The Annals of Applied Probability
2002, Vol. 12, No. 3, 1071–1095

ASYMPTOTICS OF HITTING PROBABILITIES FOR GENERAL ONE-DIMENSIONAL PINNED DIFFUSIONS[1]

BY PAOLO BALDI AND LUCIA CARAMELLINO

University of Rome–Tor Vergata

We consider a general one-dimensional diffusion process and we study the probability of crossing a boundary for the associated pinned diffusion as the time at which the conditioning takes place goes to zero. We provide asymptotics for this probability as well as a first order development. We consider also the cases of two boundaries possibly depending on the time. We give applications to simulation.

1. Introduction. Recently, simple formulas concerning level crossing probabilities for the Brownian bridge have been widely used to improve the performance of numerical simulation [see, e.g., Andersen and Brotherton-Ratcliffe (1996), Baldi (1995), Baldi, Caramellino and Iovino (1999), Beaglehole, Dybvig and Zhou (1997), Caramellino and Iovino (2001)]. Actually it has been remarked that the usual discretization schemes perform poorly when the process of interest is to be killed at the crossing of a prescribed boundary and the knowledge of the level crossing probability for the pinned process allows a more efficient procedure to be devised. See also Gobet (2000), where rigorous results are given concerning the improvement which can be obtained by this procedure.

If $x, y, a \in \mathbb{R}$ are such that $x < a$ and $y < a$, then the probability of crossing the level a for the Brownian motion starting at x and pinned by $B_\varepsilon = y$ is equal to

$$\exp\biggl(-\frac{2}{\varepsilon}(a-x)(a-y)\biggr).$$

A similar exact formula also holds for the probability of crossing a linear boundary, instead of the constant boundary a. However, as soon as one needs to estimate the probability of crossing a more general boundary, exact formulas are not available. To overcome this drawback, Baldi, Carmellino and Iovino (1999) directed attention toward the determination of the asymptotics of this probability, as $\varepsilon \to 0$. The estimates obtained have been applied to the numerical computation of knock-out and knock-in options, with results to be considered satisfactory, in the light of numerical evidence.

Received November 2000; revised October 2001.

[1]Supported by MURST-COFIN funds.

AMS 2000 subject classifications. Primary 60F10; secondary 60J60.

Key words and phrases. Conditioned diffusions, sharp large deviation estimates, exit time probabilities.

The situation is not yet satisfactory since the above-mentioned applications to simulation naturally require the computation of the crossing probability for the process arising from the conditioning of a diffusion process more general than the Brownian motion. The natural idea consisting of freezing the coefficients at, say, the starting point x and approximating the crossing probability with the corresponding crossing probability of the bridge of the diffusion with the frozen coefficients has proved to be useful in Gobet (2000), but gives incorrect asymptotics and it has been pointed out, by the numerical treatment of some precise examples, that the approximations produced by this method can be far from the true ones. See for this point Giraudo and Sacerdote [(2000), Section 5], who also suggest some formulas for the computation of the crossing probability.

In this paper we give the asymptotics as $\varepsilon \to 0$ of the crossing probability of a general one-dimensional diffusion process with respect to one or two, possibly time-dependent, boundaries. For the case of a constant boundary the formula is very simple (see Corollary 2.3) and gives approximations for the crossing probability which are in accordance with the numerical example treated in Giraudo and Sacerdote (2000) (see Example 2.5 below).

The main results are stated in Section 2; the ideas of the proof are introduced in Section 3, whereas the more technical points are developed in Section 4. The idea actually is simple: one first makes a nonlinear change of variable which reduces the problem to a new diffusion with a constant diffusion coefficient (and a more complicated drift). One proves then that the asymptotics for the crossing probability of the conditioned process is independent of the drift; we are, however, also able to compute a first-order approximation [Theorem 2.1(b)] which is easily computed as a function of the drift and of the diffusion coefficient.

2. Main results. Let us consider a one-dimensional diffusion process Z on an interval $\mathcal{I} \subset \mathbb{R}$, that is, $\{Z_t\} \subset \mathcal{I}$, satisfying

$$\begin{aligned} dZ_t &= b(Z_t)\,dt + \sigma(Z_t)\,dB_t, \\ Z_0 &= x \end{aligned} \tag{1}$$

on $\mathring{\mathcal{I}}$ for some suitable coefficients b and σ.

Throughout this paper we implicitly assume that Z has a transition density.

We denote by $\hat{Z}^{y,\varepsilon}$ the associated conditioned diffusion pinned by $Z_\varepsilon = y$. Take $f:[0,1] \to \mathring{\mathcal{I}}$ and let $\hat{\tau}_f^\varepsilon$ be the hitting time on f of the conditioned diffusion $\hat{Z}^{y,\varepsilon}$:

$$\hat{\tau}_f^\varepsilon = \inf\{t > 0 : \hat{Z}_t^{y,\varepsilon} \geq f(t)\}.$$

In the following statements, the points x and y always refer to the starting and pinning point, respectively. Moreover, from now on $p_\varepsilon \sim q_\varepsilon$ means $p_\varepsilon / q_\varepsilon \to 1$ as $\varepsilon \to 0$.

Then we have the following theorem.

THEOREM 2.1. *Assume that $b \in \mathcal{C}^1(\mathring{\mathcal{I}})$, $\sigma \in \mathcal{C}^2(\mathring{\mathcal{I}})$ and $\sigma(x) > 0$. Let $x, y \in \mathring{\mathcal{I}}$ both be smaller* (*or larger*) *than $f(0)$*:

(a) *If f is continuous with Lipschitz continuous derivative*, *then*

$$\mathrm{P}(\hat{\tau}_f^\varepsilon \le \varepsilon) \sim C_f \exp\left(-\frac{2}{\varepsilon}\int_x^{f(0)} \frac{dr}{\sigma(r)} \cdot \int_y^{f(0)} \frac{dr}{\sigma(r)}\right),$$

where

$$C_f = \exp\left(-2\frac{f'(0)}{\sigma(f(0))}\int_x^{f(0)} \frac{dr}{\sigma(r)}\right). \tag{2}$$

(b) *If*, *additionally*, *f has a Lipschitz continuous second derivative and $\lambda = \sigma \cdot (\frac{b}{\sigma} - \frac{1}{2}\sigma')' + (\frac{b}{\sigma} - \frac{1}{2}\sigma')^2$ is locally Lipschitz continuous on $\mathring{\mathcal{I}}$* [*in particular if $b \in C^2(\mathring{\mathcal{I}})$ and $\sigma \in C^3(\mathring{\mathcal{I}})$*], *then*

$$\mathrm{P}(\hat{\tau}_f^\varepsilon \le \varepsilon) = C_f \exp\left(-\frac{2}{\varepsilon}\int_x^{f(0)} \frac{dr}{\sigma(r)} \cdot \int_y^{f(0)} \frac{dr}{\sigma(r)}\right)\left(1 + \varepsilon(\Psi_f + \Phi_f) + \varepsilon \mathcal{R}_\varepsilon\right),$$

where

$$\Psi_f = \frac{\sigma'(f(0)) f'^2(0) - \sigma(f(0)) f''(0)}{\sigma^2(f(0))} \cdot \frac{(\int_x^{f(0)}(dr/\sigma(r)))^2}{\int_x^{f(0)}(dr/\sigma(r)) + \int_y^{f(0)}(dr/\sigma(r))}, \tag{3}$$

$$\Phi_f = \begin{cases} \dfrac{1}{2}\Biggl(\dfrac{\int_x^y (\lambda(r)/\sigma(r))\,dr}{\int_x^y (dr/\sigma(r))} \\ \qquad - \dfrac{\int_x^{f(0)}(\lambda(r)/\sigma(r))\,dr + \int_y^{f(0)}(\lambda(r)/\sigma(r))\,dr}{\int_x^{f(0)}(dr/\sigma(r)) + \int_y^{f(0)}(dr/\sigma(r))}\Biggr), & \text{if } x \ne y, \\ \dfrac{1}{2}\left(\lambda(x) - \dfrac{\int_x^{f(0)}(\lambda(r)/\sigma(r))\,dr}{\int_x^{f(0)}(dr/\sigma(r))}\right), & \text{if } x = y, \end{cases} \tag{4}$$

and $\lim_{\varepsilon \to 0} \mathcal{R}_\varepsilon = 0$, *uniformly for (x, y) in a compact subset of $\mathring{\mathcal{I}} \times \mathring{\mathcal{I}}$.*

One should remark that the asymptotics of Theorem 2.1(a) do not depend of the drift b, which only affects the first-order approximation of Theorem 2.1(b) through the term Φ.

We can also give the asymptotics of the passage probability through two moving barriers.

THEOREM 2.2. *Assume that $b \in \mathcal{C}^1(\mathring{\mathcal{I}})$, $\sigma \in \mathcal{C}^2(\mathring{\mathcal{I}})$ and $\sigma(x) > 0$. Let $f_1, f_2 : [0, 1] \to \mathring{\mathcal{I}}$ and let $x, y \in \mathring{\mathcal{I}}$ be points such that $f_1(0) < x, y < f_2(0)$. Let $\hat{\tau}^\varepsilon = \inf\{t; \hat{Z}_t^{y,\varepsilon} \le f_1(t)$ or $\hat{Z}_t^{y,\varepsilon} \ge f_2(t)\}$ be the hitting time of $\hat{Z}^{y,\varepsilon}$ on the* *barriers* f_1 *and* f_2. *Let us set*

$$\psi_1(x,y)=\int_{f_1(0)}^{x}\frac{dr}{\sigma(r)}\cdot\int_{f_1(0)}^{y}\frac{dr}{\sigma(r)}\quad\textit{and}\quad\psi_2(x,y)=\int_{x}^{f_2(0)}\frac{dr}{\sigma(r)}\cdot\int_{y}^{f_2(0)}\frac{dr}{\sigma(r)}$$

and $\psi(x,y)=\min(\psi_1(x,y),\psi_2(x,y))$.

(a) *If* f_1 *and* f_2 *are both continuous with Lipschitz continuous derivative*, *then*

$$\mathrm{P}(\hat\tau^\varepsilon\le\varepsilon)\sim C\exp\biggl(-\frac{2}{\varepsilon}\psi(x,y)\biggr),$$

where, C_f *being defined in* (2),

$$C=\begin{cases}C_{f_1}, & \textit{if }\ \psi_1(x,y)<\psi_2(x,y),\\ C_{f_2}, & \textit{if }\ \psi_1(x,y)>\psi_2(x,y),\\ C_{f_1}+C_{f_2}, & \textit{if }\ \psi_1(x,y)=\psi_2(x,y).\end{cases}$$

(b) *If*, *additionally*, f_1 *and* f_2 *both have a Lipschitz continuous second derivative and* $\lambda=\sigma\cdot(\frac{b}{\sigma}-\frac12\sigma')'+(\frac{b}{\sigma}-\frac12\sigma')^2$ *is locally Lipschitz continuous on* $\mathring{\mathcal{I}}$, *then*

$$\mathrm{P}(\hat\tau^\varepsilon\le\varepsilon)=C\exp\biggl(-\frac{2}{\varepsilon}\psi(x,y)\biggr)\bigl(1+\varepsilon(\Psi+\Phi)+\varepsilon\mathcal{R}_\varepsilon\bigr),$$

where, Ψ_f *and* Φ_f *being defined in* (3) *and* (4), *respectively*,

$$\Psi=\begin{cases}\Psi_{f_1}, & \textit{if }\ \psi_1(x,y)<\psi_2(x,y),\\ \Psi_{f_2}, & \textit{if }\ \psi_1(x,y)>\psi_2(x,y),\\ \dfrac{C_{f_1}\Psi_{f_1}+C_{f_2}\Psi_{f_2}}{C_{f_1}+C_{f_2}}, & \textit{if }\ \psi_1(x,y)=\psi_2(x,y),\end{cases}$$

$$\Phi=\begin{cases}\Phi_{f_1}, & \textit{if }\ \psi_1(x,y)<\psi_2(x,y),\\ \Phi_{f_2}, & \textit{if }\ \psi_1(x,y)>\psi_2(x,y),\\ \dfrac{C_{f_1}\Phi_{f_1}+C_{f_2}\Phi_{f_2}}{C_{f_1}+C_{f_2}}, & \textit{if }\ \psi_1(x,y)=\psi_2(x,y),\end{cases}$$

and $\lim_{\varepsilon\to0}\mathcal{R}_\varepsilon=0$, *uniformly for* (x,y) *in a compact subset of* $\mathring{\mathcal{I}}\times\mathring{\mathcal{I}}$.

