_k(n, u; v_1, \ldots, v_k; s_1, \ldots, s_{k-1}) \\ &\qquad = \{\pi^{-1} d \sin(d\pi)\}^k C_k(u; v_1, \ldots, v_k; s_1, \ldots, s_{k-1}), \end{aligned}$$

where

$$\begin{aligned} &C_k(u; v_1, \ldots, v_k; s_1, \ldots, s_{k-1}) \\ &\quad := \left\{ \prod_{m=1}^{k} \frac{1}{(v_m)^{1-d}} \right\} \frac{1}{(v_k + s_{k-1} + 1 + u)^{1+d}} \\ &\qquad \times \left\{ \prod_{m=1}^{k-2} \frac{1}{(v_{m+1} + s_{m+1} + s_m + 1)^{1+d}} \right\} \frac{1}{(v_1 + s_1 + 1)^{1+d}}. \end{aligned}$$

On the other hand, it follows from (3.2) and (2.11) that, for $n \geq N_2$,

$$\begin{aligned} &|B_k(n, u; v_1, \ldots, v_k; s_1, \ldots, s_{k-1})| \\ &\qquad \leq \{\pi^{-1} r^2 d \sin(d\pi)\}^k C_k(u; v_1, \ldots, v_k; s_1, \ldots, s_{k-1}). \end{aligned}$$

Using (3.3), we see that the integral

$$\int_0^\infty dv_k \cdots \int_0^\infty dv_1 \int_0^\infty ds_{k-1} \cdots \int_0^\infty ds_1 C_k(u; v_1, \ldots, v_k; s_1, \ldots, s_{k-1})$$

is equal to $(\pi/d)^k f_k(u)$, hence in particular is finite. Therefore, the dominated convergence theorem yields (4.6), and so (4.5). $\square$

The next theorem gives the asymptotic behavior of $\varepsilon(\cdot)$. Compare Inoue [(2000), Theorem 6.4]. See also Inoue and Kasahara (1999) and (2000) for relevant work on prediction errors of continuous-time stationary processes.

THEOREM 4.3. *We have*

$$\varepsilon(n) \sim \frac{d^2}{n}, \qquad n \to \infty. \tag{4.9}$$

PROOF. Using Theorem 4.1, we obtain

$$n\varepsilon(n) = \sum_{k=1}^{\infty} \int_0^{\infty} \{nd_k(n, [nu])\}^2 \, du, \qquad n \geq N_2.$$

By (3.4), we have

$$\sum_{k=1}^{\infty} \int_0^{\infty} \big[\{r^3 \sin(d\pi)\}^k f_k(u)\big]^2 \, du = \sum_{k=1}^{\infty} A_k \{r^3 \sin(d\pi)\}^{2k} < \infty.$$

Therefore, using Lemma 4.2, (3.4), (3.17) and the dominated convergence theorem, we let $n \to \infty$ to conclude

$$\lim_{n\to\infty} n\varepsilon(n) = \sum_{k=1}^{\infty} A_k \sin^{2k}(d\pi) = d^2.$$

Thus the result follows. □

As in Inoue (2000), we define

$$\delta(n) := \varepsilon(n) - \varepsilon(n+1), \qquad n \geq 1.$$

Then it readily follows that

$$\sum_{k=n}^{\infty} \delta(k) = \varepsilon(n), \qquad n \geq 1. \tag{4.10}$$

The next proposition, which serves as the necessary Tauberian condition to deduce the asymptotic behavior of $\delta(\cdot)$ from that of $\varepsilon(\cdot)$, is an essential ingredient in the proof of Theorem 1.1.

PROPOSITION 4.4. *For $\lambda > 1$, we have*

$$\limsup_{n\to\infty} \sup_{n \leq m \leq \lambda n} n^2\{\delta(m) - \delta(n)\} \leq 0 \qquad (\textit{hence } = 0). \tag{4.11}$$

PROOF. From Theorem 4.1, we have, for $n \geq N_2$,

$$\begin{aligned} \delta(n) &= \sum_{k=1}^{\infty} \sum_{p=0}^{\infty} \{d_k(n, p)^2 - d_k(n+1, p)^2\} \\ &= \mathrm{I}(n) + 2\mathrm{II}(n) + 2\mathrm{III}(n) + \mathrm{IV}(n), \end{aligned}$$

where

$$\mathrm{I}(n) := \sum_{k=1}^{\infty} \sum_{p=0}^{\infty} \{g_k(n, p) - g_k(n+1, p)\}\{g_k(n, p) + g_k(n+1, p)\},$$

$$\mathrm{II}(n) := \sum_{k=1}^{\infty}\sum_{p=0}^{\infty}\{g_k(n,p) - g_k(n+1,p)\}e_k(n,p),$$

$$\mathrm{III}(n) := \sum_{k=1}^{\infty}\sum_{p=0}^{\infty} g_k(n+1,p)\{e_k(n,p) - e_k(n+1,p)\},$$

$$\mathrm{IV}(n) := \sum_{k=1}^{\infty}\sum_{p=0}^{\infty}\{e_k(n,p) - e_k(n+1,p)\}\{e_k(n,p) + e_k(n+1,p)\}.$$

