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THE VARIANCE CONSTANT FOR THE ACTUAL WAITING TIME OF THE PH/PH/1 QUEUE

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In this paper we consider the asymptotic time average variance of the actual waiting time of the PH/PH/1 queue. To this end Poisson's equation plays a major role, since the solution to Poisson's equation is shown to be intimately connected with the desired variance. We derive explicit solutions to Poisson's equation using phase-type methodology, and present some numerical examples.

1. Introduction. Let W_n denote the actual waiting time in the PH/PH/1 queue, that is, the waiting time the nth arriving customer faces upon arrival until service is initiated. We are interested in the asymptotic behavior of

$$\frac{1}{n}\sum_{k=0}^{n-1} f(W_k).$$

To this end we apply Poisson's equation to show that this functional converges to the same limit as a related martingale; this martingale, in turn, converges to a normally distributed random variable. In particular, we are interested in calculating the variance of this limiting normal distribution, and we shall refer to this value as the (*time average*) *variance constant*.

The work of this paper can be viewed as an extension of Glynn [8]. For discrete-time Markov chains, the relevant basic facts can be found in Neveu [13], Revuz [15], Nummelin [14] or Glynn [8]. For the continuous-time case and its application to calculating the variance constant of the virtual waiting time, see Bladt and Asmussen [7] for details.

All results in this paper have been proved in [8] for the $M/G/1$ queue (and in [7] for the MAP/MMPH/1 virtual waiting time; MAP stands for Markovian arrival process and MMPH for service times that are Markov modulated phase-type, that is, phase-type distributed, where the parameters in the distribution depend on an underlying Markovian environment). In order to extend these results to the PH/PH/1 queue (interarrival times and service times are phase-type distributed), we need to put the theory into a slightly more general framework.

The paper is organized as follows. In Section 2 we introduce the PH/PH/1 queue, and present a central limit theorem in order to obtain the asymptotic variance. In Section 3 explicit calculations are performed using ladder heights and renewal theory for phase-type distributions. In Section 4 we present two numerical examples.

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2. Poisson's equation and a central limit theorem for the PH/PH/1 queue. We use the following notation for phase-type distributions. A phase-type distribution has representation $(\boldsymbol{\alpha}, \boldsymbol{S})$ if it has a density of the form

$$f(x) = \boldsymbol{\alpha} e^{\boldsymbol{S}x}\boldsymbol{s}, \qquad x \geq 0.$$

Here $e^{\boldsymbol{S}x}$ is a matrix exponential, $\boldsymbol{\alpha}$ a row vector and $\boldsymbol{s}$ a column vector that satisfies $-\boldsymbol{S}\boldsymbol{e} = \boldsymbol{s}$, where $\boldsymbol{e}$ is the column vector, the components of which are equal to 1. A phase-type distribution is the distribution of the life-time of a terminating Markov jump process with finitely many states, and in this context $\boldsymbol{\alpha}$ plays the role of the initial distribution, $\boldsymbol{S}$ the intensity matrix among the nonabsorbing states (subintensity matrix) and $\boldsymbol{s}$ the exit rate vector. For more details about phase-type distributions, see, for example, Asmussen [2], Bladt and Asmussen [6], Neuts [11] or Neuts [12].

The assumption of phase-type distributed service times and interarrival times is necessary in Section 4, where we rely on the probabilistic interpretation that a phase-type distribution is the life-time of a terminating Markov jump process. Extensions to matrix-exponential distributions are not immediate, and will require some new ideas as replacements for the probabilistic reasoning (e.g., transform methods).

Consider the actual waiting time sequence $W = \{W_n : n \geq 0\}$ in the PH/PH/1 queue. Then

$$W_{n+1} = [W_n + U_n - T_n]^+,$$

where $W_0 = 0$, U_n denotes the i.i.d. sequence of service times with common phase-type distribution A and T_n denotes the i.i.d. sequence of interarrival times with common phase-type distribution B.

Let $\rho = \mathbb{E}(U_n)/\mathbb{E}(T_n)$ be the traffic intensity. Then W_n is recurrent if $\rho \leq 1$, since 0 will be visited infinitely often. Thus W_n is a Harris chain, as it follows from $\{0\}$ being a regeneration set; see [1], page 150. Throughout this paper we will assume $\rho < 1$.

Let μ be a measure on $\mathbb{R}_+$ and let $f\colon \mathbb{R}_+ \to \mathbb{R}$ be a measurable function. Moreover, let P^n denote the transition kernel of the Harris chain, $P^n(x, A) = \mathbb{P}(W_n \in A \mid W_0 = x)$, and define $P = P^1$. Then we define

$$\begin{aligned} \mu f &= \int_0^\infty f(y)\mu(dy), \\ (\mu P^n)(\cdot) &= \int_0^\infty P^n(y, \cdot)\mu(dy), \qquad (1) \\ (P^n f)(\cdot) &= \int_0^\infty f(y) P^n(\cdot, dy). \end{aligned}$$

