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RECURRENCE RELATIONS FOR GENERALIZED HITTING TIMES FOR SEMI-MARKOV PROCESSES[1]

BY DMITRII S. SILVESTROV

Umeå University

Recurrence relations and upper bounds are obtained for power moments of generalized hitting times for semi-Markov processes. General necessary and sufficient conditions for the existence of these moments are also found. Applications to hitting times for semi-Markov dynamical systems of linear type, semi-Markov random walks, diffusion processes and queuing systems are discussed.

1. Introduction. Let $\eta(t)$, $t \geq 0$, be a semi-Markov process with arbitrary phase space X, that is, a process with stepwise trajectories possessing the Markov property at moments of jumps. We consider random functionals τ defined on trajectories of $\eta(t)$ and called generalized hitting times. Ordinary hitting times in domains $D \subseteq X$ for a semi-Markov process $\eta(t)$ are included in this class, as well as more general place-dependent hitting times, first record times, additive functionals accumulated on trajectories of a semi-Markov process $\eta(t)$ up to the moment τ and many others. The concept of this class of functionals arose from the model of terminal stopping times (place-dependent hitting times) for discrete-time Markov chains investigated by Pitman (1977).

In this paper we focus on recurrence relations, upper bounds and necessary and sufficient conditions for the existence (meaning finiteness) of the moments of generalized hitting times $E_\pi \tau^r$.

Moments of this type play an important role in limit and ergodic theorems for Markov-type processes. As a rule, first and second order moments are used in the conditions of the theorems; higher order moments are used in the rates of convergence and asymptotic expansions.

There is also another important area of application. Hitting times are often interpreted as transition times for different stochastic systems described by Markov-type processes (occupation times or waiting times in queuing systems, lifetimes in reliability models, extinction times in population dynamic models, etc.). The problem of estimating the tails of the corresponding distributions leads one (due to Chebyshev-type inequalities) to the use of different order moments for hitting times.

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Our results are based on the development and expansion of two groups of results concerning hitting times for Markov chains to generalized hitting times for semi-Markov processes.

The first group of results is concerned with recurrence relations for the moments of integer order for ordinary and place-dependent hitting times. Recurrence relations of this type have been obtained for Markov chains by Chung (1954, 1960). Further development was achieved by Lamperti (1963), Kemeny and Snell (1961a, b) and subsequently by Pitman (1974, 1976, 1977).

In this paper a new approach to this problem is developed. Instead of direct calculations for moments, we first obtain trajectory stochastic recurrence representations for powers of generalized hitting times τ^r (Theorem 1). Then simple calculations for the first-order moments yield desirable recurrence relations for the higher order moments and the description of these moments as minimal solutions of the corresponding recurrent integral equations (Theorem 2). This theorem presents, for generalized hitting times and semi-Markov processes, results similar to those given for ordinary and place-dependent hitting times for Markov chains in the cited papers.

The second group of results is concerned with upper bounds for the expectations of ordinary hitting times, which can be obtained in terms of test functions with the use of the so-called *sweeping* technique. Upper bounds for expectations $E_\pi \tau$ and criteria for positive recurrence of Markov chains with discrete phase space have been obtained in terms of a test function of one space variable by Foster (1953). These results have been extended to Markov chains with general phase space by Lamperti (1963) and Tweedie (1975, 1976). Related problems have been investigated by Hasminskii (1969), Kalashnikov (1973, 1977, 1978), Nummelin (1984) and Kalashnikov and Rachev (1990). The latest results have been obtained by Meyn (1989) and Meyn and Tweedie (1992, 1993).

As far as the moments of higher order are concerned, the works of Kalashnikov (1973, 1978) have to be mentioned, in which the corresponding upper bounds have been obtained for Markov chains with the use of more complicated bivariate test functions of time and space variables.

It should be noted that it is useful to have upper bounds for moments of hitting times in terms of test functions from different classes. In the work of Silvestrov (1980), the idea of recurrent construction of upper bounds based on test functions of only one space variable has been introduced, and the corresponding upper bounds have been obtained for higher order moments of ordinary hitting times for semi-Markov processes with a discrete phase space. Similar results for Markov chains and semi-Markov processes with an arbitrary phase space have been obtained independently by Nummelin and Tuominen (1983), Tweedie (1983) and Silvestrov (1983a, b).

In this paper we consider more general functionals, called generalized hitting times, and present the corresponding recursive test function techniques in a more advanced operator version (Theorem 3). These techniques can be applied to minimal solutions of recurrent operator equations and depend more on some general properties of the operator model than on specific features connected with the structure of hitting time functionals. It seems to us that such a method clarifies the real mechanism for the corresponding recurrent upper bounds in a wider mathematical context.

The upper bounds obtained in Theorem 3 possess a natural minimal property which makes it possible to reformulate them in the form of necessary and sufficient conditions for the existence of moment-type functionals of generalized hitting times. We do this in the general form (Theorem 4) which permits us cover many different Foster-type criteria for higher order moments of generalized hitting times in the frame of one model. As examples, we formulate necessary and sufficient conditions for so-called uniform recurrence of generalized hitting times (Theorem 5) as well as a version of these conditions for time-dependent generalized hitting times (Theorem 6). These theorems correspond to the well-known results concerned with the uniform recurrence and regularity of ordinary hitting times for Markov chains. References and discussion related to this question can be found in the books by Nummelin (1984) and Meyn and Tweedie (1993) and in the recent papers by Tuominen and Tweedie (1994) and Silvestrov (1993, 1994a).

Applications to some concrete models of semi-Markov processes are also considered in order to demonstrate the techniques of recurrent test functions and the relevance of the semi-Markov setting. We obtain recurrent upper bounds for higher order moments of hitting times for semi-Markov dynamical systems of linear type (Theorems 7 and 8), semi-Markov random walks (Theorems 9 and 10), diffusion processes (Theorems 11 and 12) and M/G-type queuing systems (Theorems 13 and 14), and we show how these results can be used to obtain rates of convergence in the corresponding ergodic theorems (Theorem 15).

2. Definition of the model. Let $\bar{\beta} = \{\beta_n = (\eta_n, \alpha_n),\ n = 0, 1, \dots\}$ be a random sequence with phase space $X \times [0, +\infty]$. (Here X is an arbitrary space with σ-algebra of measurable subsets B_X, and $[0, +\infty]$ is an extended half-line with the Borel σ-algebra of measurable subsets B_+.) Let $\tau_0 = 0$, $\tau_n = \alpha_1 + \cdots + \alpha_n$, $n = 0, 1, \dots$, and $\tau_\infty = \alpha_1 + \alpha_2 + \cdots$. We can connect with the sequence $\bar{\beta}$ the stepwise random process $\eta(t)$ with phase space X, considering random variables η_n, α_n and τ_n as states of this process at moments of jumps, times between jumps and moments of jumps, respectively, and defining $\eta(t) = \eta_n$ for $\tau_n \le t < \tau_{n+1}$, $n = 0, 1, \dots$.

The process $\eta(t)$ is a *semi-Markov process* if the random sequence $\bar{\beta}$ is a *Markov renewal process*, that is, a homogeneous Markov chain with phase space $X \times [0, +\infty]$, an initial distribution $\pi(A) = P\{\eta_0 \in A\} = P\{\eta_0 \in A, \alpha_0 = 0\}$ and transition probabilities

$$P\{\eta_{n+1} \in A,\ \alpha_{n+1} \le u \mid \eta_n = x,\ \alpha_n = v\} = P(x, A, u). \tag{1}$$

In this case the first component $\bar{\eta} = \{\eta_n,\ n = 0, 1, \dots\}$ is also a homogeneous Markov chain with transition probabilities $P(x, A) = P(x, A, \infty)$.

We do not use the random variable α_0 to define the process $\eta(t)$. That is why it is convenient to assign $\alpha_0 \equiv 0$ and consider, together with $\bar{\beta}$, the reduced version of this sequence $\bar{\beta}' = \{\eta_0, \beta_1, \beta_2, \dots\}$.

Let $h(\cdot)$ be a measurable functional acting from the space of trajectories of the sequence $\bar{\beta}'$ to $[0, +\infty]$, and let $h(\bar{\beta}')$ be the corresponding random functional defined on trajectories of the random sequence $\bar{\beta}'$. We denote by θ_n and Θ_n the shift operators translating the random sequence $\bar{\beta}'$ in the sequence $\theta_n \bar{\beta}' = \{\eta_n, \beta_{n+1}, \dots\}$ and the nonnegative random functional $h(\bar{\beta}')$ in the random functional $\Theta_n h(\bar{\beta}') = h(\theta_n \bar{\beta}')$, respectively.

As usual, the notation P_x and E_x is used to denote probabilities and expectations calculated when the initial distribution is concentrated in point x. If some equality or inequality holds almost everywhere in measure P_x, we use the notation $P_x 1$ beside the corresponding symbol (e.g., $=_{P_x 1}$).

Now, let $\mu = g(\bar{\beta}')$ and $\lambda = h(\bar{\beta}')$ be two nonnegative random functionals. We call a random functional ν a *generalized hitting time* (for the Markov chain $\bar{\eta}$) generated by a random functional μ if ν takes values $1, 2, \dots$ and is defined by the formula

$$(2) \qquad \nu =_{P_x 1} \min(n \ge 1 : \Theta_{n-1} \mu \ge 1), \qquad x \in X.$$

Similarly, we call a random functional τ a *generalized hitting time* [for a semi-Markov process $\eta(t)$] generated by random functionals μ and λ if τ takes values in the interval $[0, \infty]$ and is defined by the formula

$$(3) \qquad \tau =_{P_x 1} \sum_{n=0}^{\nu-1} \Theta_n \lambda =_{P_x 1} \sum_{n=0}^{\infty} \chi_n \Theta_n \lambda, \qquad x \in X,$$

where $\chi_n = I(\nu > n)$ and, in the right-hand side of (3), one takes the products $\chi_n \Theta_n \lambda$ as 0 if the corresponding indicators χ_n take the value 0.

We say that τ is a *Markovian generalized hitting time* if the functionals $\mu = g(\eta_0, \eta_1, \alpha_1)$ and $\lambda = h(\eta_0, \eta_1, \alpha_1)$ depend on the triple of random variables η_0, η_1, α_1 but not on the whole sequence $\bar{\beta}'$.

Let us consider some examples to show that this class is sufficiently rich and to illustrate the term *generalized hitting time*.

If $\mu = g(\eta_1)$, then $\nu =_{P_x 1} \min(n \ge 1 : \eta_n \in D)$, where $D = \{y : g(y) \ge 1\}$. In this case ν is an ordinary hitting time for the Markov chain $\bar{\eta}$. If $\mu = g(\eta_0, \eta_1)$, then $\nu =_{P_x 1} \min(n \ge 1 : \eta_n \in D_{\eta_{n-1}})$, where $D_x = \{y : g(x, y) \ge 1\}$, $x \in X$. One can call such a functional a place-dependent hitting time. Let $\mu = t^{-1}\alpha_1$, where $t > 0$. Then $\nu =_{P_x 1} \min(n \ge 1 : \alpha_n \ge t)$. One can call such a functional the first record time.

Let us mention two of the most important examples involving the variation of functionals λ. If $\lambda = \alpha_1$, then $\tau =_{P_x 1} \tau_\nu$ is an ordinary generalized hitting time for the semi-Markov process $\eta(t)$. If $\lambda = h(\eta_0, \eta_1)$, then τ is an additive functional accumulated on trajectories of the Markov chain $\bar{\eta}$ up to the moment ν. The list of examples can be continued.

For Markovian generalized hitting times we can effectively use the Markov property of the initial semi-Markov process $\eta(t)$ and obtain desirable recurrent relations and upper bounds for the moments $E_\pi \tau^r$.