In the case of constant barriers, and if one is satisfied with the simple asymptotics of the crossing time, the previous results become the following corollaries.

COROLLARY 2.3. *Assume that* $b\in\mathcal{C}^1(\mathring{\mathcal{I}})$, $\sigma\in\mathcal{C}^2(\mathring{\mathcal{I}})$ *and* $\sigma(x)>0$. *Let* $a,x,y\in\mathring{\mathcal{I}}$ *be such that* a *is larger* (*or smaller*) *than both* x *and* y, *and denote by* $\hat\tau_a^\varepsilon$ *the passage time in* a *for* $\hat Z^{y,\varepsilon}$. *Then*

$$\mathrm{P}(\hat\tau_a^\varepsilon\le\varepsilon)\sim\exp\biggl(-\frac{2}{\varepsilon}\int_x^a\frac{dr}{\sigma(r)}\cdot\int_y^a\frac{dr}{\sigma(r)}\biggr).$$

COROLLARY 2.4. *Assume that* $b \in \mathcal{C}^1(\mathring{\mathcal{I}})$, $\sigma \in \mathcal{C}^2(\mathring{\mathcal{I}})$ *and* $\sigma(x) > 0$. *Let* $a_1, a_2 \in \mathring{\mathcal{I}}$ *and* $x, y \in]a_1, a_2[$, *and denote by* $\hat{\tau}^\varepsilon$ *the exit time of* $\hat{Z}^{y,\varepsilon}$ *from* $]a_1, a_2[$. *Then*

$$P(\hat{\tau}^\varepsilon \le \varepsilon) \sim \begin{cases} \exp\Bigl(-\dfrac{2}{\varepsilon}\displaystyle\int_{a_1}^{x}\frac{dr}{\sigma(r)}\cdot\int_{a_1}^{y}\frac{dr}{\sigma(r)}\Bigr), & \textit{if } \displaystyle\int_{a_1}^{x}\frac{dr}{\sigma(r)} < \int_{y}^{a_2}\frac{dr}{\sigma(r)},\\ \exp\Bigl(-\dfrac{2}{\varepsilon}\displaystyle\int_{x}^{a_2}\frac{dr}{\sigma(r)}\cdot\int_{y}^{a_2}\frac{dr}{\sigma(r)}\Bigr), & \textit{if } \displaystyle\int_{a_1}^{x}\frac{dr}{\sigma(r)} > \int_{y}^{a_2}\frac{dr}{\sigma(r)},\\ 2\exp\Bigl(-\dfrac{2}{\varepsilon}\displaystyle\int_{x}^{a_2}\frac{dr}{\sigma(r)}\cdot\int_{y}^{a_2}\frac{dr}{\sigma(r)}\Bigr), & \textit{if } \displaystyle\int_{a_1}^{x}\frac{dr}{\sigma(r)} = \int_{y}^{a_2}\frac{dr}{\sigma(r)}. \end{cases}$$

EXAMPLE 2.5. Let us consider the interest rate model studied by Cox, Ingersoll and Ross, called the CIR process, whose driving stochastic differential equation (SDE) is

$$\begin{aligned} dZ_t &= (\alpha Z_t + \beta)\,dt + \sqrt{2Z_t}\,dB_t,\\ Z_0 &= x. \end{aligned} \tag{5}$$

Table 1 shows the exit probability from an upper constant barrier $a = 3$ for the bridge of this process as ε and the pinning point y vary. In this table the parameter set is $x = 2$, $\alpha = -1$, $\beta = 2$; p stands for a presumably exact value of the exit probability, obtained by solving numerically a Volterra equation [Giraudo and Sacerdote (2000), Section 5]; $\hat{p}$ is the approximation of Corollary 2.3; $\hat{p}_{fr}$ is the rough exit probability obtained by freezing the diffusion coefficients; $\hat{p}_\psi$ and $\hat{p}_j$ are the approximations of the exit probability obtained in Giraudo and Sacerdote (2000).

It is apparent that the approximation formula of Corollary 2.3 shows a good accordance. Also, the approximations $\hat{p}_\psi$ and $\hat{p}_j$ look very good (particularly $\hat{p}_\psi$); however, to compute them, the knowledge of quantities related to the diffusion which are much more difficult to obtain is required, such as the value of the transition function and of some of its derivatives. For instance $\hat{p}_j$ is obtained in

TABLE 1
CIR process

$\boldsymbol{\varepsilon}$	$\boldsymbol{y}$	$\boldsymbol{p}$	$\hat{\boldsymbol{p}}$	$\hat{\boldsymbol{p}}_{\boldsymbol{fr}}$	$\hat{\boldsymbol{p}}\cdot(\mathbf{1}+\boldsymbol{\varepsilon}\boldsymbol{\Phi})$	$\hat{\boldsymbol{p}}_{\boldsymbol{\psi}}$	$\hat{\boldsymbol{p}}_{\boldsymbol{j}}$
0.05	2.5	2.15×10^{-2}	2.15×10^{-2}	6.74×10^{-3}	2.15×10^{-2}	2.15×10^{-2}	2.06×10^{-2}
0.1	1.5	1.56×10^{-3}	1.58×10^{-3}	5.53×10^{-4}	1.56×10^{-3}	1.56×10^{-3}	1.59×10^{-3}
0.1	2.5	1.46×10^{-1}	1.47×10^{-1}	8.21×10^{-2}	1.46×10^{-1}	1.46×10^{-1}	1.35×10^{-1}
0.2	1.5	3.90×10^{-2}	3.98×10^{-2}	2.35×10^{-2}	3.88×10^{-2}	3.90×10^{-2}	3.54×10^{-2}
0.2	2.5	3.79×10^{-1}	3.83×10^{-1}	2.86×10^{-1}	3.78×10^{-1}	3.80×10^{-1}	3.26×10^{-1}

Giraudo and Sacerdote (2000) as the integral

$$\int_0^{\varepsilon} \left(b(a)p(0,t,x,a) - \frac{1}{2}\frac{\partial}{\partial a}\big(\sigma^2(a)p(0,t,x,a)\big)\right) \frac{p(t,\varepsilon,x,y)}{p(0,\varepsilon,x,y)}\,dt,$$

where the function $p(s,t,x,y)$ denotes the transition density of the diffusion (5). The transition density of the CIR process is known explicitly in terms of special functions [see, e.g., Lamberton and Lapeyre (1996)]; notice that the integrand is actually a reasonable approximation of the density of the passage time.

The column labeled $\hat{p}\cdot(1+\varepsilon\Phi)$ provides the sharper approximation produced by Theorem 2.1(b) (here $\Psi = 0$, the barrier being constant). This approximation improves the estimates, albeit not dramatically.

EXAMPLE 2.6. Let us consider the constant elasticity of variance (CEV) model, first studied by Cox in 1975. The underlying asset price satisfies, under risk neutral probability, the SDE

$$dS_t = rS_t\,du + \sigma {S_t}^{\alpha/2}\,dB_t, \tag{6}$$

where r and σ are constant and $0 < \alpha \le 2$ (the elasticity factor). The quantity of interest we consider is the price of a knock-out double barrier call option, that is,

$$\mathrm{E}[e^{-rT}(S_T - K)_+ 1_{\{\tau > T\}}].$$

Here T and K are respectively the maturity and the strike price and r is the (constant) spot rate; τ is the hitting time on the barriers. In Table 2 we compare the estimates for the price of Boyle and Tian [B–T; determined by means of a numerical procedure in Boyle and Tian (1997)] with Monte Carlo estimates obtained using different simulation procedures: the Euler and the Milstein schemes [see Kloeden and Platen (1992), also for the comparison between the associated orders of convergence below], combined with the estimate of the exit probabilities between consecutive discretization times first considering frozen coefficient (p_{fr}) and then using Corollary 2.4 ($\hat{p}$). The barriers a_1, a_2 are taken to be constant.

It appears that the Milstein scheme can take full advantage of the sharper estimates of Corollary 2.4 in order to give better results, even with a larger discretization step.

TABLE 2
Double knock-out call option prices with time-to-maturity 6 months ($T = 0.5$): CEV model with $\alpha = 1.5$ ($r = 0.1$, $K = 105$, $S_0 = 100$, $\sigma = 0.790$, $a_1 = 95$, $a_2 = 140$)

Method	**B–T**	**Euler $\hat{p}_{fr}$**	**Euler $\hat{p}$**	**Milstein $\hat{p}_{fr}$**	**Milstein $\hat{p}$**
Step	—	1/365	1/365	2.5/365	2.5/365
Price	2.4379	2.4321	2.4303	2.4331	2.4379
St. dev.	—	0.0733	0.0535	0.0646	0.0635
95% conf. int.	—	[2.4177, 2.4465]	[2.4198, 2.4408]	[2.4204, 2.4458]	[2.4254, 2.4503]

But it is also clear that the simulations derive a benefit from the use of the Milstein scheme, a fact giving some suggestions. Indeed, it is well known that for approximating the law of the process at a fixed time T (i.e., for weak approximations) the Milstein and the Euler schemes have the same order of convergence, while the Milstein scheme turns out to be faster than the Euler one in the L^2-norm approximation of the process at time T (i.e., strong approximations). Here the quantity of interest is a killed diffusion, whose weak approximation order of convergence has been studied only in the context of the Euler scheme [see, e.g., Gobet (2000) and references quoted therein]. The empirical results in Table 2 thus suggest that if a boundary is considered and then the pathwise behaviour of the process needs to be taken into account, the Milstein scheme seems to be more appropriate than the Euler one also for weak approximations. This would explain why the simulations record an improvement in the combination of the Milstein scheme to the use of $\hat{p}$: $\hat{p}$ takes into account the behaviour of the path during all the infinitesimal time interval, while p_{fr} depends on the values of the process only at the end points.

3. Main arguments. The proof rests on two main ideas. We first shift to the auxiliary diffusion process $Y_t = F(Z_t)$, where

$$F(z) = \int_x^z \frac{dr}{\sigma(r)}. \tag{7}$$

Under the assumptions of Theorem 2.3, F is twice differentiable on $\mathring{\mathcal{I}}$. If $\mathcal{J} = F(\mathcal{I})$, by the Itô formula Y satisfies on $\mathring{\mathcal{J}}$ the SDE

$$\begin{aligned} dY_t &= \tilde{b}(Y_t)\,dt + dB_t, \\ Y_0 &= \xi, \end{aligned} \tag{8}$$

where $\xi = F(x)$ and

$$\tilde{b}(y) = \frac{b(F^{-1}y)}{\sigma(F^{-1}y)} - \frac{1}{2}\sigma'(F^{-1}y). \tag{9}$$

The probability for the pinned process $\hat{Z}^{y,\varepsilon}$ to cross the level $f = f(t)$ is obviously the same as the probability of crossing the level $g(t) = F(f(t))$ for the conditioned process $\hat{Y}^{\eta,\varepsilon}$, with $\eta = F(y)$. As follows from Baldi, Caramellino and Iovino (1999), this argument already gives the result if $\tilde{b} \equiv 0$ and $\mathring{\mathcal{I}} = \mathbb{R}$, since then Y would be a Brownian motion. The condition $\tilde{b} \equiv 0$ is equivalent to $b = \frac{1}{2}\sigma\sigma'$; that is, the equation for X is $dX_t = \sigma(X_t) \circ dB_t$, $\circ$ denoting Stratonovitch differential.

The second argument of the proof is that actually the drift $\tilde{b}$ does not affect the asymptotics of the crossing probability for the conditioned process (but it affects its development of order 1). The influence of $\tilde{b}$ is the question we investigate in the rest of the proof.