It is known from the theory of Harris chains (see, e.g., [1]) that there exists a σ-finite stationary measure ν in the sense that $\nu = \nu P^n$ for all n, and that ν can be represented by

$$\nu(\cdot) = \mathbb{E}_0\left(\sum_{k=0}^{T-1} I\{W_k \in \cdot\}\right),$$

where $\mathbb{E}_0$ is the conditional expectation given $W_0 = 0$ and $T = \inf\{n \geq 1: W_n = 0\}$. Similarly, $\mathbb{P}_x$ and $\mathbb{E}_x$ refer to the case where $W_0 = x$. Moreover [see, e.g., [1], Theorem 2.3(b)],

$$\nu(\cdot) = U_+(\cdot) = \sum_{k=0}^{\infty} G_+^{*k}(\cdot), \tag{2}$$

where $U_+(\cdot)$ is the renewal measure of the ascending ladder process that corresponds to the random walk S_n, G_+ denotes the ascending ladder height distribution and G_+^{*k} is its k-fold convolution. In our case, where $\rho < 1$, the distribution of $G_+(\cdot)$ is defective. The mass of ν, defined as $||\nu|| = \int_0^\infty \nu(dy)$, is

$$||\nu|| = \sum_{k=0}^{\infty} ||G_+||^k = \frac{1}{1 - ||G_+||}.$$

Introduce the distribution $\pi(\cdot) = (1-||G_+||)\nu(\cdot)$ as the normalized distribution of ν. We notice that $\pi(\cdot)$ is in fact the distribution function of the maximum of the random walk S_n. Indeed, $G_+^{*k}(x)(1 - ||G_+||)$ is the probability that the maximum is attained in exactly k ladder steps and is less than or equal to x, since $(1 - ||G_+||)$ is the probability of no further ladder steps. Thus the result follows by summing over k.

Poisson's equation is an equation of the form

$$(I - P)g = f.$$

We want to find a solution g to this equation. Thus we will look for a *solution kernel* Γ, which is a kernel such that $g(x) = (\Gamma f)(x)$. By a kernel we understand a family $\{\Gamma(x, \cdot): x \in \mathbb{R}_+\}$ of σ-finite measures such that $x \to \Gamma(x, A)$ is a measurable function for any measurable set A.

Poisson's equation can only have π-integrable solutions if f is such that $\pi f = 0$ or, equivalently, $\nu f = 0$. Define

$$\Gamma_x(\cdot) = \Gamma(x, \cdot) = \mathbb{E}_x \sum_{k=0}^{T-1} I(W_k \in \cdot).$$

Defining $(\Gamma f)(x) = \Gamma_x f$ similarly to (1), we can prove the following theorems in a similar way as in Glynn [8].

THEOREM 2.1. *Assume that f is π-integrable. If $\pi f = 0$, then $(\Gamma f)(x)$ solves Poisson's equation.*

We note that if g solves Poisson's equation, then so does $g + c$, where c is an arbitrary constant. The next theorem, the proof of which is similar to that in [8], states that the solutions of the form $g + c$ are in fact the only solutions.

THEOREM 2.2. *Let g be a π-integrable solution to Poisson's equation that is finite everywhere, $|g(x)| < \infty$ for all x. Then $g(\cdot) = (\Gamma f)(\cdot) + c$ for some constant c.*

In the proof of Theorem 2.1 it is assumed that $\Gamma_x|g| < \infty$ for all $x \geq 0$. To establish this rigorously, let $f_x(y) = \mathbb{P}(W_1 \in dy|W_0 = x)$ be a transition density with respect to the Lebesgue measure. Assume that for all $x, z \geq 0$ there exists a constant $0 < c = c(x, z) < \infty$ such that

$$f_x(y) \geq c f_z(y) \tag{3}$$

for all y. This assumption is satisfied for the PH/PH/1 queue. Indeed, we have that $f_x(u) = c_x \delta_0(u) + h(u-x)$ for $u \geq 0$, where $\delta_0(\cdot)$ is the degenerate distribution at 0 and h is the density of $U_n - T_n$. Assume that representations of U_n and T_n are, respectively, $(\boldsymbol{\alpha}, \boldsymbol{S})$ and $(\boldsymbol{\beta}, \boldsymbol{T})$. Then for $z \geq 0$ we have that

$$\begin{aligned} h(z) &= \int_0^\infty \boldsymbol{\alpha} e^{\boldsymbol{S}(t+z)} \boldsymbol{s} \boldsymbol{\beta} e^{\boldsymbol{T}t} \boldsymbol{t}\, dt \\ &= -(\boldsymbol{\alpha} e^{\boldsymbol{S}z} \otimes \boldsymbol{\beta})(\boldsymbol{S} \oplus \boldsymbol{T})^{-1}(\boldsymbol{s} \otimes \boldsymbol{t}), \end{aligned}$$

where $\otimes$ and $\oplus$ denote, respectively, the Kronecker product and the Kronecker sum; see Graham [9] for details. Similarly for $z < 0$. Thus $h(z)$ decays as $c_1 e^{-\alpha z}$ as $z \to \infty$ for some positive constants c_1 and α. Thus it is clear that assumption (3) is satisfied.

Using (3) and the assumptions in Theorem 2.2, we then prove that $\Gamma_x|g| < \infty$ for all $x \geq 0$ in the following way. Assume without loss of generality that $g \geq 0$. Then

$$\begin{aligned} \Gamma_x(g) &= g(x) + \mathbb{E}_x \sum_{n=1}^{T-1} g(W_n) \\ &= g(x) + \int_0^\infty f_x(y) \Gamma_y(g)\, dy \\ &\geq c \int_0^\infty f_z(y) \Gamma_y(g)\, dy, \end{aligned}$$

so $\Gamma_x(g) < \infty$ implies that $\int_0^\infty f_z(y) \Gamma_y(g)\, dy < \infty$ and hence $\Gamma_z(g) < \infty$, since

$$\Gamma_z(g) = g(z) + \int_0^\infty f_z(y) \Gamma_y(g)\, dy.$$

Thus, if there exists an x such that $\Gamma_x(g) < \infty$, then $\Gamma_x(g) < \infty$ for all x, but this is ensured by $\pi(g) = \Gamma_0(g) < \infty$.

