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ITÔ FORMULA FOR AN ASYMPTOTICALLY 4-STABLE PROCESS[1]

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We study an asymptotically 4-stable process. The main result is an Itô type formula.

1. Introduction. The purpose of this article is to give a rigorous version of the Itô formula for a stochastic process introduced in mathematical physics by Mądrecki and Rybaczuk (1989, 1993). Our model approaches the equation $\partial_t u = -\partial_x^4 u$ from a probabilistic point of view by means of a process which is 4-stable in an asymptotic sense. The mathematical foundations of the model have been laid in our previous article [Burdzy and Mądrecki (1995)]. We plan to develop further elements of stochastic calculus for the asymptotically 4-stable process in a future paper.

The "squared Laplacian" appears in many equations of mathematical physics. The research presented in Mądrecki and Rybaczuk (1993) was motivated by several concrete examples. Here is a review of two of those examples.

The first example is concerned with the Hamiltonian in the "momentum representation"

$$H = \frac{p^2}{2m} + V\left(-i\hbar\frac{\partial}{\partial p}\right),$$

where p represents momentum and m stands for mass. If we consider the anharmonic oscillator described by

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2 + \lambda x^4 = \frac{p^2}{2m} + P(x),$$

with $P(x) = \lambda(x^2 - \eta^2) - \lambda\eta^4$, $\lambda < 0$, $\eta^2 = m\omega^2/(4\lambda)$, we obtain an equation with potential discussed in Vainstein, Zakharov, Novikov and Shifman (1982).

Another example is provided by a relativistic particle with mass m moving at high velocity. The relativistic kinetic energy E is given by

$$E = \sqrt{p^2c^2 + m^2c^4} - mc^2 \approx \frac{p^2}{2m} - \frac{p^4}{8m^3c^3} + \cdots,$$

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where p is momentum, m is mass and c is the speed of light. If the particle is moving in the potential field $V(x)$, then, retaining the first two terms, we get

$$H = \frac{1}{8m^3c^3}\frac{\partial^4}{\partial x^4} + \frac{1}{2m}\frac{\partial^2}{\partial x^2} + V(x).$$

This leads to a Schrödinger equation of the form

$$c_4\frac{\partial^4 u(z,t)}{\partial z^4} + c_2\frac{\partial^2 u(z,t)}{\partial z^2} + V(z)u(z,t) = \frac{\partial u(z,t)}{\partial t}.$$

See Mądrecki and Rybaczuk (1993) for more details and bibliography.

This article is part of a larger project which aims at developing a usable stochastic calculus which may be successfully applied to equations involving the squared Laplacian in the same way the classical stochastic calculus is applied to equations involving the classical Laplacian. Next we will review other attempts at "fourth order" stochastic calculus and point out their limitations.

There were at least three other attempts at constructing a "4-stable process" (we use the quotation marks since it is a classical fact that a genuine 4-stable process does not exist). The oldest construction, due to Krylov (1960), used a signed finitely additive measure with infinite variation on $C[0,1]$ as an analog of the Wiener measure. The model was further developed by Hochberg (1978), Hochberg and Orsingher (1994) and later by Nishioka (1985, 1987, 1994) but the finite additivity of the underlying measure put strong limitations on the model.

Funaki (1979) used a composition of two Brownian motions to represent some solutions of $\partial_t u = \partial_x^4 u$. The idea was developed in a slightly different direction by Burdzy (1993, 1994) who considered "iterated Brownian motion" (IBM) but so far there is no stochastic calculus or potential theory for IBM.

Another model, quite similar to ours in many respects, has been recently proposed by Sainty (1992). However, his Definition 3.1 can only be interpreted as that of an n-stable process with homogeneous, independent increments, which does not exist for $n > 2$.

One can also approach the problem of "squared Laplacian" probabilistically without introducing a 4-stable process. Helms (1967) used only standard Brownian motion in his paper but it seems that his ideas have not been developed any further.

Our model has been inspired by a sequential approach to distributions [see, e.g., Antosiewicz, Mikusiński and Sikorski (1973)].

The next section contains a review of the basic definitions and results from Burdzy and Mądrecki (1995). The stochastic integral with respect to an asymptotically 4-stable process is defined in Section 3. An Itô formula is proved in Section 4.

2. An asymptotically 4-stable process: a review. All definitions and results in this section are taken from Burdzy and Mądrecki (1995). The reader is asked to consult that paper for the proofs. The sets of all natural, real and complex numbers will be denoted $\mathbf{N}$, $\mathbf{R}$ and $\mathbf{C}$, respectively. It will be convenient to identify the complex plane $\mathbf{C}$ with $\mathbf{R}^2$ and occasionally switch from complex notation to vector notation.

Let $(\Omega, \mathscr{F}, P)$ be a probability space. We will concentrate on $\mathbf{C}^{\mathbf{N}}$-valued random variables $Z\colon (\Omega, \mathscr{F}, P) \to (\mathbf{C}^{\mathbf{N}}, \mathscr{B}^{\mathbf{N}})$, where $\mathscr{B}^{\mathbf{N}}$ is the Borel σ-algebra of $\mathbf{C}^{\mathbf{N}}$. Clearly, Z is a $\mathbf{C}^{\mathbf{N}}$-valued random variable iff Z is a sequence $\{Z_n\}$ of complex random variables.

If $Z = \{Z_n\}$ is a $\mathbf{C}^{\mathbf{N}}$-valued random variable and f is a function defined on $\mathbf{C}$, then $f(Z)$ will stand for the sequence $\{f(Z_n)\}$.

DEFINITION 2.1. If $Z = \{Z_n\}$ is a $\mathbf{C}^{\mathbf{N}}$-valued random variable on $(\Omega, \mathscr{F}, P)$ and $\lim_{n\to\infty} EZ_n$ exists, then the limit will be denoted $\mathscr{E}Z$ and called the first (asymptotic) moment of Z or (asymptotic) expected value of Z. For example, the kth (asymptotic) moment of Z is given by $\mathscr{E}Z^k = \lim_{n\to\infty} EZ_n^k$ (if the limit exists).

DEFINITION 2.2. Suppose that a $\mathbf{C}^{\mathbf{N}}$-valued random variable $Z = \{Z_n\}$ satisfies the following two conditions:

(i) Write $Z_n = U_n + iV_n = (U_n, V_n)$ and let μ_n be the distribution of V_n. Then μ_n has a two-sided Laplace transform. That is,

$$\int_{\mathbf{R}} e^{-tv} \mu_n(dv) < \infty$$

for every $t \in \mathbf{R}$ and $n \in \mathbf{N}$.

(ii) The complex sequence $\{E \exp(itZ_n)\}$ is convergent.

Then we define the (asymptotic) characteristic function $\psi_Z(t) = \psi(Z, t)$ of Z by the formula

$$\psi_Z(t) = \psi(Z, t) = \mathscr{E}e^{itZ} = \lim_{n\to\infty} E\exp(itZ_n), \qquad t \in \mathbf{R}.$$

The set of Z satisfying (i) and (ii) will be denoted $\mathscr{R}(\Omega, \mathbf{C}^{\mathbf{N}})$.

Suppose that X and X_m are random elements with values in $\mathbf{C}^{\mathbf{N}}$. We will say that X_m converges to X in $L^p_{\mathscr{E}}$ if $\lim_{m\to\infty} \mathscr{E}(X_m - X)^p = 0$.

The distribution of a $\mathbf{C}^{\mathbf{N}}$-valued random variable Z will be denoted $\mathscr{L}_Z$; that is $\mathscr{L}_Z(B) = P\{Z^{-1}(B)\}$ for all Borel sets $B \in \mathbf{C}^{\mathbf{N}}$.