3. Stochastic recurrence relations. The results given in this section show that recurrence relations for powers of generalized hitting times (without the additional Markovian supposition) can be obtained, not only for the moments, but directly for these functionals as trajectory stochastic recurrence relations which take place with probability 1.

Let us introduce the random functionals

$$\lambda_r =_{P_x 1} \lambda^r + \sum_{l=1}^{r-1} C_r^l \lambda^{r-l} \chi_1(\Theta_1 \tau)^l, \qquad x \in X,\ r = 1, 2, \dots, \tag{4}$$

where one takes the products $\lambda^{r-l}\chi_1(\Theta_1\tau)^l$ as 0 if at least one of the factors takes the value 0.

THEOREM 1. *Let τ be a generalized hitting time generated by nonnegative random functionals μ and λ. Then, for any $r = 1, 2, \dots,$ the random functional τ^r is also a generalized hitting time generated by the same functional μ [used to define the random index ν in* (3)] *for all $r = 1, 2, \dots$ and the functional $\lambda = \lambda_r$ which is specified for every $r = 1, 2, \dots$ and defined in* (4).

PROOF. Let us introduce doubly truncated hitting times which are finite random variables

$$\tau(N, T) = \sum_{n=1}^{(\nu \wedge N)-1} \Theta_n \lambda \wedge T,$$

where $N = 0, 1, \dots,\ T \geq 0$. By definition, the random functional $\tau(N, T)$ can be represented as the sum of two summands

$$\tau(N, T) =_{P_x 1} \lambda \wedge T + \chi_1 \Theta_1 \tau(N-1, T), \qquad x \in X. \tag{5}$$

Taking equality (5) to the rth power, we obtain the equivalent stochastic equality

$$\tau(N, T)^r =_{P_x 1} \lambda_r(N, T) + \chi_1 \Theta_1 \tau(N-1, T)^r, \qquad x \in X, \tag{6}$$

where

$$\lambda_r(N, T) =_{P_x 1} (\lambda \wedge T)^r + \sum_{l=1}^{r-1} C_r^l (\lambda \wedge T)^{r-l} \chi_1(\Theta_1 \tau(N-1, T))^l. \tag{7}$$

Iterating (6) and taking into account that, by definition, $\tau_{0,T} = 0$, we can obtain the stochastic equality

$$\tau(N, T)^r =_{P_x 1} \sum_{n=0}^{N-1} \chi_n \Theta_n \lambda_r(N-n, T), \qquad x \in X. \tag{8}$$

By definition, one has

$$0 \leq \tau(N, T) \leq_{P_x 1} \tau, \qquad x \in X, \tag{9}$$

and, as $N, T \to +\infty$ in an arbitrary way,

$$\tau(N,T) \to_{P_x 1} \tau, \qquad x \in X. \tag{10}$$

It follows from (7) that

$$0 \leq \lambda_r(N,T) \leq_{P_x 1} \lambda_r, \qquad x \in X, \tag{11}$$

and, using (9) and (10), we also obtain that, as $N, T \to +\infty$ in an arbitrary way,

$$\lambda_r(N,T) \to_{P_x 1} \lambda_r, \qquad x \in X. \tag{12}$$

To evaluate the limit in (12), we use (7). In this formula the random variables $\lambda \wedge T$ and $\Theta_1 \tau(N-1, T)$ are finite, and so the products $(\lambda \wedge T)^{r-l} \chi_1 (\Theta_q \tau(N-1,T))^l$ are 0 if at least one of the factors takes the value 0. Therefore, the limit in (12) can be calculated with the use of the product rule described in (4).

From (11) and (12) it follows that similar relations also hold for the shifted random variables $\Theta_n \lambda_r(N-n, T)$. Taking this into consideration, we obtain by (11) and (12) that, as $N, T \to +\infty$ in an arbitrary way,

$$\sum_{n=0}^{N-1} \chi_n \Theta_n \lambda_r(N-n,T) \to_{P_x 1} \sum_{n=0}^{\infty} \chi_n \Theta_n \lambda_r, \qquad x \in X. \tag{13}$$

In (13) the random variables $\Theta_n \lambda_r(N-n,T)$ are finite and so the products $\chi_n \Theta_n \lambda_r(N-n,T)$ are 0, if the corresponding indicators χ_n are 0. Therefore, the limit in (13) can also be calculated with the use of the product rule described in (3).

Evaluating the limits for the expressions on the left-hand and right-hand sides in (8) and using relations (10) and (13), we finally get

$$\tau^r =_{P_x 1} \sum_{n=0}^{\infty} \chi_n \Theta_n \lambda_r, \qquad x \in X, \tag{14}$$

where we use the product rule described in (3). □

REMARK 1. We formulated Theorem 1 as an assertion concerning generalized hitting times for semi-Markov processes. However, analyzing the proof of Theorem 1, one can observe that only the general structural properties of the random sequence $\bar{\beta}$ were used, and the semi-Markov properties of this sequence were not used at all. This fact allows us to conclude, somewhat unexpectedly, that the assertion of Theorem 1 is justified for an arbitrary random sequence $\bar{\beta}$ with phase space $X \times [0, +\infty]$ and any generalized hitting time defined by formulas similar to (2) and (3) as a random functional on trajectories of the random sequence $\bar{\beta}$.

4. Recurrence relations for moments. In this section we obtain recurrence relations and equations for moments of generalized hitting times. The Markovian property of generating functionals plays an essential role. Therefore, we consider the Markovian generalized hitting time τ generated by random functionals $\mu = g(\eta_0, \eta_1, \alpha_1)$ and $\lambda = h(\eta_0, \eta_1, \alpha_1)$.

Let L be a space of functions acting from X to $[0, +\infty]$ that are B_+-measurable with respect to the σ-algebra B_X.

Let us introduce, for each $r = 0, 1, \dots,$ the linear integral operator P_r acting from L to L according to

$$\begin{aligned} P_r v(x) &= E_x \lambda^r \chi_1 v(\eta_1) \\ &= \int_X \int_{[0, +\infty]} P(x, dy, dt) h(x, y, t)^r I(g(x, y, t) < 1) v(y), \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x \in X, \end{aligned} \tag{15}$$

where λ^r and $h(x, y, t)^r$ are counted as 1 if r is 0, and the products $\lambda^r \chi_1 v(\eta_1)$ and $h(x, y, t)^r I(g(x, y, t) < 1) v(y)$ as 0 if at least one of the factors takes the value 0. As usual, one takes the integral of $+\infty$ over a set A to be 0 or $+\infty$ according to whether A has zero or positive measure.

The operator P_0 plays the most important role. It is not difficult to show that P_0^n, the nth iteration of this operator, acts by the formula

$$P_0^n v(x) = E_x \chi_n v(\eta_n), \qquad x \in X, \tag{16}$$

where the rule of calculation is as for (15).

We use the potential-type operator U mapping L to L and defined by the formula

$$Uv(x) = v(x) + P_0 v(x) + P_0^2 v(x) + \cdots, \qquad x \in X, \tag{17}$$

where the rules are as for (15) and (16).

We also need to consider the linear integral operator functionals $m^{[r]}$, $r = 1, 2, \dots,$ mapping $L \times \cdots \times L$ ($r - 1$ times) to L by the formula

$$\begin{aligned} m^{[r]}(v_1, \dots, v_{r-1})(x) &= E_x \lambda^r + \sum_{l=1}^{r-1} C_r^l E_x \lambda^{r-l} \chi_1 v_l(\eta_1) \\ &= m^{(r)}(x) + \sum_{l=1}^{r-1} C_r^l P_{r-l} v_l(x), \end{aligned} \tag{18}$$

where $m^{(r)} = \langle m^{(r)}(x) = E_x \lambda^r,\ x \in X \rangle$ and, once again, one has rules as for (15).

To distinguish the function of $x \in X$ defined in (18) from the corresponding operator functional, we use the notation $m_{[r]} = m_{[r]}(v_1, \dots, v_{r-1}) = \langle m^{[r]}(v_1, \dots, v_{r-1})(x),\ x \in X \rangle$. Let us also write $m_r = \langle E_x \lambda_r,\ x \in X \rangle$ and $M_r = \langle E_x \tau^r,\ x \in X \rangle$. Note that, by definition, $m^{(1)} = m_{[1]} = m_1$.

THEOREM 2. *Let τ be a Markovian generalized hitting time generated by nonnegative random functionals μ and λ. Then the functions m_r, M_r, $r = 1, 2, \dots,$ belong to the space L and are defined by the recurrence formulas*

$$m_r = m_{[r]}(M_1, \dots, M_{r-1}), \qquad M_r = U m_r. \tag{a}$$

Moreover, the function M_r *satisfies the linear integral equation*

$$M_r = m_r + P_0 M_r \tag{b}$$

and is the minimal solution of this equation in L [$M_r \le M$ *for any other solutions of* (b) *from* L].

PROOF. We are going to use Theorem 1. First, let us calculate $E_x \lambda_r$ using (4). In this formula the random variables $\lambda^{r-l}\chi_1$ depend on the random variables η_0, η_1, α_1, but the shifted generalized hitting time $\Theta_1 \tau$ depends on the random variables $\eta_1, \eta_2, \alpha_2, \dots$. Therefore, using the Markov property of the initial Markov renewal process $\bar{\beta}$, we get

$$\begin{aligned} E_x \lambda_r &= E_x \lambda^r + \sum_{l=1}^{r-1} C_r^l E_x \lambda^{r-l} \chi_1 E\{\Theta_1 \tau^l | \eta_1\} \\ &= m^{(r)}(x) + \sum_{l=1}^{r-1} E_x \lambda^{r-l} \chi_1 M_l(\eta_1) \\ &= m_{[r]}(M_1, \dots, M_{r-1}), \qquad x \in X. \end{aligned} \tag{19}$$

The rule of calculation in (19) is the same as in (18), which follows from those of (4).

Now, we can calculate $E_x \tau^r$ using (3). In this formula the random indicators χ_n depend on the random variables $\eta_0, \eta_1, \alpha_1, \dots, \eta_n, \alpha_n$, but the shifted generalized hitting time $\Theta_n \tau$ depends on the random variables $\eta_n, \eta_{n+1}, \alpha_{n+1}, \dots$. Therefore, using the Lebesgue theorem to change the order of computation for expectation and sum of series and then using the Markov property of the initial Markov renewal processes $\bar{\beta}$, we get

$$\begin{aligned} E_x \tau^r &= \sum_{n=0}^{\infty} E_x \chi_n \Theta_n \lambda_r \\ &= \sum_{n=0}^{\infty} E_x \chi_n E\{\Theta_n \lambda_r | \eta_n\} \\ &= m_r(x) + P_0 m_r(x) + P_0^2 m_r(x) + \cdots \\ &= U m_r(x), \qquad x \in X. \end{aligned} \tag{20}$$

The rule of calculation in (20) is the same as in (16) and (17). It follows from the rule of calculation used in (3).

The function $M_1 = Um_1$ belongs to L since the function $m_1 = m^{(1)}$ belongs to L and this space is closed with respect to the *operation of finite and countable summation* ($v = v_1 + v_2 + \cdots$ belongs to L for any functions $v_1, v_2, \dots$ from L). Then (19) and (20) let us conclude that the functions $m_2 = m_{[2]}(M_1)$ and $M_2 = Um_2$ belong to L and so on. By induction, we can conclude that the functions m_r, M_r belong to L for all $r = 1, 2, \dots$.

The linear integral operator P_0 is a *countably additive operator* $[P_0(v_1 + v_2 + \cdots) = P_0 v_1 + P_0 v_2 + \cdots$ for any finite or countable sequence of functions $v_1, v_2, \dots$ from $L]$. Therefore, using (a), we can write

$$\begin{aligned} m_r + P_0 M_r &= m_r + P_0(m_r + P_0 m_r + \cdots) \\ &= m_r + P_0 m_r + P_0^2 m_r + \cdots = M_r. \end{aligned}$$

Therefore, M_r is a solution of the operator equation (b).