Next we link the solution of Poisson's equation to a central limit theorem, and prove that the variance constant can be found in terms of this solution.

Define $D_k = g(W_k) - (Pg)(W_{k-1})$, and consider the martingale

$$M_n = \sum_{k=1}^n D_k + g(W_0).$$

Since $D_k = g(W_k) - \mathbb{E}_{W_{k-1}} g(W_k)$ we see that D_k is stationary when the initial distribution is π, and ergodicity of the sequence follows from Asmussen [1], page 182. Then, from Billingsley [5], page 206, we get the following result.

THEOREM 2.3. *Let g be a π-square-integrable solution to Poisson's equation $(I-P)g=\hat{f}$, where f is π-integrable. Then, independently of the initial distribution*,

$$\frac{1}{\sqrt{n}}(M_n - \mathbb{E}M_0) \to_{\mathscr{D}} N(0, \sigma^2(f)),$$

where $\sigma^2(f) = 2\,\pi(\hat{f}g) - \pi\hat{f}^2$.

Note that the square-integrability condition of the martingale has been removed, the reason being as follows. The π-square-integrability of g implies that the limit theorem is valid for the initial distribution π. Since $\rho < 1$, W_n is an aperiodic regenerative Harris chain (see, e.g., [1], page 182) and hence it admits a coupling to its stationary distribution π (see, e.g., Asmussen [1], Proposition 3.13). Thus, at least from a certain time S (which is finite a.s.), the square-integrability is ensured by $\pi g^2 < \infty$ for sufficiently large n.

It is clear that any polynomial g will satisfy the conditions of the theorem for the PH/PH/1 queue (perform repeated partial integrations), which is sufficient for our purposes.

Let us link the asymptotic behavior of the martingale M_n with the asymptotic variance of $\sum_{i=0}^{n-1} f(W_i)$. First note that

$$M_{n+1} = \sum_{i=0}^{n} f(W_i) + g(W_{n+1}).$$

Since W_n converges weakly to a limit W, say, then $g(W_n)/\sqrt{n} \to 0$ weakly as $n \to \infty$, so the asymptotic behavior of $\sum_{i=0}^{n-1} f(W_i)/\sqrt{n}$ is identical to that of $M_n/\sqrt{n}$.

3. Explicit calculation of the variance constant. In this section we shall repeatedly make use of the following integration rules for matrix exponentials: for any matrix $\boldsymbol{A}$ we have that

$$\boldsymbol{A}\int_0^x e^{\boldsymbol{A}t}\,dt = e^{\boldsymbol{A}x} - \boldsymbol{I}, \tag{4}$$

where $\boldsymbol{I}$ is the identity matrix of the same dimension as $\boldsymbol{A}$. Moreover, for any two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ such that any sum of an eigenvalue of $\boldsymbol{A}$ and an eigenvalue of $\boldsymbol{B}$ is negative, and any vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ of appropriate dimensions, we have that

$$\int_0^x \boldsymbol{a}e^{\boldsymbol{A}t}\boldsymbol{\alpha}\boldsymbol{b}e^{\boldsymbol{B}t}\boldsymbol{\beta}\,dt = (\boldsymbol{a}\otimes\boldsymbol{b})(\boldsymbol{A}\oplus\boldsymbol{B})^{-1}\big(e^{(\boldsymbol{A}\oplus\boldsymbol{B})x} - \boldsymbol{I}\big)(\boldsymbol{\alpha}\otimes\boldsymbol{\beta}), \tag{5}$$

where $\boldsymbol{I}$ is the identity matrix of the same dimension as $\boldsymbol{A}\oplus\boldsymbol{B}$.

For the PH/PH/1 queue we are able to find an explicit formula for the stationary measure ν.

PROPOSITION 3.1. *For all* y,

$$\nu(dy) = \delta_0(dy) + I_{(0,\infty)}(y)\boldsymbol{\alpha}_+\exp((\boldsymbol{S}+\boldsymbol{s}\boldsymbol{\alpha}_+)y)\boldsymbol{s}\,dy, \tag{6}$$

where $\boldsymbol{\alpha}_+$ *is the solution to the nonlinear matrix equation*

$$\boldsymbol{\alpha}_+ = \boldsymbol{\alpha}\hat{A}[\boldsymbol{S} + \boldsymbol{s}\boldsymbol{\alpha}_+] \tag{7}$$

and $\hat{A}(\boldsymbol{B}) = \int_0^\infty e^{\boldsymbol{B}x} A(dx)$.