PROPOSITION 2.1. (i) *If $\alpha_1, \alpha_2 \in \mathbf{R}$ and Z_1, Z_2 are two independent random elements from $\mathscr{R}(\Omega, \mathbf{C}^{\mathbf{N}})$, then $\alpha_1 Z_1 + \alpha_2 Z_2$ also belongs to $\mathscr{R}(\Omega, \mathbf{C}^{\mathbf{N}})$ and*

$$\psi(\alpha_1 Z_1 + \alpha_2 Z_2, t) = \psi(Z_1, \alpha_1 t)\psi(Z_2, \alpha_2 t)$$

for $t \in \mathbf{R}$. Moreover, $\psi_Z(0) = 1$ for any $Z \in \mathscr{R}(\Omega, \mathbf{C}^{\mathbf{N}})$. If $Z \in \mathscr{R}(\Omega, \mathbf{C}^{\mathbf{N}})$ is such that $Z\colon \Omega \to \mathbf{R}^{\mathbf{N}}$, then $\sup_{t\in\mathbf{R}} |\psi_Z(t)| = 1$.

(ii) *Let $X = \{X_n\}$ be a sequence of real random variables. If the sequence $\{X_n\}$ tends to Y in the sense of distribution, then $\psi_X(t) = Ee^{itY}$; that is, $\psi_X(t)$ is the characteristic function of Y in the classical sense.*

Proposition 2.1 shows that the (asymptotic) characteristic function ψ_Z has properties similar to those of the classical characteristic functions of real-valued random variables. However, one can show that ψ_Z does not determine $\mathscr{L}_Z$.

DEFINITION 2.3. We will say that a $\mathbf{C}^{\mathbf{N}}$-valued random variable Z has a p-stable (asymptotic) distribution with $0 < p \leq 4$ if there exist two complex numbers m and σ and a real number $\tau > 0$ such that

$$\psi_Z(t) = \exp\bigl(imt + \sigma|t|^{p/2} + \tau|t|^p\bigr), \qquad t \in \mathbf{R}. \tag{2.1}$$

The asymptotic distribution $\mathscr{L}_Z^a$ of such Z will be denoted $S_p(m, \sigma, \tau)$. We will also write $Z \sim S_p(m, \sigma, \tau)$.

The set $S_p(m, \sigma, \tau)$ is nonempty for each p with $0 < p \leq 4$ and each triplet $(m, \sigma, \tau) \in \mathbf{C}^2 \times \mathbf{R}_+$.

PROPOSITION 2.2. *Suppose that* $Z \sim S_p(m_1, \sigma_1, \tau_1)$, $Y \sim S_p(m_2, \sigma_2, \tau_2)$ *and* $\alpha_1, \alpha_2 \in \mathbf{R}$. *If* Z *and* Y *are independent, then*

$$(\alpha_1 Z + \alpha_2 Y) \sim S_p\bigl(\alpha_1 m_1 + \alpha_2 m_2, \alpha_1^{p/2}\sigma_1 + \alpha_2^{p/2}\sigma_2, \alpha_1^p\tau_1 + \alpha_2^p\tau_2\bigr).$$

Proposition 2.2 and (2.1) justify the name "stable distribution" for $S_p(m, \sigma, \tau)$ as they show that these distributions have properties similar to the classical stable distributions. It will be convenient to renormalize the parameters m, σ and τ. Namely, we will write $N_4(m, \sigma, \tau)$ instead of $S_4(m, \sqrt{3}\,\sigma/2, \tau/8)$. That is, $Z \sim N_4(m, \sigma, \tau)$ iff

$$\psi_Z(t) = \exp\bigl(imt + \sqrt{3}\,\sigma t^2/2 + \tau t^4/8\bigr), \qquad t \in \mathscr{R}.$$

With this normalization of parameters, m and σ may be interpreted as the mean and variance of $N_4(m, \sigma, \tau)$ [see Theorem 2.1(ii) below]. The third parameter, τ, is normalized to give the simplest form to the statement of the "central limit theorem" [see Theorem 2.1(iii) below]. The distribution $N_4(m, \sigma, \tau)$ may be looked upon as an analog of the normal distribution.

THEOREM 2.1. (i) *Assume that* Z *and* Y *are independent*, $Z \sim N_4(m_1, \sigma_1, \tau_1)$, $Y \sim N_4(m_2, \sigma_2, \tau_2)$ *and* $\alpha_1, \alpha_2 \in \mathbf{R}$. *Then*

$$(\alpha_1 Z + \alpha_2 Y) \sim N_4\bigl(\alpha_1 m_1 + \alpha_2 m_2, \alpha_1^2\sigma_1 + \alpha_2^2\sigma_2, \alpha_1^4\tau_1 + \alpha_2^4\tau_2\bigr).$$

(ii) *Suppose that* $Z \sim N_4(m, \sigma, \tau)$. *Then, for each* $j \in \mathbf{N}$, *there exists a* j-*asymptotic moment* $\mathscr{E}(Z^j)$ *and* $\mathscr{E}Z = m$, $\mathscr{E}(Z - m)^2 = \sigma$ *and* $\mathscr{E}((Z - m)^2 - \sigma)^2 = 3\tau$. *Moreover, if* $m = \sigma = 0$, *then the jth derivative of* $\psi_Z(t)$ *at* 0 *exists and*

$$\psi_Z^{(j)}(0) = i^j\mathscr{E}(Z^j).$$

(iii) (*Central limit theorem*) *Let* $\{Z_k^n\colon k, n \in \mathbf{N}\}$ *be a family of complex random variables and let* $Z^n = (Z_1^n, Z_2^n, \dots, Z_k^n, \dots)$. *Assume that, for all* $m, n \in \mathbf{N}$: (1) Z^m *and* Z^n *have the same asymptotic distribution, that is*,

$\mathscr{L}^a_{Z^m} = \mathscr{L}^a_{Z^n}$; (2) *for a fixed* n, *the random variables* $\{Z^k_n\}_{k \ge 1}$ *are jointly independent*; (3) *the first four asymptotic moments of* Z^n *exist*,

$$\mathscr{E}(Z^n) = \mathscr{E}(Z^n)^2 = \mathscr{E}(Z^n)^3 = 0$$

and, for some $\tau > 0$ *independent of* n,

$$\mathscr{E}(Z^n)^4 = \tau > 0;$$

(4) *the fourth derivatives* $\psi^{(4)}_{Z^n_k}$ *of the characteristic functions* $E\exp(itZ^n_k)$ *are uniformly convergent on some open neighborhood of* 0 *as* $k \to \infty$.

Then

$$\lim_{n\to\infty} \psi([Z^1 + Z^2 + \cdots + Z^n]/n^{1/4}, u) = \exp(\tau u^4/8), \qquad u \in \mathbf{R};$$

that is, the distributions of the normalized sums $S_n = (Z^1 + \cdots + Z^n)/n^{1/4}$ *tend in a very weak sense to the "normal" distribution* $N_4(0, 0, \tau)$.

In order to define an asymptotically 4-stable process, it will be convenient to work with a product of two probability spaces $(\Omega, \mathscr{F}, P) = (\Omega_1, \mathscr{F}_1, P_1) \times (\Omega_2, \mathscr{F}_2, P_2)$. First we define on $(\Omega_2, \mathscr{F}_2, P_2)$ a standard Brownian motion $\{b_t\colon t \ge 0\}$ and two families $\{b^+_t(n)\colon t \ge 0\}_{n \ge 1}$ and $\{b^-_t(n)\colon t \ge 0\}_{n \ge 1}$ of processes which satisfy the following conditions [the space $(\Omega_2, \mathscr{F}_2, P_2)$ has to be sufficiently rich]:

1. For each $n \in \mathbf{N}$, $b^+_0(n) = b^-_0(n) = 0$ and the processes $b^+(n)$ and $-b^-(n)$ have nondecreasing paths, that is, $0 \le b^+_s(n) \le b^+_t(n)$ and $b^-_t(n) \le b^-_s(n) \le 0$ for all $0 < s < t < \infty$.
2. For all $t \ge 0$,
$$\lim_{n\to\infty} (b^+_t(n) + b^-_t(n)) = b_t \quad P_2\text{-a.s.}$$
3. Let $b_t(n) = b^+_t(n) + b^-_t(n)$. For every pair $(n, t) \in \mathbf{N} \times \mathbf{R}_+$, the random variable $b_t(n)$ has a Gaussian density on $\mathbf{R}$.

