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NORMAL CONVERGENCE PROBLEM? TWO MOMENTS AND A RECURRENCE MAY BE THE CLUES[1]

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To Irina, my wife of forty years

For various global characteristics of large size combinatorial structures, such as graphs, trees, one can usually estimate the mean and the variance, and also obtain a recurrence for the generating function, with the structure size n serving as the recursive parameter. As a heuristic principle based on our experience, we claim that such a characteristic is asymptotically normal if the mean and the variance are "nearly linear" in n. The technical reason is that in such a case the moment generating function (the characteristic function) of the normal distribution with the same two moments "almost" satisfies the recurrence. Of course, an actual proof may well depend on a magnitude of the relative error, and the latter is basically determined by degree of nonlinearity of the mean and the variance. We provide some new illustrations of this paradigm. The uniformly random tree on n-labelled vertices is studied. Using and strengthening the earlier results of Meir and Moon, we show that the independence number is asymptoticaly normal, with mean ρn and variance $\sigma^2 n$, $(\sigma^2 = \rho(1-\rho-\rho^2)(1+\rho)^{-1})$; here $\rho \approx 0.5671$ is the root of $xe^x = 1$. As a second example, we prove that—in the rooted tree—the counts of vertices with total progeny of various sizes form an asymptotically Gaussian sequence. This is an extension of Rényi's result on asymptotic normality of the number of leaves in the random tree. In both cases we establish convergence of the generating function. Finally we show that the overall number of ways to extend, totally, the tree–induced partial order is lognormal in the limit, with mean and variance roughly $\log n! - an$ and $bn \log n$. The proof is based on convergence of the cumulants of the generating function.

1. Introduction. It is not uncommon that a global characteristic of a large random structure (like a graph) is asymptotically normal, even though it seems unlikely that such a random variable, albeit additive in nature, can be constructively represented as a sum of many independent, or even weakly dependent, terms. In the author's experience, the argument usually involves sharp asymptotic estimates of the mean and variance, which turn out to be nearly linear in n, the structure size, and a demonstration that the normal distribution with that same parameters very nearly satisfies a recurrence equation for the moment generating function (m.g.f.). (See, for instance, [18], [19], [12], [6].) Such a demonstration appears to be possible largely due to the exponential form of the m.g.f. for the normal distribution *and* the asymptotic

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linearity of the mean and the variance. In fact, sometimes the simplest way to estimate the first two moments sharply is by making a m.g.f. of a normal distribution fit closely the recurrence for the actual m.g.f.; see [18], for example.

In this paper we use the approach to analyze some characteristics of the *uniformly* random tree on n vertices labelled $1, 2, \dots, n$. One such parameter, X_n, is the cardinality of a largest independent subset of vertices of the random tree. [A set of vertices is called independent (stable) if no two of them are adjacent.] To introduce other parameters we transform our random tree into the random *rooted* tree by marking uniformly at random one of its vertices. Let us orient the tree edges away from the root. The orientation induces a partial order ($\prec$) on the vertex set, so that $i \prec j$ iff j is a descendant of i. Introduce $\mathbf{Z}_n = \{Z_{nk}\}_{1\le k\le n}$, where Z_{nk} is the number of vertices each having total progeny (including itself) of size k; $Z_{nn} = 1$ and Z_{n1} is the number of leaves. Combinatorially, Z_{nk} is the total number of *filters* of cardinality k in our partially ordered set. We also define Y_n as the the overall number of ways to extend the partial order to a total order. It is known ([9], Chapter 5, Exercise 20) that

$$Y_n = \frac{n!}{\prod_{k=1}^{n} k^{Z_{nk}}}. \tag{1.1}$$

In a nutshell, our aim is to show that X_n and $\log Y_n$ are asymptotically normal and that $\mathbf{Z}_n$ is asymptotically Gaussian.

Meir and Moon [13] were the first to study the independence number X_n. They proved that

$$\mathbf{E}\, X_n = \rho n + O\big(n^{1/2}\big), \qquad \rho e^{\rho} = 1 \tag{1.2}$$

and that

$$\frac{X_n}{n} \to_{\mathrm{P}} \rho, \qquad n \to \infty. \tag{1.3}$$

Later they obtained similar results for a broader class of random trees known as “simple trees” and also for unlabelled random trees [14], [15], [16]. A simple tree (on n-labelled vertices) appears naturally when one considers the “genealogical” tree of a branching process, conditioned on the event “ tree size is n.” If the immediate family size distribution is Poisson, that tree is uniform. This important case is the subject of our present study. The techniques we develop can be used, however, for any simple tree.

Here are some preliminaries from [13] which we will need in this paper. Let T be a rooted tree, and let $X(T)$ denote its independence number. If every independent set of $X(T)$ vertices contains the root, T is classified as a type I tree; all other trees become type II trees. Introduce g_{nk} (f_{nk}, resp.) the total number of rooted trees T type I (type II, resp.) on n vertices such that $X(T) = k$; $1 \le k \le n$. Then $t_{nk} := g_{nk} + f_{nk}$ is the total number of the rooted trees with $X(T) = k$; $g_n := \sum_{k=1}^{n} g_{nk}$ ($f_n := \sum_{k=1}^{n} f_{nk}$, resp.) is the total number of the rooted trees type I (type II, resp.). In particular, $\sum_{k=1}^{n} t_{nk} = n^{n-1}$. Set

$$t(z,x) = \sum_{n\ge 1}\left(\sum_{k=1}^{n} t_{nk} z^k\right)\frac{x^n}{n!},$$

$$G(z,x) = \sum_{n\ge 1}\left(\sum_{k=1}^{n} g_{nk} z^k\right)\frac{x^n}{n!},$$

$$F(z,x) = \sum_{n\ge 1}\left(\sum_{k=1}^{n} f_{nk} z^k\right)\frac{x^n}{n!}.$$

(All three series converge for $|z| \le 1$, $|x| \le e^{-1}$, since

$$\sum_{n\ge 1} \frac{n^{n-1}e^{-n}}{n!} = 1.)$$

LEMMA.

$$G = zxe^F, \tag{1.4}$$

$$F = x(e^G - 1)e^F. \tag{1.5}$$

The proof of these remarkable identities is based on an ingenious observation that a rooted tree T is type I iff all the rooted subtrees $T_1, \dots, T_j$ obtained by the deletion of the root of T are type II, in which case

$$X(T) = 1 + \sum_{i=1}^{j} X(T_j).$$

The summand 1 is dropped when T is type II.

Note 1. Let $\tilde{f}_{nk}$ be the total number of rooted trees type II such that there is at least one largest independent set which does contain the root. For such a tree T, exactly one of the subtrees $T_1, \dots, T_j$ is type I. So, introducing

$$\tilde{F}(z,x) = \sum_{n\ge 1}\left(\sum_{k=1}^{n} \tilde{f}_{nk} z^k\right)\frac{x^n}{n!},$$

instead of (1.5) we obtain

$$\tilde{F} = xGe^F. \tag{1.6}$$

It follows from this lemma that $g(x) := G(1, x)$ satisfies

$$ge^g = \mathscr{T}(x), \tag{1.7}$$

where $\mathscr{T}(x) = G(1,x) + F(1,x)$ is the well-familiar tree function $\mathscr{T}(x) = \sum_{n\ge 1} n^{n-1}x^n/n!$, ($|x| \le e^{-1}$), known to satisfy

$$t = xe^t. \tag{1.8}$$

Since each tree on n vertices can be rooted in the same number of ways, X_n has the same distribution as $X(T)$, where T is the uniformly random rooted tree on n vertices. It should be clear then that

$$t(z,x) = \sum_{n\ge 1} \mathbf{E}(z^{X_n}) n^{n-1}\frac{x^n}{n!}.$$

So, differentiating (1.4), (1.5) with respect to z at $z = 1$ and adding the results we get

$$\sum_{n\ge 1} (\mathbf{E}\, X_n) n^{n-1}\frac{x^n}{n!} = \frac{g(x)}{1-\mathscr{T}(x)}, \tag{1.9}$$

the relation which led Meir and Moon [13] (via repeated application of Lagrange's inversion formula) to

$$\mathbf{E}\, X_n = \sum_{k=1}^{n} \binom{n}{k}\left(\frac{-k}{n}\right)^{k-1} \quad (!).$$

[The authors attribute the lemma and (1.8) to de Bruijn.] The relation (1.9) and an analogous identity for the second factorial moment $\mathbf{E}[X_n]_2$, namely

$$\sum_{n\ge 1} (\mathbf{E}[X_n]_2) n^{n-1}\frac{x^n}{n!} = \frac{\mathscr{T} g^2}{(1-\mathscr{T})^3} + \frac{2\mathscr{T} g^2}{(1+g)(1-\mathscr{T})^2}, \tag{1.10}$$

were used in [13] to prove the key relations

$$\mathbf{E}\, X_n = n\rho + O(n^{1/2}), \qquad \mathbf{E}[X_n]_2 = n^2\rho^2 + O(n^{3/2}). \tag{1.11}$$

Here are the results of the present paper. We sharpen (1.11) to

$$\mathbf{E}\, X_n = n\rho + \frac{\rho^2(\rho+2)}{2(\rho+1)^3} + O(n^{-1}), \tag{1.12}$$

$$\mathbf{E}[X_n]_2 = n^2\rho^2 - n\rho\big(1-(\rho+1)^{-3}\big) + O(1); \tag{1.13}$$

thus

$$\operatorname{Var} X_n = n\sigma^2 + O(1), \qquad \sigma^2 = \frac{\rho(1-\rho-\rho^2)}{(\rho+1)^2}. \tag{1.14}$$

The total number of the rooted trees type I is given by

$$g_n = n^{n-1}\frac{\rho}{\rho+1} + n^{n-2}\frac{\rho^2(\rho+4)}{2(\rho+1)^5} + O(n^{n-3}); \tag{1.15}$$

therefore, since $f_n = n^{n-1} - g_n$,

$$\mathbf{P}\{T \text{ is type I}\} = \frac{\rho}{\rho+1} + O(n^{-1}),$$
$$\mathbf{P}\{T \text{ is type II}\} = \frac{1}{\rho+1} + O(n^{-1}). \tag{1.16}$$

Using (1.6) we show that $\tilde{f}_n := \sum_{k=1}^{n} \tilde{f}_{nk}$ (see Note 1) is asymptotic to $2n^{n-1}\rho^2/(1+\rho)$. So, conditioned on the event $\{T \text{ is type II}\}$, the probability that at least one largest independent set contains the root of T is asymptotic to $2\rho^2 \approx 0.643$.

Our main result for the independence number is that, conditioned on the event $\{T \text{ is type I}\}$ ($\{T \text{ is type II}\}$ resp.), $X(T)$ is asymptotically Gaussian with mean $n\rho$ and variance $n\sigma^2$, with σ^2 defined in (1.14). This means that the tree T's type and $(X(T) - n\rho)/(\sigma n^{1/2})$ are asymptotically independent. Consequently, X_n is also Gaussian in the limit, with parameters ρn and $\sigma^2 n$.

As Meir and Moon notice, any result on $X(T)$ can be translated into the related statement for $M(T)$, the maximum matching number of the tree T, because $X(T) + M(T) = n$. Thus we obtain that in the limit the distribution of $M(T)$ is the normal $\mathcal{N}(n(1-\rho), n\sigma^2)$.

Note 2. (i) It is well known that, for $c < 1$, the uniformly random graph $G(n, cn/2)$ on n vertices with $cn/2$ edges is, with high probability, a forest plus a few unicyclic components bounded in size. This allowed us [19] to use the Moon–Meir results [13] to prove that the independence number I_n is asymptotically normal with mean $n\alpha(c)$ and variance $n\beta(c)$. Because of the connection between the independence number and the matching number of a tree, the matching number M_n of the random graph is also normal in the limit, with mean $n(1-\alpha(c))$ and the same variance $n\beta(c)$.

(ii) Dyer, Frieze and Pittel [6] studied an average-case performance of two greedy matching algorithms for $G(n, cn/2)$ and random trees. It was shown that in each case the resulting matching number is asymptotically Gaussian, with both mean and variance linear in n. For a (statistically) better algorithm out of those two, in the case of random trees the matching number is close to $n(e-1)/(2e-1) \approx 0.387n$, which is about 89 percent of the likely maximum $M(T)(\approx (1-\rho)n)$.

(iii) The salient feature of the greedies in [6] is that at each step an edge to be added to the current matching set is chosen at random among all edges present. Karp and Sipser [8] (see also [3]) designed and studied a more elaborate algorithm for the graphs that turns out to be asymptotically optimal with high probability for the random graph $G(n, cn/2)$. M_n was shown to be asymptotic, in probability, to $f(c)n$, $[f(c) = 1 - \alpha(c)$, for $c < 1]$. It was conjectured in [3] that M_n remains asymptotically normal for every $c > 0$. One of the rules is " include into the matching set an edge incident to a randomly chosen pendant vertex, if any is present." Such an algorithm applied to a tree would certainly determine $M(T)$ precisely. Is there a version of this algorithm that can be used for an alternative proof of asymptotic normality of $M(T)$ based on a recurrence? Any such recurrence may not be simpler than those we use in this paper, since after the very first step of any algorithm from this class we end up, usually, with a forest.

Turn to $\mathbf{Z}_n$ and Y_n. Introduce

$$p(k) = \frac{k^{k-1}e^{-k}}{k!}, \qquad q(k) = \frac{k^k e^{-k}}{k!}. \tag{1.17}$$

($\{p(k)\}$ is the probability distribution of the total population in the critical Poisson branching process.) We will prove that the sequence $\{n^{-1/2}(Z_{nk} - np(k))\}_{k\geq 1}$ is Gaussian in the limit, with zero means and covariance function,

$$K(j,k) = p(j)\big(q(k-j) - (j+k)p(k)\big), \qquad j \leq k. \tag{1.18}$$

As for Y_n, the number of total orders on the vertex set compatible with the tree-induced partial order, we will prove that $\log Y_n$ is asymptotically normal with mean $\log n! - an$ and variance $bn \log n$, where

$$a = \sum_{k\geq 1} p(k)\log k, \qquad b = 4\log\frac{e}{2}. \tag{1.19}$$

We prove this result using (1.1). The double series $\sum_{j,k} K(j,k)\log j \log k$ diverges, which makes the formula $\operatorname{Var}\log Y_n \approx bn\log n$ less surprising. This divergence is also the reason why we will have to prove the normality of $\log Y_n$ separately, not directly invoking the limiting distribution of $\mathbf{Z}_n$. What is more, in contrast to X_n and $\mathbf{Z}_n$, we prove convergence of $\log Y_n$ by showing convergence of its scaled cumulants (semiinvariants) rather than of its m.g. function. However, it is a properly chosen recurrence for the m.g. function that makes the cumulants an efficient tool.