Set $U_t^\varepsilon = Y_{\varepsilon t}$. U^ε solves the SDE

$$dU_t^\varepsilon = \varepsilon \tilde{b}(U_t^\varepsilon)\,dt + \sqrt{\varepsilon}\,dB_t, \qquad U_s^\varepsilon = \xi. \tag{10}$$

Of course it is the same to look at the crossing probability for the conditioned process $\hat{U}^{\eta,\varepsilon}$, pinned by $U_1^\varepsilon = \eta$. However, note that, under our assumptions, U^ε satisfies (10) only up to the exit from $\mathring{\mathcal{J}}$. We assume first $\mathring{\mathcal{J}} = \mathbb{R}$; let us set

$$W_{\xi,s}^\varepsilon(t) = \xi + \sqrt{\varepsilon}(B_t - B_s) \tag{11}$$

(which would be the same as U^ε if $\tilde{b} \equiv 0$). We are going to compare the law of $\hat{U}^{\eta,\varepsilon}$ and the law of the Brownian bridge $\hat{W}^{\eta,\varepsilon}$ by writing a Girsanov-type density of the first with respect to the second one.

The endpoint η is fixed from now on. Let us denote by $\mathcal{C} = \mathcal{C}([0,1],\mathbb{R})$ the space of continuous paths and define, on $\mathcal{C}$, $X_t(w) = w_t$, $\mathcal{F}_t = \sigma(X_s, s \le t)$. On $(\mathcal{C}, \mathcal{F}_1)$ we consider the probability laws

$$\begin{aligned}
\mathrm{P}_{\xi,s}^\varepsilon &= \text{the law of } W_{\xi,s}^\varepsilon,\\
\hat{\mathrm{P}}_{\xi,s}^{\eta,\varepsilon} &= \text{the law of } W_{\xi,s}^\varepsilon \text{ pinned by } W_{\xi,s}^\varepsilon(1) = \eta,\\
\mathrm{Q}_{\xi,s}^\varepsilon &= \text{the law of } U_{\xi,s}^\varepsilon,\\
\hat{\mathrm{Q}}_{\xi,s}^{\eta,\varepsilon} &= \text{the law of } U_{\xi,s}^\varepsilon \text{ pinned by } U_{\xi,s}^\varepsilon(1) = \eta.
\end{aligned}$$

In the following $\mathrm{E}_{\xi,s}^\varepsilon$ and $\hat{\mathrm{E}}_{\xi,s}^{\eta,\varepsilon}$ denote the expectations taken with respect to $\mathrm{P}_{\xi,s}^\varepsilon$ and $\hat{\mathrm{P}}_{\xi,s}^{\eta,\varepsilon}$, respectively. We now write the density of $\hat{\mathrm{Q}}_{\xi,s}^{\eta,\varepsilon}$ with respect to $\hat{\mathrm{P}}_{\xi,s}^{\eta,\varepsilon}$. The main idea of the proof, besides some technical points, is that $\hat{\mathrm{Q}}_{\xi,s}^{\eta,\varepsilon}$ has a density with respect to $\hat{\mathrm{P}}_{\xi,s}^{\eta,\varepsilon}$ that goes to 1 as $\varepsilon \to 0$.

If

$$\zeta_t = \exp\biggl(\int_s^t \tilde{b}(X_u)\,dX_u - \frac{\varepsilon}{2}\int_s^t \tilde{b}(X_u)^2\,du\biggr),$$

then by Girsanov's theorem, for every $A \in \mathcal{F}_1$,

$$\mathrm{Q}_{\xi,s}^\varepsilon(A) = \mathrm{E}_{\xi,s}^\varepsilon[\zeta_1 1_{\{X \in A\}}]. \tag{12}$$

Let G denote a primitive of $\tilde{b}$: $G(\xi) = \int_{\xi_0}^{\xi} \tilde{b}(z)\,dz$, for some ξ_0. Then, by Itô's formula, $\mathrm{P}_{\xi,s}^{\eta,\varepsilon}$-a.s.,

$$\int_s^1 \tilde{b}(X_u)\,dX_u = G(X_1) - G(\xi) - \frac{\varepsilon}{2}\int_s^1 \tilde{b}'(X_u)\,du$$

so that

$$\zeta_1 = \exp\biggl(G(X_1) - G(\xi) - \frac{\varepsilon}{2}\int_s^1 [\tilde{b}'(X_u) + \tilde{b}(X_u)^2]\,du\biggr). \tag{13}$$

The following elementary lemma allows us to compare the laws $\hat{\mathrm{Q}}_{\xi,s}^{\eta,\varepsilon}$ and $\hat{\mathrm{P}}_{\xi,s}^{\eta,\varepsilon}$.

LEMMA 3.1. *Let* $(E, \mathcal{E})$, $(H, \mathcal{H})$ *be measurable spaces and let* ν, μ *be measures on* $(E, \mathcal{E})$. *Let* $\pi : (E, \mathcal{E}) \to (H, \mathcal{H})$ *be a measurable map and let* $\bar{\nu}$ *and* $\bar{\mu}$ *be the measures on* $(H, \mathcal{H})$ *induced by* π, *that is, for any* $\Gamma \in \mathcal{H}$,

$$\bar{\nu}(\Gamma) = \nu(\pi^{-1}\Gamma), \qquad \bar{\mu}(\Gamma) = \mu(\pi^{-1}\Gamma).$$

Suppose that the following hold:

(a) *both* ν *and* μ *admit a disintegration on* $(E, \mathcal{H})$: *there exist kernels* $\nu_y(dz)$ *and* $\mu_y(dz)$ *on* $H \times \mathcal{E}$ *such that*

$$\nu(dz) = \int_H \bar{\nu}(dy)\nu_y(dz), \qquad \mu(dz) = \int_H \bar{\mu}(dy)\mu_y(dz);$$

(b) ν *and* $\bar{\nu}$ *are absolutely continuous with respect to* μ *and* $\bar{\mu}$, *with densities* g *and* $\bar{g}$, *respectively*.

Then for almost every fixed $y \in H$, *the kernel* $\nu_y(dz)$ *is absolutely continuous with respect to* $\mu_y(dz)$ *and*

$$\nu_y(dz) = \frac{g(z)}{\bar{g}(y)}\mu_y(dz).$$

PROOF. The proof is immediate. For every measurable map $\psi : E \to \mathbb{R}$ one has

$$\int_E \psi(z)\,\nu(dz) = \int_H \bar{\nu}(dy) \int_E \psi(z)\nu_y(dz)$$

but also

$$\begin{aligned}\int_E \psi(z)\nu(dz) &= \int_E \psi(z)g(z)\mu(dz) = \int_H \bar{\mu}(dy) \int_E \psi(z)g(z)\mu_y(dz) \\ &= \int_H \bar{\nu}(dy) \int_E \psi(z)\frac{g(z)}{\bar{g}(y)}\mu_y(dz).\end{aligned}$$

□

Let us compute the density of $\hat{\mathrm{Q}}^{\eta,\varepsilon}_{\xi,s}$ with respect to $\hat{\mathrm{P}}^{\eta,\varepsilon}_{\xi,s}$ on $\mathcal{F}_1$. Let $\pi : \mathcal{C}([0,1], \mathbb{R}) \to \mathbb{R}$ be the map defined by $\pi(\varphi) = \varphi(1)$. Thus, by Lemma 3.1 and (12), for $A \in \mathcal{F}_1$,

$$\hat{\mathrm{Q}}^{\eta,\varepsilon}_{\xi,s}(A) = \frac{1}{\bar{g}(\eta)}\hat{\mathrm{E}}^{\eta,\varepsilon}_{\xi,s}[\zeta_1 1_{\{X \in A\}}],$$

where $\bar{g}$ stands for the density of the law of $U^{\varepsilon}_{\xi,s}(1)$ with respect to the law of $W^{\varepsilon}_{\xi,s}(1)$. Since these (real) random variables are both absolutely continuous, such a density turns out to be the ratio between the respective densities with respect to the Lebesgue measure: if by $q_\varepsilon(t-s, \xi, \eta)$ and $p_\varepsilon(t-s, \xi, \eta)$ we denote the transition densities of U^{ε} and W^{ε}, respectively, then

$$\bar{g}(\eta) = \frac{q_\varepsilon(1-s, \xi, \eta)}{p_\varepsilon(1-s, \xi, \eta)}.$$

It then follows that, for any $A \in \mathcal{F}_1$,

$$\hat{\mathrm{Q}}^{\eta,\varepsilon}_{\xi,s}(A) = \frac{p_\varepsilon(1-s,\xi,\eta)}{q_\varepsilon(1-s,\xi,\eta)} \hat{\mathrm{E}}^{\eta,\varepsilon}_{\xi,s}[\zeta_1 1_{\{X\in A\}}]$$

and, by (13),

$$\hat{\mathrm{Q}}^{\eta,\varepsilon}_{\xi,s}(A) = \frac{p_\varepsilon(1-s,\xi,\eta)}{q_\varepsilon(1-s,\xi,\eta)} e^{G(\eta)-G(\xi)} \hat{\mathrm{E}}^{\eta,\varepsilon}_{\xi,s}[e^{-(\varepsilon/2)\int_s^1 [\tilde{b}'(X_u)+\tilde{b}(X_u)^2]\,du} 1_{\{X\in A\}}]. \tag{14}$$

In the next statement we use this representation to deduce that, roughly speaking, the asymptotics of $\hat{\mathrm{Q}}^{\eta,\varepsilon}_{\xi}(A)$ does not depend of $\tilde{b}$.

Note, however, that if $\tilde{b}' + \tilde{b}^2 = \text{const}$, then (14) gives $\hat{\mathrm{Q}}^{\eta,\varepsilon}_{\xi,s}(A) = c(\xi,\eta,\varepsilon,s) \times \hat{\mathrm{P}}^{\eta,\varepsilon}_{\xi,s}(A)$. Since both $\hat{\mathrm{Q}}^{\eta,\varepsilon}_{\xi,s}$ and $\hat{\mathrm{P}}^{\eta,\varepsilon}_{\xi,s}$ are probabilities, this means that $\hat{\mathrm{Q}}^{\eta,\varepsilon}_{\xi,s} = \hat{\mathrm{P}}^{\eta,\varepsilon}_{\xi,s}$. This remark is contained in Benjamini and Lee (1997), where the equation $\tilde{b}' + \tilde{b}^2 = \text{const}$ is studied [it has solutions $\tilde{b} = \text{const}$ or $\tilde{b}(\zeta) = k\tanh(k\zeta + c)$ for some constants k, c].

PROPOSITION 3.2. *Suppose that $\tilde{b}$ is a bounded and continuously differentiable function on $\mathring{\mathcal{J}} = \mathbb{R}$, with bounded derivative. Then if $A \in \mathcal{F}_1$ is an event, possibly depending on ε,*

$$\hat{\mathrm{Q}}^{\eta,\varepsilon}_{\xi,s}(A) \sim \hat{\mathrm{P}}^{\eta,\varepsilon}_{\xi,s}(A)$$

uniformly for (ξ,η) in a compact subset of $\mathbb{R}^2$ as $\varepsilon \to 0$. If moreover A is contained in a set of paths taking values in a bounded subset of $\mathring{\mathcal{J}}$ and independent of ε, then the assumptions that $\mathring{\mathcal{J}} = \mathbb{R}$ and that $\tilde{b}$ and $\tilde{b}'$ are bounded can be dropped.

PROOF. First let us remark that, for $s \in [0,1[$,

$$\frac{p_\varepsilon(1-s,\xi,\eta)\, e^{G(\eta)-G(\xi)}}{q_\varepsilon(1-s,\xi,\eta)} \to 1 \tag{15}$$

as $\varepsilon \to 0$. This follows from Lemma 4.3 below. This relationship might also be derived by the results of Molchanov [(1975), Theorem 2.1], concerning the behavior of the transition density in a short time [see also Elie (1980)]. Molchanov estimates actually hold in a much broader generality but require regularity assumptions on $\tilde{b}$ (derivatives of order 3 at least) which are stronger than those we are considering here. If A is a bounded set of paths, one could write

$$e^{\varepsilon M_1} \hat{\mathrm{P}}^{\eta,\varepsilon}_{\xi,s}(A) \le \hat{\mathrm{E}}^{\eta,\varepsilon}_{\xi,s}[e^{-(\varepsilon/2)\int_0^1 [\tilde{b}'(X_s)+\tilde{b}(X_s)^2]\,ds} 1_{\{X\in A\}}] \le e^{\varepsilon M_2} \hat{\mathrm{P}}^{\eta,\varepsilon}_{\xi,s}(A),$$

M_1 and M_2 being respectively the supremum and the infimum of $\frac{1}{2}(\tilde{b}'(x) + \tilde{b}^2(x))$ over the set $\{x, |x| \le \sup_{w\in A} \|w\|_\infty\}$. This completes the proof if A is contained, for every ε, in a bounded set of paths or the function $\tilde{b}' + \tilde{b}^2$ is bounded over $\mathbb{R}$. □

If f is as in the statement of Theorem 2.1 and $\tau_f^\varepsilon = \inf\{t; X_t \geq f(\varepsilon t)\}$, by applying Proposition 3.2 to the set $A = \{\tau_f^\varepsilon < 1\}$, this already concludes the proof of Theorem 2.1(a), under the additional assumption that $\tilde{b}$ and its derivative are bounded and $\mathring{\mathcal{J}} = \mathbb{R}$. It is, however, apparent that the asymptotics of the crossing probability as $\varepsilon \to 0$ should depend only on the behavior of the coefficients near x, y and the barrier. Thus a localization argument will allow us to conclude the proof of Theorem 2.1. This point is investigated in the next section.