Here is one way to construct such processes. We will limit ourselves to the interval $[0, 1]$. Recall that a Haar function is given by

$$H(\tau) = H(m, k, \tau) = \begin{cases} 2^{(m-1)/2}, & \text{if } (k-1)/2^m \le \tau < k/2^m, \\ -2^{(m-1)/2}, & \text{if } k/2^m \le \tau < (k+1)/2^m, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\{H_n\}$ be an ordering of all Haar functions for $m \ge 0$ and $k = 1, 3, 5, \dots, 2^m - 1$, such that $H(m_1, k_1, \cdot)$ comes earlier in the sequence $\{H_n\}$ than $H(m_2, k_2, \cdot)$ if $m_1 < m_2$. Let X_n be i.i.d. real standard normal random variables. Let

$$b_t(\omega) = \sum_{n=0}^{\infty} X_n(\omega) \int_0^t H_n(\tau)\, d\tau, \qquad 0 \le t \le 1.$$

Thus defined, b_t is a Brownian motion. Let $\operatorname{sgn}(x)$ denote the sign of x with the convention $\operatorname{sgn} 0 = 0$ and let $\chi_1(\cdot)$, $\chi_{-1}(\cdot)$ be the indicator functions of the sets $\{0, 1\}$ and $\{-1\}$, respectively. Then let

$$b_t^+(n) = \sum_{k=0}^{2^n} X_k \left[\chi_1(\operatorname{sgn} X_k) \int_0^t \max(H_k(\tau), 0)\, d\tau \right.$$
$$\left. + \chi_{-1}(\operatorname{sgn} X_k) \int_0^t \min(H_k(\tau), 0)\, d\tau \right]$$

and

$$b_t^-(n) = \sum_{k=0}^{2^n} X_k \left[\chi_1(\operatorname{sgn} X_k) \int_0^t \min(H_k(\tau), 0)\, d\tau \right.$$
$$\left. + \chi_{-1}(\operatorname{sgn} X_k) \int_0^t \max(H_k(\tau), 0)\, d\tau \right].$$

These processes satisfy conditions 1 to 3. A typical Brownian path has unbounded variation and, therefore, it cannot be represented as a sum of two monotone functions. The functions $b_t^+(n)$ and $b_t^-(n)$ provide an "approximate" decomposition of this type as they represent the increasing and decreasing parts of the truncated series in the Haar function representation of Brownian motion.

We will also need two independent Brownian motions $w_x^+ = (w_x^+(t)\colon t \geq 0)$ and $w_x^- = (w_x^-(t)\colon t \geq 0)$ starting from $x \in \mathbf{R}$ and defined on $(\Omega_1, \mathscr{F}_1, P_1)$. The processes w^+, w^- and b may be looked upon as three independent processes defined on the product space $(\Omega, \mathscr{F}, P)$. A generic element of Ω will be denoted $\omega = (\omega_1, \omega_2)$.

Definition 2.4. Suppose that $a = a_1 + ia_2 \in \mathbf{C}$. A $\mathbf{C}^{\mathbf{N}}$-valued process $\{Z_t^a\colon t \geq 0\}$ defined by the formula

$$Z_t^a(\omega, n) = Z_t^a(\omega_1, \omega_2, n) = w_{a_1}^+(\omega_1, b_t^+(\omega_2, n)) + i w_{a_2}^-(\omega_1, -b_t^-(\omega_2, n))$$

will be called 4-stable motion starting from a.

In the sequel we will write Z instead of Z^0 and we will call Z the standard 4-stable motion.

Since our probability space is a product space, the expected value functional $\mathscr{E}$ for $\mathbf{C}^{\mathbf{N}}$-valued random variables may be written as

$$\mathscr{E} Z = \lim_{n \to \infty} \left(E_2(E_1 Z(n)) \right),$$

where E_j is the expected value on the probability space Ω_j.

Theorem 2.2. *The* 4-*stable motion* $Z^a = \{Z_t^a\colon t \geq 0\}$ *has the following properties*:

(i) *For each* $(t, n) \in \mathbf{R}_+ \times \mathbf{N}$, *the function* $\omega \to Z_t^a(\omega, n)$ *is* $\mathscr{F}$-*measurable*; *that is*, *it is a random variable*.

(ii) $Z_0^a(n) = a$ *a.s.*

(iii) *For any $t > s \geq 0$, the $\mathbf{C}^{\mathbf{N}}$-valued random variable $(Z_t^a - Z_s^a)$ has the 4-stable asymptotic distribution $N_4(0, 0, t - s)$.*

(iv) *For each natural number $l \in \mathbf{N}$ and every $0 \leq s < t$, the expectation $\mathscr{E}(Z_t^a - Z_s^a)^l$ exists. If l is divisible by 4 and $l = 4p$, then*

$$\mathscr{E}(Z_t^a - Z_s^a)^l = (4p - 1)!!(2p - 1)!!(t - s)^p,$$

where $k!! = 1 \cdot 3 \cdot 5 \cdot \dots \cdot k$. If l is not divisible by 4, then $\mathscr{E}(Z_t^a - Z_s^a)^l = 0$. In particular, $\mathscr{E}(Z_t^a - Z_s^a)^j = 0$ if $j = 1, 2, 3$ and $\mathscr{E}(Z_t^a - Z_s^a)^4 = 3(t - s)$.

(v) *For each $p \in \mathbf{N}$, for every sequence $t_0 < t_1 < \cdots < t_p$, for arbitrary multi-indices $\alpha = (\alpha_1, \dots, \alpha_p) \in \mathbf{N}^p$ and for arbitrary complex numbers c_α, the asymptotic expectation*

$$\mathscr{E}\left(\sum_{\alpha = (\alpha_1, \dots, \alpha_p)} c_\alpha \prod_{i=1}^p \left(Z_{t_{i+1}}^a - Z_{t_i}^a \right)^{\alpha_i} \right)$$

exists, provided the sum extends over a finite set of multi-indices α.

(vi) *The increments of Z^a are asymptotically uncorrelated; that is, for every sequence $t_0 < t_1 < \cdots < t_p$ and each multi-index $\alpha = (\alpha_1, \dots, \alpha_p)$,*

$$\mathscr{E}\left(\prod_{i=1}^p \left(Z_{t_{i+1}}^a - Z_{t_i}^a \right)^{\alpha_i} \right) = \prod_{i=1}^p \mathscr{E}\left(Z_{t_{i+1}}^p - Z_{t_i}^a \right)^{\alpha_i}.$$

(vii) *For every $\beta_1, \beta_2 \in \mathbf{N}$ and all disjoint intervals (s, t) and (u, v), we have*

$$\mathscr{E}\left[(Z_t^a - Z_s^a)^{\beta_1} (b_v - b_u)^{\beta_2} \right] = \mathscr{E}(Z_t^a - Z_s^a)^{\beta_1} \mathscr{E}(b_v - b_u)^{\beta_2}.$$

(viii) *The paths $t \to Z_t^a(\omega_1, \omega_2) \in \mathbf{C}^{\mathbf{N}}$ are continuous, assuming that $\mathbf{C}^{\mathbf{N}}$ is endowed with the product topology.*

(ix) *For each fixed $(\omega_2, n) \in \Omega_0 \times \mathbf{N}$, the stochastic process $\{Z_t^a(\cdot, \omega_2, n): t \geq 0\}$ has independent increments.*

3. Stochastic integral with respect to *Z*. We will continue to work with the probability space $(\Omega, \mathscr{F}, P) = (\Omega_1, \mathscr{F}_1, P_1) \times (\Omega_2, \mathscr{F}_2, P_2)$ introduced in the previous section. Recall the Brownian motion b used in the construction of Z. From now on, we will use the letter B to denote it; that is, $B_t(\omega) = b_t(\omega_2)$ for $\omega = (\omega_1, \omega_2)$. For notational convenience we will consider only the process Z starting from 0, that is, $Z_0 = 0$.