The curious reader may wish to show, via (1.1) and the exponential generating functions, that

$$\mathbf{E}\,Y_n = \frac{(n!)^2}{n^n}, \qquad \mathbf{E}\,Y_n^2 = \frac{2(n!)^3}{2^n n^n}.$$

(That $\mathbf{E}\,Y_n^2 \gg \mathbf{E}^2\,Y_n$ explains why we need to deal with $\log Y_n$ in order to obtain the distributional results.)

Note 3. (i) The asymptotic normality of Z_{n1} was established by Rényi [21]. Later Steele [22] proved, via Harper's method, a *local* limit theorem for Z_{n1}, in the more general case of a Gibbs' distributed random tree. Devroye [5] proved, using a different approach, the limit theorems for Z_{nk} in a case of a random binary search tree.

(ii) Alon, Bollobás, Brightwell and Janson [2] proved an analogous result for a partial order induced by the random acyclic graph. (In their case, the variance was asymptotically linear in n, the number of vertices.)

(iii) Conceptually close is a study of the total number of the Young tableaux for a random diagram (frame) of size n, [23] and [20]. We conjectured that the logarithm of this number is also asymptotically normal, with variance of order about $n^{3/2}$.

(iv) Fill [7] obtained the asymptotics of $\mathbf{E}\log Y_n$ and $\mathrm{Var}\log Y_n$ for a binary search tree, either grown from a random permutation of n keys, or uniform. Very recently Meir and Moon [17] did the same for a broad class of random simple trees, of which the uniformly random tree on n-labelled vertices and the uniform binary tree are the special cases. Fill also proved asymptotic normality of $\log Y_n$ for the tree from the random permutation.

2. Independence number.

2.1. *Asymptotics for tree counts, expectations and variances.* Consider first $\mathbf{E}\,X_n$ and $\mathbf{E}[X_n]_2$.

Observe that $(ze^z)' = 0$ at $z = -1$ only. Therefore the relation $he^h = t$ defines a function $h(t)$ analytic in t plane with a cut $\{t = u + iv\colon v = 0, u \le -e^{-1}\}$. Clearly, $h(1) = \rho$ where $\rho e^\rho = 1$, and elementary computations show

$$h' = \frac{h}{t(1+h)}, \qquad h'' = -(h')^2 \frac{2+h}{1+h},$$
$$h''' = (h')^3 \frac{1+2(2+h)^2}{(1+h)^2}, \tag{2.1}$$

so that

$$h'(1) = \frac{\rho}{\rho+1}, \qquad h''(1) = -\frac{\rho^2(\rho+2)}{(\rho+1)^3},$$
$$h'''(1) = \frac{\rho^3\big[1+2(\rho+2)^2\big]}{(\rho+1)^5}. \tag{2.2}$$

Since $|te^{-t}| > e^{-1}$ for all t from the cut, and

$$\big|\mathscr{T}(x)e^{-\mathscr{T}(x)}\big| = |x| \le e^{-1},$$

for $|x| \le e^{-1}$, from (1.7) we obtain that $g(x) = h(\mathscr{T}(x))$ for $|x| \le e^{-1}$. Introduce

$$\mathscr{L} = \big\{x = te^{-t}\colon t = re^{i\theta}, \theta \in (-\pi, \pi]\big\};$$

it is easy to see that, for $re^r < e^{-1}$, $\mathscr{L}$ is a simple closed contour in the complex plane x which encloses the origin and lies within the circle $\{x\colon |x| \le e^{-1}\}$. By (1.9) and Cauchy's integral formula (applicable to any such contour), we have

$$\frac{n^{n-1}}{n!}\mathbf{E}\,X_n = (2\pi i)^{-1} \oint_{\mathscr{L}} \frac{g(x)}{x^{n+1}(1-\mathscr{T}(x))}\,dx;$$

or, substituting $x = te^{-t}$, and using $\mathscr{T}\big(te^{-t}\big) = t$ for $|t| \le r$,

$$\begin{aligned}\frac{n^{n-1}}{n!}\mathbf{E}\,X_n &= (2\pi i)^{-1} \oint_{|t|=r} \frac{h(t)e^{-t}(1-t)}{(1-t)(te^{-t})^{n+1}}\,dt \\ &= (2\pi i)^{-1} \oint_{|t|=r} \frac{h(t)e^{-t}}{(te^{-t})^{n+1}}\,dt.\end{aligned} \tag{2.3}$$

With the factor $1-t$ in the denominator gone, we switch to a new contour of integration. It consists of the arc $L := \{t = e^{i\theta}:\ -\pi < \theta < \pi\}$ traversed counterclockwise and the cut $\{t = u + i0:\ u \in [-1, -e^{-1}]\}$, which is traversed twice, from left to right along the upper shore of the cut, and from right to left along the lower shore of the cut. On the cut, the integrand is of order $O(\exp(n - n/e))$ at most. Consider L. Setting $t = e^{i\theta}$, and using $1 - \cos\theta \geq a\theta^2$, $a > 0$,

$$\left|\oint_{|\theta|\geq n^{-1/3}} \frac{h(t)e^{-t}}{(te^{-t})^{n+1}}\,dt\right| \leq \operatorname{const} e^n \int_{|\theta|\geq n^{-1/3}} \exp\bigl(-n(1-\cos\theta)\bigr)\,d\theta = O\bigl(\exp(n - an^{1/3})\bigr). \tag{2.4}$$

For $|\theta| \leq n^{-1/3}$ we pick an integer l and estimate

$$\begin{aligned}
(2\pi i)^{-1}\int_{|\theta|\leq n^{-1/3}} &= \sum_{j=0}^{l} \frac{h^{(j)}(1)}{j!} I_{nj} + O(R_{nl}),\\
I_{nj} &:= (2\pi i)^{-1}\int_{|\theta|\leq n^{-1/3}} (t-1)^j \frac{e^{nt}}{t^{n+1}}\,dt,\\
R_{nl} &= e^n \int_{|\theta|\leq n^{-1/3}} \left|1 - e^{i\theta}\right|^{l+1} \exp(-an\theta^2)\,d\theta.
\end{aligned} \tag{2.5}$$

Here [cf. (2.4)],

$$\begin{aligned}
I_{nj} &= (2\pi i)^{-1}\int_L (t-1)^j \frac{e^{nt}}{t^{n+1}}\,dt + O\bigl(\exp(n - an^{1/3})\bigr)\\
&= [t^n]\bigl((t-1)^j e^{nt}\bigr) + O\bigl(\exp(n - an^{1/3})\bigr)\\
&= \sum_{k=0}^{j} (-1)^{j-k}\binom{j}{k}\frac{n^{n-k}}{(n-k)!} + O\bigl(\exp(n - an^{1/3})\bigr),
\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}
R_{nl} &\leq e^n \int_{-\infty}^{\infty} |\theta|^{l+1} \exp(-an\theta^2)\,d\theta\\
&= O\bigl(e^n n^{-(l+2)/2}\bigr).
\end{aligned} \tag{2.7}$$

Combining (2.3) and (2.5), (2.6), we get

$$\begin{aligned}
\frac{n^{n-1}}{n!}\mathbf{E}\,X_n = \sum_{j=0}^{l} \frac{h^{(j)}(1)}{j!} \sum_{k=0}^{j} (-1)^{j-k}\binom{j}{k}\frac{n^{n-k}}{(n-k)!}\\
+ O\bigl(e^n n^{-(l+2)/2}\bigr).
\end{aligned} \tag{2.8}$$

Choosing here $l = 4$ and using (2.2) we obtain (1.12). [The contributions of the terms $h^{(3)}(1)$, $h^{(4)}(1)$ turn out to be absorbed by the remainder term $O(n^{-1})$ in (1.12).]

Analogously to (2.3), it follows from (1.10) that

$$(2.9)\qquad \frac{n^{n-1}}{n!}\,\mathbf{E}[X_n]_2 = (2\pi i)^{-1}\oint_{|t|=r}\left[\frac{th^2(t)}{(1-t)^3} + \frac{2th^2(t)}{(1+h(t))(1-t)^2}\right]\frac{d(te^{-t})}{(te^{-t})^{n+1}}.$$

The relation looks quite intimidating. Fortunately, one can observe that, according to (2.1), the expression in the square brackets equals

$$\frac{t}{1-t}\frac{d}{dt}\left(h^2(t)\frac{t}{1-t}\right).$$

Plugging this into (2.9) and integrating by parts, we get a much simpler formula,

$$\frac{n^{n-1}}{n!}\,\mathbf{E}[X_n]_2 = n(2\pi i)^{-1}\oint_{|t|=r}\frac{h^2(t)}{(te^{-t})^n}\,dt.$$

With this relation at hand, we proceed as in the case of $\mathbf{E}\,X_n$ and get easily (1.13). Similarly, (1.15) [whence the relations (1.16)] follows from

$$\begin{aligned}\frac{g_n}{n!} &= [x^n]g(x) = (2\pi i)^{-1}\oint_{\mathscr{L}}\frac{g(x)}{x^{n+1}}\,dx\\ &= (2\pi i)^{-1}\oint_{|t|=r}\frac{h(t)(1-t)e^{-t}}{(te^{-t})^{n+1}}\,dt.\end{aligned}$$

[Unlike (2.3) and (2.9), however, the contribution of $h^{(3)}(1)$-term influences the second leading term in (1.15) and thus cannot be neglected, a very consequential fact brought to my attention by a thoughtful reviewer.]

Let $\tilde{f}_n := \sum_{k=1}^n \tilde{f}_{nk}$ stand for the total number of rooted trees type II with the root in at least one largest independent set (i.e., the root being "usable," so to speak). By (1.6) and (1.4), $\tilde{f}_n$ is given by

$$\frac{\tilde{f}_n}{n!} = [x^n]g^2(x) = (2\pi i)^{-1}\oint_{|t|=r}\frac{h^2(t)(1-t)e^{-t}}{(te^{-t})^{n+1}}\,dt.$$

This results in

$$\tilde{f}_n = n^{n-1}\frac{2\rho^2}{\rho+1} + O(n^{n-2}).$$

Since by (1.15),

$$f_n = n^{n-1} - g_n = n^{n-1}\frac{1}{\rho+1} + O(n^{n-2}),$$

we have proved that

$$\mathbf{P}\{\text{root is usable} \mid \text{tree is type II}\} = \frac{\tilde{f}_n}{f_n} \to 2\rho^2.$$

Let $T^{(1)}$ ($T^{(2)}$, resp.) denote the uniformly random rooted tree of type I (type II, resp.), and let $X_n^{(i)} = X(T^{(i)})$, $i = 1, 2$. From the definitions of the generating functions F and G we see that

$$\sum_{n\ge 1} f_n \,\mathbf{E}\, X_n^{(2)} \frac{x^n}{n!} = F'_z(1, x), \qquad \sum_{n\ge 1} f_n \,\mathbf{E}\big[X_n^{(2)}\big]_2 \frac{x^n}{n!} = F''_z(1, x).$$

Very straightforward, but tedious, computations, based on (1.4), (1.5), show that

$$F'_z(1, x) = \frac{g\mathscr{T}}{(1+g)(1-\mathscr{T})},$$
$$F''_z(1, x) = \frac{\mathscr{T}}{1-\mathscr{T}}\left[\frac{\mathscr{T}(2g^2+g^3)}{(1-\mathscr{T})(1+g)^3} + \frac{g^2}{(1+g)(1-\mathscr{T})^2}\right].$$

Like the square brackets part of (2.9), it can be seen that the expression in the square brackets equals [after setting $\mathscr{T} = t$ and $g = h(t)$] to

$$\frac{d}{dt}\left(\frac{t^2}{1-t}\,\frac{h^2(t)}{1+h(t)}\right) + \frac{h^2(t)}{1+h(t)}.$$

So, following the same script, we eventually get

$$\begin{aligned} \frac{f_n}{n!}\,\mathbf{E}\, X_n^{(2)} &= \frac{n^n}{n!}\,\frac{\rho}{\rho+1} + \frac{n^{n-1}}{n!}\,\frac{-\rho(\rho^2+2)}{2(\rho+1)^5} \\ &\quad + O\left(\frac{n^{n-2}}{n!}\right), \\ \frac{f_n}{n!}\,\mathbf{E}\big[X_n^{(2)}\big]_2 &= \frac{n^{n+1}}{n!}\,\frac{\rho^2}{\rho+1} - \frac{n^n}{n!}\left[\frac{3(\rho^3+2\rho^2)}{2(\rho+1)^3} + \frac{\rho^3+4\rho^2}{2(\rho+1)^5}\right] \\ &\quad + O\left(\frac{n^{n-1}}{n!}\right). \end{aligned} \tag{2.10}$$

[The reviewer alerted me to a possibility of error in (2.10), and indeed the coefficient by $n^{n-1}/n!$ in the first equation needed an important correction.]