4. Proofs. Let U^ε be a one-dimensional diffusion process satisfying

$$\begin{aligned} dU_t^\varepsilon &= \varepsilon \tilde{b}(U_t^\varepsilon)\,dt + \sqrt{\varepsilon}\,dB_t, \\ U_s^\varepsilon &= \xi, \end{aligned}$$

with $\tilde{b}$ locally Lipschitz continuous, until the exit from an interval $]\Delta_1, \Delta_2[$. We assume that U^ε has a transition density $q_\varepsilon(t-s, \xi, \cdot)$. Let us still denote by $\mathrm{Q}_{\xi,s}^\varepsilon$ the law of U^ε on the canonical space $\mathcal{C}$ with starting condition $U_s^\varepsilon = \xi$, and by $\hat{\mathrm{Q}}_{\xi,s}^{\eta,\varepsilon}$ the law of the corresponding conditioned process.

REMARK 4.1. Let us point out the following properties for the process U^ε and for its transition density q_ε.

(i) The process U^ε satisfies a SDE of the type $dU_t^\varepsilon = \beta_\varepsilon(U_t^\varepsilon)\,dt + \sqrt{\varepsilon}\alpha(U_t^\varepsilon) \times dB_t$, with $\beta_\varepsilon(u) = \varepsilon\tilde{b}(u)$ and $\alpha(u) \equiv 1$. In particular, $\beta_\varepsilon(u)$ is locally Lipschitz continuous for any ε and it converges to 0 as $\varepsilon \to 0$ uniformly on the compact subsets. Thus, by applying classical results [see, e.g., Azencott (1980) or Baldi and Chaleyat-Maurel (1986)], a large deviation principle for the processes $\{U^\varepsilon\}_\varepsilon$ can be stated, with speed $1/\varepsilon$ and rate function given by

$$I(\gamma) = \begin{cases} \frac{1}{2}\displaystyle\int_s^1 \dot{\gamma}_u^2\,du, & \text{if } \gamma \text{ is absolutely continuous,} \\ +\infty, & \text{otherwise.} \end{cases} \tag{16}$$

Let us stress that the rate function I does not depend on $\tilde{b}$; this is the main reason the zeroth-order asymptotics are not affected by the drift.

(ii) Let us recall that, by classical arguments [see Elie (1980), Section 4.2], the asymptotics of $q_\varepsilon(s, \xi, \eta)$ changes only by a quantity which is exponentially negligible if $\tilde{b}$ is modified outside an open interval containing ξ, η and whose closure is contained in $]\Delta_1, \Delta_2[$. Thus, if $\tilde{b} \in \mathcal{C}^1(]\Delta_1, \Delta_2[)$, $\tilde{b}$ can be suitably modified and extended in order to be bounded with bounded derivative on $\mathbb{R}$ and the hypotheses of Proposition 3.2 are satisfied. In particular (15) holds, so that

$$\lim_{\varepsilon\to 0} \varepsilon \log q_\varepsilon(1-u, \zeta, \xi) = \lim_{\varepsilon\to 0} \varepsilon \log p_\varepsilon(1-u, \zeta, \xi) = -\frac{1}{2(1-u)}|\zeta - \eta|^2.$$

LEMMA 4.2. *Assume that* $\tilde{b} \in \mathcal{C}^1(]\Delta_1, \Delta_2[)$. *Let* $\tilde{\Delta}_1, \tilde{\Delta}_2$ *be such that* $\Delta_1 < \tilde{\Delta}_1 < \tilde{\Delta}_2 < \Delta_2$ *and* $\xi, \eta, a \in]\tilde{\Delta}_1, \tilde{\Delta}_2[$ *with* $\xi < a$, $\eta < a$. *Let* τ_a *be the passage time in* a *and let* $\tilde{\tau}$ *be the exit time from* $]\tilde{\Delta}_1, \tilde{\Delta}_2[$. *Then we have the following*:

(a)

$$\lim_{\varepsilon \to 0} \varepsilon \log \hat{\mathrm{Q}}^{\eta,\varepsilon}_{\xi,s}(\tau_a < 1) = -\frac{2}{1-s}(a-\xi)(a-\eta).$$

(b) *As* $\varepsilon \to 0$ *the probability* $\hat{\mathrm{Q}}^{\eta,\varepsilon}_{\xi,s}(\tau_a < 1, \tilde{\tau} < 1)$ *is exponentially negligible with respect to* $\hat{\mathrm{Q}}^{\eta,\varepsilon}_{\xi,s}(\tau_a < 1)$, *that is*,

$$\lim_{\varepsilon \to 0} \varepsilon \log \hat{\mathrm{Q}}^{\eta,\varepsilon}_{\xi,s}(\tau_a < 1, \tilde{\tau} < 1) < \lim_{\varepsilon \to 0} \varepsilon \log \hat{\mathrm{Q}}^{\eta,\varepsilon}_{\xi,s}(\tau_a < 1).$$

(c) *Let* g_1, g_2 *be continuous functions* $[0,1] \to]\tilde{\Delta}_1, \tilde{\Delta}_2[$ *such that* $g_1(0) < \xi$, $\eta < g_2(0)$. *Let us define, for every* $\varepsilon > 0$, $\tau_{2,\varepsilon} = \inf\{t \geq s, X_t \geq g_2(\varepsilon t)\}$, $\tau_{1,\varepsilon} = \inf\{t \geq s, X_t \leq g_1(\varepsilon t)\}$. *Then*:

(c1) *As* $\varepsilon \to 0$, $\hat{\mathrm{Q}}^{\eta,\varepsilon}_{\xi,s}(\tau_{2,\varepsilon} < 1, \tau_{1,\varepsilon} < 1)$ *is exponentially negligible with respect to* $\hat{\mathrm{Q}}^{\eta,\varepsilon}_{\xi,s}(\tau_{2,\varepsilon} < 1)$.

(c2) *Set* $\tau_\varepsilon = \min(\tau_{1,\varepsilon}, \tau_{2,\varepsilon})$. *Then, as* $\varepsilon \to 0$, $\hat{\mathrm{Q}}^{\eta,\varepsilon}_{\xi,s}(\tau_\varepsilon < 1, \tilde{\tau} < 1)$ *is exponentially negligible with respect to* $\hat{\mathrm{Q}}^{\eta,\varepsilon}_{\xi,s}(\tau_\varepsilon < 1)$.

PROOF. (a) Let us define, for $t \geq s$,

$$M_t = \frac{q_\varepsilon(1-t, X_t, \eta)}{q_\varepsilon(1-s, \xi, \eta)},$$

q_ε being the transition density of U_ε. It is well known that M is a martingale such that $\mathrm{E}^{\mathrm{Q}^\varepsilon_{\xi,s}}(M_t) = 1$, $t \geq s$, and that, for every δ, $0 < \delta < 1$, $\hat{\mathrm{Q}}^{\eta,\varepsilon}_{\xi,s}$ has a density on $\mathcal{F}^s_{1-\delta} = \sigma(X_u, s \leq u \leq 1-\delta)$ with respect to $\mathrm{Q}^\varepsilon_{\xi,s}$ which is given by $M_{1-\delta}$. Thus

$$\hat{\mathrm{Q}}^{\eta,\varepsilon}_{\xi,s}(\tau_a < 1-\delta) = \frac{1}{q_\varepsilon(1-s,\xi,\eta)} \mathrm{E}^{\mathrm{Q}^\varepsilon_{\xi,s}}[q_\varepsilon(\delta, X_{1-\delta}, \eta) 1_{\{\tau_a < 1-\delta\}}].$$

Since M is a $\mathrm{Q}^\varepsilon_{\xi,s}$-martingale, by conditioning with respect to $\mathcal{F}_{\tau_a \wedge (1-\delta)}$ and sending $\delta \to 0$, we get

$$\begin{aligned}\hat{\mathrm{Q}}^{\eta,\varepsilon}_{\xi,s}(\tau_a < 1) &= \frac{1}{q_\varepsilon(1-s,\xi,\eta)} \mathrm{E}^{\mathrm{Q}^\varepsilon_{\xi,s}}[q_\varepsilon(1-\tau_a, X_{\tau_a}, \eta) 1_{\{\tau_a < 1\}}] \\ &= \frac{1}{q_\varepsilon(1-s,\xi,\eta)} \mathrm{E}^{\mathrm{Q}^\varepsilon_{\xi,s}}[q_\varepsilon(1-\tau_a, a, \eta) 1_{\{\tau_a < 1\}}].\end{aligned}$$

As recalled in Remark 4.1(i), $\{U^\varepsilon\}_\varepsilon$ satisfies a large deviation principle, with speed $1/\varepsilon$ and rate function I given by (16). By applying standard arguments in large deviation theory it then follows that

$$\lim_{\varepsilon \to 0} \varepsilon \log \mathrm{Q}^\varepsilon_{\xi,s}(\tau_a < 1) = -\inf_{\tau_a(\gamma) < 1, \gamma(s) = \xi} I(\gamma).$$

Moreover, recalling Remark 4.1(ii),

$$\lim_{\varepsilon\to 0}\varepsilon\log q_\varepsilon(1-u,\zeta,\eta)=-\frac{1}{2(1-u)}|\zeta-\eta|^2.$$

Thus, by using Varadhan's lemma [see Dembo and Zeitouni (1998) or Varadhan (1984); see also the Appendix], one gets

$$\begin{aligned}&\lim_{\varepsilon\to 0}\varepsilon\log \mathrm{E}^{\mathrm{Q}^{\varepsilon}_{\xi,s}}[q_\varepsilon(1-\tau_a,a,\eta)1_{\{\tau_a<1\}}]\\&\qquad=-\inf_{\tau_a(\gamma)<1,\gamma(s)=\xi}\biggl(I(\gamma)+\frac{1}{2(1-\tau_a(\gamma))}|a-\eta|^2\biggr).\end{aligned}$$

Let us compute the infimum of the r.h.s. It is clear that the minimizers are paths γ which are linear between ξ and a and are constant thereafter. Such a path is of the form

$$\gamma(u)\stackrel{\text{def}}{=}\gamma_t(u)=\frac{u-s}{t-s}a+\frac{t-u}{t-s}\xi$$

and $\gamma_t(u)\equiv a$ as $t\le u\le 1$; the parameter t here is the time at which the path γ_t reaches the level a. Then one has

$$I(\gamma_t)+\frac{1}{2(1-\tau_a(\gamma_t))}|a-\eta|^2=\frac{1}{2}\frac{(a-\xi)^2}{t-s}+\frac{1}{2}\frac{(a-\eta)^2}{1-t}.$$

A straightforward computation gives that the minimum over t, $s<t<1$, is

$$\inf_{\tau_a(\gamma)<1,\gamma(s)=\xi}\biggl(I(\gamma)+\frac{1}{2(1-\tau_a(\gamma))}|a-\eta|^2\biggr)=\frac{1}{2(1-s)}(2a-\xi-\eta)^2.$$

Thus

$$\begin{aligned}&\lim_{\varepsilon\to 0}\varepsilon\log\hat{\mathrm{Q}}^{\eta,\varepsilon}_{\xi,s}(\tau_a<1)\\&\quad=-\lim_{\varepsilon\to 0}\varepsilon\log q_\varepsilon(1-s,\xi,\eta)+\lim_{\varepsilon\to 0}\varepsilon\log\mathrm{E}^{\mathrm{Q}^{\varepsilon}_{\xi,s}}[q_\varepsilon(1-\tau_a,a,\eta)1_{\{\tau_a\le 1\}}]\\&\quad=-\frac{1}{2(1-s)}\bigl((2a-\xi-\eta)^2-(\xi-\eta)^2\bigr)=-\frac{2}{(1-s)}(a-\xi)(a-\eta)\stackrel{\text{def}}{=}-I_0.\end{aligned}$$

(b) One can write $\{\tau_a<1,\tilde\tau<1\}$ as the union of the events $\{\tilde\tau\le\tau_a<1\}$ and $\{\tau_a<\tilde\tau<1\}$. By the same arguments as in (a) one has

$$\begin{aligned}&\lim_{\varepsilon\to 0}\varepsilon\log\hat{\mathrm{Q}}^{\eta,\varepsilon}_{\xi,s}(\tilde\tau\le\tau_a<1)\\&\quad=-\lim_{\varepsilon\to 0}\varepsilon\log q_\varepsilon(1-s,\xi,\eta)+\lim_{\varepsilon\to 0}\varepsilon\log\mathrm{E}^{\mathrm{Q}^{\varepsilon}_{\xi,\eta}}[q_\varepsilon(1-\tau_a,a,\eta)1_{\{\tilde\tau\le\tau_a<1\}}]\\&\quad=\frac{1}{2(1-s)}(\xi-\eta)^2-\inf_{\gamma\in\Gamma}\biggl(I(\gamma)+\frac{1}{2(1-\tau_a(\gamma))}|a-\eta|^2\biggr),\end{aligned}$$

the infimum being taken now on the set Γ of the paths such that $\gamma(s)=\xi$, $\tilde\tau(\gamma)\le\tau_a(\gamma)<1$.