We will identify the stochastic integral with a sequence of "approximating sums." A similar idea has been applied in Hochberg (1978). Consistency requires that we define in a similar way analogs of the Itô integral with respect to Brownian motion and the Riemann integral.

Definition 3.1. Suppose that $X = \{X_t(n)\}$ is a $\mathbf{C}^{\mathbf{N}}$-valued stochastic process. For an interval $[a, b]$ and an integer $m \geq 1$, let $t_j = a + j(b - a)/m$.

(i) An (asymptotic) stochastic integral of X with respect to Z is a $\mathbf{C}^{\mathbf{N}^2}$-valued random variable $\int_a^b X_s\, dZ_s$ defined by the formula

$$\left(\int_a^b X_s\, dZ_s\right)(m, n) = \sum_{j=0}^{m-1} X_{t_j}(n)\left(Z_{t_{j+1}}(n) - Z_{t_j}(n)\right).$$

(ii) An (asymptotic) Itô integral of X with respect to B is given by

$$\left(\int_a^b X_s\, dB_s\right)(m, n) = \sum_{j=0}^{m-1} X_{t_j}(n)\left(B_{t_{j+1}} - B_{t_j}\right).$$

(iii) We define an (asymptotic) Riemann integral $\int_a^b X_s\, ds$ by

$$\left(\int_a^b X_s\, ds\right)(m, n) = \frac{(b-a)}{m} \sum_{j=0}^{m-1} X_{t_j}(n).$$

We would like to define an "asymptotic expectation" operator $\mathbf{E}$ for the integrals defined above. Since they are doubly indexed families of random variables, we have to specify the order in which we pass to the limit. We will write

$$\mathbf{E}\int_a^b X_s\, dZ_s =_{\mathrm{df}} \lim_{m\to\infty}\left(\lim_{n\to\infty} E\left(\int_a^b X_s\, dZ_s(m, n)\right)\right),$$

and similarly for the other two integrals.

PROPOSITION 3.1. (i) *The mapping* $X \to \int_a^b X_s\, dZ_s$ *is* $\mathbf{C}$*-linear.*

(ii) *If* $X_u = X \cdot \chi_{[c,d)}(u)$, *where* $[c, d) \subset [a, b)$, *the random variable* X *is* $\mathbf{C}^{\mathbf{N}}$*-valued and* $\chi_{[c,d)}$ *is the characteristic function of* $[c, d)$, *then*

$$\left(\int_a^b X_s\, dZ_s\right)(m, n) = X(n)(Z_d(n) - Z_c(n)).$$

(iii) *The mapping* $X \to \int_a^b X_s\, dZ_s$ *has the following "isometry property" for any* X *which is a polynomial in* Z:

$$\mathbf{E}\left(\int_a^b X_s\, dZ_s\right)^4 = 3\int_a^b \mathscr{E}(X_s^4)\, ds.$$

(iv) *The asymptotic Itô integral has the classical isometry property for any* X *which is a polynomial in* Z; *that is,*

$$\mathbf{E}\left(\int_a^b X_s\, dB_s\right)^2 = \int_a^b \mathscr{E}(X_s^2)\, ds.$$

PROOF. We omit the easy proofs of (i) and (ii).

(iii) We have

$$\begin{aligned}\mathbf{E}\left(\int_a^b X_s\,dZ_s\right)^4 &= \lim_m\left(\lim_n E\left(\int_a^b X_s\,dZ_s\right)(m,n)\right)^4\\ &= \lim_m\left(\lim_n E\left(\sum_{j=0}^{m-1} X_{t_j}(n)\left(Z_{t_{j+1}}(n)-Z_{t_j}(n)\right)\right)\right)^4 \qquad (3.1)\\ &= \lim_m \mathscr{E}\left(\sum_{j=0}^{m-1} X_{t_j}(n)\left(Z_{t_{j+1}}(n)-Z_{t_j}(n)\right)\right)^4.\end{aligned}$$

Since X is a polynomial in Z, the expression

$$\left(\sum_{j=0}^{m-1} X_{t_j}(n)\left(Z_{t_{j+1}}(n)-Z_{t_j}(n)\right)\right)^4$$

may be represented as a finite sum of terms of the form

$$\prod_k\left(Z_{t_{j_k+1}}(n)-Z_{t_{j_k}}(n)\right)^{r_k},$$

where the product is taken over a finite set of k. If any power r_k is less than 4, then

$$\mathscr{E}\prod_k\left(Z_{t_{j_k+1}}(n)-Z_{t_{j_k}}(n)\right)^{r_k}=0$$

by Theorem 2.2(iv) and (vi). The only terms that may have nonzero expectation have the form

$$X_{t_j}^4(n)\left(Z_{t_{j+1}}(n)-Z_{t_j}(n)\right)^4$$

and so (3.1) is equal to

$$\lim_m\sum_{j=0}^{m-1}\mathscr{E}\left(\left(X_{t_j}\right)^4\left(Z_{t_{j+1}}(n)-Z_{t_j}(n)\right)^4\right).$$

Another application of Theorem 2.2(iv) and (vi) shows that this is equal to

$$\lim_m\sum_{j=0}^{m-1}\mathscr{E}\left(X_{t_j}\right)^4\left(t_{j+1}-t_j\right)=3\int_a^b\mathscr{E}\left(X_s^4\right)ds.$$

The proof of (iv) is analogous to that of (iii). □

4. Itô formula.

THEOREM 4.1. *For all polynomials f and integers $p\ge 1$,*

$$\mathbf{E}\left(f(Z_b)-f(Z_a)-\int_a^b f'(Z_s)\,dZ_s-\frac{(1+\sqrt{2}\,i)}{2}\int_a^b f''(Z_s)\,dB_s-\frac{1}{8}\int_a^b f^{(4)}(Z_s)\,ds\right)^p=0. \qquad (4.1)$$

First we will prove two lemmas. The first one is a generalization of Theorem 2.2(vi)–(vii).

LEMMA 4.1. *Suppose that $\alpha_j, \beta_j \in \mathbf{N}$ for every $j = 1, \dots, r$. Assume that $\{(s_j, t_j)\}_{1 \le j \le r}$ is a family of pairwise disjoint intervals. Then*

$$\mathscr{E}\left(\prod_{j=1}^{r} \left(Z_{t_j} - Z_{s_j}\right)^{\alpha_j} \left(B_{t_j} - B_{s_j}\right)^{\beta_j} \right) = \prod_{j=1}^{r} \mathscr{E}\left(\left(Z_{t_j} - Z_{s_j}\right)^{\alpha_j} \left(B_{t_j} - B_{s_j}\right)^{\beta_j} \right).$$

PROOF. Recall the notation from Section 2. Recall that $B_t = b_t$ depends only on ω_2 and that $B_t(n)$ converge to B_t as $n \to \infty$. Assume for a moment that ω_2 is fixed and $j \neq k$. Since b^+ and b^- are monotone, the increments $Z_{t_j} - Z_{s_j}$ and $Z_{t_k} - Z_{s_k}$ are functions of increments of w^+ and w^- over disjoint intervals. Hence, they are conditionally independent.