Let M be an arbitrary solution of the operator equation (b). Iterating (b), we get

$$M = m_r + P_0(m_r + P_0 M) = m_r + P_0 m_r + P_0^2 M$$

and, continuing this process, we get

$$M = m_r + P_0 m_r + \cdots + P_0^n m_r + P_0^{n+1} M \ge m_r + P_0 m_r + \cdots + P_0^n m_r$$

for any $n = 1, 2, \dots$. From the last inequality it follows that $M \ge m_r + P_0 m_r + \cdots = M_r$, that is, M_r is the minimal solution of (b). □

5. Recurrent upper bounds for moments. In this section we obtain recurrent upper bounds for moments of generalized hitting times in terms of so-called test functions.

Recall the notation $m_{[r]} = m_{[r]}(v_1, \dots, v_{r-1})$ introduced in Section 4.

THEOREM 3. *Let τ be a Markovian generalized hitting time. If there exist test functions v_r, $r = 1, \dots, k$, from L that satisfy the test inequality*

(c) $$v_r \ge m_{[r]} + P_0 v_r, \qquad x \in X,\ r = 1, \dots, k,$$

then, for functions M_r, $r = 1, \dots, k$, the following upper bounds hold:

(d) $$M_r \le m_{[r]} + P_0 v_r \le v_r, \qquad x \in X,\ r = 1, \dots, k.$$

PROOF. Let us first prove that

(21) $$M_r \le v_r, \qquad x \in X,\ r = 1, \dots, k.$$

The space L is closed with respect to the *computation of monotonic difference* (the function $v = v_1 - v_2$ [by definition $v(x) = 0$ if $v_1(x) = v_2(x) = +\infty$] belongs to L for any two functions $v_1 \ge v_2$ from L). Therefore, a countably additive operator P_0 is a *partial monotonic operator* ($P_0 v_1 \ge P_0 v_2$ for any functions $v_1 \ge v_2$). Indeed, if $v_1 \ge _2$, then the function v_1 can be represented in the form $v_1 = v_2 + v$, where $v \in L$. Therefore, we have $P_0 v_1 = P_0 v_2 + P_0 v \ge P_0 v_2$. It is also obvious that the property of partial monotonicity of the operator P_0 imply the same property for the iterates P_0^n.

Using these properties of the operator P_0, the equality $m_{[1]} = m_1$ and the test inequality (c), we get

$$\begin{aligned} v &\ge m_1 + P_0 v \ge m_1 + P_0(m_1 + P_0 v) \\ &= m_1 + P_0 m_1 + P_0^2 v \\ &\ge \cdots \ge m_1 + P_0 m_1 + \cdots + P_0^n m_1 + P_0^{n+1} v, \qquad x \in X. \end{aligned}$$

It follows from the last relation and (a) that

$$v \geq m_1 + P_0 m_1 + \cdots = M_1, \qquad x \in X.$$

Therefore, inequality (21) is valid for $r = 1$.

Let us assume that inequality (21) holds for all $l \leq r - 1$ for some $r \leq k$, and check that under this assumption inequality (21) also holds for $l = r$.

It is not difficult to check that the operators P_r are also partially monotonic for all $r = 1, \dots$. Therefore, the operator functionals $m^{[r]}$ are also partially monotonic $[m_{[r]}(v_1, \dots, v_{r-1}) \geq m_{[r]}(v'_1, \dots, v'_{r-1})$ for any functions $v_l \geq v'_l$, $l = 1, \dots, r-1$, $x \in X]$. Using this fact and the induction hypothesis made above, we have

$$\begin{aligned} m_{[r]} &= m_{[r]}(v_1, \dots, v_{r-1}) \\ &\geq m_{[r]}(M_1, \dots, M_{r-1}) = m_r, \qquad x \in X. \end{aligned}$$

Also, using the partial monotonicity of the operators P_0^n and the test inequality (c), we get

$$\begin{aligned} v_r &\geq m_{[r]} + P_0 v_r \\ &\geq m_{[r]} + P_0\big(m_{[r]} + P_0 v_r\big) \\ &= m_{[r]} + P_0 m_{[r]} + P_0^2 v_r \\ &\geq \cdots \geq m_{[r]} + P_0 m_{[r]} + \cdots + P_0^n m_{[r]} + P_0^{n+1} v_r, \qquad x \in X. \end{aligned}$$

It follows from the last two inequalities and (a) that

$$\begin{aligned} v_r &\geq m_{[r]} + P_0 m_{[r]} + P_0^2 m_{[r]} + \cdots \\ &\geq m_r + P_0 m_r + P_0^2 m_r + \cdots = M_r, \qquad x \in X. \end{aligned}$$

We can now conclude by induction that inequality (21) is valid for all $r = 1, \dots, k$.

To complete the proof and to get the upper bounds (d), we use the fact that the functions M_r are solutions of (b) together with (21), the partial monotonicity of P_0 and of the $m^{[r]}$ and the test inequality (c). We finally get

$$\begin{aligned} M_r &= m_{[r]}(M_1, \dots, M_r) + P_0 M_r \\ &\leq m_{[r]}(v_1, \dots, v_{r-1}) + P_0 v_r \leq v_r, \qquad x \in X,\ r = 1, \dots, k. \end{aligned} \tag{22}$$

The proof is complete. □

6. Necessary and sufficient conditions for the existence of moments. The moments $E_\pi \tau^k$ may take finite values or the value $+\infty$. Usually, one says that these moments *exist* in the first case. The recurrent upper bounds represented in Theorem 3 allow us to formulate effective necessary and sufficient conditions for the existence of these moments and of some more general moment functionals.

The measurable functional f acting from $L_k = L \times \cdots \times L$ (k times) to $[0, +\infty]$ is said to be *nonnegative partly monotonic* if $f(v_1, \ldots, v_k) \le f(v'_1, \ldots, v'_k)$ for any functions from L such that $v_r \le v'_r$, $x \in X$, $r = 1, \ldots, k$.

The following theorem supplements Theorem 3.

THEOREM 4. *Let f be a nonnegative partly monotonic functional and let τ be a Markovian generalized hitting time. Then there exist test functions v_r, $r = 1, \ldots, k$, from L that satisfy the test inequality* (c), *and*

$$F_k = f\big(m_{[1]} + P_0 v_1, \ldots, m_{[k]} + P_0 v_k\big) < \infty \tag{e}$$

is the necessary and sufficient condition for the relation

$$f(M_1, \ldots, M_k) < \infty \tag{f}$$

to hold. In this case

$$f(M_1, \ldots, M_k) \le F_k. \tag{g}$$

PROOF. The assertion of sufficiency and inequality (g) follow directly from the assertions of Theorem 3 and the partial monotonicity of f. We obtain necessity by choosing the moments M_r, $r = 1, \ldots, k$, as test functions. In this case the test inequality (c) holds since the moments M_r, $r = 1, \ldots, k$, satisfy (b), replacing the test inequality (c). The test inequality (e) coincides with (f), also because of (b). □

REMARK 2. Analyzing the proofs of Theorems 2–4, one may note that the recurrent explicit formulas (a) play a key role. We obtain these formulas using the special structure of Markovian generalized hitting times and the Markov property of the initial Markov renewal process. After that, we obtain the description of the moments M_r as minimal solutions of the recurrent integral equations (b) and the recurrent upper bounds (d), as well as the necessary and sufficient conditions of existence and upper bounds (g) using only general properties of the space L, the operator P_0 and the operator functionals $m^{[r]}$. Specifically, one uses the fact that the space L is closed with respect to the operations of *finite and countable summation* and *computation of monotonic differences*, the *countable additivity* and *partial monotonicity* of the operator P_0, and the *partial monotonicity* of the operator functionals $m^{[r]}$. Assertions similar to those formulated in Theorems 2–4 can be proved under similar general assumptions for any space of functions L acting from X to $[0, \infty]$ and any P_0, $m^{[r]}$ and minimal series $M_r = m_r + P_0 m_r + \cdots$ associated with the functions m_r by the recurrence relations $m_r = m_{[r]}(M_1, \ldots, M_{r-1})$.

7. Uniform recurrence. Many examples illustrating Theorem 4 can be generated if one takes into consideration the fact that the class of nonnegative partly monotonic functionals is closed with respect to linear combination with nonnegative coefficients, multiplication, maximum, minimum and so forth.

Let us consider one typical example of a nonnegative partly monotonic functional

$$f(v_1,\dots,v_k) = \sup_{\pi\in\Pi} E_\pi v_k(\eta_0) = \int_X v_k(x)\pi(dx),$$

where π is a σ-finite measure on B_X and Π is some class of such measures. [We retain the notation $E_\pi\xi = \int_X E_x\xi\pi(dx)$ for expectations of random functionals averaged by a measure π, even if π is not a probability measure.] Applying Theorem 3, we obtain in this case necessary and sufficient conditions for the relation

$$M_{k\Pi} = \sup_{\pi\in\Pi} E_\pi\tau^k < \infty$$

to hold, as well as the corresponding upper bounds for $M_{k\Pi}$. One can interpret the last relation as the definition of some kind of uniform recurrence of order k for the semi-Markov process $\eta(t)$ with respect to a generalized hitting time τ and a class of initial measures Π.

Note that in the case of ordinary hitting times the typical choice of the class of initial measures Π is the class of all distributions concentrated at points $x \in D$. Another important case is when the family Π includes only one measure which is a stationary measure for the Markov chain η_n or its truncation to the same set D.

To simplify the notation, let us define

$$V_r(x) = E_x\lambda^r + \sum_{l=1}^{r} C_r^l E_x\lambda^{r-l}\chi_1 v_l(\eta_1), \qquad x\in X,\ r = 1,\dots,k,$$

where the rule of calculation is as for (15).

THEOREM 5. *Let τ be a Markovian generalized hitting time. Then the existence of test functions $v_1,\dots,v_k$ from L that satisfy the test inequalities*

(h) $$v_r(x) \ge V_r(x), \qquad x\in X,\ r = 1,\dots,k,$$

and

(i) $$\sup_{\pi\in\Pi} E_\pi V_k(\eta_0) = V_{k\Pi} < \infty$$

is a necessary and sufficient condition for the relation

(j) $$M_{k\Pi} < \infty$$

to hold. In this case

(k) $$M_{k\Pi} \le V_{k\Pi}.$$

The model considered in Theorem 4 can easily be extended to the case $k = \infty$, when $f(v_1, v_2, \dots)$ is a measurable nonnegative partly monotonic functional acting from the space $L_\infty = L\times L\times\cdots$ to $[0, +\infty]$. One can keep the formulation of Theorem 4 without any changes.

Taking this into consideration, one can also formulate necessary and sufficient conditions for exponential recurrence of semi-Markov processes by considering the nonnegative partly monotonic functional

$$f(v_1, v_2, \dots) = \sup_{\pi \in \Pi} \Big(1 + \sum_{k \geq 1} (a^k/k!) E_\pi v_k(\eta_0)\Big),$$

where $a > 0$. We refer to the paper by Silvestrov (1994).

8. *D*-invariant generalized hitting times. A very useful simplification in the conditions of Theorems 3 and 4 can be achieved in the important case in which the hitting time ν satisfies the following condition of D-invariance for some domain $D \in B_X$:

$$I(\nu > 1, \eta_1 \in D) =_{P_x 1} 0, \qquad x \in X. \tag{23}$$

For example, condition (23) is satisfied in the case of ordinary hitting times when

$$\nu =_{P_x 1} \min(n \geq 1 : \eta_n \in D), \qquad x \in X.$$

It is also satisfied in the case of place-dependent hitting times when

$$\nu =_{P_x 1} \min\big(n \geq 1 : \eta_n \in D_{\eta_{n-1}}\big), \qquad x \in X,$$

if all domains $D_x \supseteq D$, $x \in X$.