Then we get analogous formulas for the moments of $X_n^{(1)}$ via

$$g_n \,\mathbf{E}\, X_n^{(1)} + f_n \,\mathbf{E}\, X_n^{(2)} = n^{n-1}\,\mathbf{E}\, X_n,$$
$$g_n \,\mathbf{E}[X_n^{(1)}]_2 + f_n \,\mathbf{E}[X_n^{(2)}]_2 = n^{n-1}\,\mathbf{E}[X_n]_2$$

and (1.12), (1.13). Using (1.15), we arrive finally at

$$\begin{aligned} \mathbf{E}\, X_n^{(1)} &= n\rho + \frac{\rho^3+2\rho^2+2}{2(1+\rho)^3} + O(n^{-1}), \\ \mathbf{E}\, X_n^{(2)} &= n\rho + \frac{\rho^3+2\rho^2-2\rho}{2(1+\rho)^3} + O(n^{-1}), \end{aligned} \tag{2.11}$$

and

$$\mathbf{E}\big[X_n^{(1)}\big]_2 = n^2\rho^2 + n\left[-\frac{\rho^2(3+2\rho)}{2(1+\rho)^2} + \frac{\rho(\rho+4)(1-\rho)}{2(1+\rho)^3}\right] + O(1),$$
$$\mathbf{E}\big[X_n^{(2)}\big]_2 = n^2\rho^2 + n\left[-\frac{3\rho^2(2+\rho)}{2(1+\rho)^2} - \frac{\rho^2(\rho+4)(1-\rho)}{2(1+\rho)^3}\right] + O(1).$$

The difference between the expectations is $O(1)$, but the second-order factorial moments differ by const n. However, using

$$\operatorname{Var} X = \mathbf{E}[X]_2 + \mathbf{E}\,X - (\mathbf{E}\,X)^2,$$

we find after some algebra that, contrary to our intuition (and the initial formulas),

$$\operatorname{Var} X_n^{(i)} = \sigma^2 n + O(1), \qquad i = 1, 2,$$
$$\sigma^2 = \frac{\rho(1-\rho-\rho^2)}{(1+\rho)^2}, \tag{2.12}$$

the same σ^2 as in the formula (1.14) for $\operatorname{Var} X_n$!

2.2. *Asymptotics for distributions.* Introduce

$$g_n(z) = \sum_{k=1}^{n} z^k g_{nk}, \qquad f_n(z) = \sum_{k=1}^{n} z^k f_{nk}.$$

Probabilistically,

$$g_n(z) = g_n\,\mathbf{E}\,z^{X_n^{(1)}}, \qquad f_n(z) = f_n\,\mathbf{E}\,z^{X_n^{(2)}},$$

where g_n and f_n is the total number of trees type I and type II, respectively ($X_1^{(2)} = 0$, by definition).

Consider $X_n^{(2)}$ first. Since $f_\nu(z)/\nu! = [x^\nu]F(z,x)$ and (see the lemma)

$$F(z,x) = x\Big(\exp\big(zx\exp\big(F(z,\,x)\big)\big) - 1\Big)\exp\big(F(z,x)\big),$$

we have a recurrence for $f_\nu(z)$: $f_1(z) = 0$ and for $\nu \ge 2$,

$$\begin{aligned}\frac{f_\nu(z)}{\nu!} &= [x^\nu]\bigg(x\sum_{j\ge1}\frac{(zxe^F)^j}{j!}e^F\bigg)\\ &= [x^\nu]\bigg(x\sum_{j\ge1}\frac{z^jx^j}{j!}e^{(j+1)F}\bigg)\\ &= \sum_{j=1}^{\nu-1}\frac{z^j}{j!}\sum_{k\ge0}\big[x^{\nu-1-j}\big]\frac{(j+1)^kF^k}{k!}\\ &= \frac{z^{\nu-1}}{(\nu-1)!} + \sum_{j=1}^{\nu-2}\frac{z^j}{j!}\sum_{k\ge1}\frac{(j+1)^k}{k!}\sum_{l_1+\cdots+l_k=\nu-1-j}\prod_{t=1}^{k}\frac{f_{l_t}(z)}{l_t!};\end{aligned} \tag{2.13}$$

$F = F(z, x)$. In particular, setting $z = 1$,

$$\text{(2.14)} \qquad \frac{f_\nu}{\nu!} = \frac{1}{(\nu-1)!} + \sum_{j=1}^{\nu-2} \frac{1}{j!} \sum_{k\ge 1} \frac{(j+1)^k}{k!} \sum_{l_1+\cdots+l_k=\nu-1-j} \prod_{t=1}^{k} \frac{f_{l_t}}{l_t!}.$$

Notice that, for ν *fixed*,

$$f_\nu(e^u) = f_\nu \mathbf{E}\exp(uX_\nu) = f_\nu\big(1 + ua_\nu + u^2 b_\nu/2 + O(|u|^3)\big),$$
$$a_\nu := \mathbf{E}\, X_\nu^{(2)}, \qquad b_\nu := \mathbf{E}\big(X_\nu^{(2)}\big)^2.$$

So, setting $z = e^u$ in (2.13), expanding both of its sides in powers of u and equating first the coefficients by u and second by u^2 we obtain the recurrencies for a_ν and b_ν, the latter involving a_μ, $\mu \le \nu$, of course. We do not have to solve those complicated recurrencies though, since the asymptotics of a_n, b_n have already been obtained in Section 2a. Instead we observe that

$$\phi_\nu(u) := \exp\big(ua_\nu + u^2 \operatorname{Var} X_\nu^{(2)}/2\big) = 1 + ua_\nu + u^2 b_\nu/2 + O\big(|u|^3\big),$$

as well. Therefore if in (2.13) we replace $f_\nu(e^u)$ by $h_\nu(u) := f_\nu \phi_\nu(u)$, the ratio of the sides is $1 + O(|u|^3)$, as $u \to 0$. The combination of the recurrence and "almost linearity" of $\mathbf{E}\, X_\nu^{(2)}$, $\operatorname{Var} X_\nu^{(2)}$ will be seen as a key reason why, uniformly for $\nu \le n$,

$$\text{(2.15)} \qquad \mathbf{E}\exp\left(\frac{vX_\nu^{(2)}}{n^{1/2}}\right) = (1 + o(1))\exp\left(\frac{v\,\mathbf{E}\, X_\nu^{(2)}}{n^{1/2}} + \frac{v^2 \operatorname{Var} X_\nu^{(2)}}{2n}\right),$$

for a fixed $v \in \mathbb{R}$. By a theorem due to Curtiss [4], the relation (2.15) (with $\nu = n$) implies that

$$\frac{X_n^{(2)} - a_n}{\operatorname{Var}^{1/2} X_\nu^{(2)}} \Rightarrow \mathcal{N}(0, 1).$$

To fulfill the program, we must find a way to sharply estimate $f_\nu(u)$ for $u = O(n^{-1/2})$ in a full diapason of ν, from 1 to n.

Our first step is the following lemma.

LEMMA 1. *Let* $s_n = (n/\log\log n)^{1/2}$. *The overall contribution to the sum in* (2.14) *made by the summands with* $k + j \ge s_n$ *is* $O((f_\nu/\nu!)\exp(-c\sqrt{n}))$ *for every positive* $c > 0$, *uniformly for* $\nu \in [3, n]$.

PROOF. Motivated by the derivation of (2.14), we introduce $F(x)=F(1,x)$ $=\sum_{\nu\ge 1} f_\nu x^\nu/\nu!$ and compute

$$\begin{aligned}
&[x^\nu y_1^j y_2^k]\Big(x\big(\exp(xy_1\exp(y_2F(x)))-1\big)\exp(y_2F(x))\Big)\\
&\quad=[x^\nu y_2^k]\bigg(x\frac{(x\exp(y_2F(x)))^j}{j!}\exp(y_2F(x))\bigg)\\
&\quad=[x^\nu y_2^k]\bigg(x^{j+1}\frac{\exp(y_2F(x)(j+1))}{j!}\bigg)\\
&\quad=[x^\nu]\bigg(\frac{x^{j+1}}{j!}[y_2^k]\exp(y_2F(x)(j+1))\bigg)\\
&\quad=[x^\nu]\bigg(\frac{x^{j+1}}{j!}\,\frac{F^k(x)}{k!}(j+1)^k\bigg)=\frac{(j+1)^k}{j!k!}[x^{\nu-1-j}]F^k(x)\\
&\quad=\frac{(j+1)^k}{j!k!}\sum_{l_1+\cdots+l_k=\nu-1-j}\prod_{t=1}^k\frac{f_{l_t}}{l_t!}.
\end{aligned}$$

This is precisely the generic (k,j)th term of the sum in (2.14). Consequently the sum of the terms with $j+k=r$ (r given), equals

$$a_{\nu r}:=[x^\nu y^r]\Big(x\big(\exp(xye^{yF(x)})-1\big)\exp(yF(x))\Big). \tag{2.16}$$

So, using the idea of Chernoff's method, we bound the sum of the terms with $j+k\ge s_n$ (denote it $S_{n\nu}$) as follows: $\forall\, x\in(0,e^{-1}]$, $\forall\, y>1$,

$$S_{n\nu}\le\frac{x\exp(yF(x))\exp(xye^{yF(x)})}{x^\nu y^{s_n}}\le\frac{x\exp(y\mathscr{T}(x))\exp(xye^{y\mathscr{T}(x)})}{x^\nu y^{s_n}}.$$

Selecting $x=e^{-1}$ and using $\mathscr{T}(e^{-1})=1$, we have

$$S_{n\nu}\le e^\nu\frac{\psi e^\psi}{y^{s_n}},\qquad \psi:=ye^{y-1}.$$

To get the most out of this bound we need to select y that (almost) minimizes it. A near optimal choice turns out to be $y=\log s_n-2\log\log s_n$, for which

$$S_{n\nu}\le\exp\big(\nu-(1/2)n^{1/2}(\log\log n)^{1/2}\big).$$

It remains to recall that f_ν is of order $\nu^{\nu-1}$, so that $f_\nu/\nu!$ is of order $e^\nu/\nu^{3/2}$. □

The next estimate shows that when the generic (k,j)th term in (2.14) is multiplied by $(j+k)^3$, the new sum remains of essentially the same order.

LEMMA 2. *Using the notation* (2.16),

$$\sum_{r\ge 1} r^3 a_{\nu r}=O\bigg(\frac{f_\nu}{\nu!}\bigg). \tag{2.17}$$

PROOF. According to (2.16),

$$
\begin{aligned}
\sum_{r\ge1} r^3 a_{\nu r} &\le \textstyle\sum_{r\ge1}(r+1)^3 a_{\nu r}\\
&= [x^\nu]\left(y\frac{\partial}{\partial y}\right)^3 \sum_{\nu\ge1,\, r\ge1} a_{\nu r}x^\nu y^{r+1}\Big|_{y=1}\\
&= [x^\nu]\left(y\frac{\partial}{\partial y}\right)^3 \Big[xy\exp(yF(x))\big(\exp(xy\exp(yF(x)))-1\big)\Big]\Big|_{y=1}\\
&\le [x^\nu]\left(y\frac{\partial}{\partial y}\right)^3 \exp(2H(x,y))\Big|_{y=1},
\end{aligned}
\tag{2.18}
$$

where

$$H(x,\eta) := x\eta\exp(\eta\mathscr{T}(x)),$$

and $|x|\le e^{-1}$. Since $\mathscr{T}(x)=xe^{\mathscr{T}(x)}$,

$$H(x,\eta)=\eta x^{1-\eta}\mathscr{T}^{\eta}(x). \tag{2.19}$$

Notice that

$$
\begin{aligned}
y\frac{\partial}{\partial y}H(x,y) &= xy\big[\exp(y\mathscr{T}(x)) + y\mathscr{T}(x)\exp(y\mathscr{T}(x))\big]\\
&\prec xy\big[\exp(y\mathscr{T}(x)) + \exp(2y\mathscr{T}(x))\big]\\
&\prec x(2y)\exp(2y\mathscr{T}(x))\\
&= H(x,2y).
\end{aligned}
\tag{2.20}
$$

[Given two power series $B(x,y)=\sum_{\nu,r\ge1} b_{\nu r}x^\nu y^r$, $C(x,y)=\sum_{\nu,\, r\ge1} c_{\nu r}x^\nu y^r$ with nonnegative coefficients, we write $B(x,y)\prec C(x,y)$ if $b_{\nu r}\le c_{\nu r}$ for all ν, $r\ge1$.] Applying (2.20) repeatedly to (2.18), and using (2.19) at the end, we get

$$
\begin{aligned}
\sum_{r\ge1} r^3 a_{\nu r} &\le 20[x^\nu]\exp(2H(x,1))\big(H^3(x,2)+H(x,2)H(x,4)+H(x,8)\big)\\
&= 160\big([x^{\nu+5}]\mathscr{T}^8(x) + [x^{\nu+6}]\mathscr{T}^8(x) + [x^{\nu+9}]\mathscr{T}^{10}(x)\big)\\
&= O\left(\frac{\nu^{\nu-1}}{\nu!}\right)\\
&= O\left(\frac{f_\nu}{\nu!}\right).
\end{aligned}
$$

[For the penultimate estimate, we have used a well-known formula

$$\left(\sum_{m\ge1} x^m\frac{m^{m-1}}{m!}\right)^\mu = \sum_{m\ge1} x^m\frac{\mu m^{m-\mu-1}[m]_\mu}{m!},$$

which is essentially equivalent to the Cayley formula for the number of forests of μ trees on m vertices, each tree containing one of specified μ vertices.] □

Lemmas 1 and 2 allow us to show that our preliminary estimates done for a fixed ν hold also for all $\nu \le n$, if $u = O(n^{-1/2})$.