First consider the case when, for every j, $\alpha_j = 2\gamma_j$ where γ_j is an integer. Then the conditional independence discussed above and (4.6) of Burdzy and Mądrecki (1995) give

$$\begin{aligned} &E_1\left(\prod_{j=1}^{r} \left(Z_{t_j}(n) - Z_{s_j}(n)\right)^{\alpha_j} \left(B_{t_j} - B_{s_j}\right)^{\beta_j} \right) \\ &\quad = \prod_{j=1}^{r} E_1\left[\left(Z_{t_j}(n) - Z_{s_j}(n)\right)^{\alpha_j} \left(B_{t_j} - B_{s_j}\right)^{\beta_j} \right] \\ &\quad = \prod_{j=1}^{r} (2\gamma_j - 1)!! \left(B_{t_j}(n) - B_{s_j}(n)\right)^{\gamma_j} \left(B_{t_j} - B_{s_j}\right)^{\beta_j}. \end{aligned}$$

One can check that these random variables are E_2-uniformly integrable by calculating their higher moments. Thus one can pass to the limit in the following formula:

$$\begin{aligned} &\mathscr{E}\left(\prod_{j=1}^{r} \left(Z_{t_j} - Z_{s_j}\right)^{\alpha_j} \left(B_{t_j} - B_{s_j}\right)^{\beta_j} \right) \\ &\quad = \lim_{n\to\infty} E_2\left(E_1\left(\prod_{j=1}^{r} \left(Z_{t_j}(n) - Z_{s_j}(n)\right)^{\alpha_j} \left(B_{t_j} - B_{s_j}\right)^{\beta_j} \right) \right) \\ &\quad = \lim_{n\to\infty} E_2\left(\prod_{j=1}^{r} (2\gamma_j - 1)!! \left(B_{t_j}(n) - B_{s_j}(n)\right)^{\gamma_j} \left(B_{t_j} - B_{s_j}\right)^{\beta_j} \right) \\ &\quad = E_2\left(\prod_{j=1}^{r} (2\gamma_j - 1)!! \left(B_{t_j} - B_{s_j}\right)^{\gamma_j} \left(B_{t_j} - B_{s_j}\right)^{\beta_j} \right) \\ &\quad = \prod_{j=1}^{r} E_2\left[(2\gamma_j - 1)!! \left(B_{t_j} - B_{s_j}\right)^{\gamma_j} \left(B_{t_j} - B_{s_j}\right)^{\beta_j} \right]. \end{aligned}$$

If we take $r = 1$ in this identity, we obtain

$$\mathscr{E}\left[\left(Z_{t_j} - Z_{s_j}\right)^{\alpha_j} \left(B_{t_j} - B_{s_j}\right)^{\beta_j} \right] = E_2\left[(2\gamma_j - 1)!! \left(B_{t_j} - B_{s_j}\right)^{\gamma_j} \left(B_{t_j} - B_{s_j}\right)^{\beta_j} \right]$$

and so

$$\prod_{j=1}^{r} \mathscr{E}\Big[\big(Z_{t_j} - Z_{s_j}\big)^{\alpha_j}\big(B_{t_j} - B_{s_j}\big)^{\beta_j}\Big]$$
$$= \prod_{j=1}^{r} E_2\Big[(2\gamma_j - 1)!!\big(B_{t_j} - B_{s_j}\big)^{\gamma_j}\big(B_{t_j} - B_{s_j}\big)^{\beta_j}\Big].$$

The right-hand side of this identity is the same as the right-hand side of the identity obtained in the previous paragraph. It follows that the left-hand sides are equal as well and this proves the lemma.

It remains to consider the case when one of the α_j's is not even. In this case, the product

$$\prod_{j=1}^{r} \big(Z_{t_j}(n) - Z_{s_j}(n)\big)^{\alpha_j}\big(B_{t_j} - B_{s_j}\big)^{\beta_j}$$

contains a factor of the form $(w^+)^k$ or $(w^-)^l$, where either k or l is odd, and, therefore, the E_1-expectation of the product must be 0. The same remark applies to a factor on the right-hand side of the equation given in the statement of the lemma. □

The following lemma is a slight generalization of Lemma 5.1 of Burdzy and Mądrecki (1995).

LEMMA 4.2. *Suppose that* $\beta_1, \beta_2 \in \mathbf{N}$. *Then, for* $s < t$,

$$\mathscr{E}\Big[(Z_t - Z_s)^{\beta_1}(B_t - B_s)^{\beta_2}\Big] = (\beta_1/2 + \beta_2 - 1)!!(t - s)^{\beta_1/4 + \beta_2/2}$$

if β_1 *and* $\beta_1/2 + \beta_2$ *are even. Otherwise the expectation is equal to* 0.

PROOF. If β_1 and $\beta_1/2 + \beta_2$ are even, then the result follows from Lemma 5.1 of Burdzy and Mądrecki (1995). It remains to discuss what happens when one of these conditions is not satisfied.

If β_1 is even but $\beta_1/2 + \beta_2$ is odd, then the proof of Lemma 5.1 in Burdzy and Mądrecki (1995) shows that

$$\mathscr{E}\Big[(Z_t - Z_s)^{\beta_1}(B_t - B_s)^{\beta_2}\Big] = E_2(B_t - B_s)^{\beta_1/2 + \beta_2}.$$

Since the distribution of $B_t - B_s$ is centered normal, the expectation is equal to 0.

If β_1 is odd, then we argue as in the proof of Lemma 4.1. The expression $(Z_t(n) - Z_s(n))^{\beta_1}(B_t - B_s)^{\beta_2}$ must contain a factor of the form $(w^+)^k$ or $(w^-)^l$, where either k or l is odd, and, therefore, the E_1-expectation of this expression must be 0. When we apply the E_2-expectation and pass to ∞ with n, we obtain 0 for the value of the expectation in the statement of the lemma. □

PROOF OF THEOREM 4.1. Fix some interval $[a, b]$ and an integer $m \geq 1$. Let $t_j = a + j(b-a)/m$ and

$$\begin{aligned}
\Delta t_j &= t_{j+1} - t_j, \\
\Delta B_j &= B_{t_{j+1}} - B_{t_j}, \\
\Delta Z_j &= Z_{t_{j+1}} - Z_{t_j}, \\
\Delta U_j &= \left(Z_{t_{j+1}} - Z_{t_j}\right)^2 - (1 + i\sqrt{2})\left(B_{t_{j+1}} - B_{t_j}\right), \\
\Delta V_j &= \left(Z_{t_{j+1}} - Z_{t_j}\right)^4 - 3\left(t_{j+1} - t_j\right).
\end{aligned}$$

Recall that f is a polynomial and $Z_0 = 0$. Let d be the degree of f. We have

$$Z_{t_j} = \sum_{k=0}^{j-1} (Z_{t_{k+1}} - Z_{t_k})$$

and

$$f\left(Z_{t_{j+1}}\right) - f\left(Z_{t_j}\right) - f'\left(Z_{t_j}\right) \Delta Z_j = \sum_{k=2}^{d} \frac{f^{(k)}\left(Z_{t_j}\right)}{k!} \left(Z_{t_{j+1}} - Z_{t_j}\right)^k.$$