PROOF. By (2), it is enough to find U_+. To this end we use a result for the GI/PH/1 queue in Asmussen [2] which states that, for a random walk $S_n = X_1 + \cdots + X_n$, $S_0 = 0$ with $X_i \sim X = U - T$, U being phase-type with representation $(\boldsymbol{\alpha}, \boldsymbol{S})$ and T having some arbitrary distribution A, the ascending ladder height distribution G_+ is phase-type with representation $(\boldsymbol{\alpha}_+, \boldsymbol{S})$, and $\boldsymbol{\alpha}_+$ is the solution of the nonlinear matrix equation

$$\boldsymbol{\alpha}_+ = \boldsymbol{\alpha}\hat{A}[\boldsymbol{S} + \boldsymbol{s}\boldsymbol{\alpha}_+],$$

which is known to exist; see Asmussen [2] for details. Since U_+ is the renewal measure for the ascending ladder height process,

$$U_+(dy) = \delta_0(dy) + \boldsymbol{\alpha}_+ \exp((\boldsymbol{S} + \boldsymbol{s}\boldsymbol{\alpha}_+)y)\boldsymbol{s}\, dy$$

for all $y \geq 0$. Thus, for all y,

$$\nu(dy) = \delta_0(dy) + I_{(0,\infty)}(y)\boldsymbol{\alpha}_+ \exp((\boldsymbol{S} + \boldsymbol{s}\boldsymbol{\alpha}_+)y)\boldsymbol{s}\, dy. \qquad \square$$

Equation (7) can be solved iteratively by choosing $\boldsymbol{\alpha}_+^{(0)} = 0$ and $\boldsymbol{\alpha}_+^{(n+1)} = \boldsymbol{\alpha}\hat{A}[\boldsymbol{S} + \boldsymbol{s}\boldsymbol{\alpha}_+^{(n)}]$; see [2] for details.

Let $U_x(\cdot) = \mathbb{E}_x(\sum_{k=0}^\infty I\{S_{\eta_k} \in \cdot\})$, where η_k denotes the *descending* ladder epochs for S_n.

PROPOSITION 3.2. *Defining* $\boldsymbol{\beta}_+$ *similarly to* $\boldsymbol{\alpha}_+$ *as the solution to the nonlinear equation* $\boldsymbol{\beta}_+ = \boldsymbol{\beta}\hat{B}[\boldsymbol{T} + \boldsymbol{t}\boldsymbol{\beta}_+]$ *and* $\hat{B}(\boldsymbol{C}) = \int_0^\infty e^{\boldsymbol{C}x} B(dx)$, *we have that*

$$U_x(dy) = \delta_x(dy) + I_{(-\infty,x)}(y)\boldsymbol{\beta}_+ \exp((\boldsymbol{T} + \boldsymbol{t}\boldsymbol{\beta}_+)(x - y))\boldsymbol{t}\, dy.$$

PROOF. We can rewrite U_x as

$$U_x(A) = \mathbb{E}_x\left(\sum_{k=0}^\infty I\{S_{\eta_k} \in A\}\right) = \mathbb{E}_0\left(\sum_{k=0}^\infty I\{-S_{\eta_k} \in x - A\}\right). \tag{8}$$

Now apply the result of [2] again. The distribution of U_n does not matter with respect to the following argument, and it may even help to consider it as general for a moment. Thus, considering U_n as general, we consider a PH/G/1 queue. Since η_k are *descending* ladder epochs for S_n, they are also *ascending* ladder epochs for the random walk $-S_n$. (Note that there are no problems concerning *weak* versus *strict* ladder heights, since phase-type densities are positive everywhere and cannot have an atom at 0.) Now $-S_n$ corresponds to the GI/PH/1 case, and we may then calculate U_x using this result. For $-S_n$ we know that the ascending ladder height distribution is phase-type with representation $(\boldsymbol{\beta}_+, \boldsymbol{T})$. Thus, by (8), U_x is the renewal measure of the ascending ladder process for $-S_n$ evaluated on the set $x - A$, and hence

$$U_x(dy) = \delta_x(dy) + I_{(-\infty,x)}(y)\boldsymbol{\beta}_+ \exp((\boldsymbol{T} + \boldsymbol{t}\boldsymbol{\beta}_+)(x - y))\boldsymbol{t}\, dy. \qquad \square$$

The formulas for U_x and ν are now used to find an explicit form of the solution kernel for Poisson's equation. Define

$$\boldsymbol{Q} = ((\boldsymbol{S} + \boldsymbol{s\alpha}_+) \oplus (\boldsymbol{T} + \boldsymbol{t\beta}_+))^{-1}(\boldsymbol{s} \otimes \boldsymbol{t}).$$

PROPOSITION 3.3. *The solution kernel* $\Gamma(x, dz)$ *has the following form*:

$$\begin{aligned}\Gamma(x, dz) = \delta_x(dz) &+ I_{(0,x)}(z)\boldsymbol{\beta}_+ \exp((\boldsymbol{T} + \boldsymbol{t\beta}_+)(x - z))\boldsymbol{t}\, dz \\ &+ I_{(x,\infty)}(z)\boldsymbol{\alpha}_+ \exp((\boldsymbol{S} + \boldsymbol{s\alpha}_+)(z - x))\boldsymbol{s}\, dz \\ &+ (\boldsymbol{\alpha}_+ \exp((\boldsymbol{S} + \boldsymbol{s\alpha}_+)z) \otimes \boldsymbol{\beta}_+ \exp((\boldsymbol{T} + \boldsymbol{t\beta}_+)x))\boldsymbol{Q}\, dz \\ &- I_{(0,x)}(z)(\boldsymbol{\alpha}_+ \otimes \boldsymbol{\beta}_+ \exp((\boldsymbol{T} + \boldsymbol{t\beta}_+)(x - z)))\boldsymbol{Q}\, dz \\ &- I_{(x,\infty)}(z)(\boldsymbol{\alpha}_+ \exp((\boldsymbol{S} + \boldsymbol{s\alpha}_+)(z - x)) \otimes \boldsymbol{\beta}_+)\boldsymbol{Q}\, dz,\end{aligned}$$

where all vectors and matrices are described above.