LEMMA 3. *Let* $z = e^u$, $u = vn^{-1/2}$, $v \neq 0$ *being fixed. Let* $c_n \to \infty$, $c_n = O(n^{1/2})$. *Denote by* $\mathscr{S}_\nu^{\pm}(u)$ *the right-hand side of* (2.13) *with* $f_l(z)$ *replaced by*

$$h_l^{\pm}(u) = f_l \phi_l^{\pm}(u),$$
$$\phi_l^{\pm}(u) := \exp\big(ua_l + u^2 \operatorname{Var} X_l^{(2)}/2 \pm c_n|u|^3(l-1)\big).$$

Then for all sufficiently large n and for all $\nu \in [4, n]$,

$$\frac{h_\nu^+(u)}{\nu!} \ge \mathscr{S}_\nu^+(u), \qquad \frac{h_\nu^-(u)}{\nu!} \le \mathscr{S}_\nu^-(u).$$

PROOF. Consider, for instance, the *plus* case. Let us have a look at the u-dependent factors of a generic term in the sum in $\mathscr{S}_\nu^+(u)$. By the definition of $h_l^+(u)$, for $l_1, \dots, l_k \ge 1$ such that $\sum_{t=1}^k l_t = \nu - 1 - j$, we have

$$e^{ju} \prod_{t=1}^k h_{l_t}^+(u) = \exp\bigg[u\bigg(j + \sum_{t=1}^k a_{l_t}\bigg) + (u^2/2)\sum_{t=1}^k \operatorname{Var} X_{l_t}^{(2)} + c_n|u|^3(\nu - 1 - j - k)\bigg] \prod_{t=1}^k f_{l_t}.$$

Here, applying (2.11) to each $\mathbf{E}\, X_{l_t}^{(2)} (= a_{l_t})$ and (2.12) to $\operatorname{Var} X_{l_t}^{(2)}$,

$$\begin{aligned}\sum_{t=1}^k a_{l_t} &= \rho \sum_{t=1}^k l_t + O(k) = \rho(\nu - 1 - j) + O(k)\\ &= a_\nu + O(j + k),\\ \sum_{t=1}^k \operatorname{Var} X_{l_t}^{(2)} &= \sigma^2 \sum_{t=1}^k l_t + O(k) = \sigma^2(\nu - 1 - j) + O(k)\\ &= \operatorname{Var} X_\nu^{(2)} + O(j+k).\end{aligned}$$

So, recalling the definion of $h_\nu^+(u)$, $\phi_\nu^+(u)$,

$$e^{ju} \prod_{t=1}^k h_{l_t}^+(u) = \phi_\nu^+(u)\bigg(\prod_{t=1}^k f_{l_t}\bigg) \times \exp\big(uR_\nu(j, l) + (u^2/2)\mathscr{R}_\nu(l) - c_n|u|^3(j+k)\big), \tag{2.21}$$

where $c_n(j+k) = O(n^{3/2})$, and $R_\nu(j,l)$, $\mathscr{R}_\nu(l) = O(j+k) = O(n)$ are defined by

$$R_\nu(j,l) = j + \sum_{t=1}^{k} a_{l_t} - a_\nu,$$

$$\mathscr{R}_\nu(l) = \sum_{t=1}^{k} \operatorname{Var} X_{l_t}^{(2)} - \operatorname{Var} X_\nu^{(2)}.$$

Since $u = O(n^{-1/2})$ and $c_n = O(n^{1/2})$, the exponent is of order $O(n^{1/2})$. Then, by Lemma 1, the total contribution to $\mathscr{S}_\nu^+(u)$ coming from the terms with $j+k \geq s_n$ is at most

$$\phi_\nu^+(u) S_{n\nu} \exp\bigl(O(n^{1/2})\bigr) = O\biggl(\frac{h_\nu^+(u)}{\nu!}\exp(-dn^{1/2})\biggr), \tag{2.22}$$

for every $d > 0$.

Consider now the u-dependent factors of a generic term in $\mathscr{S}_\nu^+(u)$ with $j+k \leq s_n$. Since in this case $j+k = o(n^{1/2})$, the exponent in (2.21) is $o(1)$. So we can expand the exponential function and get for this term a formula,

$$\begin{aligned}\phi_\nu^+(u)\biggl(\prod_{t=1}^{k} f_{l_t}\biggr)\Bigl(&1 + uR_\nu(j,l) + (u^2/2)\bigl(R_\nu^2(j,l) + \mathscr{R}_\nu(l)\bigr)\\ &-c_n(j+k)|u|^3 + O\bigl(|u|^3(j+k)^3\bigr)\Bigr).\end{aligned} \tag{2.23}$$

[For the attentive reader, the remainder term comes from an intermediate bound $O\bigl(|u|^3(j+k)^3(1+c_n|u|)\bigr)$, and the condition $c_n = O(n^{1/2})$.] We notice upfront that by Lemma 2 the contribution of the remainder term to $\mathscr{S}_\nu^+(u)$, even when the restriction $j+k \leq s_n$ is dropped, is

$$O\biggl(|u|^3 \frac{f_\nu \phi_\nu^+(u)}{\nu!}\biggr) = O\biggl(|u|^3 \frac{h_\nu^+(u)}{\nu!}\biggr). \tag{2.24}$$

Furthermore, by Lemma 1, the overall contribution of the explicit terms in (2.23) for $j+k > s_n$ is (for every $d > 0$) of order

$$\begin{aligned}&\phi_\nu^+(u)\frac{f_\nu}{\nu!}\bigl(1 + |u|\nu + |u|^2\nu^2\bigr)\exp(-d\sqrt{n})\\ &\quad = O\biggl(\frac{h_\nu^+(u)}{\nu!}\exp(-d'n^{1/2})\biggr) \qquad \forall\, d' < d.\end{aligned} \tag{2.25}$$

It remains to evaluate the overall contribution of the explicit terms, with the restriction $j+k \leq s_n$ being dropped. First, the contribution of $-c_n|u|^3(j+k)$ is, by (2.14), at least

$$-2c_n\alpha|u|^3\frac{h_\nu^+(u)}{\nu!}, \qquad \alpha := 1 - \max_{j\geq 4}\frac{j}{f_j} > 0,$$

which certainly outweighs, for n large enough, the bound given in (2.24). Moreover, since $|u|^3$ being of order $n^{-3/2}$ dwarfs $\exp(-\delta n^{1/2})$, $(\delta > 0)$, the above expression absorbs the remainder terms in (2.22) and (2.25). So, for those n's, the overall contribution of $|u|^3$ plus those remainder terms is at most $-c_n\alpha|u|^3 h_\nu^+(u)/\nu!$.

Let us compute the *unrestricted* sum of the lower (zero, first, second) degree terms. Like $h_l(u)$, the functions $h_l^{\pm}(u)$ [substituted for $f_\nu(e^u)$] satisfy, for a *fixed* ν, (2.13) within a factor $1+O(|u|^3)$. So, dividing both sides of this *approximate* equality by $\phi_\nu^+(u)$, we have

$$\frac{f_\nu}{\nu!} \approx \frac{\exp\bigl(u(\nu-1-a_\nu)-u^2 \operatorname{Var} X_\nu^{(2)}/2\bigr)}{(\nu-1)!} + \sum_{j=1}^{\nu-2}\frac{1}{j!}\sum_{k\ge 1}\frac{(j+1)^k}{k!}$$
$$\times \sum_{\mathbf{l}}\biggl(\prod_{t=1}^{k}\frac{f_{l_t}}{l_t!}\biggr)\Bigl(1+uR_\nu(j,\mathbf{l})+(u^2/2)\bigl(R_\nu^2(j,\mathbf{l})+\mathscr{R}_\nu(\mathbf{l})\bigr)\Bigr).$$

The symbol "$\approx$" means that the second degree Taylor polynomials of both sides are the same. (Equating the coefficients of u and then the coefficients of u^2, we get the recurrencies for $a_\nu = \mathbf{E}\ X_\nu^{(2)}$ and $\operatorname{Var} X_\nu^{(2)}$.) Therefore the above sum equals

$$\frac{f_\nu}{\nu!} - \frac{1}{(\nu-1)!}\biggl[1+u(\nu-1-a_\nu)+\frac{u^2}{2}\bigl((\nu-1-a_\nu)^2-\operatorname{Var} X_\nu^{(2)}\bigr)\biggr].$$

Combining the bounds and the last relation, we conclude that

$$\begin{aligned}\mathscr{S}_\nu^+(u) \le{}& \frac{\exp(u(\nu-1))}{(\nu-1)!}\\ &+\phi_\nu^+(u)\biggl\{\frac{f_\nu}{\nu!}-\frac{1}{(\nu-1)!}\Bigl[1+u(\nu-1-a_\nu)+(u^2/2)\bigl((\nu-1-a_\nu)^2\\ &\qquad\qquad -\operatorname{Var} X_\nu^{(2)}\bigr)\Bigr]\biggr\}\\ &-c_n\alpha|u|^3\frac{h_\nu^+(u)}{\nu!}.\end{aligned}$$

By the definition of $\phi_\nu^+(u)$, this expression is *at most* $h_\nu^+(u)/\nu!$ times

$$\begin{aligned}1-c_n\alpha|u|^3+\frac{\nu}{f_\nu}\biggl\{&\exp\biggl[u(\nu-1-a_\nu)-\frac{u^2}{2}\operatorname{Var} X_\nu^{(2)}\biggr]\\ &-\biggl[1+u(\nu-1-a_\nu)+\tfrac{u^2}{2}\bigl((\nu-1-a_\nu)^2-\operatorname{Var} X_\nu^{(2)}\bigr)\biggr]\biggr\}.\end{aligned}$$

Let us prove that, for all $4\le\nu\le n$, this can be made less than 1. Recall that a_ν, $\operatorname{Var} X_\nu^{(2)} = O(\nu)$, and u is of order $n^{-1/2}$ exactly. If $\nu \le n^{1/2}\log^{-1} n$, then the exponent is $o(1)$ and, approximating the exponential function, we transform the last expression into

$$1-c_n\alpha|u|^3+O\biggl(\frac{\nu}{f_\nu}|u|^3\nu^3\biggr) = 1-c_n\alpha|u|^3+O\bigl(|u|^3\bigr)\le 1-(c_n\alpha/2)|u|^3,$$

the last inequality holding for large n. If $\nu \geq n^{1/2}\log^{-1} n$, then we get

$$1 - c_n\alpha|u|^3 + O\biggl(\frac{\exp\bigl(O(\nu n^{-1/2})\bigr)}{\nu^{\nu-2}}\biggr) \leq 1 - (c_n\alpha/2)|u|^3,$$

for all large n. Therefore, for sufficiently large n and all $4 \leq \nu \leq n$,

$$\frac{\mathscr{S}_\nu^{(+)}(u)}{h_\nu^+(u)/\nu!} \leq 1 - (c_n\alpha/2)|u|^3.$$

The *minus* case is completely analogous. □

COROLLARY 1. *In the notation of Lemma* 3, *for* $\nu \in [1, n]$,

$$h_\nu^-(u) \leq f_\nu(e^u) \leq h_\nu^+(u). \tag{2.26}$$

PROOF (By induction on ν). We trivially have $f_1(e^u) = h_1^\pm(u)$, since both sides are zero. Furthermore, $X_2^{(2)} = 1$ and $X_3^{(2)} = 2$, so $\operatorname{Var} X_\nu^{(2)} = 0$ in both cases, whence

$$f_\nu(e^u) = f_\nu \exp\bigl(u\,\mathbf{E}\,X_\nu^{(2)}\bigr) \in \bigl[h_\nu^-(u),\ h_\nu^+(u)\bigr], \qquad \nu = 2, 3.$$

The induction step follows easily from (2.13) and Lemma 3. □

Let $c_n = \log n$; setting $\nu = n$ in (2.26) we arrive at the theorem.

THEOREM 1. *For every fixed* $v \in \mathbb{R}$,

$$\mathbf{E}\exp\biggl(\frac{vX_n^{(2)}}{n^{1/2}}\biggr) = \bigl(1 + O(n^{-1/2}\log n)\bigr)\exp\biggl(\frac{v\,\mathbf{E}\,X_n^{(2)}}{n^{1/2}} + \frac{v^2\operatorname{Var} X_n^{(2)}}{2n}\biggr);$$

consequently $(X_n^{(2)} - \mathbf{E}\,X_n^{(2)})/\sqrt{\operatorname{Var} X_n^{(2)}}$ *converges*, *in distribution and with all its moments*, *to the standard normal variable*.

Next we have Theorem 2.

THEOREM 2. *The analogous assertion holds for* $X_n^{(1)}$.

SKETCH OF THE PROOF OF THEOREM 2. Since the argument is close and simpler, we just describe the basic steps.

From (1.4), (1.5),

$$G(z, x) = zx\exp\bigl(z^{-1}\bigl(\exp(G(z, x)) - 1\bigr)G(z, x)\bigr),$$

so, for $\nu \geq 2$,

$$\begin{aligned}
\frac{g_\nu(z)}{\nu!} &= z\big[x^{\nu-1}\big]\exp\big(z^{-1}\big(\exp(G(z,x))-1\big)G(z,x)\big)\\
&= z\sum_{j\geq 1}\frac{z^{-j}}{j!}\big[x^{\nu-1}\big]\bigg(\sum_{k\geq 1}\frac{G^{k+1}(z,x)}{k!}\bigg)^j\\
&= z\sum_{j\geq 1}\frac{z^{-j}}{j!}\sum_{\mathbf{k}}\prod_{r=1}^{j}\frac{1}{k_r!}\big[x^{\nu-1}\big]G^{|\mathbf{k}|+j}(z,x), \qquad |\mathbf{k}| := \sum_{r=1}^{j} k_r,\\
&= z\sum_{j\geq 1}\frac{z^{-j}}{j!}\sum_{\mathbf{k},\mathbf{m}}\prod_{r=1}^{j}\frac{1}{k_r!}\prod_{s=1}^{|\mathbf{k}|+j}\frac{g_{m_s}(z)}{m_s!};
\end{aligned} \tag{2.27}$$

here

$$k_r \geq 1, \quad 1 \leq r \leq j, \qquad m_s \geq 1, \quad 1 \leq s \leq |\mathbf{k}|+j,$$
$$|\mathbf{m}| := \sum_{s=1}^{|\mathbf{k}|+j} m_s = \nu - 1,$$

so $j \leq (\nu-1)/2$, in particular. For $z = 1$ we get

$$\frac{g_\nu}{\nu!} = \sum_{j\geq 1}\frac{1}{j!}\sum_{\mathbf{k},\mathbf{m}}\prod_{r=1}^{j}\frac{1}{k_r!}\prod_{s=1}^{|\mathbf{k}|+j}\frac{g_{m_s}}{m_s!}. \tag{2.28}$$

LEMMA 1a. *Let $s_n = (n/\log\log n)^{1/2}$ and $S_{n\nu}$ be the contribution to the sum in* (2.28) *of the terms with $|\mathbf{k}| + j \geq s_n$. Then $S_{n\nu} = O((g_\nu/\nu!)\exp(-c\sqrt{n}))$, for every $c > 0$, uniformly for $\nu \in [2, n]$.*

The proof is based on the observation that

$$S_{n\nu} = \sum_{r\geq s_n} a_{\nu r},$$

where

$$\begin{aligned}
a_{\nu,r} &= \big[x^{\nu-1}y^r\big]\exp\big(yG(x)\big(\exp(yG(x))-1\big)\big)\\
&\leq \frac{\exp\big(y\mathscr{T}(x)(\exp(y\mathscr{T}(x))-1)\big)}{x^{\nu-1}y^r} \qquad \forall\, x \in (0, e^{-1}],\ \forall\, y > 1.
\end{aligned}$$

The next lemmas are the counterparts of Lemmas 2 and 3.