Therefore, (4.1) is a limit of

$$\begin{aligned}
&\mathscr{E}\Bigg[\sum_{j=0}^{m} \Bigg(\frac{f''\left(Z_{t_j}\right)}{2!} \left[\left(Z_{t_{j+1}} - Z_{t_j}\right)^2 - (1 + i\sqrt{2})\left(B_{t_{j+1}} - B_{t_j}\right)\right] \\
&\qquad + \frac{f^{(3)}\left(Z_{t_j}\right)}{3!} \left(Z_{t_{j+1}} - Z_{t_j}\right)^3 \\
&\qquad + \frac{f^{(4)}\left(Z_{t_j}\right)}{4!} \left[\left(Z_{t_{j+1}} - Z_{t_j}\right)^4 - 3\left(t_{j+1} - t_j\right)\right] \\
&\qquad\qquad + \sum_{k=5}^{d} \frac{f^{(k)}\left(Z_{t_j}\right)}{k!} \left(Z_{t_{j+1}} - Z_{t_j}\right)^k \Bigg)\Bigg]^p
\end{aligned}$$

$$\begin{aligned}
&= \mathscr{E}\Bigg[\sum_{j=0}^{m} \Bigg(\frac{f''\left(\sum_{k=0}^{j-1}\left(Z_{t_{k+1}} - Z_{t_k}\right)\right)}{2!} \\
&\qquad \times \left[\left(Z_{t_{j+1}} - Z_{t_j}\right)^2 - (1 + i\sqrt{2})\left(B_{t_{j+1}} - B_{t_j}\right)\right] \\
&\qquad + \frac{f^{(3)}\left(\sum_{k=0}^{j-1}\left(Z_{t_{k+1}} - Z_{t_k}\right)\right)}{3!} \left(Z_{t_{j+1}} - Z_{t_j}\right)^3 \\
&\qquad + \frac{f^{(4)}\left(\sum_{k=0}^{j-1}\left(Z_{t_{k+1}} - Z_{t_k}\right)\right)}{4!} \left[\left(Z_{t_{j+1}} - Z_{t_j}\right)^4 - 3\left(t_{j+1} - t_j\right)\right] \\
&\qquad\qquad + \sum_{r=5}^{d} \frac{f^{(r)}\left(\sum_{k=0}^{j-1}\left(Z_{t_{k+1}} - Z_{t_k}\right)\right)}{r!} \left(Z_{t_{j+1}} - Z_{t_j}\right)^r \Bigg)\Bigg]^p.
\end{aligned} \tag{4.2}$$

The last expression is equal to

$$\mathscr{E}\left(\sum_{\{\alpha_j\},\{\beta_j\},\{\gamma_j\}} c(\{\alpha_j\},\{\beta_j\},\{\gamma_j\}) \times \sum_{k_1,\dots,k_N} \prod_{j=0}^{N} (\Delta Z_{k_j})^{\alpha_j}(\Delta U_{k_j})^{\beta_j}(\Delta V_{k_j})^{\gamma_j}\right), \tag{4.3}$$

where:

(i) the first sum is taken over all sequences of integers $\{\alpha_j\},\{\beta_j\},\{\gamma_j\}$ between (and including) 0 and $N = pd$;

(ii) the second sum is taken over a subset (depending on $\{\alpha_j\},\{\beta_j\},\{\gamma_j\}$) of all sequences $k_1,\dots,k_N$ of integers between (and including) 0 and m;

(iii) the coefficients $c(\{\alpha_j\},\{\beta_j\},\{\gamma_j\})$ can be equal to 0 for some $\{\alpha_j\},\{\beta_j\},\{\gamma_j\}$.

Let us consider the expectation of a single term in (4.3). It follows easily from Lemma 4.1 that

$$\mathscr{E}\left(\prod_{j=0}^{N} (\Delta Z_{k_j})^{\alpha_j}(\Delta U_{k_j})^{\beta_j}(\Delta V_{k_j})^{\gamma_j}\right) = \prod_{j=0}^{N} \mathscr{E}\left[(\Delta Z_{k_j})^{\alpha_j}(\Delta U_{k_j})^{\beta_j}(\Delta V_{k_j})^{\gamma_j}\right]. \tag{4.4}$$

We will analyze expectations of the form $\mathscr{E}[(\Delta Z_{k_j})^{\alpha_j}(\Delta U_{k_j})^{\beta_j}(\Delta V_{k_j})^{\gamma_j}]$. First suppose that $\gamma_j \ge 1$ and write

$$\mathscr{E}\left[(\Delta Z_{k_j})^{\alpha_j}(\Delta U_{k_j})^{\beta_j}(\Delta V_{k_j})^{\gamma_j}\right] = \mathscr{E}\left[(\Delta Z_{k_j})^{\alpha_j}(\Delta U_{k_j})^{\beta_j}(\Delta V_{k_j})^{\gamma_j-1}\Delta V_{k_j}\right]. \tag{4.5}$$

Observe that $(\Delta Z_{k_j})^{\alpha_j}(\Delta U_{k_j})^{\beta_j}(\Delta V_{k_j})^{\gamma_j-1}$ is a sum of terms of the form $c(Z(t_{k_j+1}) - Z(t_{k_j}))^a$ or $c(B(t_{k_j+1}) - B(t_{k_j}))^b$ or $c(Z(t_{k_j+1}) - Z(t_{k_j}))^a(B(t_{k_j+1}) - B(t_{k_j}))^b$, where a and b are integers. Here c may contain the (nonrandom) factor Δt_{k_j}. If a is divisible by 4 and $a > 0$, then, according to Theorem 2.2(iv),

$$\begin{aligned}
&\mathscr{E}\left[c\left(Z(t_{k_j+1}) - Z(t_{k_j})\right)^a \Delta V_{k_j}\right] \\
&\quad = \mathscr{E}\Big[c\left(Z(t_{k_j+1}) - Z(t_{k_j})\right)^a \\
&\qquad \times \left(\left(Z(t_{k_j+1}) - Z(t_{k_j})\right)^4 - 3\left(t_{k_j+1} - t_{k_j}\right)\right)\Big] \\
&\quad = c_1\left(\Delta t_{k_j}\right)^{1+a/4} - c_2\left(\Delta t_{k_j}\right)^{1+a/4} = c_3\left(\Delta t_{k_j}\right)^{1+a/4} = c_3\left(\Delta t_{k_j}\right)^{a_1},
\end{aligned} \tag{4.6}$$

where $a_1 = 1 + a/4 \ge 2$. If $a > 0$ but a is not a multiple of 4, then the same theorem implies that the expectation in (4.6) is equal to 0. Another application of the same theorem shows that $\mathscr{E}\Delta V_{k_j} = 0$ and so we conclude that the expectation in (4.6) is equal to 0 if $a = 0$. We see that (4.6) holds for all a if we allow c_3 to be equal to 0.

A similar argument based on Lemma 4.2 shows that

$$
\begin{aligned}
&\mathscr{E}\Big[c\big(B(t_{k_j+1}) - B(t_{k_j})\big)^b \,\Delta V_{k_j}\Big]\\
&\quad= \mathscr{E}\Big[c\big(B(t_{k_j+1}) - B(t_{k_j})\big)^b\\
&\qquad\times\Big(\big(Z(t_{k_j+1}) - Z(t_{k_j})\big)^4 - 3\big(t_{k_j+1} - t_{k_j}\big)\Big)\Big]\\
&\quad= c_1\big(\Delta t_{k_j}\big)^{1+b/2} - c_2\big(\Delta t_{k_j}\big)^{1+b/2} = c_3\big(\Delta t_{k_j}\big)^{1+b/2} = c_3\big(\Delta t_{k_j}\big)^{a_2},
\end{aligned}
\tag{4.7}
$$

with $a_2 = 1 + b/2 \geq 2$ provided b is even and greater than 0. Otherwise the expectation is 0 so (4.7) holds with $c_3 = 0$.

Finally, we analyze the most complicated case. We observe that

$$
\begin{aligned}
&\mathscr{E}\Big[c\big(Z(t_{k_j+1}) - Z(t_{k_j})\big)^a\big(B(t_{k_j+1}) - B(t_{k_j})\big)^b \,\Delta V_{k_j}\Big]\\
&\quad= \mathscr{E}\Big[c\big(Z(t_{k_j+1}) - Z(t_{k_j})\big)^a\big(B(t_{k_j+1}) - B(t_{k_j})\big)^b\\
&\qquad\times\Big(\big(Z(t_{k_j+1}) - Z(t_{k_j})\big)^4 - 3\big(t_{k_j+1} - t_{k_j}\big)\Big)\Big]\\
&\quad= c_1\big(\Delta t_{k_j}\big)^{1+a/4+b/2} - c_2\big(\Delta t_{k_j}\big)^{1+a/4+b/2}\\
&\quad= c_3\big(\Delta t_{k_j}\big)^{1+a/4+b/2} = c_3\big(\Delta t_{k_j}\big)^{a_3},
\end{aligned}
\tag{4.8}
$$

with $a_3 = 1 + a/4 + b/2 \geq 2$ assuming a is divisible by 4 and b is even (see Lemma 4.2). In other cases the expression is equal to 0.