Under condition (23) the operator P_0 and its iterates P_0^n, $n = 1, 2, \dots,$ the operators P_r, $r = 1, 2, \dots,$ and their iterates and also the operator functionals $m^{[r]}$, $r = 1, 2, \dots,$ are D-invariant. [The functions $P_r^n v$ and $m_{[r]}(v_1, \dots, v_{r-1})$ depend on the values of the functions v and $v_1, \dots, v_{r-1}$ on $\overline{D}$ only.]

Using this property, we can weaken the test inequality (c) in Theorems 3 and 4, demanding that it be satisfied only for $x \in \overline{D}$.

We can still obtain, as in the proof of Theorem 3, the upper bounds $M_r \leq v_r$, $r = 1, \dots, k$, but for $x \in \overline{D}$ only. Then, using (b), we can obtain the recurrent upper bounds $M_r \leq m_{[r]} + P_0 v_r$ for all $x \in X$, $r = 1, \dots, k$.

Therefore, if we simplify the initial test inequality as described above, we lose only the possibility of bounding moments M_r directly by test functions v_r on D in Theorem 3 and we lose nothing in Theorem 4.

9. First-order moments. Taking into consideration Remark 2, one can note that, in the case of first-order moments, Theorems 2–4 can be proved for generalized hitting times with weaker Markovian properties. Specifically, the generating functional $\mu = g(\eta_0, \eta_1, \alpha_1)$ has to possess the Markovian property but $\lambda = h(\eta_0, \eta_1, \alpha_1, \dots)$ may be an arbitrary nonnegative random functional.

The calculations given in (19) can be omitted since $m_1 = m^{(1)}$. However, (a) can still be obtained by a calculation as for (20).

In the case of higher order moments, the Markovian property of both functionals μ and λ is essential to carry out the calculations given in (19) and (20) and to obtain (a).

10. Time-dependent generalized hitting times. The results obtained in Theorems 1–5 may also be generalized to a nonhomogeneous Markov renewal process $\beta_n = (\eta_n, \alpha_n)$, differing from a homogeneous Markov renewal process only by transition probabilities depending on a time parameter n:

$$P\{\eta_{n+1} \in A,\ \alpha_{n+1} \le u \mid \eta_n = x,\ \alpha_n = v\} = P_n(x, A, u).$$

We may use the well-known method of transforming a Markov renewal process β_n into a homogeneous Markov chain by adding a supplementary time-parameter component $n_0 + n$, where n_0 is some initial shift of time. If we define $\hat{\beta}_n = (\hat{\eta}_n, \hat{\alpha}_n)$, where $\hat{\eta}_n = (n_0 + n, \eta_{n_0+n})$, $\hat{\alpha}_n = \alpha_{n_0+n}$, $n = 0, 1, \dots$, then $\hat{\beta}_n$ is a homogeneous Markov renewal process with phase space $\hat{X} = N \times X$, where $N = \{0, 1, \dots\}$.

In the definition of generalized hitting times given in (2) and (3), generating functionals can now depend on time. For example, in the case of Markovian generalized hitting times, shifts of generating functionals in (2) and (3) take the form $\Theta_{n-1}\mu = g(n_0 + n - 1, \eta_{n_0+n-1}, \eta_{n_0+n}, \alpha_{n_0+n})$ and $\Theta_n \lambda = h(n_0 + n, \eta_{n_0+n}, \alpha_{n_0+n+1}, \eta_{n_0+n+1})$.

Test functions $v(n, x)$ acting from the space $\hat{X}$ to $[0, +\infty]$ have to be used in this case and the operators P_r defined in (15) must be interpreted as operators acting from $\hat{X}$ to $\hat{X}$ according to

$$\begin{aligned} P_r v(n, x) = \int_X \int_{[0, +\infty]} & P_n(x, dy, dt) h(n, x, y, t)^r I(g(n, x, y, t) < 1) \\ & \times v(n+1, y), \qquad x \in X,\ n = 0, 1, \dots, \end{aligned}$$

where the rule of calculation is as for (15).

We do not reformulate Theorems 1–5 for the nonhomogeneous case. Instead, we consider a homogeneous Markov renewal process β_n with *time-dependent generalized hitting times* defined by generating functionals depending on time.

We need to consider different initial points $(n, x) \in \hat{X}$. Due to the homogeneity of β_n, one way do this is by operating with a family of time-dependent generalized hitting times corresponding to different initial shifts of time

$$\nu_m =_{P_x 1} \min(n \ge 1 \colon g(m + n - 1, \eta_{n-1}, \eta_n, \alpha_n) \ge 1), \qquad x \in X,\ m \ge 0,$$

and

$$\tau_m =_{P_x 1} \sum_{n=0}^{\nu_m - 1} h(m + n, \eta_n, \eta_{n+1}, \alpha_{n+1}), \qquad x \in X,\ m = 0, 1, \dots .$$

Let $\pi = \{\pi_m,\ m = 0, 1, \dots\}$ be a sequence of probability measures on B_X, $\bar{p} = \{p_m,\ m = 0, 1, \dots\}$ be a discrete distribution, $\overline{\Pi}$ be some family of pairs $(\bar{p}, \bar{\pi})$ and

$$M_{k\Pi} = \sup_{(\bar{p}, \bar{\pi}) \in \overline{\Pi}} \sum_{m=0}^{\infty} p_m E_{\pi_m} \tau_m^k .$$

We formulate only the analog of Theorem 5. To simplify the notation, let us define, for $x \in X$, $n = 0, 1, \dots, r \geq 1$,

$$\begin{aligned} V_r(n, x) &= E_x h(n, \eta_0, \eta_1, \alpha_1)^r \\ &\quad + \sum_{l=1}^{r} C_r^l E_x h(n, \eta_0, \eta_1, \alpha_1)^{r-l} \\ &\quad \times I(g(n, \eta_0, \eta_1, \alpha_1) < 1) v_l(n+1, \eta_1), \end{aligned}$$

where the rule of calculation is as for (15).

THEOREM 6. *Let τ_m, $m = 0, 1, \dots,$ be a family of time-dependent Markovian generalized hitting times. Then there exist test functions $v_1(n, x), \dots, v_k(n, x)$ that satisfy the test inequalities*

(l) $$v_r(n, x) \geq V_r(n, x), \qquad x \in X,\ n = 0, 1, \dots,\ r = 1, \dots, k,$$

and

(m) $$\sup_{(\bar{p}, \bar{\pi}) \in \overline{\Pi}} \sum_{m \geq 0} p_m E_{\pi_m} V_k(m, \eta_0) = V_{k\overline{\Pi}} < \infty$$

is a necessary and sufficient condition for the relation

(n) $$M_{k\overline{\Pi}} < \infty$$

to hold. In this case

(o) $$M_{k\overline{\Pi}} \leq V_{k\overline{\Pi}}.$$

As an example, let us consider the model with $\nu_m = \nu$ for all $m = 0, 1, \dots,$ where

$$\nu =_{P_x 1} \min(n \geq 1: \eta_n \in D), \qquad x \in X,$$

with $h(n, x, y, t) = r_n f(x)$ nonnegative, so that

$$\tau_m =_{P_x 1} \sum_{n \leq \nu - 1} r_{m+n} f(\eta_n), \qquad x \in X,\ m = 0, 1, \dots .$$

Furthermore, let all distributions $\bar{p}$ be concentrated at 0, and let Π be some class of probability measures on B_X. In this case Theorem 6 gives necessary and sufficient conditions for $M_{k, \Pi}(D, r., f(\cdot)) = \sup_{\pi \in \Pi} E_\pi \tau_0^k$ to be finite and provides the corresponding upper bounds. Note that, according to the remarks made in Section 8, test inequality (l) can be assumed for $x \in D$, $n = 0, 1, \dots$ only. If $k = 1$ we get conditions of so-called (r, f)-regularity from the recent paper by Tuominen and Tweedie (1994) where these conditions were used to obtain rates of convergence in ergodic theorems for Markov chains.

11. Random potentials. The results obtained above can also be applied to a model with generating functionals $g(\cdot) < 1$ so that possibly $\nu =_{P_x 1} +\infty$, $x \in X$.

Again, we consider only the time-dependent generalized hitting times for a homogeneous Markov renewal process β_n and comment only on the analog of Theorem 6. The shifted time-dependent generalized hitting times defined in the previous section take the form

$$\tau_m =_{P_x 1} \sum_{n=0}^{\infty} h(m+n, \eta_n, \eta_{n+1}, \alpha_{n+1}), \qquad x \in X,\ m = 0, 1, \dots .$$

The formulation of Theorem 6, including the formulas defining functions $V_r(n, x)$, carries over without change.

Note that we call the functional τ a random potential because the expectation $E_x \tau$ in the case $h(n, x, y, t) = r_n f(x)$ is really the (r, f)-potential of the Markov chain η_n.

12. Fractional and mixed moments. Upper bounds for fractional-type moment functionals for ordinary hitting times based on test functions have been investigated in the papers by Nummelin and Tuominen (1983) and Tweedie (1983) for Markov chains, as well as in the recent paper by Silvestrov (1993) for semi-Markov processes. Similar results can be developed for generalized hitting times on the basis of a stochastic representation analogous to that in Theorem 1.

Let $G_{k+1} = \{g\}$ be a class of functions which, by definition, act from $[0, \infty)$ to $[0, \infty)$ and satisfy the following conditions:

(p) the derivative $g^{(k)}(t)$ exists and is a continuous, monotonic and convex function for all t

and

(q) $$g^{(r)}(0) \geq 0, \qquad r = 0, \dots, k,$$

where $g^{(0)}(t) \equiv g(t)$. Additionally, we define the values at $+\infty$ by $g^{(r)}(+\infty) = \lim_{t \to \infty} g^{(r)}(t)$, $r = 0, \dots, k$.

The polynomials of order $k+1$ with nonnegative coefficients, as well as $t^{k+\delta}$ and $t^{k+\delta} \ln t$ for $0 < \delta \leq 1$, belong to G_{k+1}. These facts explain the term *power-type function* for functions in this class.

Let the generalized hitting time τ be generated by nonnegative random functionals μ and λ. One can obtain the following stochastic inequality generalizing the stochastic representation given in Theorem 1:

(24) $$g(\tau) - g(0) \leq_{P_x 1} \tau_{k+1, g}, \qquad x \in X,$$

where $\tau_{k+1, g}$ is a generalized hitting time generated by the same functional μ used to define the random index ν in (3), and the functional $\lambda = \lambda_{k+1, g}$ is defined by

(25) $$\lambda_{k+1, g} =_{P_x 1} g(\lambda) - g(0) + \sum_{r=1}^{k} (r!)^{-1} \big(g^{(r)}(\lambda) - g^{(r)}(0)\big) \chi_1(\Theta_1 \tau)^r,$$

where one counts products $(g^{(r)}(\lambda) - g^{(r)}(0)) \chi_1(\Theta_1 \tau)^l$ as 0 if at least one of the factors takes the value 0.

If the function g is a polynomial of order $k + 1$ with nonnegative coefficients, then the symbol $\leq$ can be replaced by the symbol $=$ in (24).

The proof is similar to that of Theorem 1. However, instead of taking equality (5) to the rth power, one now calculates g and uses the inequality (equality for polynomials of order $k + 1$)

$$g(a+b) \leq \sum_{0 \leq r \leq k} (r!)^{-1}\big(g^{(r)}(a) - g^{(r)}(0)\big)b^r + g(b), \qquad a, b \geq 0,$$

which can be obtained by the use of the Taylor expansion for $g(a + t) - g(t)$ and properties of functions from the class G_{k+1}. Further details can be found in the paper by Silvestrov (1993).