LEMMA 2a.

$$\sum_{r\geq 1} r^3 a_{\nu r} = O\bigg(\frac{g_\nu}{\nu!}\bigg),$$

LEMMA 3a. *Let $\mathscr{S}_\nu^{\pm}(u)$ denote the sum in* (2.27) *with $g_m(z)$ replaced by*

$$h_m^{\pm}(u) = g_m\phi_m^{\pm}(u), \qquad \phi_m^{\pm}(u) := \exp\bigg(u\,\mathbf{E}\,X_m^{(1)} + \frac{u^2}{2}\operatorname{Var} X_m^{(1)} \pm c_n(m-1)|u|^3\bigg),$$

with $c_n \to \infty$, $c_n = O(n^{1/2})$. *Let* $u = vn^{-1/2}$, $v \neq 0$ *being fixed. Then for all sufficiently large* n *and* $2 \le \nu \le n$,

$$\frac{h_\nu^+(u)}{\nu!} \ge \mathscr{S}_\nu^+(u), \qquad \frac{h_\nu^-(u)}{\nu!} \le \mathscr{S}_\nu^-(u).$$

From the last lemma and the recurrence (2.27) *we obtain the corollary.*

COROLLARY 1a. *In the notation of Lemma* 3a, *for* n *large enough*,

$$h_\nu^-(u) \le g_\nu(e^u) \le h_\nu^+(u), \qquad 1 \le \nu \le n.$$

So Theorem 2 follows. □

3. Number of total orders.

3.1. *Two moments of vertex counts by progeny.* Given a rooted tree on n labelled vertices, let $Z_j(T)$ denote the number of vertices each having progeny of size j; so $Z_{nj} = Z_j(T)$ when T is random . If the root has degree d and the subtrees rooted at its neighbors are $T_1, \dots, T_d$ then, for $j < n$,

$$Z_j(T) = \sum_{i=1}^{d} Z_j(T_i). \tag{3.1}$$

For a fixed $k \in \mathbb{N}$, and $z_1, \dots, z_k \in \mathbb{R}$, introduce a multivariate generating function

$$f_n(\mathbf{z}) = \mathbf{E}\left(\prod_{j=1}^{k} z_j^{Z_{nj}}\right).$$

Clearly,

$$f_n(\mathbf{z}) = \frac{F_n(\mathbf{z})}{n^{n-1}},$$
$$F_n(\mathbf{z}) := \sum_T \prod_{j=1}^{k} z_j^{Z_j(T)},$$

and, crudely,

$$\left|F_n(\mathbf{z})\right| \le z^n n^{n-1} \quad \text{if } z := \max_{1\le i\le k} |z_i| \ge 1. \tag{3.2}$$

A standard argument, based on (3.1), shows that, for $n > k$,

$$\begin{aligned} \frac{F_n(\mathbf{z})}{n!} &= S_n(\mathbf{z}), \\ S_n(\mathbf{z}) &:= \sum_{j\ge 1} \frac{1}{j!} \sum_{\substack{l_1,\dots,l_j \ge 1 \\ l_1+\cdots+l_j = n-1}} \prod_{i=1}^{j} \frac{F_{l_i}(\mathbf{z})}{l_i!}; \end{aligned} \tag{3.3}$$

for $n \le k$,

$$\frac{F_n(\mathbf{z})}{n!} = z_n S_n(\mathbf{z}); \tag{3.4}$$

[cf. (2.13), (2.27)]; $S_1(\mathbf{z}) = 1$ by definition. Introduce

$$F(\mathbf{z}, x) = \sum_{n\ge 1} F_n(\mathbf{z}) \frac{x^n}{n!};$$

by (3.2), the series definitely converges if $1 \le z \le (e|x|)^{-1}$, which requires that $|x| \le e^{-1}$. From (3.3) and (3.4),

$$F(\mathbf{z}, x) = x \exp(F(\mathbf{z}, x)) + \sum_{i=1}^{k} (1 - z_i^{-1}) F_i(\mathbf{z}) \frac{x^i}{i!}. \tag{3.5}$$

For $z_i \equiv 1$, $F(\mathbf{z}, x) = \mathscr{T}(x)$ and (3.5) reduces to (1.8). This equation is our source for the moments formula. Let $|x| < e^{-1}$. Differentiating both sides of (3.5) with respect to z_k at $\mathbf{z} = \mathbf{1}$, and using $F_k(\mathbf{1}) = k^{k-1}$, we obtain

$$\sum_{n=1}^{\infty} \left(\sum_T Z_k(T) \right) \frac{x^n}{n!} = \left. \frac{\partial F}{\partial z_k} \right|_{\mathbf{z}=\mathbf{1}} = \frac{k^{k-1}}{k!} \frac{x^k}{1 - \mathscr{T}(x)}. \tag{3.6}$$

Since

$$\frac{1}{1 - \mathscr{T}(x)} = \sum_{i=0}^{\infty} i^i \frac{x^i}{i!}, \tag{3.7}$$

we get from (3.6)

$$\sum_T Z_k(T) = \binom{n}{k} k^{k-1} (n-k)^{n-k},$$

whence

$$\begin{aligned} \mathbf{E}\, Z_{nk} &= \binom{n}{k} \frac{k^{k-1}(n-k)^{n-k}}{n^{n-1}} \\ &= np(k) + \frac{1}{2} q(k) + O(n^{-1}); \end{aligned} \tag{3.8}$$

see (1.17) for the definitions of $p(\cdot)$, $q(\cdot)$. Differentiating (3.5) twice with respect to z_k and using (3.6), we obtain

$$\sum_{n\ge 1} \left(\sum_T Z_k(T)(Z_k(T) - 1) \right) \frac{x^n}{n!} = \left. \frac{\partial^2 F}{\partial^2 z_k} \right|_{\mathbf{z}=\mathbf{1}} = \left(\frac{k^{k-1}}{k!} \right)^2 \frac{x^{2k} \mathscr{T}(x)}{(1 - \mathscr{T}(x))^3}.$$

And using (1.17) and (3.7),

$$\begin{aligned} \frac{\mathscr{T}(x)}{(1 - \mathscr{T}(x))^3} &= x \frac{\mathscr{T}(x)}{x(1 - \mathscr{T}(x))^3} = x \frac{d}{dx} \left(\frac{1}{1 - \mathscr{T}(x)} \right) \\ &= \sum_{i\ge 0} i^{i+1} \frac{x^i}{i!}. \end{aligned}$$

So, like (3.8),

$$\mathbf{E}\big(Z_{nk}(Z_{nk}-1)\big) = \frac{1}{n^{n-1}}\binom{n}{k}\binom{n-k}{k}\big(k^{k-1}\big)^2(n-2k)^{n-2k+1} \\ = n^2p^2(k) - nkp^2(k) + O(1). \tag{3.9}$$

Differentiating (3.5) with respect to z_k and z_j, $(j < k)$, we compute similarly

$$\begin{aligned}\mathbf{E}\big(Z_{nj}Z_{nk}\big) &= \frac{1}{n^{n-1}}\binom{n}{k}\binom{n-k}{j} j^{j-1}k^{k-1}(n-k-j)^{n-k-j+1} \\ &\quad + \frac{1}{n^{n-1}}\binom{n}{k}\binom{k}{j} j^{j-1}(k-j)^{k-j}(n-k)^{n-k} \\ &= n^2p(j)p(k) - n\frac{(j+k)}{2}p(j)p(k) + np(j)q(k-j) + O(1).\end{aligned} \tag{3.10}$$

It follows from (3.8)–(3.10) that, for $j \le k$,

$$\begin{aligned}\operatorname{Cov}\big(Z_{nj}, Z_{nk}\big) &= nK(j,k) + O(1), \\ K(j,k) &:= p(j)\big(q(k-j) - (k+j)p(k)\big).\end{aligned} \tag{3.11}$$

3.2. *Limit distribution of vertex counts.* For a fixed $k \ge 1$, denote $(Z_{n1}, \dots, Z_{nk}) = \mathbf{Z}_n^t$, t standing for " transpose," and let $\mathbf{K} = \{K(\mu, \nu)\}_{1\le\mu,\nu\le k}$ denote the symmetric $k \times k$ matrix, with

$$K(\mu,\nu) = p(\mu)\big(q(\nu-\mu) - (\mu+\nu)p(\nu)\big), \qquad \mu \le \nu.$$

THEOREM 3. *For a fixed* $\mathbf{v}^t = (v_1, \dots, v_k) \in \mathbb{R}^k$,

$$\mathbf{E}\exp\big(n^{-1/2}(\mathbf{v}^t\mathbf{Z}_n)\big) = \big(1 + O(n^{-1/2}\log n)\big) \\ \times \exp\big(n^{-1/2}(\mathbf{v}^t\,\mathbf{E}\,\mathbf{Z}_n) + (2n)^{-1}\mathbf{v}^t\mathbf{K}\mathbf{v}\big),$$

so, consequently, $n^{-1/2}(\mathbf{Z}_n - n\,\mathbf{E}\,\mathbf{Z}_n)$ *is asymptotically Gaussian with the mean* $\mathbf{0}$ *and the covariance matrix* $\mathbf{K}$.

COROLLARY 2. *The random vector* $\{n^{-1/2}(Z_{nk} - np(k))\}_{1\le k\le n}$ *converges, in terms of finite-dimensional distributions, to a Gaussian sequence* $\mathbf{Z} = \{Z_k\}_{k\ge1}$ *with* $\mathbf{E}\,Z_k \equiv 0$ *and the covariance function* $K(i, j)$.

Here is a brief proof sketch. Given $\mathbf{u} \in \mathbb{R}^k$, define $f_\nu(\mathbf{u}) = \mathbf{E}\exp(\mathbf{u}^t\mathbf{Z}_\nu)$. Let T be a uniformly random rooted tree with ν vertices. Analogously to (3.3), for $\nu > k$,

$$f_\nu(\mathbf{u}) = \frac{\nu!}{\nu^{\nu-1}}\sum_{j\ge1}\frac{1}{j!}\sum_{\substack{l_1,\dots,l_j\ge1 \\ l_1+\cdots+l_j=\nu-1}}\prod_{i=1}^{j}\frac{l_i^{l_i-1}f_{l_i}(\mathbf{u})}{l_i!}.$$

The counterparts of Lemma 1 (1a) and Lemma 2 (2a) are two estimates:

$$\sum_{j\ge(n/\log\log n)^{1/2}}\frac{1}{j!}\sum_{\substack{l_1,\dots,l_j\\ l_1+\dots+l_j=\nu-1}}\prod_{i=1}^{j}\frac{l_i^{l_i-1}}{l_i!}=O\biggl(\frac{\nu^{\nu-1}}{\nu!}\exp(-cn^{1/2})\biggr)\qquad\forall\,c>0,$$

$$\sum_{j\ge1}\frac{j^3}{j!}\sum_{\substack{l_1,\dots,l_j\\ l_1+\dots+l_j=\nu-1}}\prod_{i=1}^{j}\frac{l_i^{l_i-1}}{l_i!}=O\biggl(\frac{\nu^{\nu-1}}{\nu!}\biggr).$$

The bounds can be established as in the proofs of Theorems 1 and 2. Much more simply, we can reformulate the estimates as

$$\mathbf{P}\Biggl\{D_\nu\ge\sqrt{\frac{n}{\log\log n}}\Biggr\}=O\bigl(\exp(-cn^{1/2})\bigr)\qquad\forall\,c>0,\tag{3.12}$$
$$\mathbf{E}\,D_\nu^3=O(1),$$

where D_ν is the degree of the root, and prove them by using the fact that $D_\nu=1+\mathrm{Bin}\bigl(\nu-2,\nu^{-1}\bigr)$. Using the estimates *and* linearity (within the remainder of $O(1)$) of $\mathbf{a}_\nu$, $\mathbf{b}_\nu$, the mean and the covariance matrix of the vector $\mathbf{Z}_\nu$, we show finally that for $\mathbf{u}=n^{-1/2}\mathbf{v}$, $\mathbf{v}\in\mathbb{R}^k$, $\mathbf{v}\ne\mathbf{0}$ and $c_n\to\infty$, $c_n=O(n^{1/2})$,

$$\phi_\nu^+(\mathbf{u})\ge\mathscr{S}_\nu^+(\mathbf{u}),\qquad\phi_\nu^-(\mathbf{u})\le\mathscr{S}_\nu^-(\mathbf{u}).$$

Here

$$\phi_\nu^\pm(\mathbf{u}):=\exp\Bigl(\mathbf{u}^t\mathbf{a}_\nu+\tfrac12\mathbf{u}^t\mathbf{b}_\nu\mathbf{u}\pm c_n(\nu-1)\bigl(\mathbf{u}^t\mathbf{u}\bigr)^{3/2}\Bigr),$$

and $\mathscr{S}_\nu^\pm(\mathbf{u})$ is the right-hand side of the recurrence with $\phi_\nu^\pm(\mathbf{u})$ instead of $f_\nu(\mathbf{u})$. The proof is completed then just like Theorems 1 and 2. □

3.3. *Limit distribution of* Y_n. By (1.1),

$$\log Y_n=\log n!-L_n,\qquad L_n:=\sum_{k=1}^{n}Z_{nk}\log k.$$

Now, in view of Theorem 3 and (3.8), (3.11), one would expect that L_n is asymptotically Gaussian with mean na and variance $n\sigma^2$, where

$$a=\sum_{k\ge1}^{\infty}p(k)\log k,\qquad\sigma^2=\sum_{j,k\ge1}K(j,k)\log j\log k.$$

However, the double series slowly diverges to infinity, so the variance of L_n must grow faster than n. It is proved in Appendix that

$$\operatorname{Var}L_n=bn\log n+\xi_1n+\xi_2n^{1/2}\log^2n+O\bigl(n^{1/2}\log n\bigr),\tag{3.13}$$

and we will use this to show that $b=4\log(e/2)$. So indeed $\operatorname{Var}L_n$ is growing a bit faster than in a linear fashion. Moreover, unlike the counts of vertices by progeny, $\mathbf{E}\,L_n$ is not nearly linear either. We also show in the Appendix that

$$\mathbf{E}\,L_n=an-\sqrt{2\pi n}+s_1\log n+s_2+O\bigl(n^{-1/2}\log n\bigr),\tag{3.14}$$

and it is the sharpness of this formula which allows obtaining (3.13). The actual values of the constants ξ_i, s_i are immaterial though. (Fill [7] and Meir and Moon [17] obtained the formulas $\mathbf{E}\, L_n = \alpha n - \beta n^{1/2} + O(\log n)$, $\operatorname{Var} L_n = \gamma n + O(n)$, respectively, for the uniform binary tree, and the whole class of simply generated random trees.) In a certain technical sense, it is this $n^{1/2}$ term in $\mathbf{E}\, L_n$ which is "responsible" for the slight superlinearity of $\operatorname{Var} L_n$.