PROOF. By (6) we see that

$$\begin{aligned}\nu(A - x) &= \int_{A-x} \nu(du) \\ &= \delta_x(A) + \int_A I_{(x,\infty)}(u)\boldsymbol{\alpha}_+ \exp((\boldsymbol{S} + \boldsymbol{s\alpha}_+)(u - x))\boldsymbol{s}\, du.\end{aligned}$$

Using Glynn [8], Proposition 4.1, we get

$$\begin{aligned}\Gamma(x, A) &= \int_0^\infty \nu(A - y)U_x(dy) \\ &= \int_0^\infty \left(\delta_y(A) + \int_A I_{(y,\infty)}(z)\boldsymbol{\alpha}_+ \exp((\boldsymbol{S} + \boldsymbol{s\alpha}_+)(z - y))\boldsymbol{s}\, dz\right) \\ &\quad \times \left(\delta_x(dy) + I_{(-\infty,x)}(y)\boldsymbol{\beta}_+ \exp((\boldsymbol{T} + \boldsymbol{t\beta}_+)(x - y))\boldsymbol{t}\, dy\right).\end{aligned} \tag{9}$$

The integral can be calculated in a straightforward manner using the integration rules (4) and (5). □

Note that the condition for (5) to hold is satisfied for $\boldsymbol{A} = \boldsymbol{S} + \boldsymbol{s\alpha}_+$ and $\boldsymbol{B} = \boldsymbol{T} + \boldsymbol{t\beta}_+$, since $\boldsymbol{B}$ has exactly one eigenvalue that is 0 and the rest are negative, whereas all eigenvalues of $\boldsymbol{A}$ are negative ($\boldsymbol{B}$ and $\boldsymbol{A}$ being, respectively, an intensity matrix and subintensity matrix). Let $f_k(x) = x^k$ and put $\hat{f}_k = f - \pi f_k$. The mass $||G_+||$ is given by

$$||G_+|| = \int_0^\infty \boldsymbol{\alpha}_+ e^{\boldsymbol{S}x}\boldsymbol{s}\, dx = -\boldsymbol{\alpha}_+\boldsymbol{S}^{-1}\boldsymbol{s}$$

such that

$$\pi(\cdot) = (1 + \boldsymbol{\alpha}_+\boldsymbol{S}^{-1}\boldsymbol{s})\nu(\cdot).$$

Put $r = 1 + \boldsymbol{\alpha}_+\boldsymbol{S}^{-1}\boldsymbol{s}$ and $s_i = \boldsymbol{\alpha}_+(\boldsymbol{S} + \boldsymbol{s\alpha}_+)^{-i}\boldsymbol{s}$. We want to find the solutions g to the equations $(I - P)g = \hat{f}_1$. First note that simple partial integration gives that $\pi f_1 = rs_2$, so, in particular, $\hat{f}_1(y) = y - rs_2$.

Next note that we cannot integrate the term $\exp((\boldsymbol{T}+\boldsymbol{t}\boldsymbol{\beta}_+)z)$ in the usual way since the matrix $\boldsymbol{T}+\boldsymbol{t}\boldsymbol{\beta}_+$ is singular. This is due to the fact that the traffic intensity $\rho<1$, which implies that S_n has negative drift. Therefore $\boldsymbol{T}+\boldsymbol{t}\boldsymbol{\beta}_+$ is the intensity matrix of a nondefective renewal process (pure renewal process). This also implies that $\boldsymbol{\beta}_+\boldsymbol{e}=1$. So in order to integrate $\exp((\boldsymbol{T}+\boldsymbol{t}\boldsymbol{\beta}_+)z)$, we use a trick involving generalized inverses. Let $\boldsymbol{\pi}$ be the stationary distribution of the Markov jump process with intensity matrix $\boldsymbol{T}+\boldsymbol{t}\boldsymbol{\beta}_+$; that is, $\boldsymbol{\pi}$ satisfies $\boldsymbol{\pi}(\boldsymbol{T}+\boldsymbol{t}\boldsymbol{\beta}_+)=0$. Put

$$\boldsymbol{D}=(\boldsymbol{T}+\boldsymbol{t}\boldsymbol{\beta}_+-\boldsymbol{e}\boldsymbol{\pi})^{-1}+\boldsymbol{e}\boldsymbol{\pi}.$$

Then the indefinite integral

$$\int \exp((\boldsymbol{T}+\boldsymbol{t}\boldsymbol{\beta}_+)y)\,dy = x\boldsymbol{e}\boldsymbol{\pi}+\boldsymbol{D}\exp((\boldsymbol{T}+\boldsymbol{t}\boldsymbol{\beta}_+)x),$$

and, in particular,

$$\int_0^x \exp((\boldsymbol{T}+\boldsymbol{t}\boldsymbol{\beta}_+)y)\,dy = x\boldsymbol{e}\boldsymbol{\pi}+\boldsymbol{D}(\exp((\boldsymbol{T}+\boldsymbol{t}\boldsymbol{\beta}_+)x)-\boldsymbol{I}). \tag{10}$$