When we combine (4.6)–(4.8) and assume that $\gamma_j \geq 1$, (4.5) shows that

$$
\mathscr{E}\Big[\big(\Delta Z_{k_j}\big)^{\alpha_j}\big(\Delta U_{k_j}\big)^{\beta_j}\big(\Delta V_{k_j}\big)^{\gamma_j}\Big] = c\big(\Delta t_{k_j}\big)^{a_4},
\tag{4.9}
$$

where $a_4 \geq 2$ and c is a constant which may be equal to 0.

Next we assume that $\gamma_j = 0$. Hence, the expectation in (4.5) becomes $\mathscr{E}[(\Delta Z_{k_j})^{\alpha_j}(\Delta U_{k_j})^{\beta_j}]$. Assume first that $\beta_j \geq 2$ and write, as in (4.5),

$$
\mathscr{E}\Big[\big(\Delta Z_{k_j}\big)^{\alpha_j}\big(\Delta U_{k_j}\big)^{\beta_j}\Big] = \mathscr{E}\Big[\big(\Delta Z_{k_j}\big)^{\alpha_j}\big(\Delta U_{k_j}\big)^{\beta_j-2}\big(\Delta U_{k_j}\big)^2\Big].
\tag{4.10}
$$

If $\alpha_j = 0$ and $\beta_j - 2 = 0$, then we can show just as in the proof of Theorem 5.1 of Burdzy and Mądrecki (1995) that

$$
\mathscr{E}\Big[\big(\Delta Z_{k_j}\big)^{\alpha_j}\big(\Delta U_{k_j}\big)^{\beta_j}\Big] = \mathscr{E}\Big[\big(\Delta U_{k_j}\big)^2\Big] = 0.
\tag{4.11}
$$

Now suppose that either $\alpha_j > 0$ or $\beta_j - 2 > 0$. Then $(\Delta Z_{k_j})^{\alpha_j}(\Delta U_{k_j})^{\beta_j - 2}$ is a sum of terms of the form $c(\Delta Z_{k_j})^a(\Delta B_{k_j})^b$ where a and b are integers. We have

$$
\begin{aligned}
&\mathscr{E}\Big[c\big(\Delta Z_{k_j}\big)^a\big(\Delta B_{k_j}\big)^b\big(\Delta U_{k_j}\big)^2\Big]\\
&\quad = \mathscr{E}\Big[c\big(\Delta Z_{k_j}\big)^a\big(\Delta B_{k_j}\big)^b\big(\Delta Z_{k_j}\big)^4\Big]\\
&\qquad - 2(1 + i\sqrt{2})\mathscr{E}\Big[c\big(\Delta Z_{k_j}\big)^a\big(\Delta B_{k_j}\big)^b\big(\Delta Z_{k_j}\big)^2\,\Delta B_{k_j}\Big]\\
&\qquad + (1 + i\sqrt{2})^2\mathscr{E}\Big[c\big(\Delta Z_{k_j}\big)^a\big(\Delta B_{k_j}\big)^b\big(\Delta B_{k_j}\big)^2\Big].
\end{aligned}
\tag{4.12}
$$

Since either a or b is greater than 0, Lemma 4.2 and Theorem 2.2(iv) show that each expectation on the right-hand side of (4.12) is either equal to 0 or is equal to $c(\Delta t_{k_j})^r$ with $r \geq 2$. It follows from this and (4.11) that if $\beta_j \geq 2$, then

$$
\mathscr{E}\Big[\big(\Delta Z_{k_j}\big)^{\alpha_j}\big(\Delta U_{k_j}\big)^{\beta_j}\Big] = c\big(\Delta t_{k_j}\big)^r, \tag{4.13}
$$

with $r \geq 2$ and some c which may be equal to 0.

Suppose that $\beta_j = 1$. Then

$$
\begin{aligned}
&\mathscr{E}\Big[\big(\Delta Z_{k_j}\big)^{\alpha_j}\big(\Delta U_{k_j}\big)^{\beta_j}\Big]\\
&\quad = \mathscr{E}\Big[\big(\Delta Z_{k_j}\big)^{\alpha_j}\big(\Delta Z_{k_j}\big)^2\Big] - (1 + i\sqrt{2})\mathscr{E}\Big[\big(\Delta Z_{k_j}\big)^{\alpha_j}\,\Delta B_{k_j}\Big],
\end{aligned}
\tag{4.14}
$$

and another application of Lemma 4.2 and Theorem 2.2(iv) shows that the expectations on the right-hand side are either equal to 0 or equal to $c(\Delta t_{k_j})^r$ with $r \geq 2$, provided $\alpha_j \neq 2$. Thus (4.13) is true also in the case $\beta_j = 1$ and $\alpha_j \neq 2$.

If $\beta_j = 1$ and $\alpha_j = 2$, then

$$
\begin{aligned}
\mathscr{E}\Big[\big(\Delta Z_{k_j}\big)^{\alpha_j}\big(\Delta U_{k_j}\big)^{\beta_j}\Big] &= \mathscr{E}\Big[\big(\Delta Z_{k_j}\big)^2\big(\Delta Z_{k_j}\big)^2\Big]\\
&\quad - (1 + i\sqrt{2})\mathscr{E}\Big[\big(\Delta Z_{k_j}\big)^2\,\Delta B_{k_j}\Big] = c\,\Delta t_{k_j}.
\end{aligned}
\tag{4.15}
$$

By analyzing all possible products in (4.2), we see that we may remove from formula (4.3) all the products $\prod_{j=0}^{N}(\Delta Z_{k_j})^{\alpha_j}(\Delta U_{k_j})^{\beta_j}(\Delta V_{k_j})^{\gamma_j}$ with $\gamma_j = 0$, $\beta_j = 1$ and $\alpha_j = 2$ unless there is $m \neq j$ such that $\gamma_m \geq 1$ or $\alpha_m \geq 3$, $\alpha_m \neq 4$.

Finally, assume that both β_j and γ_j are equal to 0. Then Theorem 2.2(iv) yields

$$
\mathscr{E}\big(\Delta Z_{k_j}\big)^{\alpha_j} = c\big(\Delta t_{k_j}\big)^r \tag{4.16}
$$

for some $r \geq 2$ if $\alpha_j \neq 4$ (c may be equal to 0). If $\alpha_j = 4$, then

$$
\mathscr{E}\big(\Delta Z_{k_j}\big)^{\alpha_j} = c\,\Delta t_{k_j}. \tag{4.17}
$$

However, if $\alpha_j = 4$, then we may assume that there exists $m \neq j$ such that $\alpha_m \geq 3$, $\alpha_m \neq 4$. The products which do not satisfy this condition do not contribute anything to the expectation in (4.2).

Now we combine the estimates contained in (4.9), (4.13) and (4.16) to obtain

$$\mathscr{E}\left[\left(\Delta Z_{k_j}\right)^{\alpha_j}\left(\Delta U_{k_j}\right)^{\beta_j}\left(\Delta V_{k_j}\right)^{\gamma_j}\right] = c\left(\Delta t_{k_j}\right)^r \tag{4.18}$$

for some $r \geq 2$ unless one of the following conditions is true:

(i) $\gamma_j = 0$, $\beta_j = 1$ and $\alpha_j = 2$, or
(ii) $\gamma_j = 0$, $\beta_j = 0$ and $\alpha_j = 4$.