THEOREM 4. *The random variable* $(L_n - an)/\big(n \log n\big)^{1/2}$ *converges in distribution and with all its moments, to the normal random variable, with zero mean, and variance equal* $4\log(e/2)$.

PROOF. A natural way to proceed would seem to be the following. In the notations of Section 3.1, let

$$L(T) = \sum_k Z_k(T) \log k,$$

so $L(T) = L_\nu$ if T is a uniformly random rooted tree with ν vertices. Clearly,

$$L(T) = \log \nu + \sum_{t=1}^{D_\nu} L(T_t), \tag{3.15}$$

where $D_\nu = D(T)$ is the root's degree. Conditioned on D_ν and the vertex sets V_t of the subtrees T_t, $(t \le D_\nu)$, each $L(T_t)$ is distributed as $L_{|V_t|}$, independently of other subtrees. Introducing $f_\nu(u) = \mathbf{E}\, e^{uL_\nu}$, we obtain a recurrence

$$\begin{aligned} f_\nu(u) &= \exp(u \log \nu)\, \mathbf{E}\left(\prod_{t=1}^{D_\nu} f_{|V_t|}(u) \right) \\ &= \exp(u \log \nu) \frac{\nu!}{\nu^{\nu-1}} \sum_{j \ge 1} \frac{1}{j!} \sum_{\substack{l_1, \dots, l_j \\ l_1 + \cdots + l_j = \nu - 1}} \prod_{t=1}^{j} \frac{l_t^{l_t - 1} f_{l_t}(u)}{l_t!} \end{aligned} \tag{3.16}$$

[cf. (3.3), (3.4)]. The crucial step is to prove that for u of order $(n \log n)^{-1/2}$,

$$\phi_l^{\pm}(u) := \exp\left(u a_l + \frac{u^2}{2} b_l \pm c_n (\nu - 1) |u|^3 \right)$$

(a_l, b_l being the mean and variance of L_l) satisfy

$$\phi_\nu^+(u) \ge \mathscr{S}_\nu^+(u), \qquad \phi_\nu^- \le \mathscr{S}_\nu^-(u),\ 2 \le \nu \le n, \tag{3.17}$$

provided that $c_n \to \infty$ not too fast. [Needless to say, $\mathscr{S}_\nu^{\pm}(u)$ is obtained from the right-hand side of (3.16) by replacing $f_l(u)$ with $\phi_l^{\pm}(u)$.] If successful, we would quickly complete the proof of Theorem 4. However, all our attempts to prove (3.17) have failed, and now we feel that, for the full range of ν, (3.17) is not even true. Basically, the roadblock is the "insufficient linearity" of $\mathbf{E}\, L_\nu$ and $\operatorname{Var} L_\nu$.

To prove the theorem we derive a much simpler recurrrence for $f_\nu(u)$, and use it to show that the *cumulants* (semiinvariants) of L_ν scaled by its standard deviation converge, as $\nu \to \infty$, to those of the standard normal distribution.

Initially we thought that this second recurrence was sufficient to show convergence of the m.g. function, but the careful reviewer found an error in the argument. The problem in question appears to be an interesting case for the normal convergence, when the m.g. function does not converge, but the cumulants still do. We emphasize, though, that it is the recurrence for the m.g. function that makes the cumulants such an efficient tool.

An unrooted tree on ν labelled vertices can be construed as a terminal state of a process of $\nu - 1$ successive insertions of its $\nu - 1$ edges, one edge at a time. After t steps ($t < \nu - 1$), we have a forest of $\nu - t$ trees, and the $(t + 1)$-th step results in two of those trees getting joined by an edge, thus forming a larger tree. If all $\nu^{\nu-2}(\nu-1)!$ realizations of this process are equally likely, the terminal tree is uniformly distributed. In the literature, the random sequence of forests $\{F_t\}_{t<\nu}$ is known as *the random spanning tree model*, [24], [25], [10], [11]. (See [1] for the functional limit theorem for this process.) Let $\pi(F)$ denote the partition of $[\nu]$ into the vertex sets of trees in a forest F. Yao discovered that the random sequence $\{\pi(F_t)\}_{t<\nu}$ is a Markov chain. More precisely, if π is a partition with $\nu - t$ subsets, and π' is obtained from π by merging two of its subsets, of sizes i and j say, then

$$\mathbf{P}\big(\pi(F_{t+1}) = \pi' \mid \pi(F_t) = \pi\big) = \frac{i + j}{\nu(\nu - t - 1)}.$$

Given such a merger, all ij choices of the contact vertices are equally likely. (For the classic random graph process, due to Erdős and Rényi, the analogous probability is proportional to the product ij.) Implicit in [24] is another discovery, namely that the (joint) size distribution for the $\nu - t$ trees in F_t is the same as the size distribution for the uniformly random forest of $\nu - t$ *rooted* trees. By Cayley's formula, the number of of such forests is

$$\binom{\nu}{r} r\nu^{\nu-r-1}\bigg|_{r=\nu-t} = \binom{\nu}{t}(\nu - t)\nu^{t-1};$$

hence if the trees in F have sizes $i_1, \dots, i_{\nu-t}$, then

$$\mathbf{P}(F_t = F) = \frac{\prod_{j=1}^{\nu-t} i_j}{\binom{\nu}{t}(\nu - t)\nu^{t-1}}. \tag{3.18}$$

Consequently ([10], [25]) $p_{\nu k}$, the probability that the two trees of the penultimate forest $F_{\nu-2}$ have sizes k, $\nu - k$, is given by

$$p_{\nu k} = \frac{\binom{\nu}{k} k^{k-1}(\nu - k)^{\nu-k-1}}{2\nu^{\nu-2}(\nu - 1)}, \qquad 1 \le k \le \nu - 1. \tag{3.19}$$

(The two trees are randomly ordered, and k is the size of the first tree.) It should be clear that, conditioned on the their vertex sets, the trees are independent, each being uniformly distributed.

To use this construction as a basis for the recurrence we seek, it is only natural to interpret the random *contact* vertices of these two trees as their respective roots, which makes each of them a uniformly random rooted tree. And we certainly want one of the roots to become the root of the whole tree. A root selection rule should be such that the resulting rooted tree is uniformly random. We have come up with the following rule: with probability k/ν (resp. $1-k/\nu$) the root of the terminal tree is the root of the tree of size $\nu-k$ (resp. k). Not the other way around. (For $k=\nu/2$ we simply flip a fair coin.) Let us prove that the rule delivers a uniformly random rooted tree. If T is a rooted tree, let d be the root's degree, and let $k_1,\dots,k_d$ be the sizes of subtrees rooted at the root's neighbors; clearly, $\sum_{j=1}^{d} k_j=\nu-1$. Let $F(j)$ denote a forest of two trees obtained by deletion of the edge joining the root and its jth neighbor. By (3.18),

$$\mathbf{P}\big(F_{\nu-2}=F(j)\big)=\frac{k_j(\nu-k_j)}{\nu^{\nu-2}(\nu-1)},$$

and, according to the root-selection rule, we obtain

$$\begin{aligned}\mathbf{P}(F_{\nu-1}=T)&=\sum_{j=1}^{d}\mathbf{P}\big(F_{\nu-2}=F(j)\big)\frac{1}{k_j(\nu-k_j)}\frac{k_j}{\nu}\\&=\frac{1}{\nu^{\nu-1}(\nu-1)}\sum_{j=1}^{d}k_j\\&=\frac{1}{\nu^{\nu-1}}\quad(!).\end{aligned}$$

Thus the random tree $F_{\nu-1}$ is indeed uniformly distributed.

In view of this fact, we obtain, conditioned on the vertex sets of the two trees in $F_{\nu-2}$,

$$L_\nu=\begin{cases}L_k+L_{\nu-k}+\log\nu-\log(\nu-k), & \text{with probability } k/\nu,\\ L_k+L_{\nu-k}+\log\nu-\log k, & \text{with probability } (\nu-k)/\nu;\end{cases}$$

here k, $\nu-k$ are the cardinalities of the vertex sets, and L_k, $L_{\nu-k}$ are independent. Consequently, using $p_{\nu k}=p_{\nu,\,\nu-k}$, we write

$$\begin{aligned}f_\nu(u)&=\exp(u\log\nu)\sum_{k=1}^{\nu-1}p_{\nu k}\bigg[\frac{k}{\nu}\frac{f_k(u)f_{\nu-k}(u)}{\exp(u\log(\nu-k))}\\&\qquad\qquad+\frac{\nu-k}{\nu}\frac{f_{\nu-k}(u)f_k(u)}{\exp(u\log k)}\bigg]\\&=\sum_{k=1}^{\nu-1}q_{\nu k}\exp\bigg(u\log\frac{\nu}{\nu-k}\bigg)f_k(u)f_{\nu-k}(u);\end{aligned}\tag{3.20}$$

here

$$q_{\nu k} = \binom{\nu}{k} \frac{k^k(\nu-k)^{\nu-k-1}}{\nu^{\nu-1}(\nu-1)}. \tag{3.21}$$

Like $\{p_{\nu k}\}$, $\{q_{\nu k}\}$ is a probability distribution. Let X_ν be a random variable such that $\mathbf{P}(X_\nu = k) = q_{\nu k}$. Then (3.20) becomes

$$f_\nu(u) = \mathbf{E}\left[\exp\left(u \log \frac{\nu}{\nu - X_\nu}\right) f_{X_\nu}(u) f_{\nu - X_\nu}(u)\right]. \tag{3.22}$$

This is the desired recurrence equation.

Introduce $\{s_{\nu j}\}_{j\ge 1}$, the sequence of cumulants of L_ν, so that

$$f_\nu(u) = \exp\left(\sum_{j\ge 1} \frac{s_{\nu j}}{j!} u^j\right);$$

in particular

$$s_{\nu 1} = \mathbf{E}\, L_\nu, \qquad s_{\nu 2} = \operatorname{Var} L_\nu,$$

and there are general formulas expressing the cumulants through the moments, and vice versa. Equation (3.22) becomes

$$\begin{aligned} \exp\left(\sum_{j\ge 1} \frac{s_{\nu j}}{j!} u^j\right) = \sum_{k=1}^{\nu-1} q_{\nu k}\Bigg[\exp\Bigg(& u\big(\log \nu(\nu-k)^{-1} + s_{k1} + s_{\nu-k,1}\big) \\ & + \sum_{j\ge 2} \frac{s_{kj} + s_{\nu-k,j}}{j!} u^j\Bigg)\Bigg]. \end{aligned} \tag{3.23}$$

So, expanding both sides in powers of u and equating the coefficients by the same powers, we get (quasilinear) recurrence equations for the cumulants of any given order. Specificallly, equating the linear terms,

$$s_{\nu 1} = \sum_{k=1}^{\nu-1} q_{\nu k}\left(\log \frac{\nu}{\nu-k} + s_{k1} + s_{\nu-k,1}\right),$$

and we already know that

$$\begin{aligned} s_{\nu 1} &= \sum_{k=1}^{\nu} \binom{\nu}{k} \frac{k^{k-1}(\nu-k)^{\nu-k}}{\nu^{\nu-1}} \\ &= a\nu - \sqrt{2\pi\nu} + O(\log \nu). \end{aligned} \tag{3.24}$$

Introduce

$$\Delta_j(\nu, k) = \begin{cases} \log \dfrac{\nu}{\nu-k} + s_{k1} + s_{\nu-k,1} - s_{\nu 1}, & \text{if } j = 1, \\ s_{kj} + s_{\nu-k,j} - s_{\nu j}, & \text{if } j \ge 2. \end{cases}$$

Let $t \geq 2$. Multiplying both sides of (3.23) by $\exp(-\sum_{j<t}(s_{\nu j}u^j)/j!)$, and equating the coefficients by u^t in the resulting equation, we obtain

$$\begin{aligned} \frac{s_{\nu t}}{t!} &= \text{coeff}_{u^t} \sum_{k=1}^{\nu-1} q_{\nu k} \sum_{r=1}^{t} \frac{1}{r!}\bigg(\sum_{j<t} u^j \Delta_j(\nu,\ k) + u^t \frac{s_{kt} + s_{\nu-k,\,t}}{t!}\bigg)^r \\ &= \sum_{k=1}^{\nu-1} q_{\nu k}\bigg[\frac{1}{t!}\big(s_{kt} + s_{\nu-k,\,t}\big) + \sum_{r=2}^{t}\frac{1}{r!}\sum_{j_1+\cdots+j_r=t} \Delta_{j_1}(\nu, k)\cdots\Delta_{j_r}(\nu, k)\bigg]. \end{aligned} \tag{3.25}$$

So, given the values of the first $t-1$ cumulants s_{mj} $(m \geq 1,\ j < t)$, (3.25) is a (nonhomogeneous) linear recurrence for the tth order cumulants s_{mt}, subject to an initial condition $s_{1t} = 0$. (The latter follows from $L_1 = 0$.)