If $\gamma\in\Gamma$, then

$$I(\gamma)>\frac{1}{2}\int_{\tilde\tau(\gamma)}^1\dot\gamma_u^2\,du,$$

so that

$$\begin{aligned}\inf_{\gamma\in\Gamma}&\biggl(I(\gamma)+\frac{1}{2(1-\tau_a(\gamma))}|a-\eta|^2\biggr)\\&\ge\inf_{\gamma\in\Gamma}\biggl(\frac{1}{2}\int_{\tilde\tau(\gamma)}^1\dot\gamma_u^2\,du+\frac{1}{2(1-\tau_a(\gamma))}|a-\eta|^2\biggr)\end{aligned}$$

and, by the same argument as in (a), one gets

$$\begin{aligned}\inf_{\gamma\in\Gamma}&\biggl(I(\gamma)+\frac{1}{2(1-\tau_a(\gamma))}|a-\eta|^2\biggr)\\&\ge-\frac{1}{2(1-s)}(\xi-\eta)^2+\frac{1}{2(1-s)}(2a-\tilde\Delta_1-\eta)^2.\end{aligned}$$

Thus,

$$\begin{aligned}\lim_{\varepsilon\to0}\varepsilon\log\hat{\mathrm{Q}}^{\eta,\varepsilon}_{\xi,s}(\tilde\tau\le\tau_a<1)&\le\frac{1}{2(1-s)}(\xi-\eta)^2-\frac{1}{2(1-s)}(2a-\tilde\Delta_1-\eta)^2\\&<\frac{1}{2(1-s)}(\xi-\eta)^2-\frac{1}{2(1-s)}(2a-\xi-\eta)^2\\&=-\frac{2}{(1-s)}(a-\xi)(a-\eta)=-I_0.\end{aligned}$$

With similar arguments one can estimate the quantity $\hat{\mathrm{Q}}^{\eta,\varepsilon}_{\xi,s}(\tau_a<\tilde\tau<1)$. Splitting this into the sum of $\hat{\mathrm{Q}}^{\eta,\varepsilon}_{\xi,s}(\tau_a<\tilde\tau<1,X_{\tilde\tau}=\tilde\Delta_1)$ and $\hat{\mathrm{Q}}^{\eta,\varepsilon}_{\xi,s}(\tau_a<\tilde\tau<1,X_{\tilde\tau}=\tilde\Delta_2)$ it is not difficult to prove that

$$\lim_{\varepsilon\to0}\varepsilon\log\hat{\mathrm{Q}}^{\eta,\varepsilon}_{\xi,s}(\tau_a<\tilde\tau<1)=-I_1<-I_0.$$

(c1) Again we split $\{\tau_{2,\varepsilon}<1,\tau_{1,\varepsilon}<1\}$ into the union of $\{\tau_{1,\varepsilon}<\tau_{2,\varepsilon}<1\}$ and $\{\tau_{2,\varepsilon}<\tau_{1,\varepsilon}<1\}$.

For every $\delta>0$ and $t\in[0,1]$, one has, for ε small, $g_i(0)-\delta\le g_i(\varepsilon t)\le g_i(0)+\delta$, $i=1,2$. Let $\tilde\tau_{2+}$ and $\tilde\tau_{2-}$ denote the passage times at $g_2(0)+\delta$ and $g_2(0)-\delta$, respectively, and let $\tilde\tau_{1+}$ be the passage time at $g_1(0)+\delta$. Obviously

$$\mathrm{Q}^{\eta,\varepsilon}_{\xi,s}(\tau_{2,\varepsilon}<1)\le\mathrm{Q}^{\eta,\varepsilon}_{\xi,s}(\tilde\tau_{2+}<1),$$

$$\mathrm{Q}^{\eta,\varepsilon}_{\xi,s}(\tau_{2,\varepsilon}<1,\tau_{1,\varepsilon}<1)\ge\mathrm{Q}^{\eta,\varepsilon}_{\xi,s}(\tau_{2-}<1,\tau_{1+}<1).$$

We know from (a) and the proof of (b) that

$$\begin{aligned}\lim_{\varepsilon\to 0} &\varepsilon \log \hat{Q}^{\eta,\varepsilon}_{\xi,s}(\tilde{\tau}_{2+} < 1)\\ &= -\frac{2}{1-s}\big(g_2(0)+\delta-\xi\big)\big(g_2(0)+\delta-\eta\big) \stackrel{\text{def}}{=} -I_0^\delta,\\ \lim_{\varepsilon\to 0} &\varepsilon \log \hat{Q}^{\eta,\varepsilon}_{\xi,s}(\tau_{2-} < 1, \tau_{1+} < 1)\\ &= -\frac{2}{1-s}\big(g_2(0)-\delta-g_1(0)-\delta\big)\big(g_2(0)-\delta-\eta\big) \stackrel{\text{def}}{=} -I_1^\delta.\end{aligned}$$

Since $g_1(0) < \xi$, it is clear that for δ small enough $I_0^\delta < I_1^\delta$, so that $\hat{Q}^{\eta,\varepsilon}_{\xi,s}(\tau_{1,\varepsilon} < \tau_{2,\varepsilon} < 1)$ is exponentially negligible with respect to $\hat{Q}^{\eta,\varepsilon}_{\xi,s}(\tau_{2,\varepsilon} < 1)$. Similarly one proves the same for $\hat{Q}^{\eta,\varepsilon}_{\xi,s}(\tau_{2,\varepsilon} < \tau_{1,\varepsilon} < 1)$.

(c2) The proof follows much the same pattern as (a) and (b). Here of course $\tau_\varepsilon < \tilde{\tau}$ so that $\{\tau_\varepsilon < 1, \tilde{\tau} < 1\} = \{\tilde{\tau} < 1\}$. Now arguing as in (b) one gets easily that

$$\lim_{\varepsilon\to 0} \varepsilon \log \hat{Q}^{\eta,\varepsilon}_{\xi,s}(\tilde{\tau} < 1) = -\frac{1}{1-s} J_1,$$

where $J_1 = \min((\tilde{\Delta}_2 - \xi)(\tilde{\Delta}_2 - \eta), (\xi - \tilde{\Delta}_1)(\eta - \tilde{\Delta}_1))$. To show that $\hat{Q}^{\eta,\varepsilon}_{\xi,s}(\tilde{\tau} < 1)$ is exponentially negligible with respect to $\hat{Q}^{\eta,\varepsilon}_{\xi,s}(\tau_\varepsilon < 1)$, let $\tilde{\Delta}_1', \tilde{\Delta}_2'$ be such that $\tilde{\Delta}_1 < \tilde{\Delta}_1' < g_1(t) < g_2(t) < \tilde{\Delta}_2' < \tilde{\Delta}_2$ for every $t \le \varepsilon_0$ for some ε_0 small. Let $\tilde{\tau}'$ be the exit time from $]\tilde{\Delta}_1', \tilde{\Delta}_2'[$. Of course $\hat{Q}^{\eta,\varepsilon}_{\xi,s}(\tau_\varepsilon < 1) \ge \hat{Q}^{\eta,\varepsilon}_{\xi,s}(\tilde{\tau}' < 1)$ and

$$\lim_{\varepsilon\to 0} \varepsilon \log \hat{Q}^{\eta,\varepsilon}_{\xi,s}(\tilde{\tau}' < 1) = -\frac{1}{1-s} J_1',$$

where $J_1' = \min((\tilde{\Delta}_2' - \xi)(\tilde{\Delta}_2' - \eta), (\xi - \tilde{\Delta}_1')(\eta - \tilde{\Delta}_1'))$. Since $J_1' < J_1$, (c2) is proved. $\square$

PROOF OF THEOREM 2.1(a). For F as in (7), set $F(\mathring{\mathcal{I}}) = \mathring{\mathcal{J}} =]\Delta_1, \Delta_2[$ and $g(t) = F(f(t))$. If $Y_t = F(Z_t)$ and $U_t^\varepsilon = Y_{\varepsilon t}$, since U^ε solves (10) on $\mathring{\mathcal{J}}$, clearly

$$P(\hat{\tau}_f^\varepsilon \le \varepsilon) = \hat{Q}^{\eta,\varepsilon}_{\xi,0}(\tau_\varepsilon < 1),$$

where $\hat{Q}^{\eta,\varepsilon}_{\xi,0}$ is the law on $\mathcal{C}$ of U^ε with starting condition $U_0^\varepsilon = \xi = F(x)$ and pinned by $U_1^\varepsilon = \eta = F(y)$ and $\tau_\varepsilon = \inf\{u > 0, X_u \ge g(\varepsilon u)\}$. By Lemma 4.2(c1), with $g_2 = g$ and $g_1 \equiv \tilde{\Delta}_1$, where $\tilde{\Delta}_1 > \Delta_1$ and $\tilde{\Delta}_1 < \xi, \eta$, we obtain

$$\hat{Q}^{\eta,\varepsilon}_{\xi,0}(\tau_\varepsilon < 1) \sim \hat{Q}^{\eta,\varepsilon}_{\xi,0}(\tau_\varepsilon < 1, \tilde{\tau} > 1).$$

The sets of paths $A_\varepsilon = \{\tau_\varepsilon < 1, \tilde{\tau} > 1\}$ is such that $|\tilde{b}'(\gamma_u) + \tilde{b}^2(\gamma_u)| \le K$ for every $u \in [0, 1]$ and $\gamma \in A_\varepsilon$, for some K. Then Proposition 3.2 yields

$$\hat{Q}^{\eta,\varepsilon}_{\xi,0}(\tau_\varepsilon < 1, \tilde{\tau} > 1) \sim P^{\eta,\varepsilon}_{\xi,0}(\tau_\varepsilon < 1, \tilde{\tau} > 1),$$

so that, again by Lemma 4.2(c),

$$\hat{\mathrm{Q}}^{\eta,\varepsilon}_{\xi,0}(\tau_\varepsilon < 1) \sim \hat{\mathrm{P}}^{\eta,\varepsilon}_{\xi,0}(\tau_\varepsilon < 1).$$

Now, taking into account Proposition 5.3 of Baldi, Caramellino and Iovino (1999),

$$\hat{\mathrm{P}}^{\eta,\varepsilon}_{\xi,0}(\tau_\varepsilon < 1) \sim e^{-2g'(0)(g(0)-\xi)} e^{-(2/\varepsilon)(g(0)-\xi)(g(0)-\eta)}.$$

By replacing $\xi = F(x)$, $\eta = F(y)$, $g(t) = F \circ f(t)$, the statement is proved. $\square$

Corollaries 2.3 and 2.4 are immediate consequences Theorems 2.1(a) and 2.2, respectively.