Using the stochastic inequality (24), one can easily obtain upper bounds analogous to those given in Theorems 3–6 for the more general power-type moments $E_\pi g(\tau)$.

To conclude the general discussion, we would like to mention that the results of the previous sections can also be generalized to the more general product moments $E_\pi \tau_1 \times \cdots \times \tau_k$. It can be shown that, for any generalized hitting times τ_r, $r = 1, \dots, k$, the product $\tau_1 \times \cdots \times \tau_k$ is also a generalized hitting time. One can also obtain recurrent stochastic representations similar to those in Theorem 1, as well as recurrent relations and upper bounds for the moments similar to those given in Theorems 2–6.

13. Semi-Markov dynamical systems of linear type. We use this term for a semi-Markov process $\eta(t)$, $t \geq 0$, with phase space $X = [0, \infty)$, initial state $\eta_0 = \text{const}$ and the corresponding embedded Markov renewal process $\beta_n = (\eta_n, \alpha_n)$, defined in the following recurrent dynamical form:

$$\begin{aligned} \eta_{n+1} &= \big[\eta_n + a(\eta_n) + b(\eta_n)\xi_n\big]^+, \\ \alpha_{n+1} &= c(\eta_n) + d(\eta_n)\gamma_n, \qquad n = 0, 1 \dots, \end{aligned} \tag{26}$$

where:

1. $\rho_n = (\xi_n, \gamma_n)$, $n = 0, 1, \dots,$ is a sequence of i.i.d. random vectors taking values in $R_1 \times [0, \infty)$;
2. $a(x), b(x)$ and $c(x), d(x)$ are measurable functions mapping X to R_1 and X to $[0, \infty)$, respectively.

We are interested in the ordinary hitting times $\tau(0) = \tau_{\nu(0)}$, where $\nu(0) = \min(n \geq 1\colon \eta_n = 0)$, and we are going to obtain upper bounds for the moments $E_x \tau(0)^r$ using the technique of recurrent test functions.

The model (26) may be considered as a semi-Markov analog of a nonhomogeneous random walk as well as a semi-Markov nonlinear autoregressive time series. We refer to the paper by Borovkov (1991) and the book by Tong (1990), which contains a survey of results concerning hitting times for nonhomogeneous random walks and nonlinear autoregressive time series. These results are mainly related to expectations of hitting times and the corresponding conditions of ergodicity. The object of our interest is, however, not expectations but power-type moments of high order for hitting times. Our results were partially presented in the papers by Silvestrov (1993, 1996) where one can also find more detailed references.

First, we consider the basic case with $a(x)$, $b(x)$, $c(x)$ and $d(x)$ bounded for $x > 0$, and then we discuss some generalizations to the case of unbounded functional coefficients.

The natural choice of test functions in this model are polynomials with nonnegative coefficients. Two types of upper bounds can be developed.

The first corresponds to the model with the function $E[a(x) + b(x)\xi_1]$ negative and bounded away from 0 for large x. In this case we choose test functions $v_r(x) = v_{0r} + v_{1r}x^r$ with nonnegative constants v_{0r}, v_{1r}, $r = 1, \dots, k$.

Let us write $A_z = \{x > 0: x + a(x) + zb(x) \le 0\}$, $\tilde{A}_z = \{x > 0: x + a(x) + zb(x) > 0\}$. The following are obvious:

1. $A_z \supseteq A$, where $A = \{x > 0: b(x) = 0,\ x + a(x) \le 0\}$ for any real z;
2. for nonnegative $b(x)$ one has $A_{z'} \supseteq A_{z''}$ if $z' \le z''$;
3. for bounded $a(x)$ and $b(x)$, the sets A_z are bounded, that is, $A_z \subseteq (0, h_z]$, where $h_z < \infty$ for any real z.

Let us also write $e = \inf(x: P\{\xi_1 \le x\} > 0) \ge -\infty$ and

$$f_r(x) = \frac{x^r - E[x + a(x) + b(x)\xi_1]^r}{1 + x^{r-1}}, \qquad x > 0, r = 1, \dots, k.$$

THEOREM 7. *Let $b(x) \ge 0$, let $a(x)$, $b(x)$, $c(x)$ and $d(x)$ be bounded for $x > 0$ and let the following conditions hold for some natural $k \ge 1$*:

$$\text{(r)} \qquad E\left[\gamma_1^k + |\xi_1|^k\right] < \infty$$

and

$$\text{(s)} \qquad \lim_{z > e,\, z \to e} \inf_{x \in \tilde{A}_z} f_r(x) > 0, \qquad r = 1, \dots, k.$$

Then there exist constants $v_{0r}, v_{1r} > 0$, $r = 1, \dots, k$, such that the following inequality holds:

$$\text{(t)} \qquad E_x\tau(0)^r \le v_{0r} + v_{1r}x^r, \qquad x > 0, r = 1, \dots, k.$$

PROOF. We are going to use Theorem 3. The test inequality (c) takes the following form if one places all terms related to the function $v_r(x)$ on the left-hand side:

$$\text{(27)} \qquad v_{0r}P_{00}(x) + v_{1r}\left(x^r - E[x + a(x) + b(x)\xi_1]^r + P_{0r}(x)\right) \ge m_r(x), \\ x > 0, r = 1, \dots, k,$$

where

$$\begin{aligned}m_r(x) &= m_r(v_{01}, v_{11}, \dots, v_{0r-1}, v_{1r-1}; x)\\ &= E[c(x) + d(x)\gamma_1]^r + \sum_{l=1}^{r-1} C_r^l v_{0l}\Big(E[c(x) + d(x)\gamma_1]^{r-l} - P_{r-l0}(x)\Big)\\ &\quad + \sum_{l=1}^{r-1} C_r^l v_{1l}\Big(E\big\{[c(x) + d(x)\gamma_1]^{r-l}\\ &\qquad\qquad \times [x + a(x) + b(x)\xi_1]^l\big\} - P_{r-ll}(x)\Big)\end{aligned}$$

and

$$\begin{aligned}P_{rl}(x) = E\big\{&[c(x) + d(x)\gamma_1]^r\\ &\times [x + a(x) + b(x)\xi_1]^l I(x + a(x) + b(x)\xi_1 \le 0)\big\}.\end{aligned}$$

Note that, by definition, $P_{00}(x) = P\{x + a(x) + b(x)\xi_1 \le 0\}$ is the probability of hitting the sequence η_n into 0 from the point x after the first step.

Let us show that under condition (r), for any $1 \le l \le r \le k$,

$$\lim_{z>e,\, z\to e} \sup_{x\in\tilde{A}_z} |P_{r-ll}(x)| = 0. \tag{28}$$

We have to consider two cases: $e = -\infty$ and $e > -\infty$.

If $x \in \tilde{A}_z \cap \{x\colon b(x) = 0\} = \{x > 0\colon x + a(x) > 0,\ b(x) = 0\}$, then for both cases

$$P_{r-ll}(x) = E[c(x) + d(x)\gamma_1]^{r-l}[x + a(x)]^l I(x + a(x) \le 0) = 0. \tag{29}$$

Let us write

$$b = \sup_{x>0} b(x), \qquad q = \sup_{x>0}(c(x) + d(x)), \qquad u(x) = -b(x)^{-1}(x + a(x)).$$

If $e = -\infty$ and $x \in \tilde{A}_z \cap \{x\colon b(x) > 0\} = \{x > 0\colon z > u(x),\ b(x) > 0\}$, then

$$\begin{aligned}&|P_{r-ll}(x)|\\ &\quad = \big|b(x)^l E[c(x) + d(x)\gamma_1]^{r-l}[\xi_1 - u(x)]^l I(\xi_1 \le u(x))\big|\\ &\quad \le b^l q^{r-l} \sum_{m=0}^{l} C_l^m |u(x)|^{l-m} E[1 + \gamma_1]^{r-l} |\xi_1|^m I(\xi_1 \le u(x))\\ &\quad \le b^l q^{r-l} \sum_{m=0}^{l} C_l^m \sup_{y<z} |y|^{l-m} E[1 + \gamma_1]^{r-l} |\xi_1|^m I(\xi_1 \le y)\\ &\quad = Q_{r-ll}(z).\end{aligned} \tag{30}$$

Using condition (r), we get from (29) and (30), in the case $e = -\infty$,

$$\sup_{x\in\tilde{A}_z} |P_{r-ll}(x)| \le Q_{r-ll}(z) \to 0 \qquad \text{as } z \to -\infty. \tag{31}$$

Let $z > e > -\infty$ and $x \in \tilde{A}_z \cap \{x\colon b(x) > 0\}$. For x such that $u(x) < e$ the definition of e gives

$$
\begin{aligned}
&P_{r-ll}(x) \\
(32)\qquad &= E[c(x) + d(x)\gamma_1]^{r-l}[x + a(x) + b(x)\xi_1]^l I(\xi_1 \le u(x)) \\
&= 0.
\end{aligned}
$$

For x such that $u(x) \ge e$ we have

$$|x + a(x) + vb(x)| = |v - u(x)|b(x) \le |z - e|b$$

for $e \le v \le u(x)$. Therefore,

$$
\begin{aligned}
&|P_{r-ll}(x)| \\
&= \left|b(x)^l E[c(x) + d(x)\gamma_1]^{r-l}[\xi_1 - u(x)]^l I(e \le \xi_1 \le u(x))\right| \\
(33)\qquad &\le |z - e|^l b^l q^{r-l} E[1 + \gamma_1]^{r-l} I(e \le \xi < z) \\
&= Q'_{r-ll}(z).
\end{aligned}
$$

Using condition (r), we get from (29), (32) and (33), in the case $e > -\infty$,

$$(34)\qquad \sup_{x \in \tilde{A}_z} |P_{r-ll}(x)| \le Q'_{r-ll}(z) \to 0 \qquad \text{as } z > e,\ z \to e.$$

Both relations (31) and (34) are stronger than (28).

From the boundedness of the sets A_z, the boundedness on finite intervals of the functions $a(x), \ldots, d(x)$, condition (r) and relation (28), it follows that the functions $P_{r-ll}(x)$, $0 \le l \le r \le k$, are bounded for $x > 0$. From this fact and the boundedness of $a(x), \ldots, d(x)$ for $x > 0$, we get the following relations:

$$(35)\qquad \sup_{x > 0} \frac{m_r(x)}{1 + x^{r-1}} < \infty, \qquad r = 1, \ldots, k,$$

and

$$(36)\qquad \sup_{x > 0} \frac{x^r - E[x + a(x) + b(x)\xi_1]^r + P_{0r}(x)}{1 + x^{r-1}} < \infty, \qquad r = 1, \ldots, k.$$

Using condition (s) and relation (28), we can also find $\hat{z}_k > e$ such that, for any $z \in (e, \hat{z}_k]$,

$$(37)\qquad \inf_{x \in \tilde{A}_z} \frac{x^r - E[x + a(x) + b(x)\xi_1]^r + P_{0r}(x)}{1 + x^{r-1}} > 0, \qquad r = 1, \ldots, k.$$