For $t = 2$ (3.25) becomes

$$\begin{aligned} s_{\nu 2} &= \sum_{k=1}^{\nu-1} q_{\nu k}\big(s_{k2} + s_{\nu-k,\,2}\big) + R_{\nu 2}, \\ R_{\nu 2} &:= \sum_{k=1}^{\nu-1} q_{\nu k}\Delta_1^2(\nu, k). \end{aligned} \tag{3.26}$$

Let us use this equation to determine the coefficient b in the asymptotic formula (3.13) for $s_{j2}(=\operatorname{Var} L_j)$. To do so, notice first that by (3.21) and Stirling's formula,

$$\begin{aligned} q_{\nu k} &= q^0_{\nu k} + O\bigg(\frac{\nu^{3/2}}{k^{3/2}(\nu-k)^{5/2}}\bigg) = O\bigg(\frac{\nu^{1/2}}{k^{1/2}(\nu-k)^{3/2}}\bigg); \\ q^0_{\nu k} &:= \frac{\nu^{1/2}}{(2\pi k)^{1/2}(\nu-k)^{3/2}}. \end{aligned} \tag{3.27}$$

We will also use an elementary bound

$$\begin{aligned} &\sum_{k=1}^{\nu-1} \nu^{-1}F(k/\nu) - \int_0^1 F(x)\,dx \\ &\quad = O\bigg(\nu^{-1}\big|F(\nu^{-1})\big| + \nu^{-1}\big|F(1-\nu^{-1})\big| \\ &\qquad + \int_{x\notin[\nu^{-1},\,1-\nu^{-1}]} |F(x)|\,dx + \nu^{-1}\int_{\nu^{-1}}^{1-\nu^{-1}} |F'(x)|\,dx\bigg), \end{aligned} \tag{3.28}$$

which follows from the Euler–Maclaurin summation formula.

By (3.26), the definition of Δ's and (3.24),

$$\begin{aligned} R_{\nu 2} &= 2\pi \sum_{k=1}^{\nu-1} q_{\nu k}\big(k^{1/2} + (\nu-k)^{1/2} - \nu^{1/2}\big)^2 \\ &\quad + O\bigg(\log\nu \sum_{k=1}^{\nu-1} q_{\nu k}\min\big\{k^{1/2}, (\nu-k)^{1/2}\big\} + \log^2\nu\bigg), \end{aligned} \tag{3.29}$$

since

$$0 \le k^{1/2} + (\nu - k)^{1/2} - \nu^{1/2} \le \min\{k^{1/2}, (\nu - k)^{1/2}\}.$$

Using the cruder bound for $q_{\nu k}$ in (3.27) and considering the cases $k \le \nu/2$ and $k > \nu/2$ separately, we see that the remainder term in (3.29) is $O(\log^2 \nu)$. Likewise, replacing $q_{\nu k}$ in the first sum in (3.29) by the leading term $q^0_{\nu k}$ in (3.27) produces an error of order

$$\sum_{k=1}^{\nu-1} \frac{\nu^{3/2}}{k^{3/2}(\nu - k)^{5/2}} \min\{k, (\nu - k)\} = O(1).$$

In addition, by (3.28), with

$$F(x) = \frac{\left(x^{1/2} + (1 - x)^{1/2} - 1\right)^2}{x^{1/2}(1 - x)^{3/2}},$$

we see that the difference

$$2\pi \sum_{k=1}^{\nu-1} q^0_{\nu k}\left(k^{1/2} + (\nu - k)^{1/2} - \nu^{1/2}\right)^2 - (2\pi\nu)^{1/2} \int_0^1 F(x)\, dx$$

is of order

$$\begin{aligned} \nu^{1/2} \int_0^{\nu^{-1}} F(x)\, dx &+ \nu^{1/2} \int_{1-\nu^{-1}}^1 F(x)\, dx + \nu^{-1/2} F(\nu^{-1}) \\ &+ \nu^{-1/2} F(1 - \nu^{-1}) + \nu^{-1/2} \int_{\nu^{-1}}^{1-\nu^{-1}} \left|F'(x)\right| dx \\ &= O(1), \end{aligned}$$

the term $O(1)$ owing to $F(x) \sim \text{const}\ (1 - x)^{-1/2}$ for $x \to 1$. Here, substituting $x = \sin^2 \theta,\ \theta \in [0, \pi/2]$, and doing elementary integration,

$$\int_0^1 F(x)\, dx = 4 \log \frac{e}{2}.$$

Thus the nonhomogeneous term in (3.26) is given by

$$R_{\nu 2} = (2\pi\nu)^{1/2} 4 \log \frac{e}{2} + O\left(\log^2 \nu\right). \tag{3.30}$$

Let us see how closely we can satisfy (3.26) by using

$$S_\nu = \alpha\nu \log \nu + \beta\nu + \gamma\nu^{1/2} \log \nu, \qquad \nu \ge 1,$$

instead of $s_{\nu 2}$. Using (3.27) and (3.28), for $\nu \geq 2$, we compute

$$\sum_{k=1}^{\nu-1} q_{\nu k}\big[\nu \log \nu - k \log k - (\nu-k)\log(\nu-k)\big]$$
$$= \left(\frac{\nu}{2\pi}\right)^{1/2} \sum_{k=1}^{\nu-1} \nu^{-1} \frac{(k/\nu)\log(k/\nu)^{-1} + (1-k/\nu)\log(1-k/\nu)^{-1}}{(k/\nu)^{1/2}(1-k/\nu)^{3/2}} + O(\log \nu)$$
$$= \left(\frac{\nu}{2\pi}\right)^{1/2} \int_0^1 \frac{x\log x^{-1} + (1-x)\log(1-x)^{-1}}{x^{1/2}(1-x)^{3/2}}\,dx + O(\log \nu)$$
$$= (2\pi\nu)^{1/2} + O(\log \nu),$$

the last integral being computed via substitution $x = \sin^2\theta$. The sum corresponding to the linear term in S_ν is zero. Furthermore,

$$\sum_{k=1}^{\nu-1} q_{\nu k}\big[\nu^{1/2}\log \nu - k^{1/2}\log k - (\nu-k)^{1/2}\log(\nu-k)\big]$$
$$= -\sum_{k=1}^{\nu-1} q_{\nu k}(\nu-k)^{1/2}\log(\nu-k) + O(\log \nu)$$
$$= -\left(\frac{\nu}{2\pi}\right)^{1/2} \sum_{k=1}^{\nu-1} \frac{\log(\nu-k)}{k^{1/2}(\nu-k)} + O(\log \nu)$$
$$= -(2\pi)^{-1/2} \sum_{k=1}^{\nu-1} \frac{\log(\nu-k)}{\nu-k} + O(\log \nu)$$
$$= -2^{-1}(2\pi)^{-1/2}\log^2 \nu + O(\log \nu).$$

Therefore, recalling (3.30),

$$S_\nu - \sum_{k=1}^{\nu-1} q_{\nu k}(S_k + S_{\nu-k}) - R_{\nu 2}$$
$$= \alpha(2\pi\nu)^{1/2} - 4(2\pi\nu)^{1/2}\log(e/2) - \gamma 2^{-1}(2\pi)^{-1/2}\log^2 \nu$$
$$+ O(\log^2 \nu + \alpha \log \nu + \gamma \log \nu)$$
$$= -\gamma 2^{-1}(2\pi)^{-1/2}\log^2 \nu + O(\log^2 \nu + \gamma \log \nu),$$

provided that we choose $\alpha = 4\log(e/2)$, which we do. Note that the bounded factor implicit in the term $O(\log^2 \nu)$ is independent of γ, and, due to cancellation of linear terms, the parameter β is not present. So, given $\gamma < 0$ ($\gamma > 0$, resp.), there exists $\nu(\gamma) \in \mathbf{N}$ such that the last expression is positive (negative, resp.) for all $\nu \geq \nu(\gamma)$, regardless of β. Furthermore, for $\gamma < 0$, we select $\beta > 0$ large enough to guarantee that $s_{\nu 2} \leq S_\nu$ for all $\nu < \nu(\gamma)$. Using these inequalities as the induction basis and the positivity of $S_\nu - \sum_{k=1}^{\nu-1} q_{\nu k}(S_k + S_{\nu-k}) - R_{\nu 2}$ ($\nu \geq \nu(\gamma)$) for the inductive step, we obtain that thus defined S_ν is an upper bound for $s_{\nu 2}$ for all $\nu \ge 1$. Likewise, for $\gamma > 0$, we can select $\beta < 0$ such that $s_{\nu 2} \ge S_\nu$ for all $\nu \ge 1$. Hence we have proved that

$$s_{\nu 2} = 4\log(e/2)\nu\log\nu + O(\nu).$$

Thus [see (3.13)],

$$s_{\nu 2} = 4\log(e/2)\nu\log\nu + \xi_1\nu + \xi_2\nu^{1/2}\log^2\nu + O\big(\nu^{1/2}\log\nu\big). \tag{3.31}$$

It remains to show that, for every fixed $j \ge 3$,

$$s_{\nu j} = o\big(s_{\nu 2}^{j/2}\big),$$

since then the cumulants of $(L_\nu - s_{\nu 1})/s_{\nu 2}^{1/2}$ converge to the cumulants of the standard normal variable. We achieve this by showing that, for every fixed $j \ge 3$, there exists a constant c_j such that

$$s_{\nu j} = c_j\nu^{j/2} + O\big(\nu^{(j-1)/2}\log^j 2\nu\big), \tag{3.32}$$

uniformly for $\nu \ge 1$. Let $t \ge 3$ and suppose (3.32) holds for all $j < t$. (For $t = 3$, the assumption holds true by default.) By (3.25), for $\nu \ge 2$,

$$\begin{aligned} s_{\nu t} &= \sum_{k=1}^{\nu-1} q_{\nu k}\big(s_{kt} + s_{\nu-k,\,t}\big) + R_{\nu t}, \\ R_{\nu t} &:= \sum_{k=1}^{\nu-1} q_{\nu k}\sum_{r=2}^{t}\frac{t!}{r!}\sum_{j_1+\cdots+j_r=t}\Delta_{j_1}(\nu,k)\cdots\Delta_{j_r}(\nu,k). \end{aligned} \tag{3.33}$$

Here [see (3.24), (3.31), (3.32)]

$$\Delta_j(\nu,k) = c_j\nu^{j/2}\big[f_j(k/\nu) + \delta_j(\nu,k/\nu)\big], \qquad \nu \ge 2,$$

$$c_1 = (2\pi)^{1/2}, \qquad c_2 = 4\log(e/2),$$

$$f_j(x) = \begin{cases} 1 - x^{1/2} - (1-x)^{1/2}, & \text{if } j = 1, \\ x\log x + (1-x)\log(1-x), & \text{if } j = 2, \\ x^{j/2} + (1-x)^{j/2} - 1, & \text{if } j \ge 3, \end{cases}$$

and

$$\delta_j(\nu,x) = \begin{cases} O\big(\nu^{-1/2}\log^j\nu\big), & \text{if } j = 1 \text{ or } j \ge 3, \\ \Big(-\dfrac{\xi_2}{b}f_1(x)\Big)\nu^{-1/2}\log^2\nu + O\big(\nu^{-1/2}\log\nu\big), & \text{if } j = 2. \end{cases}$$

To estimate $R_{\nu t}$, we need to bound the overall error obtained from dropping the remainder term in the expression for $\Delta_j(\nu,k)$. Considering (3.27) and the fact that $f_1(x)$ is least smooth among $f_j(x)$ at $x = 1$, we see that the dominant contribution to such a bound comes from summands with $r = 2$ and $\{j_1, j_2\} =$

$\{1, t-1\}$. For $t > 3$, it is of order

$$\nu^{(t-1)/2}(\log \nu)^{t-1} \sum_{k=1}^{\nu-1} q_{\nu k}|f_1(k/\nu)| = O\big(\nu^{(t-2)/2} \log^t \nu\big).$$

The estimate holds for $t = 3$ as well. Importantly, we can do better in this case, and get (see the definition of δ_2) a bound of order

$$\nu^{(3-1)/2}(\log \nu)^{3-1} \sum_{k=1}^{\nu-1} q_{\nu k} f_1^2(k/\nu) = O\big(\nu^{1/2} \log^2 \nu\big).$$

Likewise, for $t > 3$,

$$\begin{aligned}
&\sum_{k=1}^{\nu-1} (q_{\nu k} - q_{\nu k}^0) \sum_{r=2}^{t} \frac{t!}{r!} \sum_{j_1+\cdots+j_r=t} \prod_{m=1}^{r} \big(\nu^{j_m/2} c_{j_m} f_{j_m}(k/\nu)\big) \\
&\quad = O\bigg(\sum_{k=1}^{\nu-1} \frac{\nu^{3/2}}{k^{3/2}(\nu-k)^{5/2}} \nu^{t/2} f_1^t(k/\nu)\bigg) \\
&\quad = O\big(\nu^{(t-3)/2}\big).
\end{aligned}$$

If $t = 3$ then the total order of magnitude is determined by $r = 2$ and $\{j_1, j_2\} = \{1, 2\}$, and it is $O(\log^2 \nu)$. Furthermore, applying (3.28),

$$\sum_{k=1}^{\nu-1} q_{\nu k}^0 \nu^{t/2} \prod_{m=1}^{r} c_{j_m} f_{j_m}(k/\nu) = \gamma(\mathbf{j})\nu^{(t-1)/2} + \begin{cases} O(\log \nu), & \text{if } t = 3, \\ O(\nu^{(t-3)/2}), & \text{if } t > 3; \end{cases}$$

$$\gamma(\mathbf{j}) := (2\pi)^{-1/2} \int_0^1 x^{-1/2}(1-x)^{-3/2} \prod_{m=1}^{r} c_{j_m} f_{j_m}(x)\, dx,$$

with $O(\log \nu)$ for $t = 3$ coming from $r = 2$ and $\{j_1, j_2\} = \{1, 2\}$. Therefore,

$$\begin{aligned}
R_{\nu t} &= \gamma(t)\nu^{(t-1)/2} + O\big(\nu^{(t-2)/2} \log^{m(t)} \nu\big), \qquad t \ge 3, \\
\gamma(t) &:= \sum_{r=2}^{t} \frac{t!}{r!} \sum_{j_1+\cdots+j_r=t} \gamma(\mathbf{j})
\end{aligned} \tag{3.34}$$

[compare with (3.30)]. Here $m(t) = t$ for $t > 3$, and $m(3) = 2$.