If any of these two conditions holds, then

$$\mathscr{E}\left[\left(\Delta Z_{k_j}\right)^{\alpha_j}\left(\Delta U_{k_j}\right)^{\beta_j}\left(\Delta V_{k_j}\right)^{\gamma_j}\right] = c\,\Delta t_{k_j} \tag{4.19}$$

and there exists $m \neq j$ such that

$$\mathscr{E}\left[\left(\Delta Z_{k_m}\right)^{\alpha_m}\left(\Delta U_{k_m}\right)^{\beta_m}\left(\Delta V_{k_m}\right)^{\gamma_m}\right] = c\left(\Delta t_{k_m}\right)^r, \tag{4.20}$$

with $r \geq 2$. Hence, in view of (4.3), estimates (4.18)–(4.20) give

$$\begin{aligned}
&\mathscr{E}\left(\sum_{k_1,\dots,k_N}\prod_{j=0}^{N}\left(\Delta Z_{k_j}\right)^{\alpha_j}\left(\Delta U_{k_j}\right)^{\beta_j}\left(\Delta V_{k_j}\right)^{\gamma_j}\right)\\
&\quad= \sum_{k_1,\dots,k_N}\mathscr{E}\left(\prod_{j=0}^{N}\left(\Delta Z_{k_j}\right)^{\alpha_j}\left(\Delta U_{k_j}\right)^{\beta_j}\left(\Delta V_{k_j}\right)^{\gamma_j}\right)\\
&\quad= \sum_{k_1,\dots,k_N}\prod_{j=0}^{N}\mathscr{E}\left[\left(\Delta Z_{k_j}\right)^{\alpha_j}\left(\Delta U_{k_j}\right)^{\beta_j}\left(\Delta V_{k_j}\right)^{\gamma_j}\right]\\
&\quad= \sum_{k_1,\dots,k_N}\prod_{j=0}^{N}c_j\left(\Delta t_{k_j}\right)^{r_j}\\
&\quad\leq \prod_{j=0}^{N}c_j\sum_{k_j=0}^{m}\left(\Delta t_{k_j}\right)^{r_j},
\end{aligned} \tag{4.21}$$

where at least one r_j is greater than 1. The inequality sign in (4.21) is due to the fact that the summation in (4.3) is taken over some but not necessarily all k_j's between 0 and m. Let $\Delta^* t = \max_j \Delta t_j$. Then the right-hand side in (4.21) is less than or equal to

$$(b-a)^N(\Delta^* t)^\rho,$$

with some $\rho \geq 1$ and, therefore,

$$\mathscr{E}\left(\sum_{k_1,\dots,k_N}\prod_{j=0}^{N}\left(\Delta Z_{k_j}\right)^{\alpha_j}\left(\Delta U_{k_j}\right)^{\beta_j}\left(\Delta V_{k_j}\right)^{\gamma_j}\right)$$

converges to 0 as $\Delta^* t \to 0$. The first sum in (4.3) extends over a finite set of sequences, so the expectation in (4.3) goes to 0 as $\Delta^* t \to 0$. $\square$

For many functions f, the process $f'(B_t)$ is a martingale and then

$$Ef(B_b) - Ef(B_a) = \tfrac{1}{2}\int_a^b Ef''(B_s)\,ds.$$

We have the following analog of this identity for the asymptotically 4-stable process Z.

COROLLARY 4.1. *If f is a polynomial, then*

$$\mathbf{E}f(Z_b) - \mathbf{E}f(Z_a) = \tfrac{1}{8}\int_a^b \mathscr{E}f^{(4)}(Z_s)\,ds.$$

PROOF. If we apply Theorem 4.1 with $p = 1$, we see that it will suffice to show that $\int_a^b f'(Z_s)\,dZ_s = 0$ and $\int_a^b f''(Z_s)\,dB_s = 0$. It will be enough to show that

$$\mathscr{E}\sum_{j=0}^{m-1} f'\big(X_{t_j}(n)\big)\big(Z_{t_{j+1}}(n) - Z_{t_j}(n)\big) = 0$$

and

$$\mathscr{E}\sum_{j=0}^{m-1} f''\big(X_{t_j}(n)\big)\big(B_{t_{j+1}} - B_{t_j}\big) = 0.$$

We obtain these identities from the fact that the increments of Z and B are "asymptotically uncorrelated" [recall that f is a polynomial and use Theorem 2.2(vi) and Lemma 4.1]. □

We owe the following remark to an anonymous referee.

REMARK 4.1. (i) One may consider an asymptotic integral given by

$$\sum_{j=0}^{m-1} X_{t_j}(n)\big(Z_{t_{j+1}}(n) - Z_{t_j}(n)\big)^3;$$

cf. Definition 3.1(i). This integral is analogous to an integral considered by Hochberg (1978). In a sense, this is "an integral with respect to a process composed of a 3/2-stable process and a Brownian motion."

(ii) It seems that Theorem 4.1 should hold for holomorphic functions.

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REFERENCES

ANTOSIEWICZ, P., MIKUSIŃSKI, J. and SIKORSKI, R. (1973). *Theory of Distributions* (*The Sequential Approach*). North-Holland, Amsterdam.

BURDZY, K. (1993). Some path properties of iterated Brownian motion. In *Seminar on Stochastic Processes 1992* (E. Çinlar, K. L. Chung and M. Sharpe, eds.) 67–87. Birkhäuser, Boston.

BURDZY, K. (1994). Variation of iterated Brownian motion. In *Measure-Valued Processes, Stochastic Partial Differential Equations and Interacting Systems* (D. A. Dawson, ed.) 35–53. Amer. Math. Soc., Providence, RI.

BURDZY, K. and MĄDRECKI, A. (1995). An asymptotically 4-stable process. *J. Fourier Anal. Appl. Special Issue. Proceedings of the Conference in Honor of J.-P. Kahane* 97–117.

FUNAKI, T. (1979). Probabilistic construction of the solution of some higher order parabolic differential equations. *Proc. Japan Acad. Ser. A Math. Sci.* **55** 176–179.

HELMS, L. L. (1967). Biharmonic functions and Brownian motion. *J. Appl. Probab.* **4** 130–136.

HOCHBERG, K. J. (1978). A signed measure on path space related to Wiener measure. *Ann. Probab.* **6** 433–458.

HOCHBERG, K. J. and ORSINGHER, E. (1994). The arc-sine law and its analogues for processes governed by signed and complex measures. *Stochastic Process. Appl.* To appear.

KRYLOV, V. YU. (1960). Some properties of the distribution corresponding to the equation $\partial u/\partial t = (-1)^{q+1}\, \partial^{2q} u/\partial x^{2q}$. *Soviet Math. Dokl.* **1** 760–763.

MĄDRECKI, A. and RYBACZUK, M. (1989). On a Feynman–Kac formula. In *Stochastic Methods in Experimental Sciences. Proceedings of COSMEX Conference 1989* (W. Kasprzak and A. Weron, eds.) 312–321. World Scientific, Singapore.

MĄDRECKI, A. and RYBACZUK, M. (1993). New Feynman–Kac type formula. *Rep. Math. Phys.* **32** 301–327.

NISHIOKA, K. (1985). Stochastic calculus for a class of evolution equations. *Japan J. Math.* (*N.S.*) **11** 59–102.

NISHIOKA, K. (1987). A stochastic solution of a high order parabolic equation. *J. Math. Soc. Japan* **39** 209–231.

NISHIOKA, K. (1994). The first hitting time and place of a biharmonic pseudo process. Preprint.

SAINTY, PH. (1992). Construction of a complex-valued fractional Brownian motion of order *N*. *J. Math. Phys.* **33** 3128–3149.

VAINSTEIN, A. I., ZAKHAROV, V. I., NOVIKOV, V. A. and SHIFMAN, M. A. (1982). ABC of instantons. *Uspekhi Fiz. Nauk* **136** 553–591.

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