It also follows from the definitions of A_z and e that, for any $z > e$,

$$(38)\qquad \inf_{x \in A_z} P_{00}(x) \ge P\{\xi_1 \le z\} > 0.$$

Using relations (35)–(38), we can define, for any $z_r \in (e, \hat{z}_k]$, $r = 1, \dots, k$, nonnegative finite constants $v_{0r}(z_r), v_{1r}(z_r)$, $r = 1, \dots, k$, and v_{0r}, v_{1r}, $r = 1, \dots, k$, by the following recurrence formulas:

$$
\begin{aligned}
v_{1r} \ge v_{1r}(z_r) &= v_{1r}(v_{01}, v_{11}, \dots, v_{0r-1}, v_{1r-1}; z_r) \\
&= \sup_{x \in \tilde{A}_{z_r}} \frac{m_r(v_{01}, v_{11}, \dots, v_{0r-1}, v_{1r-1}; x)}{x^r - E[x + a(x) + b(x)\xi_1]^r + P_{0r}(x)}, \\
v_{0r} \ge v_{0r}(z_r) &= v_{0r}(v_{01}, v_{11}, \dots, v_{0r-1}, v_{1r-1}, v_{1r}; z_r) \\
&= 0 \vee \sup_{x \in A_{z_r}} \left\{ \frac{m_r(v_{01}, v_{11}, \dots, v_{0r-1}, v_{1r-1}; x)}{P_{00}(x)} \right. \\
&\qquad \left. - \frac{v_{1r}\big(x^r - E[a(x) + b(x)\xi_1]^r + P_{0r}(x)\big)}{P_{00}(x)} \right\}.
\end{aligned} \tag{39}
$$

For any constants v_{0r}, v_{1r}, $r = 1, \dots, k$, taken from recurrence relations (39), the test inequality (27) holds. Taking sequentially $r = 1, \dots, k$ and using (39), we have, for $x \in \tilde{A}_{z_r}$,

$$
\begin{aligned}
&v_{0r} P_{00}(x) + v_{1r}\big(x^r - E_x[x + a(x) + b(x)\xi_1]^r + P_{0r}(x)\big) \\
&\quad \ge v_{1r}(z_r)\big(x^r - E_x[x + a(x) + b(x)\xi_1]^r + P_{0r}(x)\big) \\
&\quad \ge m_r(x),
\end{aligned}
$$

and, for $x \in A_{z_r}$,

$$
\begin{aligned}
v_{0r} P(x) &\ge v_{0r}(z_r) P_{00}(x) \\
&\ge m^{[r]}(x) - v_{1r}\big(x^r - E_x[x + a(x) + b(x)\xi_1]^r + P_{0r}(x)\big).
\end{aligned}
$$

The proof of the theorem is complete. □

The second type of upper bound corresponds to the model with the function $E[a(x) + b(x)\xi_1]$ vanishing, but the function $xE[a(x) + b(x)\xi_1]$ negative and bounded away from 0 for large x. In this case we choose test functions $v_r'(x) = v_{0r}' + v_{1r}' x^{2r}$, $r = 1, \dots, k$, with nonnegative constants $v_{0r}', v_{1r}' \ge 0$, $r = 1, \dots, k$. The proof of the following theorem is similar to that of Theorem 7. Let

$$
f_r'(x) = \frac{x^{2r} - E[x + a(x) + b(x)\xi_1]^{2r}}{1 + x^{2r-2}}, \qquad x > 0,\ r = 1, \dots, k.
$$

THEOREM 8. *Let $b(x) \ge 0$, let $a(x)$, $b(x)$, $c(x)$ and $d(x)$ be bounded for $x > 0$ and let the following conditions hold for some natural $k \ge 1$:*

$$
E\big[\gamma_1^k + |\xi_1|^{2k}\big] < \infty \tag{u}
$$

and

(v) $$\lim_{z>e,\, z\to e} \inf_{x\in\tilde{A}_z} f_r'(x) > 0, \qquad r = 1,\dots,k.$$

Then there exist constants $v_{0r}', v_{1r}' > 0$, $r = 1,\dots,k$, *such that the following inequality holds*:

(w) $$E_x\tau(0)^r \le v_{0r} + v_{1r}x^{2r}, \qquad x > 0,\, r = 1,\dots,k.$$

REMARK 3. By definition, one has $\tilde{A}_z \supseteq [h_z, \infty)$, where $h_z < \infty$ for all real z. Therefore, conditions (r) and (s) entail the following relation:

(x) $$\limsup_{x\to\infty} E[a(x) + b(x)\xi_1] < 0,$$

which means that the function $E[a(x) + b(x)\xi_1]$ has to be negative and bounded away from 0 for large x.

At the same time, if the function $b(x)$ is bounded away from 0 on finite intervals, that is, $\inf_{x\in(0,h]} b(x) > 0$, $h > 0$, then for any $h > 0$ the sets $\tilde{A}_z \subseteq [h, \infty)$ for $z \le z(h)$ small enough. It follows from this fact that, in this case, relation (x) and condition (s) are equivalent if $e = -\infty$ and (r) holds.

By similar reasoning we get that conditions (u) and (v) entail the following relation:

(y) $$\limsup_{x\to\infty}\left(2xE[a(x) + b(x)\xi_1] + (2k-1)E[a(x) + b(x)\xi_1]^2\right) < 0,$$

which means that the function $E[a(x) + b(x)\xi_1]$ can vanish in this case but the function $2xE[a(x) + b(x)\xi_1] + (2k-1)E[a(x) + b(x)\xi_1]^2$ has to be negative and bounded away from 0 for large x. Therefore, if the function $b(x)$ is bounded away from 0 on finite intervals, $e = -\infty$ and condition (u) holds, then relation (y) and condition (v) are equivalent.

REMARK 4. Relations (37) and (39) give explicit formulas for the constants in (t). By definition, $v_{1r}(z_r)$ and $v_{0r}(z_r)$ are nonincreasing and nondecreasing functions of z_r, respectively. If $\bar{v}_{1r} = \lim_{z_r>e,\ z_r\to e} v_{1r}(z_r)$ and $\bar{v}_{0r} = \lim_{z_r>e,\ z_r\to e} v_{0r}(z_r) \ge v_{0r}(z_r)$, then any constants $v_{1r} > \bar{v}_{1r}$ and $v_{0r} \ge \bar{v}_{0r}$ satisfy (39) for all $z_r \in (e, \hat{z}_k]$ close enough to e. Therefore, such constants can be used in inequality (t). Such choices minimize the values of the constants v_{1r}, $r = 1,\dots,k$.

To illustrate this assertion, let us consider the case in which $e = -\infty$ and $a(x)$, $b(x)$, $c(x)$ and $d(x)$ have limits a, b, c and d, respectively, as $x\to\infty$. Then a simple calculation in (39) yields formulas for the constants

$$\bar{v}_{1r} = v_r = \left[\frac{a + bE\xi_1}{c + dE\gamma_1}\right]^r, \qquad r = 1,\dots,k.$$

Together with the upper bounds (t), this provides the asymptotic relations

$$\limsup_{x\to\infty} x^{-r}E_x\tau(0)^r \le v_r,\ r = 1,\dots,k.$$

This is the right asymptotic relation for moments. As is known from renewal theory, in the space-homogeneous case in which coefficients $a(x)$, $b(x)$, $c(x)$ and $d(x)$ are constants, one has $x^{-r}E_x\tau(0)^r \to v_r$ as $x \to \infty$, $r = 1,\dots,k$.

The results of Theorems 7 and 8 can be generalized to a model with a control random sequence ρ_n which switches distribution depending on the current positions of the dynamical system. We do not discuss these generalizations here and refer to the papers by Silvestrov (1993, 1996).

Instead, we would like to make some remarks concerning a model with unbounded functional coefficients. Now we suppose only that functions $a(x)$, $b(x)$, $c(x)$ and $d(x)$ are locally bounded, that is, are bounded on any finite interval $(0, h]$ for $h > 0$.

It is obvious that there always exists a positive, locally bounded function $e(x)$ bounded away from 0, such that $e(x)^{-1}a(x)$, $e(x)^{-1}b(x)$, $e(x)^{-1}c(x)$ and $e(x)^{-1}d(x)$ are all bounded for $x > 0$. For example, the function $e(x) = 1 + |a(x)| + |b(x)| + c(x) + d(x)$ possesses this property.

Let us define $e_r(x) = e(x)^r + e(x)x^{r-1}$ for $r \ge 1$. These functions are also positive, bounded away from 0 and locally bounded.

Using the definition of the functions $m^{[r]}(x)$, $x^r - E[x + a(x) + b(x)\xi_1]^r + P_{0r}(x)$ and $P_{lr}(x)$, it is easy to check that, in this case, the functions $e_r(x)^{-1}m^{[r]}(x)$ and $e_r(x)^{-1}(x^r - E[x + a(x) + b(x)\xi_1]^r + P_{0r}(x))$, $r = 1,\dots,k$, are bounded for $x > 0$, and that, under condition (r), the relation $\sup_{x \in \tilde{A}_z} e(x)^{-r}|P_{r-l\,l}(x)| \to 0$ as $z > e$, $z \to e$ holds for $1 \le l \le r \le k$ in place of relation (28). Due to these facts we can repeat the proof of Theorem 7 by replacing the functions $1 + x^{r-1}$ with the functions $e_r(x)$ in the corresponding formulas. We can retain formulas (39) for the constants v_{0r}, v_{1r}, $r = 1,\dots,k$, without change, because of the quotient structure of the expressions in (39). Of course, to do this, we also have to replace the functions $1 + x^{r-1}$ by the functions $e_r(x)$ in condition (s) which takes the form:

$$\text{(z)}\qquad \lim_{z>e,\, z\to e}\ \inf_{x\in\tilde{A}_z} \frac{x^r - E\left[x + a(x) + b(x)\xi_1\right]^r}{e_r(x)} > 0, \qquad r = 1,\dots,k.$$

Therefore, Theorem 7 is valid under the assumption of local boundedness of the functional coefficients $a(x)$, $b(x)$, $c(x)$ and $d(x)$, if condition (s) can be replaced by condition (z).

Theorem 8 can be also reformulated in a similar way. In this theorem we have to replace the functions $1 + x^{2r-2}$ by the functions $e'_r(x) = e(x)^r + e(x)x^{2r-2}$ in condition (v).

The basic model with bounded functional coefficients corresponds to the choice $e(x) \equiv 1$. Another important model, with functional coefficients possessing not more than a linear rate of growth, corresponds to the choice $e(x) \equiv 1 + |x|$. This model may be considered as a nonlinear semi-Markov analog of the classical autoregressive model having coefficients $a(x) \equiv (a - 1)x$ with $a < 1$, $b(x), c(x) \equiv 1$ and $d(x) \equiv 0$.

In conclusion, we remark that the model with $b(x) = 1$ and $d(x) = 0$ was analyzed in the recent paper by Tuominen and Tweedie (1994) where conditions of regularity of the Markov chain η_n were obtained (see Section 10) with the use of the time-dependent test functions $v(n, x) = (n + x)^k$.

14. Semi-Markov random walks. Let us consider a semi-Markov process with phase space $X = \{0, \pm 1, \dots\}$ and the transition probabilities of the corresponding Markov renewal process $\beta_n = (\eta_n, \alpha_n)$ given by

$$
\begin{aligned}
&P\{\eta_{n+1} = j,\ \alpha_{n+1} \le u \mid \eta_n = i,\ \alpha_n = v\} \\
&\qquad = \begin{cases} p_+(i)P_+(i,u), & \text{if } j = i+1,\ i \in X, \\ p_-(i)P_-(i,u), & \text{if } j = i-1,\ i \in X, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
\tag{40}
$$

The process $\eta(t)$ can be considered as a semi-Markov inhomogeneous random walk with unit jumps ± 1 on the integer line.

This model includes, as particular cases, discrete-time inhomogeneous Bernoullian random walks. Birth–death processes can also be embedded in this model by the use of standard symmetrical arguments. We shall also show how the model can be applied to hitting times for diffusion processes.

We are interested in ordinary hitting times $\tau(0) = \tau_{\nu(0)}$, where $\nu(0) = \min(n \ge 1\colon \eta_n = 0)$, and in obtaining upper bounds for the moments $E_i\tau(0)^r$. We are again going to use the method of recurrent test functions but in combination with some embedding semi-Markov procedures. Our results are analogous to those obtained for birth–death processes in the recent paper by Nilsson (1996).