First we consider $\mathrm{I}(\cdot)$. In view of (2.6), (2.8) and (2.9), both $\{a_n\}$ and $\{a_n - a_{n+1}\}$ are eventually decreasing to zero, while $c_n^0 \geq 0$ for $n \geq 0$; hence there exists an N such that, for $k \geq 1$ and $p \in \mathbb{N} \cup \{0\}$, both $\{g_k(n,p)\}_{n=N}^{\infty}$ and $\{g_k(n,p) - g_k(n+1,p)\}_{n=N}^{\infty}$ are decreasing to zero. Therefore $\{\mathrm{I}(n)\}$ is also eventually decreasing. Thus we have

$$\forall\, \lambda > 1, \qquad \limsup_{n\to\infty} \sup_{n \leq m \leq \lambda n} n^2\{\mathrm{I}(m) - \mathrm{I}(n)\} \leq 0.$$

Next we consider $\mathrm{II}(\cdot)$–$\mathrm{IV}(\cdot)$. We define a constant K_3 by

$$K_3 := (d+1)K_2 \sum_{k=1}^{\infty} k^2 A_k \{r^{5/2}\sin(d\pi)\}^{2k},$$

which is finite since (3.4) shows that the radius of convergence of $\sum_k A_k x^{2k}$ is equal to 1. By Propositions 3.4 and 3.5, we have, for $n \geq N_3$ (recall N_3 from Section 3),

$$\begin{aligned} |\mathrm{II}(n)| &\leq \sum_{k=1}^{\infty}\sum_{p=0}^{\infty} |g_k(n,p) - g_k(n+1,p)| \cdot |e_k(n,p)| \\ &= n\sum_{k=1}^{\infty}\int_0^{\infty} |g_k(n,[nu]) - g_k(n+1,[nu])| \cdot |e_k(n,[nu])|\,du \qquad (4.12) \\ &\leq n^{-2-d}K_3. \end{aligned}$$

In a similar fashion, we get, for $n \geq N_3$,

$$|\mathrm{III}(n)| \leq n^{-2-d}K_3, \qquad (4.13)$$

$$|\mathrm{IV}(n)| \leq n^{-2-2d}K_4, \qquad (4.14)$$

where the finite constant K_4 is defined by

$$K_4 := 2(d+1)(K_2)^2 \sum_{k=1}^{\infty} k^3 A_k \{r^3 \sin(d\pi)\}^{2k}.$$

From (4.12)–(4.14), it follows that, for $\lambda > 1$,

$$\limsup_{n\to\infty} \sup_{n\le m\le\lambda n} n^2\{\mathrm{II}(m) - \mathrm{II}(n)\} = 0,$$

$$\limsup_{n\to\infty} \sup_{n\le m\le\lambda n} n^2\{\mathrm{III}(m) - \mathrm{III}(n)\} = 0,$$

$$\limsup_{n\to\infty} \sup_{n\le m\le\lambda n} n^2\{\mathrm{IV}(m) - \mathrm{IV}(n)\} = 0.$$

Combining, we obtain (4.11). □

Now we are ready to prove Theorem 1.1.

PROOF OF THEOREM 1.1. In view of (4.9)–(4.11), we can apply the monotone density theorem [see Bingham, Goldie and Tengels (1989), Section 1.7.6] to show that

$$\delta(n) \sim \frac{d^2}{n^2}, \qquad n \to \infty.$$

Since it follows from the Durbin–Levinson algorithm that

$$\alpha(n)^2 \sim \delta(n-2), \qquad n \to \infty$$

[see the proof of Inoue (2000), Theorem 2.1] we obtain (1.6). □

5. Remarks.

REMARK 1. For completeness, we prove (1.2) with (1.3) for $d \in (-1/2, 0)$. See Beran [(1994), page 63] for the case $0 < d < 1/2$. Since the condition $-1/2 < d < 0$ implies $\sum_{k=0}^{\infty} c_k = 0$, we have on summing by parts that

$$\gamma(n) = \sum_{k=0}^{\infty}\left(\sum_{m=k+1}^{\infty} c_m\right)(c_{n+1+k} - c_{n+k}).$$

By (3.6),

$$\sum_{m=k+1}^{\infty} c_m \sim -\frac{K_1}{\Gamma(d+1)} k^d, \qquad k \to \infty,$$

while, by (2.7),

$$c_{n+1} - c_n \sim -\frac{K_1}{\Gamma(d-1)} n^{d-2}, \qquad n \to \infty.$$

Therefore, using, for example, Inoue [(1997), Proposition 4.3], we conclude (1.2) with

$$C = -\frac{K_1}{\Gamma(d+1)}\frac{K_1}{\Gamma(d-1)} B(1-2d, 1+d) = \frac{(K_1)^2\Gamma(1-2d)\sin(\pi d)}{\pi}.$$

REMARK 2. We suspect that, in Theorem 1.1 as well as in Inoue [(2000), Theorem 2.1], the asymptotic formula (1.6) can possibly be improved as follows:

$$\alpha(n) \sim \frac{\gamma(n)}{\sum_{k=-n}^{n} \gamma(k)}, \qquad n \to \infty. \tag{5.1}$$

REMARK 3. It is perhaps worth remarking that the hypothesis (1.5) for the fractional ARIMA(p, d, q) process is equivalent to (5.1) even if $-1/2 < d < 0$. Indeed, in this case, we have $\sum_{k=-\infty}^{\infty} \gamma(k) = 2\pi\Delta(0) = 0$, hence (1.2) with $-1/2 < d < 0$ implies

$$\frac{\gamma(n)}{\sum_{k=-n}^{n} \gamma(k)} = -\frac{\gamma(n)}{2\sum_{k=n+1}^{\infty} \gamma(k)} \sim \frac{d}{n}, \qquad n \to \infty.$$

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