To complete the proofs of Theorem 2.1 and Theorem 2.2 we need some intermediate results. Let us recall that the family of processes $\{\hat{\mathrm{P}}^{\eta,\varepsilon}_{\xi,s}\}_\varepsilon$ satisfies a large deviation principle with rate function $J = J_{\xi,\eta}$ given by

$$J(\gamma) = \tfrac{1}{2} \int_s^1 \dot{\gamma}_s^2 \, ds \tag{17}$$

if γ is absolutely continuous and $\gamma(s) = \xi$, $\gamma(1) = \eta$; otherwise $J = +\infty$. Let Φ^1_ε and Φ^2_ε be defined, as $\varepsilon > 0$, by

$$\frac{p_\varepsilon(1-s,\xi,\eta)e^{G(\eta)-G(\xi)}}{q_\varepsilon(1-s,\xi,\eta)} = 1 + \varepsilon \Phi^1_\varepsilon(s,\xi,\eta),$$

$$\hat{\mathrm{E}}^{\eta,\varepsilon}_{\xi,s}[e^{-(\varepsilon/2)\int_s^1 [\tilde{b}'(X_u)+\tilde{b}(X_u)^2]\,ds} 1_{\{X\in A\}}] = \hat{\mathrm{P}}^{\eta,\varepsilon}_{\xi,s}(A)\big(1 + \varepsilon \Phi^2_\varepsilon(s,\xi,\eta;A)\big),$$

so that, recalling (12),

$$\hat{\mathrm{Q}}^{\eta,\varepsilon}_{\xi,s}(A) = \hat{\mathrm{P}}^{\eta,\varepsilon}_{\xi,s}(A)\big(1 + \varepsilon\Phi^1_\varepsilon(s,\xi,\eta)\big)\big(1 + \varepsilon\Phi^2_\varepsilon(s,\xi,\eta;A)\big). \tag{18}$$

The next result implies, in particular, that

$$\lim_{\varepsilon\to 0} \frac{p_\varepsilon(1-s,\xi,\eta)e^{G(\eta)-G(\xi)}}{q_\varepsilon(1-s,\xi,\eta)} = 1. \tag{19}$$

LEMMA 4.3. *Assume that* $\xi, \eta \in \mathring{\mathcal{J}}$, $\tilde{b}'$ *is differentiable and* $\tilde{b}'$ *is locally Lipschitz continuous on* $\mathring{\mathcal{J}}$. *Then the following hold*:

(i) *If* $\gamma = \gamma_{\xi,\eta}$ *denotes the path joining* ξ *to* η *traveled at constant speed*,

$$\lim_{\varepsilon\to 0} \Phi^1_\varepsilon(s,\xi,\eta) = \tfrac{1}{2}\int_s^1 (\tilde{b}' + \tilde{b}^2)\big(\gamma_{\xi,\eta}(u)\big)\,du.$$

(ii) *Let* $A \in \mathcal{F}_1$ *be a set of paths* γ *such that* $\gamma(t) \in \mathring{\mathcal{J}}$ *for every* $t \in [0,1]$. *Assume that there exists a unique path* $\rho = \rho_{\xi,\eta}$ *such that*

$$J(\rho) = \inf_{\varphi\in\mathring{A}} J(\varphi) = \inf_{\varphi\in\bar{A}} J(\varphi),$$

where $J = J_{\xi,\eta}$ *is as in* (17). *Then*

$$\lim_{\varepsilon\to 0} \Phi_\varepsilon^2(s,\xi,\eta;A) = -\tfrac{1}{2}\int_s^1 (\tilde{b}' + \tilde{b}^2)\big(\rho_{\xi,\eta}(u)\big)\,du.$$

(iii) *Consider the notation of the statement of Lemma* 4.2. *Let* $g:[0,1] \to]\tilde{\Delta}_1, \tilde{\Delta}_2[$ *be a continuously differentiable function with Lipschitz continuous derivative such that* $g(t) \geq \xi, \eta$ *for every* $t \in [0,1]$. *Let* τ_g^ε *be the stopping time* $\tau_g^\varepsilon = \inf\{t; X_t \geq g(\varepsilon t)\}$ *and let* A_ε *be defined as* $A_\varepsilon = \{\tau_g^\varepsilon \leq 1, \tilde{\tau} > 1\}$. *If* $\rho = \rho_{\xi,\eta} = \arg\min_A J_{\xi,\eta}$, *with* $A = \{\varphi; \varphi(t) \geq g(0)$ *for some* $t \leq 1\}$, *then*

$$\lim_{\varepsilon\to 0} \Phi_\varepsilon^2(s,\xi,\eta;A_\varepsilon) = -\tfrac{1}{2}\int_s^1 (\tilde{b}' + \tilde{b}^2)\big(\rho_{\xi,\eta}(u)\big)\,du.$$

PROOF. We use the notation $\beta = \tilde{b}' + \tilde{b}^2$.

(i) First we use the localization argument as in Remark 4.1(ii): to study the asymptotics of q_ε, $\tilde{b}$ can be modified outside an open interval containing ξ, η and whose closure is contained in $\mathring{\mathcal{J}}$. We can thus assume that β is bounded and Lipschitz continuous on $\mathbb{R}$, so that the hypotheses of Proposition 3.2 are satisfied and in particular representation (14) holds.

For $\delta > 0$, let $B_\delta(\eta)$ denote the open interval of radius δ and centered at η. If m stands for the Lebesgue measure, one has

$$\begin{aligned} p_\varepsilon(1-s,\xi,\eta) &= \lim_{\delta\to 0} \frac{\mathrm{P}^\varepsilon_{\xi,s}(X_1 \in B_\delta(\eta))}{m(B_\delta(\eta))},\\ q_\varepsilon(1-s,\xi,\eta) &= \lim_{\delta\to 0} \frac{\mathrm{Q}^\varepsilon_{\xi,s}(X_1 \in B_\delta(\eta))}{m(B_\delta(\eta))}\\ &= \lim_{\delta\to 0} \frac{\mathrm{E}^\varepsilon_{\xi,s}[e^{G(X_1)-G(\xi)-(\varepsilon/2)\int_s^1 \beta(X_u)\,du} 1_{\{X_1\in B_\delta(\eta)\}}]}{m(B_\delta(\eta))}. \end{aligned}$$

Thus,

$$1 + \varepsilon\Phi_\varepsilon^1(s,\xi,\eta) = \lim_{\delta\to 0} \frac{1}{\mathrm{E}^\varepsilon_{\xi,s}[e^{G(X_1)-G(\eta)-(\varepsilon/2)\int_s^1 \beta(X_u)\,du} \mid X_1 \in B_\delta(\eta)]}. \tag{20}$$

We first show that

$$\begin{aligned} &\lim_{\varepsilon\to 0}\lim_{\delta\to 0} \frac{1}{\varepsilon}\big(1 - \mathrm{E}^\varepsilon_{\xi,s}[e^{G(X_1)-G(\eta)-(\varepsilon/2)\int_s^1 \beta(X_u)\,du} \mid X_1 \in B_\delta(\eta)]\big)\\ &\qquad = \frac{1}{2}\int_s^1 \beta\big(\gamma(u)\big)\,du. \end{aligned} \tag{21}$$

Let us set

$$\mathcal{A}_\varepsilon = \frac{1}{\varepsilon}(e^{G(X_1)-G(\eta)-(\varepsilon/2)\int_s^1 \beta(X_u)\,du} - 1) + \frac{1}{2}\int_s^1 \beta\big(\gamma(u)\big)\,du,$$

so that (21) holds if

$$\lim_{\varepsilon\to 0}\lim_{\delta\to 0}\mathrm{E}^{\varepsilon}_{\xi,s}[\mathcal{A}_\varepsilon \mid X_1 \in B_\delta(\eta)] = 0.$$

One has

$$\begin{aligned}\mathcal{A}_\varepsilon &= \frac{e^{G(X_1)-G(\eta)}}{\varepsilon}\Big(e^{-(\varepsilon/2)\int_s^1 \beta(X_u)\,du} - 1 + \frac{\varepsilon}{2}\int_s^1 \beta(X_u)\,du\Big)\\ &\quad + (e^{G(X_1)-G(\eta)} - 1)\Big(\frac{1}{\varepsilon} - \frac{1}{2}\int_s^1 \beta\big(\gamma(u)\big)\,du\Big)\\ &\quad + \frac{1}{2}e^{G(X_1)-G(\eta)}\int_s^1 \big[\beta\big(\gamma(u)\big) - \beta(X_u)\big]\,du.\end{aligned}$$

Since G and β are Lipschitz continuous and β is also bounded, for a suitable constant $K > 0$ one has

$$\begin{aligned}|\mathcal{A}_\varepsilon| &\le \frac{e^{K|X_1-\eta|}}{\varepsilon}\cdot\frac{\varepsilon^2 K^2}{4}e^{(K/2)\varepsilon} + K|X_1-\eta|e^{K|X_1-\eta|}\Big(\frac{1}{\varepsilon} + \frac{K}{2}\Big)\\ &\quad + \frac{K}{2}e^{K|X_1-\eta|}\int_s^1 |X_u - \gamma(u)|\,du,\end{aligned}$$

so that

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\limsup_{\delta\to 0}\mathrm{E}^{\varepsilon}_{\xi,s}[|\mathcal{A}_\varepsilon| \mid X_1 \in B_\delta(\eta)]\\ &\qquad \le \frac{K}{2}\limsup_{\varepsilon\to 0}\limsup_{\delta\to 0}\mathrm{E}^{\varepsilon}_{\xi,s}\Big[\int_s^1 |X_u - \gamma(u)|\,du\,\Big|\,X_1 \in B_\delta(\eta)\Big].\end{aligned}$$

Now, we can write

$$\begin{aligned}&\mathrm{E}^{\varepsilon}_{\xi,s}\Big[\int_s^1 |X_u - \gamma(u)|\,du\,\Big|\,X_1 \in B_\delta(\eta)\Big]\\ &\qquad = \frac{\mathrm{E}^{\varepsilon}_{\xi,s}[1_{\{X_1\in B_\delta(\eta)\}}\mathrm{E}^{\varepsilon}_{\xi,s}[\int_s^1 |X_u - \gamma(u)|\,du \mid X_1]]}{\mathrm{P}^{\varepsilon}_{\xi,s}(X_1 \in B_\delta(\eta))}.\end{aligned}$$

However,

$$\begin{aligned}\mathrm{E}^{\varepsilon}_{\xi,s}\Big[\int_s^1 |X_u - \gamma(u)|\,du\,\Big|\,X_1 = \zeta\Big] &= \hat{\mathrm{E}}^{\zeta,\varepsilon}_{\xi,s}\Big[\int_s^1 |X_u - \gamma(u)|\,du\Big]\\ &= \int_s^1 \hat{\mathrm{E}}^{\zeta,\varepsilon}_{\xi,s}\big[|X_u - \gamma(u)|\big]\,du.\end{aligned}$$

Under $\hat{\mathrm{P}}^{\zeta,\varepsilon}_{\xi,s}$, X evolves as a Brownian bridge and it is equal in law to the process $\xi + \frac{\zeta-\xi}{1-s}(u-s) + \sqrt{\varepsilon}(W_u + \frac{u-s}{1-s}W_1)$, $u \in [s,1]$, where W denotes a Brownian motion, starting in 0 at time s. Since $\gamma(u) = \xi + \frac{\eta-\xi}{1-s}(u-s)$,

$$\hat{\mathrm{E}}^{\zeta,\varepsilon}_{\xi,s}\big[|X_u - \gamma(u)|\big] \le |\zeta - \eta| + 2\sqrt{\varepsilon}.$$

Thus,

$$\mathrm{E}^{\varepsilon}_{\xi,s}\left[\int_s^1 |X_u-\gamma(u)|\,du \,\middle|\, X_1\right] \le |X_1-\eta| + 2\sqrt{\varepsilon}$$

and finally

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\limsup_{\delta\to 0}\mathrm{E}^{\varepsilon}_{\xi,s}\left[\int_s^1 |X_u-\gamma(u)|\,du \,\middle|\, X_1\in B_\delta(\eta)\right]\\
&\qquad \le \limsup_{\varepsilon\to 0}\limsup_{\delta\to 0}(\delta+2\sqrt{\varepsilon}) = 0;
\end{aligned}$$

that is, (21) holds. Now (21) implies

$$\lim_{\varepsilon\to 0}\lim_{\delta\to 0}\mathrm{E}^{\varepsilon}_{\xi,s}[e^{G(X_1)-G(\eta)-(\varepsilon/2)\int_s^1 \beta(X_u)\,du} \mid X_1\in B_\delta(\eta)] = 1,$$

which together with (20) concludes the proof of (i).