Let $J = \{\cdots < j_{-1} < 0 = j_0 < j_1 < \cdots\}$ be a subgrid of points from X, and suppose $\eta(0) \in J$ [we can always extend J by including $\eta(0)$ in J]. Also let $0 = \tau_{J0} < \tau_{J1} < \cdots$ be the sequence of times when the semi-Markov process $\eta(t)$ hits the set J. We can define the embedded process $\eta_J(t) = m$ if $\eta(\tau_{J_n}) = j_m$ for $\tau_{J_n} \le t < \tau_{J_{n+1}}$, $n = 0, 1, \dots$. The process $\eta_J(t)$ is also a semi-Markov process with phase space X, and, moreover, it is also an inhomogeneous semi-Markov random walk with unit jumps ± 1 on the integer line as the initial process $\eta(t)$. We denote the transition probabilities of the corresponding embedded Markov renewal process $(\eta_{J_n}, \alpha_{J_n})$ as in (40), but with index J, that is, $p_{J\pm}(i)P_{J\pm}(i, u)$.

Let us denote by $\tau_J(0)$ the first hitting time into state 0 for the semi-Markov process $\eta_J(t)$. By the definition of the embedded process, one has $\tau_J(0) = \tau(0)$ if $\eta(0) \ne 0$ and so

$$E_i\tau(0)^k = E_m\tau_J(0)^k \quad \text{if } i = j_m \ne 0. \tag{A}$$

To estimate hitting time moments for the initial states $j_{m-1} < i < j_m$, one can use the inequality

$$E_i\tau(0)^k \le E_{j_{m-1}}\tau(0)^k \vee E_{j_m}\tau(0)^k \tag{B}$$

and then use (A).

We give the following result without proof.

LEMMA 1. *For the process $\eta(t)$ let the following conditions hold*:

$$0 < a \le p_{\pm}(i) \le b < 1, \qquad i \in X, \tag{C}$$

and

$$E_i \alpha_1^k \le c_k < \infty, \qquad i \in X, \tag{D_k}$$

for some $k \ge 1$. *Then*

$$j_{i+1} - j_i \le d, \qquad i \in X, \tag{E}$$

implies that conditions (C) *and* (D_k) *are also satisfied for the embedded process* $\eta_J(t)$ *with some new constants* $0 < a_J$, $b_J < 1$ *and* $0 \le c_{Jk} < \infty$.

As in Section 13, the results can be developed along two lines. The first relates to the model with jump expectation $E_i[\eta_{J1} - i] = p_{J+}(i) - p_{J-}(i)$ negative and bounded away from 0 for large positive i, and positive and bounded away from 0 for large negative i. The second relates to the model for which $i[p_{J+}(i) - p_{J-}(i)]$ is negative and bounded far enough from 0 for large positive i, and positive and bounded far enough from 0 for large negative i.

THEOREM 9. *Let the following conditions hold*: (C) *and* (D_k) *for some* $k \ge 1$ *and also* (E) *and*

$$\limsup_{i \to \pm\infty} \left[\pm p_{J+}(i) \mp p_{J-}(i) \right] < 0 \tag{F}$$

for some subgrid J. Then there exist positive constants v_{0r}, v_{1r}, $r = 1, \dots, k$, *such that the following inequality holds*:

$$E_i \tau(0)^r \le v_{0r} + v_{1r} |i|^r, \qquad i \in X,\ r = 1, \dots, k. \tag{G}$$

THEOREM 10. *Let the following conditions hold*: (C) *and* (D_{2k}) *for some* $k \ge 1$ *and also* (E) *and*

$$\limsup_{i \to \pm\infty} |i| \left[\pm p_{J+}(i) \mp p_{J-}(i) \right] < (2k-1)/2 \tag{H}$$

for some subgrid J. Then there exist positive constants v'_{0r}, v'_{1r}, $r = 1, \dots, k$, *such that the following inequality holds*:

$$E_i \tau(0)^r \le v'_{0r} + v'_{1r} |i|^{2r}, \qquad i \in X,\ r = 1, \dots, k. \tag{I}$$

PROOF. We give the proof of Theorem 10, omitting some details. First, we consider only the case in which the initial state belongs to the positive half-line.

Second, note that, due to (A) and (B), if inequality (I) holds for the embedded process $\eta_J(t)$ with some constants v'_{0r}, v'_{1r}, then these inequalities also hold for the initial process $\eta(t)$ with the same constants since $m \le i$ if $j_{m-1} < i \le j_m$. This remark lets us operate with the embedded process $\eta_J(t)$ instead of the initial process $\eta(t)$.

Third, due to Lemma 1, it is clear how one can get inequality (I) for all $i > 0$ if similar polynomial inequalities can be obtained for the hitting times of the process $\eta_J(t)$ into a point N and initial states $i > N$. In reality, the hitting time $\tau(0)$ can be bounded above by the first hitting time to point N plus a sum of the geometrically distributed random number of return times to point N before the first hitting to 0. The power moments of this additional sum can be bounded above by the use of upper bounds for moments of the first exit times from the interval $[0, N]$ following from Lemma 1 and polynomial upper bounds for moments of hitting times from $N + 1$ to N.

We are going to use Theorem 3 with test functions $v_r(i) = v_r i^{2r}$ $[v_r(i) = v_r i^r$ in Theorem 9] and show that the corresponding test inequality (c) from Theorem 3 holds for $i > N$ and all N large enough. This will complete the proof. Here the test inequality (c) takes the form

$$v_r i^{2r} \geq m_{[r]}(i) + v_r\left[(i+1)^{2r} p_{J+}(i) + (i-1)^{2r} p_{J-}(i)\right], \quad i > N,\ r = 1, \ldots, k. \tag{41}$$

Note that we omitted the indicators χ_1 on the right-hand side of (41) and so made the corresponding test inequality even stronger.

We do not rewrite formula (18) but note only that in this case $m_{[r]}(i)$ is a polynomial of order $2r - 2$ with nonnegative bounded coefficients and highest-order coefficient $rv_{r-1}E_i\alpha_{J1}$ (here $v_0 = 1$).

Subtracting $v_r[i^{2r} + 2ri^{2r-1}p_{J+}(i) - 2ri^{r-1}p_{J-}(i) + r(2r-1)i^{2r-2}]$ from expressions on both sides of (41), dividing both expressions by $v_r 2ri^{2r-2}$ and calculating the upper limits on the right-hand side, one can find from (41) that these limits are less than or equal to $c_{J1}v_{r-1}/2v_r$. These expressions can be made less than any positive constant by a suitable recurrent choice of constants v_r for $r = 1, \ldots, k$. At the same time, due to condition (H), the limits of the expressions on the right-hand side are positive. Therefore, inequality (41) holds for all i large enough. □

REMARK 5. In the case when the subgrid J includes all points $i \geq i_0$, conditions (F) and (H) take the form of simple shift conditions

$$\limsup_{i \to \pm\infty} \left[\pm p_+(i) \mp p_-(i)\right] < 0 \tag{F'}$$

and

$$\limsup_{i \to \pm\infty} |i| \left[\pm p_+(i) \mp p_-(i)\right] < (2k-1)/2, \tag{H'}$$

respectively.

In the general case one can use the well-known formula

$$p_{J+}(i) = \sum_{m=j_{i-1}+1}^{j_i} \prod_{l=m}^{j_{i+1}-1} \frac{p_+(l)}{p_-(l)} \cdot \left[1 + \sum_{m=j_{i-1}+1}^{j_{i+1}-1} \prod_{l=m}^{j_{i+1}-1} \frac{p_+(l)}{p_-(l)}\right]^{-1}.$$

To recognize (F) as a space-averaged shift condition, let us consider the case when the jump probabilities $p_+(i)$ are periodic functions for $|i|$ large enough (different for positive and negative half-lines). For the positive half-line this means $p_+(i) = p_+(i + u)$ for all $i \geq v$ and some $u, v \geq 1$. In this case it is natural to choose the subgrid J in such a way that the points $v, v + u, v + 2u, \ldots$ belong to J. Using the formula for $p_{J+}(i)$ given above, one can easily transform (F) into the form $p_+(v) \cdots p_+(v + u - 1) - p_-(v) \cdots p_-(v + u - 1) < 0$. This condition was found for birth–death processes by Nilsson (1996). Under this condition the shift of a random walk can be positive instead of negative at some points in the interval $[v, v + u - 1]$. At the same time, condition (F′) yields the much stronger condition $p_+(w) - p_-(w) < 0$, $v \leq w \leq v + u - 1$.

REMARK 6. In some cases the requirements of uniform boundedness in conditions (C) and (D_k) can be weakened or removed. For example, in the case of first-order moments, one can only demand that conditions (C) and (D_1) are satisfied for each finite interval and, instead of (F), the following condition holds:

$$(\mathrm{F}'') \qquad \limsup_{i \to \pm\infty} \left[\pm p_{J+}(i) \mp p_{J-}(i) \right] / E_i \alpha_{J1} < 0.$$

15. Hitting times for diffusion processes. The semi-Markov embedding approach described in the previous section can also be applied to the model of diffusion processes and their hitting times.

Let $\eta(t)$, $t \geq 0$, be a homogeneous continuous diffusion process on the real line R_1 with coefficients $a(x)$, $b(x)$. For simplicity, we assume

$$(\mathrm{J}) \qquad a(x) \text{ and } b(x) \text{ are continuous bounded functions}$$

and

$$(\mathrm{K}) \qquad b(x) \geq b > 0, \qquad x \in R_1.$$

Again, we are interested in the hitting time $\tau(0) = \inf(t > 0: \eta(t) = 0)$ and the moments $E_x \tau(0)^k$ for $x \in R_1$. Related results can also be found in the book by Hasminskii (1969).

We fix some subgrid of real numbers $J = \{\cdots < j_{-1} < 0 = j_0 < j_1 < \cdots\}$ and then define an embedded semi-Markov process $\eta_J(t)$ in the same way as was done above for the semi-Markov random walk. That is, we define an embedded process $\eta_J(t) = m$ if $\eta(\tau_{Jn}) = j_m$ for $\tau_{Jn} \leq t < \tau_{J_{n+1}}$, $n = 0, 1, \ldots,$ where $0 = \tau_{J0} < \tau_{J1} < \cdots$ are the times when the semi-Markov process $\eta(t)$ hits the set J. The initial point $\eta(0)$ can simply be included in J and the equality

$$(\mathrm{L}) \qquad E_x \tau(0)^k = E_m \tau_J(0)^k \quad \text{if } x = j_m \neq 0$$

can be used to evaluate moments of hitting times for diffusion processes.

Under conditions (J), (K) and the additional condition

$$(\mathrm{M}) \qquad j_{m+1} - j_m \leq d, \qquad m = 0, \pm 1, \ldots,$$

conditions (C) and (D_k) for all $k \geq 1$ are satisfied for the semi-Markov process $\eta_J(t)$. Therefore, we can apply Theorems 9 and 10 to this process and obtain the following theorems.