The following constants are needed to describe the solution to Poisson's equation given in Proposition 3.4:

$$\begin{aligned}
k_1 &= \tfrac{1}{2}(\boldsymbol{\pi}\boldsymbol{t}-(\boldsymbol{\alpha}_+\otimes\boldsymbol{\pi})\boldsymbol{Q}),\\
k_2 &= 1+\boldsymbol{\beta}_+\boldsymbol{D}\boldsymbol{e}\boldsymbol{\pi}\boldsymbol{t}-\boldsymbol{\beta}_+\boldsymbol{D}\boldsymbol{t}-\boldsymbol{\alpha}_+(\boldsymbol{S}+\boldsymbol{s}\boldsymbol{\alpha}_+)^{-1}\boldsymbol{s}-(\boldsymbol{\alpha}_+\otimes\boldsymbol{\beta}_+\boldsymbol{D}\boldsymbol{e}\boldsymbol{\pi})\boldsymbol{Q}\\
&\quad+(\boldsymbol{\alpha}_+\otimes\boldsymbol{\beta}_+\boldsymbol{D})\boldsymbol{Q}+(\boldsymbol{\alpha}_+(\boldsymbol{S}+\boldsymbol{s}\boldsymbol{\alpha}_+)^{-1}\otimes\boldsymbol{\beta}_+)\boldsymbol{Q}\\
&\quad+\pi f_1((\boldsymbol{\alpha}_+\otimes\boldsymbol{\pi})\boldsymbol{Q}-\boldsymbol{\pi}\boldsymbol{t}),\\
k_3 &= -\boldsymbol{\beta}_+\boldsymbol{D}^2\boldsymbol{t}+\boldsymbol{\alpha}_+(\boldsymbol{S}+\boldsymbol{s}\boldsymbol{\alpha}_+)^{-2}\boldsymbol{s}+(\boldsymbol{\alpha}_+\otimes\boldsymbol{\beta}_+\boldsymbol{D}^2)\boldsymbol{Q}\\
&\quad-(\boldsymbol{\alpha}_+(\boldsymbol{S}+\boldsymbol{s}\boldsymbol{\alpha}_+)^{-2}\otimes\boldsymbol{\beta}_+)\boldsymbol{Q}-\pi f_1+\pi f_1\boldsymbol{\alpha}_+(\boldsymbol{S}+\boldsymbol{s}\boldsymbol{\alpha}_+)^{-1}\boldsymbol{s}\\
&\quad-\pi f_1(\boldsymbol{\alpha}_+(\boldsymbol{S}+\boldsymbol{s}\boldsymbol{\alpha}_+)^{-1}\otimes\boldsymbol{\beta}_+)\boldsymbol{Q}.
\end{aligned}$$

PROPOSITION 3.4. *Let* $\boldsymbol{M}(x)=\exp((\boldsymbol{T}_+\boldsymbol{t}\boldsymbol{\beta}_+)x)$. *In the PH/PH/1 queue* Γ_x *is a* σ*-finite measure. Thus* Γ *defines a solution kernel, and a solution to Poisson's equation* $(I-P)g=\hat{f}_1$ *is given by*

$$\begin{aligned}
g(x) &= (\Gamma\hat{f}_1)(x)\\
&= \int_0^\infty z\Gamma(x,dz)-\pi f_1\int_0^\infty \Gamma(x,dz)\\
&= k_1x^2+k_2x+k_3+\boldsymbol{\beta}_+\boldsymbol{D}^2\boldsymbol{M}(x)\boldsymbol{t}+(\boldsymbol{\alpha}_+(\boldsymbol{S}+\boldsymbol{s}\boldsymbol{\alpha}_+)^{-2}\otimes\boldsymbol{\beta}_+\boldsymbol{M}(x))\boldsymbol{Q}\\
&\quad-(\boldsymbol{\alpha}_+\otimes\boldsymbol{\beta}_+\boldsymbol{D}^2\boldsymbol{M}(x))\boldsymbol{Q}+\pi f_1(\boldsymbol{\alpha}_+(\boldsymbol{S}+\boldsymbol{s}\boldsymbol{\alpha}_+)^{-1}\otimes\boldsymbol{\beta}_+\boldsymbol{M}(x))\boldsymbol{Q}\\
&\quad+\pi f_1(\boldsymbol{\alpha}_+\otimes\boldsymbol{\beta}_+\boldsymbol{D}(\boldsymbol{M}(x)-\boldsymbol{I}))\boldsymbol{Q}-\pi f_1\boldsymbol{\beta}_+\boldsymbol{D}(\boldsymbol{M}(x)-\boldsymbol{I})\boldsymbol{t}.
\end{aligned}$$

PROOF. That Γ_x is a σ-finite measure follows by the fact that $\int_0^\infty \Gamma(x,dz)<\infty$.

The rest of the proof follows by simple integration and partial integration using the integration rules (4), (5) and (10). □

In order to find the expression for the variance constant using the formula $2\pi(\hat{f}_1 g) - \pi\hat{f}_1^2$, it is useful to notice the following result.

LEMMA 3.1.

$$\int_0^\infty x^k \boldsymbol{\alpha}_+ \exp((\boldsymbol{S} + \boldsymbol{s\alpha}_+)x)\boldsymbol{s}\,dx = (-1)^{k+1}k!s_{k+1}.$$

PROOF. The result follows by integrating by parts k times using (4). □

The following constant is needed for calculating the variance constant in Proposition 3.5.