(ii) Let us set

$$\mathcal{D}_\varepsilon = \frac{1}{\varepsilon}(e^{-(\varepsilon/2)\int_s^1 \beta(X_u)\,du} - 1) + \frac{1}{2}\int_s^1 \beta\big(\rho(u)\big)\,du.$$

We must show that

$$\lim_{\varepsilon\to 0}\frac{\hat{\mathrm{E}}^{\eta,\varepsilon}_{\xi,s}[\mathcal{D}_\varepsilon 1_{\{X\in A\}}]}{\hat{\mathrm{P}}^{\eta,\varepsilon}_{\xi,s}(A)} = 0.$$

First one can write

$$\mathcal{D}_\varepsilon = \frac{1}{\varepsilon}\left(e^{-(\varepsilon/2)\int_s^1 \beta(X_u)\,du} - 1 + \frac{\varepsilon}{2}\int_s^1 \beta(X_u)\,du\right) + \frac{1}{2}\int_s^1 [\beta\big(\rho(u)\big)-\beta(X_u)]\,du$$

and since β is bounded on A,

$$|\mathcal{D}_\varepsilon| \le \frac{1}{\varepsilon}\cdot\frac{\varepsilon^2}{4}K^2 e^{(\varepsilon/2)K} + \frac{K}{2}\int_s^1 \big|\beta(X_u)-\beta\big(\rho(u)\big)\big|\,du$$

for some $K>0$. Thus,

$$\limsup_{\varepsilon\to 0}\frac{\hat{\mathrm{E}}^{\eta,\varepsilon}_{\xi,s}[|\mathcal{D}_\varepsilon| 1_{\{X\in A\}}]}{\hat{\mathrm{P}}^{\eta,\varepsilon}_{\xi,s}(A)} \le \frac{K}{2}\limsup_{\varepsilon\to 0}\frac{\hat{\mathrm{E}}^{\eta,\varepsilon}_{\xi,s}[\int_s^1 |\beta(X_u)-\beta(\rho(u))|\,du 1_{\{X\in A\}}]}{\hat{\mathrm{P}}^{\eta,\varepsilon}_{\xi,s}(A)}.$$

Let us now fix $\delta>0$ and denote by $B_\delta(\rho)$ the open ball on $\mathcal{C}$ centered at the path ρ and with radius δ. By the boundedness and the Lipschitz continuity properties assumed for β, for a suitable constant K,

$$\int_s^1 |\beta(X_u)-\beta(\rho_u)|\,du\, 1_{\{X\in A\}} \le K\delta\hat{\mathrm{P}}^{\eta,\varepsilon}_{\xi,s}\big(A\cap B_\delta(\rho)\big) + 2K\mathrm{P}^{\eta,\varepsilon}_{\xi,s}\big(A\cap B_\delta(\rho)^c\big),$$

so that

$$\limsup_{\varepsilon\to 0} \frac{\hat{\mathrm{E}}^{\eta,\varepsilon}_{\xi,s}[\int_s^1 |\beta(X_u)-\beta(\rho_u)|\,du 1_{\{X\in A\}}]}{\hat{\mathrm{P}}^{\eta,\varepsilon}_{\xi,s}(A)}$$

$$\leq K\delta + K(\delta+2)\limsup_{\varepsilon\to 0} \frac{\hat{\mathrm{P}}^{\eta,\varepsilon}_{\xi,s}(A\cap B_\delta(\rho)^c)}{\hat{\mathrm{P}}^{\eta,\varepsilon}_{\xi,s}(A)}.$$

Since ρ is the unique path minimizing the rate function J on the interior and the closure of A, standard large deviation arguments imply that

$$\lim_{\varepsilon\to 0} \varepsilon \log \hat{\mathrm{P}}^{\eta,\varepsilon}_{\xi,s}(A) = -J(\rho),$$

$$\limsup_{\varepsilon\to 0} \varepsilon \log \hat{\mathrm{P}}^{\eta,\varepsilon}_{\xi,s}\bigl(A\cap B_\delta(\rho)^c\bigr) = -J_\delta < -J(\rho).$$

Therefore $\limsup_{\varepsilon\to 0}(\hat{\mathrm{P}}^{\eta,\varepsilon}_{\xi,s}(A\cap B_\delta(\rho)^c)/\hat{\mathrm{P}}^{\eta,\varepsilon}_{\xi,s}(A)) = 0$ and

$$\limsup_{\varepsilon\to 0} \frac{\hat{\mathrm{E}}^{\eta,\varepsilon}_{\xi,s}[\int_s^1 |\beta(X_u)-\beta(\rho_u)|\,du 1_{\{X\in A\}}]}{\hat{\mathrm{P}}^{\eta,\varepsilon}_{\xi,s}(A)} \leq K\delta.$$

Since δ can be chosen arbitrarily small, the limit is actually zero and the statement follows. Finally, let us remark that all the limits appearing above are uniform for (ξ,η) in a compact subset of $\mathring{\mathcal{J}}$.

(iii) One can reproduce the proof of (ii), since most of it can be carried out also if A is a bounded set depending on ε. The only point that needs to be handled is the fact that

$$\limsup_{\varepsilon\to 0} \frac{\hat{\mathrm{P}}^{\eta,\varepsilon}_{\xi,s}(A_\varepsilon\cap B_\delta(\rho)^c)}{\hat{\mathrm{P}}^{\eta,\varepsilon}_{\xi,s}(A_\varepsilon)} = 0.$$

Setting $\tilde g(t) = g(t) - g(0)$, then one can write

$$\begin{aligned} A_\varepsilon &= \bigl\{\tau^\varepsilon_g \leq 1, \tilde\tau > 1\bigr\} \\ &= \bigl\{X_t \geq g(\varepsilon t) \text{ for some } t\leq 1, X_t \in]\tilde\Delta_1, \tilde\Delta_2[\text{ for every } t\leq 1\bigr\} \\ &= \bigl\{X_t - \tilde g(\varepsilon t) \geq g(0) \text{ for some } t\leq 1, X_t \in]\tilde\Delta_1, \tilde\Delta_2[\text{ for every } t\leq 1\bigr\} \\ &\subset \bigl\{X_t - \tilde g(\varepsilon t) \in A'\bigr\}, \end{aligned}$$

where $A' = \{\varphi; \varphi(t) \geq g(0) \text{ for some } t\leq 1,\ \varphi(t) \in]\tilde\Delta'_1, \tilde\Delta'_2[\text{ for any } t\leq 1\}$, the inclusion holding for any ε small and for values of $\tilde\Delta'_1$ and $\tilde\Delta'_2$ such that $\tilde\Delta'_1 < \xi, \eta, \tilde g(t) < \tilde\Delta'_2$, for every $t\in[0,1]$. Under $\hat{\mathrm{P}}^{\eta,\varepsilon}_{\xi,s}$, the nonhomogeneous diffusion process $\tilde X^\varepsilon_t = X_t - \tilde g(\varepsilon t)$ has generator $\tilde L^\varepsilon$ given by

$$\phi \mapsto \tilde L^\varepsilon \phi(\zeta) = \biggl(\varepsilon \tilde g'(\varepsilon t) + \frac{\zeta-\eta}{1-t}\biggr)\frac{d\phi}{d\zeta}(\zeta) + \frac{\varepsilon}{2}\frac{d^2\phi}{d\zeta^2}(\zeta).$$

Thus, by the Freidlin–Wentzell theory of large deviations, the family of processes $\{\tilde X^\varepsilon\}_\varepsilon$ under $\hat{\mathrm{P}}^{\eta,\varepsilon}_{\xi,s}$ enjoy a large deviations principle with the same rate function as X. If $\rho_\varepsilon(t)=\rho(t)-\tilde g(\varepsilon t)$, then $\rho_\varepsilon\to\rho$ uniformly on $[0,1]$ as $\varepsilon\to 0$, so that

$$\hat{\mathrm{P}}^{\eta,\varepsilon}_{\xi,s}\big(A_\varepsilon\cap B_\delta(\rho)^c\big)\le \hat{\mathrm{P}}^{\eta,\varepsilon}_{\xi,s}\big(\tilde X^\varepsilon\in A'\cap B_\delta(\rho_\varepsilon)^c\big)\le \hat{\mathrm{P}}^{\eta,\varepsilon}_{\xi,s}\big(\tilde X^\varepsilon\in A'\cap B_{\delta'}(\rho)^c\big)$$

for some $\delta'>\delta$ for ε small. Since ρ turns out to be the unique minimizing path for J on A' too, the minimum taking place on both the closure and the interior of A', large deviation arguments now allow to conclude the proof. $\square$

PROOF OF THEOREM 2.1(b). Thanks to Baldi, Caramellino and Iovino [(1999), Proposition 5.3], we have

$$\begin{aligned}\hat{\mathrm{P}}^{\eta,\varepsilon}_{\xi,0}&(\tau^\varepsilon_g\le 1)\\ &\sim C\exp\Big(-\frac{2}{\varepsilon}\big(g(0)-\xi\big)\big(g(0)-\eta\big)\Big)\Big(1-\varepsilon\underbrace{g''(0)\,\frac{(g(0)-\xi)^2}{2g(0)-\xi-\eta}}_{=-\Psi}+o(\varepsilon)\Big)\end{aligned}$$

with $C=\exp(-2g'(0)(g(0)-\xi))$. By replacing $g=F\circ f$, $\xi=F(x)$ and $y=F(\eta)$, Ψ agrees with (3). Thus, by Lemma 4.3, recalling (18),

$$\hat{\mathrm{Q}}^{\eta,\varepsilon}_{\xi,0}(\tau^\varepsilon_g\le 1)=C\exp\Big(-\frac{2}{\varepsilon}\big(g(0)-\xi\big)\big(g(0)-\eta\big)\Big)\big(1+\varepsilon(\Phi+\Psi)+o(\varepsilon)\big),$$

where (using the notation of the statement of Lemma 4.3)

$$\Phi=\tfrac12\int_0^1\big[(\tilde b'+\tilde b^2)\big(\gamma_{\xi,\eta}(u)\big)-(\tilde b'+\tilde b^2)\big(\rho_{\xi,\eta}(u)\big)\big]\,du.$$

We only need to show that Φ can also be expressed as in (4).