THEOREM 11. *Let conditions* (J), (K) *and* (M) *hold and also condition* (F) [*for the embedded process* $\eta_J(t)$]. *Then there exist constants* v_{0r}, v_{1r}, $r \geq 1$, *such that the following inequality holds*:

$$E_x\tau(0)^r \leq v_{0r} + v_{1r}|x|^r, \qquad x \in R_1,\ r \geq 1. \tag{N}$$

THEOREM 12. *Let conditions* (J), (K) *and* (M) *hold and also condition* (H) [*for the embedded processes* $\eta_J(t)$]. *Then there exist constants* v'_{0r}, v'_{1r}, $r = 1,\dots,k$, *such that the following inequality holds*:

$$E_x\tau(0)^r \leq v'_{0r} + v'_{1r}|x|^{2r}, \qquad x \in R_1,\ r = 1,\dots,k. \tag{O}$$

The following explicit formula for transition probabilities $p_{J+}(i)$ in terms of $a(x), b(x)$ is well known:

$$p_{J+}(i) = \int_{j_i}^{j_{i+1}}\int_{j_{i-1}}^{y}\left(-2a(x)/b(x)^2\right)dx\,dy\left[\int_{j_{i-1}}^{j_{i+1}}\int_{j_{i-1}}^{y} -2a(x)/b(x)^2\,dx\,dy\right]^{-1}.$$

Conditions (F) and (H) are again types of shift conditions averaged in space. For example, in the case when the functions $a(x), b(x)$ are periodic for large $|x|$ (different on positive and negative half-lines), condition (F) and the formula for $p_{J+}(i)$ given above yield (for the positive half-line) the shift condition $\int_v^{v+u} a(x)/b(x)^2\,dx < 0$, where $[v, v+u]$ is an interval of periodicity.

16. Queuing systems of M/G type with service times depending on the queue. Let $M/G_i/1$ be a queuing system which differs from the classical $M/G/1$ system only in that the service distributions $G_i(u)$ depend on the length of the queue at the times at which services are begun. We denote the input Poisson flow parameter by λ.

Let $\zeta(t)$ be a queue at time t. The process $\zeta(t)$, $t \geq 0$, has phase space $X = \{0, 1, \dots\}$. We are interested in the first hitting times $\tau(0) = \inf(t > 0: \zeta(t) = 0)$ and in upper bounds for the moments $E_i\tau(0)^r$.

Also let $\eta(t)$ be a queue at the last before t end-of-service time. The process $\eta(t)$, $t \geq 0$, is a semi-Markov process with phase space X. By definition, $\tau(0) = \inf(t > 0: \eta(t) = 0)$ if the initial state $\zeta(t) \neq 0$. Therefore, $\tau(0)$ is an ordinary hitting time for the semi-Markov process $\eta(t)$.

We use the condition

$$m_{ik} = \int_{[0,\infty)} u^k G_i(du) \leq M_k < \infty, \qquad i \in X. \tag{P$_k$}$$

THEOREM 13. *Let conditions* (P$_k$) *and*

$$\limsup_{i\to\infty}\,[\lambda m_{i1} - 1] < 0 \tag{Q}$$

hold. Then there exist constants v_{0r}, v_{1r}, $r = 1, \dots, k$, *such that the following inequality holds*:

$$E_i \tau(0)^r \leq v_{0r} + v_{1r} i^r, \qquad i \in X,\ r = 1, \dots, k. \tag{R}$$

THEOREM 14. *Let conditions* (P_{2k}) *and*

$$\limsup_{i \to \infty} \left[2i(\lambda m_{i1} - 1) + (2k-1)\lambda^2 m_{i2} \right] < 0 \tag{S}$$

hold. Then there exist constants v'_{0r}, v'_{1r}, $r = 1, \dots, k$, *such that the following inequality holds*:

$$E_i \tau(0)^r \leq v'_{0r} + v'_{1r} i^{2r}, \qquad i \in X,\ r = 1, \dots, k. \tag{T}$$

PROOF. We give the proof of Theorem 14 and omit some details. Theorem 5 can be applied with the use of polynomial test functions. One can get inequality (T) for all $i > 0$ if similar polynomial inequalities can be obtained for the times when the process $\eta(t)$ hits point N and initial states $i > N$. In reality, the hitting time $\tau(0)$ can be bounded above by the first hitting time of point N plus a sum of the geometrically distributed random number of return times to point N before first hitting 0. The power moments of this additional sum can be bounded from above by the use of the condition that the corresponding kth-order polynomial upper bounds are valid for $i > N$, and also by the existence of the kth-order moment for the random position of the process $\eta(t)$ at the moment when it first hits to the domain $[N, \infty)$ for any initial state in the interval $[0, N]$. The existence of such a moment follows from condition (P_{2k}).

We are going to use test functions $v_r(i) = v_r i^{2r}$ and show that the test inequality (h) from Theorem 5 is satisfied for all $i \geq N$ for all N large enough. Here the inequality (h) takes the form

$$\begin{aligned} v_r i^{2r} &\geq V_r(i) \\ &= m_{ir} + \sum_{l=1}^{r} C_r^l \int_0^\infty t^{r-l} v_l (i + j - 1)^{2l} \frac{e^{-\lambda t}(\lambda t)^j}{j!} G_i(dt), \\ &\qquad i > N,\ r = 1, \dots, k. \end{aligned} \tag{42}$$

Note that we omitted the indicators χ_1 on the right-hand side of (42) and so made the corresponding test inequality even stronger.

By condition (P_{2k}) and the fact that the moments of order r of a Poisson distribution with parameter λt are polynomials of order r in λt, a moment calculation in (42) yields the following representation:

$$\begin{aligned} V_r(i) &= v_{r-1} r i^{2r-2} m_{i1} \\ &\quad + v_r \big[i^{2r} + 2r i^{2r-1}(\lambda m_{i1} - 1) \\ &\qquad + r(2r-1) i^{2r-2}(\lambda^2 m_{i2} - \lambda m_{i1} + 1) \big] + O_{2r-3}(i), \end{aligned} \tag{43}$$

where $O_{2r-3}(i)$ is some polynomial of order $2r - 3$ with bounded coefficients.

We can substitute this representation in (43), subtract from both sides the term $v_r[i^{2r} + 2ri^{2r-1}(\lambda m_{i1} - 1) + r(2r-1)i^{2r-2}(\lambda^2 m_{i2} - \lambda m_{i1} + 1)]$, divide (43) by $v_r r i^{2r-2}$ and calculate the upper limits for expressions on the left-hand side and the lower limits for expressions on the right-hand side. The limits on the left-hand side are less than or equal to $M_1 v_{r-1}/v_r$ (here $v_0 = 1$), which can be made less than any positive number by a suitable recurrent choice of constants v_r for $r = 1, \dots, k$. The limits on the right-hand side are positive due to condition (S). This proves that test inequality (42) holds for all i large enough. □

17. Rates of convergence in ergodic theorems. Explicit upper bounds for high-order moments of hitting times can be effectively used to obtain explicit rates of convergence in ergodic theorems.

It is well known that wide classes of Markov-type processes can be embedded in the model of regenerative processes in the following way: (α) the distribution of a process at time t coincides with the distribution at time t of some regenerative process, and (β) the distribution of the regeneration time for this regenerative process coincides with the distribution of some ordinary hitting time for some semi-Markov process. It is also known that (γ) rates of convergence for regenerative processes can be given in terms of high-order moments of regenerative times. Finally, as was demonstrated in this paper, (δ) effective upper bounds for moments of hitting times of semi-Markov processes can be obtained by the use of recurrent test functions. Using (α)–(δ) together, one can obtain effective explicit rates of convergence for different classes of Markov-type processes.

Let us illustrate this approach for the semi-Markov dynamical systems considered in Section 13. To do this, we need to formulate some results concerning rates of convergence in the ergodic theorems for regenerative process obtained in the papers of Silvestrov (1983a, 1984). We present them as in the survey by Silvestrov (1994b).

Let $\zeta(t)$, $t \geq 0$, be a regenerative process with phase space X, B_X and regenerative times $0 = \kappa_0 < \kappa_1 < \cdots$.

Let G and F be the distributions of the first (transition) period τ_1 and the standard regenerative period $\kappa_2 - \kappa_1$, respectively, $m_k = \int_{[0,\infty)} u^k F(du)$ and $H(t) = m_1^{-1} \int_{[0,t)} (1 - F(u))\, du$. Let $P_t(A) = P\{\zeta(t) \in A\}$ and $P(A)$ be the stationary distribution of the process $\zeta(t)$. This exists under conditions assumed below.

If

(U) the distribution F possesses a nonzero absolutely continuous component

and

$$(\mathrm{V}_k) \qquad m_k < \infty \quad \text{for some } k \geq 1,$$

then the following inequality holds:

$$d(P_t, P) = \sup_{A \in B_X} |P_t(A) - P(A)|$$

$$\leq 1 - G(t+0) + 1 - H(t+0) \tag{44}$$

$$+t^{-k} L_k \left[1 + \int_0^t u^k G(du) + \int_0^t u^k H(du) \right], \qquad t > 0,$$

where the constant $1 \leq L_k < \infty$ depends on the distribution F only. Explicit formulas are given in Silvestrov (1983a, 1984) for the calculation of the constant L_k as a continuous function of the moments m_1, m_k and other parameters of F, including parameters characterizing concentration properties of the absolutely continuous component of F.

If, in addition to (U) and (V_k),

$$e_{k-1} = E\kappa_1^{k-1} < \infty \tag{W_{k-1}}$$

holds, then it is easily shown that the expression on the right-hand side of (44) is $o(t^{-k+1})$.

The inequality (44) involves information on the distributions G and F. If one agrees to decrease the rate of convergence to $O(t^{-k+1})$, then (44) can be transformed into a form involving only the moments of G and F:

$$d(P_t, P) \leq L_k [m_k / k m_1 + e_{k-1}] t^{-k+1} + L_k t^{-k}, \qquad t > 0. \tag{45}$$

Let us apply this inequality to the semi-Markov process $\eta(t)$, $t \geq 0$, considered in Section 13. This process is a regenerative process with regenerative times which are return times at state 0. Let π be an initial distribution of the process $\eta(t)$, $t \geq 0$. Then $G(u) = P_\pi\{\tau(0) < u\}$ and $F(u) = P_0\{\tau(0) < u\}$.

Assume, for example, that the conditions of Theorem 7 are satisfied. Then

$$m_k = E_0 \tau(0)^k \leq M_k$$

$$= E[c(0) + d(0)\gamma_1]^k + v_{0k} + v_{1k} E[0 \vee (a(0) + b(0)\xi_1)]^k < \infty.$$

Let $\Pi = \{\pi\}$ be some class of initial distributions such that

$$H_{\Pi k-1} = \sup_{\pi \in \Pi} \int_{[0,\infty)} x^{k-1} \pi(dx) < \infty. \tag{X_{k-1}}$$

Then

$$e_{\Pi k-1} = \sup_{\pi \in \Pi} E_\pi \tau(0)^{k-1} \leq E_{\Pi k-1} = M_{k-1} + v_{0k-1} + v_{1k-1} H_{\Pi k-1} < \infty.$$

Assume $d(0) \neq 0$ and also

(Y) the distribution of the random variable γ_1 has a nonzero absolutely continuous component.

Obviously, in this case, the distribution $F(u)$ possesses such a component.

Let $P_{\pi t}(A) = P_\pi\{\eta(t) \in A\}$ and let $P(A)$ be the corresponding stationary distribution of the process $\eta(t)$. Applying estimate (45), we can give the explicit upper bound for the rate of convergence in the ergodic theorem for the semi-Markov process $\eta(t)$ uniformly with respect to the family of initial distributions Π.

THEOREM 15. *Suppose that* (X_{k-1}), (Y) *and the conditions of Theorem* 7 *are satisfied. Then the following inequality holds*:

$$\text{(Z)} \qquad \sup_{\pi \in \Pi} d(P_{\pi t}, P) \le L_k [M_k / k m_1 + E_{\Pi k-1}] t^{-k+1} + L_k t^{-k}, \qquad t > 0.$$

Similar results concerning rates of convergence in ergodic theorems can be developed for semi-Markov random walks, diffusion processes and $M/G_i/1$ queuing systems, since for these models the corresponding initial processes and embedded semi-Markov processes are also regenerative processes with return times at 0 as regeneration times.

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INSTITUTE OF MATHEMATICAL STATISTICS
UMEÅ UNIVERSITY
S-90187 UMEÅ
SWEDEN