LEMMA 3.2.

$$\pi\hat{f}_1^2 = r^3 s_2^2 - 2rs_3 - r^3 s_2^2 s_1 - 2r^2 s_2^2.$$

PROOF. The result follows from partially integrating $\hat{f}_1^2(x) = x^2 + r^2 s_2^2 - 2xrs_2$ and using (4). □

The variance constant in Proposition 3.5 is given in terms of the following constants.

DEFINITION 3.1.

$$\begin{aligned}
\boldsymbol{\gamma} &= \boldsymbol{\alpha}_+(\boldsymbol{S} + \boldsymbol{s\alpha}_+)^{-2} \otimes \boldsymbol{\beta}_+,\\
\boldsymbol{\gamma}_1 &= \boldsymbol{\alpha}_+ \otimes \boldsymbol{\beta}_+\boldsymbol{D}^2,\\
\boldsymbol{\gamma}_2 &= rs_2\boldsymbol{\alpha}_+(\boldsymbol{S} + \boldsymbol{s\alpha}_+)^{-1} \otimes \boldsymbol{\beta}_+,\\
\boldsymbol{\gamma}_3 &= \boldsymbol{\alpha}_+ \otimes \boldsymbol{\beta}_+\boldsymbol{D},\\
\boldsymbol{\Gamma} &= (\boldsymbol{T} + \boldsymbol{t\beta}_+) \otimes \boldsymbol{I},\\
m_1 &= 6rk_1 s_4 + 2k_1 r^2 s_2 s_3,\\
m_2 &= -2k_2 rs_3 - k_2 r^2 s_2^2,\\
m_3 &= -k_3 r^2 s_2 + rk_3 s_2 + k_3 r^2 s_2 s_1,\\
m_4 &= -r^2 s_2 \boldsymbol{\beta}_+\boldsymbol{D}^2\boldsymbol{t} + r^2 s_2(\boldsymbol{\beta}_+\boldsymbol{D}^2 \otimes \boldsymbol{\alpha}_+)((\boldsymbol{T} + \boldsymbol{t\beta}_+) \oplus (\boldsymbol{S} + \boldsymbol{s\alpha}_+))^{-1}(\boldsymbol{t} \otimes \boldsymbol{s})\\
&\quad + r(\boldsymbol{\beta}_+\boldsymbol{D}^2 \otimes \boldsymbol{\alpha}_+)((\boldsymbol{T} + \boldsymbol{t\beta}_+) \oplus (\boldsymbol{S} + \boldsymbol{s\alpha}_+))^{-2}(\boldsymbol{t} \otimes \boldsymbol{s}),\\
m_5 &= -r^2 s_2 \boldsymbol{\gamma}\boldsymbol{Q} + r(\boldsymbol{\gamma} \otimes \boldsymbol{\alpha}_+)(\boldsymbol{\Gamma} \oplus (\boldsymbol{S} + \boldsymbol{s\alpha}_+))^{-2}(\boldsymbol{Q} \otimes \boldsymbol{s})\\
&\quad + r^2 s_2(\boldsymbol{\gamma} \otimes \boldsymbol{\alpha}_+)(\boldsymbol{\Gamma} \oplus (\boldsymbol{S} + \boldsymbol{s\alpha}_+))^{-1}(\boldsymbol{Q} \otimes \boldsymbol{s}),\\
m_6 &= -r^2 s_2 \boldsymbol{\gamma}_1\boldsymbol{Q} + r^2 s_2(\boldsymbol{\gamma}_1 \otimes \boldsymbol{\alpha}_+)(\boldsymbol{\Gamma} \oplus (\boldsymbol{S} + \boldsymbol{s\alpha}_+))^{-1}(\boldsymbol{Q} \otimes \boldsymbol{s})\\
&\quad + r(\boldsymbol{\gamma}_1 \otimes \boldsymbol{\alpha}_+)(\boldsymbol{\Gamma} \oplus (\boldsymbol{S} + \boldsymbol{s\alpha}_+))^{-2}(\boldsymbol{Q} \otimes \boldsymbol{s}),\\
m_7 &= -r^2 s_2 \boldsymbol{\gamma}_2\boldsymbol{Q} + r(\boldsymbol{\gamma}_2 \otimes \boldsymbol{\alpha}_+)(\boldsymbol{\Gamma} \oplus (\boldsymbol{S} + \boldsymbol{s\alpha}_+))^{-2}(\boldsymbol{Q} \otimes \boldsymbol{s})\\
&\quad + r^2 s_2(\boldsymbol{\gamma}_2 \otimes \boldsymbol{\alpha}_+)(\boldsymbol{\Gamma} \oplus (\boldsymbol{S} + \boldsymbol{s\alpha}_+))^{-1}(\boldsymbol{Q} \otimes \boldsymbol{s}),
\end{aligned}$$