The path $\rho_{\xi,\eta}$ consists of two line segments traveled at constant speed, the first one joining ξ to $g(0)$ during the time interval $[0,t^*]$, where

$$t^*=\frac{g(0)-\xi}{2g(0)-\xi-\eta}.$$

The second one joins $g(0)$ to η in the time interval $[t^*,1]$. Thus if one writes again $\beta=\tilde b'+\tilde b^2$, then

$$\begin{aligned}\int_0^1(\tilde b'+\tilde b^2)\big(\rho_{\xi,\eta}(u)\big)\,du&=\int_0^{t^*}\beta\Big(\xi+\frac{u}{t^*}(g(0)-\xi)\Big)\,du\\ &\quad+\int_{t^*}^1\beta\Big(g(0)+\frac{u-t^*}{1-t^*}(\eta-g(0))\Big)\,du.\end{aligned}$$

Now recall the definition $\lambda=\sigma\cdot(\frac{b}{\sigma}-\frac12\sigma')'+(\frac{b}{\sigma}-\frac12\sigma')^2$ as in the statement of Theorem 2.1, so that $\beta=\lambda\circ F^{-1}$. Two elementary changes of variable then give

$$\begin{aligned}\int_0^{t^*} \beta\Big(\xi + \frac{u}{t^*}(g(0)-\xi)\Big)\,du &= \frac{t^*}{g(0)-\xi}\int_\xi^{g(0)} \beta(v)\,dv\\ &= \frac{t^*}{g(0)-\xi}\int_x^{f(0)} \frac{\lambda(r)}{\sigma(r)}\,dr\\ &= \Big(\int_x^{f(0)} \frac{dr}{\sigma(r)}\Big)^{-1}\int_x^{f(0)} \frac{\lambda(r)}{\sigma(r)}\,dr.\end{aligned}$$

Similarly one gets

$$\int_{t^*}^1 \beta\Big(g(0) + \frac{u-t^*}{1-t^*}(\eta - g(0))\Big)\,du = \Big(\int_y^{f(0)} \frac{dr}{\sigma(r)}\Big)^{-1}\int_y^{f(0)} \frac{\lambda(r)}{\sigma(r)}\,dr$$

and

$$\int_0^1 (\tilde b' + \tilde b^2)(\gamma_{\xi,\eta}(u))\,du = \Big(\int_x^y \frac{dr}{\sigma(r)}\Big)^{-1}\int_x^y \frac{\lambda(r)}{\sigma(r)}\,dr.$$

The latter expression for $x = y$ should read simply $\int_0^1 (\tilde b' + \tilde b^2)(\gamma_{\xi,\eta}(u))\,du = \lambda(x)$. $\square$

PROOF OF THEOREM 2.2. (a) To simplify the notation, let us define

$$\psi_1(x,y) = \int_{f_1(0)}^x \frac{dr}{\sigma(r)} \cdot \int_{f_1(0)}^y \frac{dr}{\sigma(r)} \quad \text{and} \quad \psi_2(x,y) = \int_x^{f_2(0)} \frac{dr}{\sigma(r)} \cdot \int_y^{f_2(0)} \frac{dr}{\sigma(r)}.$$

Consider the following inequalities, holding for any ε:

$$\max\big(\mathrm{P}(\hat\tau^\varepsilon_{f_1} \le \varepsilon), \mathrm{P}(\hat\tau^\varepsilon_{f_2} \le \varepsilon)\big) \le \mathrm{P}(\hat\tau^\varepsilon \le \varepsilon) \le \mathrm{P}(\hat\tau^\varepsilon_{f_1} \le \varepsilon) + \mathrm{P}(\hat\tau^\varepsilon_{f_2} \le \varepsilon).$$

If $\psi_1(x,y) < \psi_2(x,y)$, then one can write

$$\begin{aligned}&\frac{\mathrm{P}(\hat\tau^\varepsilon_{f_1} \le \varepsilon)}{\exp(-(2/\varepsilon)\psi_1)}\\ &\quad \le \frac{\mathrm{P}(\hat\tau^\varepsilon \le \varepsilon)}{\exp(-(2/\varepsilon)\psi_1)}\\ &\quad \le \frac{\mathrm{P}(\hat\tau^\varepsilon_{f_1} \le \varepsilon)}{\exp(-(2/\varepsilon)\psi_1)} + \frac{\mathrm{P}(\hat\tau^\varepsilon_{f_2} \le \varepsilon)}{\exp(-(2/\varepsilon)\psi_2)}\exp\Big(-\frac{2}{\varepsilon}(\psi_2 - \psi_1)\Big).\end{aligned} \tag{22}$$

Since, by Theorem 2.1(a), for $i = 1, 2$, one has $\mathrm{P}(\hat\tau^\varepsilon_{f_i} \le \varepsilon)\mathrm{e}^{(2/\varepsilon)\psi_i} \to 1$ as $\varepsilon \to 0$, we obtain $\mathrm{P}(\hat\tau^\varepsilon \le \varepsilon)/\exp(-\frac{2}{\varepsilon}\psi_1) \to 1$, as $\varepsilon \to 0$.

The case $\psi_1(x,y) > \psi_2(x,y)$ is treated similarly.

Now suppose that $\psi_1(x,y) = \psi_2(x,y)$. By using the notation previously introduced, we can write

$$\mathrm{P}(\hat\tau^\varepsilon \le \varepsilon) = \hat{\mathrm{Q}}^{\eta,\varepsilon}_{\xi,0}(\tau_{1,\varepsilon} < 1) + \hat{\mathrm{Q}}^{\eta,\varepsilon}_{\xi,0}(\tau_{2,\varepsilon} < 1) - \hat{\mathrm{Q}}^{\eta,\varepsilon}_{\xi,0}(\tau_{1,\varepsilon} < 1, \tau_{2,\varepsilon} < 1).$$

By Lemma 4.2(c1), the last term on the r.h.s. is exponentially negligible with respect the first two, so that the statement holds.

(b) If $\psi_1 < \psi_2$, the statement immediately follows from (22) and the case $\psi_1 > \psi_2$ can be treated analogously. If instead $\psi_1 = \psi_2 = \psi$, then one can write

$$\begin{aligned} \mathrm{P}(\hat{\tau}_f \le \varepsilon) &= \mathrm{P}(\hat{\tau}_{f_1} \le \varepsilon) + \mathrm{P}(\hat{\tau}_{f_2} \le \varepsilon) - \mathrm{P}(\hat{\tau}_{f_1} \le \varepsilon, \hat{\tau}_{f_2} \le \varepsilon) \\ &= e^{-(2/\varepsilon)\psi} \cdot \bigl[C_{f_1}\bigl(1 + \varepsilon(\Psi_{f_1} + \Phi f_1)\bigr) + C_{f_2}\bigl(1 + \varepsilon(\Psi_{f_2} + \Phi_{f_2})\bigr) \\ &\qquad + \varepsilon \mathcal{R}_\varepsilon - \mathrm{P}(\hat{\tau}_{f_1} \le \varepsilon, \hat{\tau}_{f_2} \le \varepsilon) e^{(2/\varepsilon)\psi}\bigr]. \end{aligned}$$

Since $\mathrm{P}(\hat{\tau}_{f_1} \le \varepsilon, \hat{\tau}_{f_2} \le \varepsilon)$ is exponentially negligible with respect to $e^{-(2/\varepsilon)\psi}$, the statement holds. $\square$

APPENDIX

We give here some details about the application of Varadhan's lemma in the proof of Lemma 4.2. The following are very simple remarks that seemed necessary to us since the results in Dembo and Zeitouni (1998) or Varadhan (1984) are not immediately applicable as they are.

Let us consider a family $(\mu_\varepsilon)_\varepsilon$ of probabilities on a metric space (E, d), satisfying a large deviations principle with rate function I. That is, $I : E \to \mathbb{R}^+ \cup \{+\infty\}$ is a lower semicontinuous (l.s.c.) function and its level sets $\{I \le \alpha\}$ are compact [it is a *good* rate function, in the notation of Dembo and Zeitouni (1998)].

PROPOSITION A.1. *Let $A \subset E$ be an open set and let $(F_\varepsilon)_\varepsilon$ be a family of functions defined on A with values in $\overline{\mathbb{R}}$. Assume that* $\lim_{\varepsilon\to 0} \varepsilon F_\varepsilon(z) = F(z)$ *uniformly for $z \in A$, where F is l.s.c. Then*

$$\liminf_{\varepsilon\to 0} \varepsilon \log \int_A e^{F_\varepsilon(z)} \mu_\varepsilon(dz) \ge \sup_{z\in A} \bigl(F(z) - I(z)\bigr).$$

PROPOSITION A.2. *Let $A \subset E$ be a closed set and let $(F_\varepsilon)_\varepsilon$ be a family of functions defined on A with values in $\overline{\mathbb{R}}$. Assume that* $\lim_{\varepsilon\to 0} \varepsilon F_\varepsilon(z) = F(z)$ *uniformly for $z \in A$, where F is upper semicontinuous (u.s.c.) and bounded from above. Then*

$$\limsup_{\varepsilon\to 0} \varepsilon \log \int_A e^{F_\varepsilon(z)} \mu_\varepsilon(dz) \le \sup_{z\in A} \bigl(F(z) - I(z)\bigr).$$

We skip the proofs, which are exact repetitions of the proofs of the corresponding Lemmas 4.3.4 and 4.3.5 of Dembo and Zeitouni (1998). An essentially stronger version of Propositions A.1 and A.2 is Theorem 2.3 in Varadhan (1984): if applied to the function $\tilde{F}^V = -F + \infty \cdot 1_{A^c}$, it basically gives Proposition A.2; Proposition A.1 can be proved with similar arguments.

In the proof of Lemma 4.2 we need to apply Varadhan's lemma to estimate the integral

$$\int_A e^{F_\varepsilon(z)}\, d\mathrm{Q}^{\varepsilon}_{\xi,s}(dz), \tag{23}$$

where $A = \{\tau \le 1\}$ and $F_\varepsilon(\gamma) = \log q_\varepsilon(1-\tau(\gamma), a, \eta)$. By (19), uniformly for $\gamma \in A$,

$$\lim_{\varepsilon\to 0} \varepsilon \log q_\varepsilon\bigl(1-\tau(\gamma), a, \eta\bigr) = -\frac{1}{2(1-\tau(\gamma))}(a-\eta)^2 \stackrel{\text{def}}{=} F(\gamma).$$

Unfortunately τ is not a continuous functional of the path γ. With our definitions τ is a l.s.c. functional on $\mathcal{C}$, so that A is closed and F turns out to be u.s.c. Proposition A.2 can thus be applied, giving the upper bound.

To obtain the lower bound we just replace τ with $\tilde\tau = \inf(u \ge s; \gamma(u) > a)$, which is now an u.s.c. functional of γ. Remark that the integral in (23) does not change if F_ε is now replaced by $F_\varepsilon(\gamma) = \log q_\varepsilon(1-\tilde\tau(\gamma), a, \eta)$ and A with $\tilde A = \{\tilde\tau < 1\}$, which is an open set.

REFERENCES

ANDERSEN, L. and BROTHERTON-RATCLIFFE, R. (1996). Exact exotics. *Risk* **9** 85–89.

AZENCOTT, R. (1980). Grandes déviations et applications. *Ecole d'Eté de Probabilités de St. Flour VIII. Lecture Notes in Math.* **774** 1–176. Springer, Berlin.

BALDI, P. (1995). Exact asymptotics for the probability of exit from a domain and applications to simulation. *Ann. Probab.* **23** 1644–1670.

BALDI, P., CARAMELLINO, L. and IOVINO, M. G. (1999). Pricing general barrier options: A numerical approach using sharp large deviations. *Math. Finance* **9** 293–322.

BALDI, P. and CHALEYAT-MAUREL, M. (1986). An extension of Ventsel–Freidlin estimates. *Stochastic Analysis and Related Topics. Lecture Notes in Math.* **1316** 305–327. Springer, Berlin.

BEAGLEHOLE, D. R., DYBVIG, P. H. and ZHOU, G. (1997). Going to extremes: Correcting simulation bias in exotic option valuation. *Financial Analysts Journal* **53** 62–68.

BENJAMINI, I. and LEE, S. (1997). Conditioned diffusions which are Brownian bridges. *J. Theoret. Probab.* **10** 733–736.

CARAMELLINO, L. and IOVINO, M. G. (2002). An exit-probability-based approach for the valuation of defaultable securities. *J. Comput. Finance*. To appear.

DEMBO, A. and ZEITOUNI, O. (1998). *Large Deviations Techniques and Applications*. Springer, Berlin.

ELIE, L. (1980). Equivalent de la densité d'une diffusion en temps petit cas des points proches. In *Géodésiques et diffusions en temps petit*. *Astérisque* **84–85** 55–71.

FREIDLIN, M. I. and WENTZELL, A. D. (1984). *Random Perturbations of Dynamical Systems*. Springer, New York.

GIRAUDO, M. T. and SACERDOTE, L. (1999). An improved technique for the simulation of first passage times for diffusion processes. *Comm. Statist. Simul. Comput.* **28** 1135–1163.

GOBET, E. (2000). Weak approximation of killed diffusion using Euler schemes. *Stochastic Process. Appl.* **87** 167–197.

KLOEDEN, P. E. and PLATEN, E. (1992). *Numerical Solutions of Stochastic Differential Equations*. Springer, Berlin.

LAMBERTON, D. and LAPEYRE, B. (1996). *Introduction to Stochastic Calculus Applied to Finance*. Chapman and Hall, London.

MOLCHANOV, S. A. (1975). Diffusion processes and Riemannian geometry. *Russian Math. Surveys* **30** 1–63.

VARADHAN, S. R. S. (1984). *Large Deviations and Applications*. SIAM, Philadelphia.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ROME–TOR VERGATA
VIA DELLA RICERCA SCIENTIFICA
I-00133 ROME
ITALY
E-MAIL: baldi@mat.uniroma2.it
caramell@mat.uniroma2.it