$$
\begin{aligned}
m_8 = {} & -r^3 s_2^2 \boldsymbol{\gamma}_3 \boldsymbol{Q} - r^3 s_2^2 (\boldsymbol{\gamma}_3 \otimes \boldsymbol{\alpha}_+)(\boldsymbol{\Gamma} \oplus (\boldsymbol{S} + \boldsymbol{s}\boldsymbol{\alpha}_+))^{-1}(\boldsymbol{Q} \otimes \boldsymbol{s}) \\
& + r^2 s_2 (\boldsymbol{\gamma}_3 \otimes \boldsymbol{\alpha}_+)(\boldsymbol{\Gamma} \oplus (\boldsymbol{S} + \boldsymbol{s}\boldsymbol{\alpha}_+))^{-2}(\boldsymbol{Q} \otimes \boldsymbol{s}) \\
& + r^3 s_2 \boldsymbol{\gamma}_3 \boldsymbol{Q} - r^2 s_s^2 \boldsymbol{\gamma}_1 \boldsymbol{Q} + r^3 s_2^2 s_1 \boldsymbol{\gamma}_3 \boldsymbol{Q}, \\
m_9 = {} & -r^2 s_2 (\boldsymbol{\beta}_+ \boldsymbol{D} \otimes \boldsymbol{\alpha}_+)((\boldsymbol{T} + \boldsymbol{t}\boldsymbol{\beta}_+) \oplus (\boldsymbol{S} + \boldsymbol{s}\boldsymbol{\alpha}_+))^{-2}(\boldsymbol{t} \otimes \boldsymbol{s}) \\
& - r^3 s_2^2 (\boldsymbol{\beta}_+ \boldsymbol{D} \otimes \boldsymbol{\alpha}_+)((\boldsymbol{T} + \boldsymbol{t}\boldsymbol{\beta}_+) \oplus (\boldsymbol{S} + \boldsymbol{s}\boldsymbol{\alpha}_+))^{-1}(\boldsymbol{t} \otimes \boldsymbol{s}) \\
& + r^3 s_2^2 s_1 \boldsymbol{\beta}_+ \boldsymbol{D}\boldsymbol{t} + r^2 s_2^2 \boldsymbol{\beta}_+ \boldsymbol{D}\boldsymbol{t}.
\end{aligned}
$$

PROPOSITION 3.5. *The variance constant for $f(x) = x$ is given by*

$$2(m_1 + m_2 + \cdots + m_9) - \pi \hat{f}_1^{\,2}.$$

PROOF. Again the proof is a simple exercise in applying the integration rules (4), (5) and (10). □

4. Examples. The formula above was implemented in Pascal on a SUN workstation, and some examples were considered. It was necessary to restrict the matrix dimensions in the program to be less than 50. The highest order of matrices appearing in the formula for the variance constant is $p_1 p_2^2$, where p_1 is the dimension of the interarrival distribution and p_2 the dimension of the service time distribution. By the dimension of a phase-type distribution we mean the dimension of its corresponding vectors (e.g., the exit rate vector). This means that our examples are restricted to lower dimensions, though in the special case of exponential service times we would be able to consider an interarrival distribution with up to 50 phases.

A free copy of the program for calculating the variance constant is available upon request from the author. A program for fitting phase-type distributions to densities or data can be obtained free of charge by e-mail at the following address: olleh@math.chalmers.se. For further information, please see Asmussen, Nerman and Olson [3] and Häggström, Asmussen and Nerman [10].

EXAMPLE 4.1. Consider the *M*/*M*/1 queue. Then it is well known (see, e.g., [16]) that the variance constant is given by

$$\rho \frac{2 + 5\rho - 4\rho^2 + \rho^3}{(1-\rho)^4},$$

where ρ is the traffic intensity.

EXAMPLE 4.2 (Erlang distributions). We consider the $E_k/E_n/1$ queue for $k = 1, 2, 3, 4, 5$, $n = 1, 2, 3$ and for traffic intensities $\rho = 0.3, 0.6, 0.9$. We fix the mean of the service time distribution to 1 and let the mean of the interarrival distribution vary accordingly. The variance constants for the $E_k/E_n/1$ queue are given in Table 1.

TABLE 1

$\rho = 0.3$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
$n = 1$	3.957	1.313	0.782	0.573	0.464
$n = 2$	1.684	0.391	0.182	0.112	0.079
$n = 3$	1.180	0.224	0.092	0.050	0.032
$\rho = 0.6$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
$n = 1$	88.50	36.36	24.98	20.25	17.70
$n = 2$	36.48	10.51	5.891	4.184	3.328
$n = 3$	25.28	5.990	2.957	1.915	1.422
$\rho = 0.9$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
$n = 1$	35,900	15,265	10,758	8,881	7,867
$n = 2$	14,993	4,476	2,602	1,901	1,549
$n = 3$	10,476	2,575	1,323	889	681

EXAMPLE 4.3 ($M/G/1$ queue). Consider the $M/G/1$ queue, where the service time distribution is a phase-type distribution with representation $(\boldsymbol{\pi}, \boldsymbol{T})$, where

$$\boldsymbol{\pi} = \begin{pmatrix} 0.9731 & 0.0152 & 0.0106 & 0.001 \end{pmatrix},$$

$$\boldsymbol{T} = \begin{pmatrix} -28.648 & 28.532 & 0.089 & 0.027 \\ 0.102 & -8.255 & 8.063 & 0.086 \\ 0.133 & 0.107 & -5.807 & 5.296 \\ 0.100 & 0.102 & 0.111 & -2.176 \end{pmatrix}$$

(this phase-type distribution was originally fitted to a log-normal distribution; see [4]). As we let the arrival intensity vary, we present the values for the asymptotic variance (as a function of the traffic intensity) in Table 2.

TABLE 2

Traffic intensity	**Variance**
0.1	0.000
0.2	0.008
0.3	0.062
0.4	0.320
0.5	1.338
0.6	5.074
0.7	18.56
0.8	66.95
0.9	233.0

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