Introduce $T_\nu = c^{(1)}\nu^{t/2} + c^{(2)}\nu^{(t-1)/2} \log^t 2\nu + c^{(3)}\nu$, with intent to make T_ν satisfy (3.33) as closely as possible. For $l \ge 2$, define

$$I_l = (2\pi)^{-1/2} \int_0^1 \frac{g_l(x)}{x^{1/2}(1-x)^{3/2}}\, dx,$$

where $g_l(x) = -f_l(x)$ for $l \geq 3$ and $g_2(x) = -3f_2(x)$. Then $I_l > 0$ for all $l \geq 2$. Using (3.27) and (3.28), we easily obtain, for $\nu \geq 2$,

$$\begin{aligned} T_\nu - \sum_{k=1}^{\nu-1} q_{\nu k}(T_k + T_{\nu-k}) - R_{\nu t} & \\ &= \big[c^{(1)} I_t - \gamma(t)\big]\nu^{(t-1)/2} + c^{(2)} I_{t-1}\nu^{(t-2)/2}\log^{m(t)}\nu \\ &\quad + O\big[\nu^{(t-2)/2}\big(c^{(1)}\log^{m(t)}\nu + c^{(2)}\log^{m(t)-1}\nu\big)\big]. \end{aligned}$$

Set

$$c^{(1)} = c_t := \frac{\gamma(t)}{I_t};$$

for $c^{(2)} > 0$ ($c^{(2)} < 0$, resp.) with $|c^{(2)}|$ sufficiently large, there exists $\nu(c^{(2)}) \in \mathbf{N}$ such that the last expression is positive (negative, resp.) for all $\nu \geq \nu(c^{(2)})$. Choose $c^{(3)} > 0$ ($c^{(3)} < 0$) with $|c^{(3)}|$ so large that $T_\nu \geq s_{\nu t}$ ($\leq s_{\nu t}$, resp.) for all $\nu < \nu(c^{(2)})$. Then, by induction, $s_{\nu t} \leq T_\nu$ ($\geq T_\nu$, resp.) for all $\nu \geq 1$. Therefore,

$$s_{\nu t} = c_t \nu^{t/2} + O\big(\nu^{(t-1)/2}\log^t 2\nu\big), \qquad \nu \geq 1.$$

Hence, for every fixed $j \geq 3$,

$$s_{\nu j} = c_j \nu^{j/2} + O\big(\nu^{(j-1)/2}\log^j 2\nu\big). \qquad \square$$

APPENDIX

Estimates of $\mathbf{E}\log L_n$ and Var $\log L_n$. We will use the following notation:

$$r(j) = \frac{e^{-j} j^{j-1}\log j}{j!}, \qquad S = \sum_{j\geq 1} r(j),$$

$$r_0(j) = \frac{\log j}{\sqrt{2\pi j^3}}, \qquad S_0 = \sum_{j\geq 1} r_0(j).$$

By the asymptotic formula

$$\text{(A.1)} \qquad m! = \sqrt{2\pi m}\left(\frac{m}{e}\right)^m \left(1 + \frac{1}{12m} + O(m^{-2})\right),$$

we have

$$\text{(A.2)} \quad \Delta(j) \overset{\text{def}}{=} r(j) - r_0(j) = -\frac{\log j}{12\sqrt{2\pi j^5}} + O\big(j^{-7/2}\log j\big), \qquad j \to \infty.$$

According to the formula for $\mathbf{E}\, Z_{nk}$ and the definition of L_n,

$$\mathbf{E}\log L_n = \frac{1}{n^{n-1}} \sum_{k=1}^{n} \binom{n}{k} k^{k-1}(n-k)^{n-k}\log k.$$

Let S_1 and S_2 denote the parts of the sum (divided by n^{n-1}) that correspond to $k \le n/2$ and $k > n/2$. Applying the asymptotic formula (A.1) to $n!$ and $(n-k)!$, we obtain

$$\begin{aligned} S_1 &= n^{3/2} \sum_{k \le n/2} (n-k)^{-1/2} r(k) \big(1 + O(kn^{-2})\big) \\ &= n^{3/2} \sum_{k \le n/2} (n-k)^{-1/2} r(k) + O\big(n^{-1/2} \log n\big), \end{aligned}$$

or

$$\begin{aligned} S_1 &= nS - n \sum_{k > n/2} r(k) + n^{3/2} \sum_{k \le n/2} r(k) \left(\frac{1}{\sqrt{n-k}} - \frac{1}{\sqrt{n}} \right) \\ &\quad + O\big(n^{-1/2} \log n\big), \\ &= nS - S_{11} + S_{12} + S_{13} + O\big(n^{-1/2} \log n\big); \end{aligned}$$

$$\text{(A.3)} \qquad \begin{aligned} S_{11} &:= n \sum_{k > n/2} r(k), \\ S_{12} &:= n^{3/2} \sum_{k \le n/2} r_0(k) \left(\frac{1}{\sqrt{n-k}} - \frac{1}{\sqrt{n}} \right), \\ S_{13} &:= n^{3/2} \sum_{k \le n/2} \big(r(k) - r_0(k)\big) \left(\frac{1}{\sqrt{n-k}} - \frac{1}{\sqrt{n}} \right). \end{aligned}$$

Let us look at each of S_{1j}. Applying the mean value theorem, we have, uniformly for $k \ge 1$,

$$\int_k^{k+1} \frac{\log x}{x^{3/2}}\, dx = \frac{\log k}{k^{3/2}} + O\left(\frac{\log 2k}{k^{5/2}} \right).$$

Therefore,

$$\text{(A.4)} \qquad S_{11} = \frac{n}{\sqrt{2\pi}} \int_{n/2}^{\infty} \frac{\log x}{x^{3/2}}\, dx + O\big(n^{-1/2} \log n\big).$$

As for S_{12}, a little algebra shows that

$$S_{12} = \frac{n}{\sqrt{2\pi}} \sum_{k \le n/2} \frac{\log k}{\sqrt{k(n-k)}\big(\sqrt{n-k} + \sqrt{n}\big)}.$$

Besides, uniformly for $x \in [k, k+1]$, $k \le n/2$,

$$\frac{d}{dx} \frac{\log x}{\sqrt{x(n-x)}\big(\sqrt{n-x} + \sqrt{n}\big)} = \frac{1 - (1/2) \log x}{2n x^{3/2}} + O\left(\frac{\log 2x}{x^{1/2} n^2} \right).$$

So, using the Euler–Maclaurin formula,

$$
\begin{aligned}
S_{12} = {} & \frac{n}{\sqrt{2\pi}}\int_1^{n/2}\frac{\log x}{\sqrt{x(n-x)}\bigl(\sqrt{n-x}+\sqrt{n}\bigr)}\,dx\\
& + \int_1^{n/2}\{x\}\frac{(1/2)\log x-1}{2\sqrt{2\pi x^3}}\,dx\\
& + O\bigl(n^{-1/2}\log n\bigr),\\
= {} & \frac{n}{\sqrt{2\pi}}\int_0^{n/2}\frac{\log x}{\sqrt{x(n-x)}\bigl(\sqrt{n-x}+\sqrt{n}\bigr)}\,dx\\
& + s_{12} + O\bigl(n^{-1/2}\log n\bigr);\\
s_{12} := {} & \int_1^{\infty}\{x\}\frac{(1/2)\log x-1}{2\sqrt{2\pi x^3}}\,dx,
\end{aligned}
\tag{A.5}
$$

with $\{x\}$ denoting the fractional part of x. Finally, with the same algebra, it is easy to see that

$$
S_{13} = s_{13} + O\bigl(n^{-1/2}\log n\bigr), \qquad s_{13} := \frac{1}{2}\sum_{k\ge 1}\left(\frac{e^{-k}k^k}{k!}-\frac{1}{\sqrt{2\pi k}}\right).
\tag{A.6}
$$

Combining (A.4)–(A.6) we get

$$
\begin{aligned}
S_1 = {} & nS + \frac{n}{\sqrt{2\pi}}\left(\int_0^{n/2}\frac{\log x}{\sqrt{x(n-x)}\bigl(\sqrt{n-x}+\sqrt{n}\bigr)}\,dx - \int_{n/2}^{\infty}\frac{\log x}{x^{3/2}}\,dx\right)\\
& + s_1 + O\bigl(n^{-1/2}\log n\bigr), \qquad s_1 := s_{12}+s_{13}.
\end{aligned}
\tag{A.7}
$$

Turn to S_2. Leaving the nth term outside of the sum and approximating the factorials in the sum by (A.1), we have

$$
\begin{aligned}
S_2 = {} & n^{3/2}\sum_{n/2<k\le n-1}(n-k)^{-1/2}r_0(k)\\
& + n^{3/2}\sum_{n/2<k\le n-1}\frac{\log k}{k^{3/2}}\left(\frac{\exp(-(n-k))(n-k)^{n-k}}{(n-k)!}-\frac{1}{\sqrt{2\pi(n-k)}}\right)\\
& + \log n + O\bigl(n^{-1/2}\log n\bigr)\\
= {} & n^{3/2}\sum_{n/2<k\le n-1}(n-k)^{-1/2}r_0(k) + s_{21}\log n + O\bigl(n^{-1/2}\log n\bigr);\\
s_{21} := {} & 1 + \sum_{j\ge 1}\left(\frac{e^{-j}j^j}{j!}-\frac{1}{\sqrt{2\pi j}}\right).
\end{aligned}
\tag{A.8}
$$

To estimate the last sum sharply, we notice that

$$
\begin{aligned}
(n-k)^{-1/2} = {} & 2\bigl((n-k+1)^{1/2}-(n-k)^{1/2}\bigr)\\
& + (n-k)^{-1/2}\bigl((n-k)^{1/2}+(n-k+1)^{1/2}\bigr)^{-2}.
\end{aligned}
\tag{A.9}
$$

The total contribution to the sum due to the second term in this identity is

$$\begin{aligned}
&\frac{\log n}{\sqrt{2\pi}} \sum_{n/2<k\le n-1} (n-k)^{-1/2}\big((n-k)^{1/2}+(n-k+1)^{1/2}\big)^{-2} \\
&\qquad + O\bigg(n^{3/2} \sum_{n/2<k\le n-1} k^{-5/2}(n-k)^{-1/2}\log k\bigg) \qquad \text{(A.10)}\\
&\quad = s_{22}\log n + O\big(n^{-1/2}\log n\big);\\
&s_{22} := (2\pi)^{-1/2}\sum_{j\ge 1} j^{-1/2}\big(j^{1/2}+(j+1)^{1/2}\big)^{-2}.
\end{aligned}$$

Summing by parts, we evaluate the contribution due to the first term in (A.9)

$$\begin{aligned}
&\frac{n^{3/2}}{\sqrt{2\pi}} \sum_{n/2<k<n-1} 2\sqrt{n-k}\bigg[\bigg(\frac{\log x}{x^{3/2}}\bigg)'\bigg|_{x=k} + O\bigg(\frac{\log k}{k^{7/2}}\bigg)\bigg] \qquad \text{(A.11)}\\
&\qquad + \frac{n^{3/2}}{\sqrt{2\pi}}\bigg[2\sqrt{n-n/2}\frac{\log(n/2)}{(n/2)^{3/2}} - 2\frac{\log n}{n^{3/2}}\bigg] + O\big(n^{-1/2}\log n\big).
\end{aligned}$$

The last sum is easily seen to be

$$\frac{n^{3/2}}{\sqrt{2\pi}}\int_{n/2}^{n} 2\sqrt{n-x}\bigg(\frac{\log x}{x^{3/2}}\bigg)' dx + O\big(n^{-1/2}\log n\big),$$

and, integrating by parts "backwards," we transform (A.11) into

$$\frac{n^{3/2}}{\sqrt{2\pi}}\int_{n/2}^{n} \frac{\log x}{x^{3/2}\sqrt{n-x}}\,dx - \frac{2}{\sqrt{2\pi}}\log n + O\big(n^{-1/2}\log n\big). \qquad \text{(A.12)}$$

Thus, combining (A.8), (A.10) and (A.12),

$$\begin{aligned}
S_2 &= \frac{n^{3/2}}{\sqrt{2\pi}}\int_{n/2}^{n} \frac{\log x}{x^{3/2}\sqrt{n-x}}\,dx + s_2\log n + O\big(n^{-1/2}\log n\big); \qquad \text{(A.13)}\\
s_2 &:= s_{21}+s_{22}-2(2\pi)^{-1/2}.
\end{aligned}$$

So, recalling (A.7), we arrive at

$$\begin{aligned}
\mathbf{E}\log L_n &= nS + \frac{n}{\sqrt{2\pi}}\bigg(\int_0^n \frac{\log x}{\sqrt{x(n-x)}\big(\sqrt{n-x}+\sqrt{n}\big)}\,dx - \int_n^\infty \frac{\log x}{x^{3/2}}\,dx\bigg)\\
&\qquad + s_1 + s_2\log n + O\big(n^{-1/2}\log n\big).
\end{aligned}$$

The value of the first integral, computed via substitution $x = n\sin^2\theta$, $\theta \in [0,\pi/2]$, and two rounds of integration by parts, turns out to be $(2\log n + 4 - 2\pi)n^{-1/2}$. The second integral equals $(2\log n + 4)n^{-1/2}$. So the logarithmic terms cancel out and we obtain

$$\mathbf{E}\log L_n = nS - \sqrt{2\pi n} + s_1 + s_2\log n + O\big(n^{-1/2}\log n\big). \qquad \text{(A.14)}$$

(The actual values of the constants s_1, s_2 will not be relevant.)

Turn now to the estimation of $\mathbf{E}(\log L_n)^2$. We limit ourselves to finding an asymptotic expansion with a remainder $O(n^{1/2}\log n)$, with explicit coefficients for the leading terms only. Not surprisingly in view of (A.14), these will be n^2 and $n^{3/2}$. For brevity, we will pay more attention to the new elements, leaving out the proofs that can be easily done by mimicking the corresponding lines in the estimation of $\mathbf{E}\log L_n$. We will also write $A(n) \approx B(n)$, or $A(n)$ is approximately $B(n)$, whenever

$$A(n) = B(n) + \xi_1 n\log n + \xi_2 n + \xi_3 n^{1/2}\log^2 n + O\big(n^{1/2}\log n\big) \tag{A.15}$$

here ξ's are the constants which may differ from case to case. Moreover, we will write $A(j,n) \approx B(j,n)$ if the (weighted) sums

$$A(n) = \sum_j \gamma(j,n)A(j,n), \qquad B(n) = \sum_j \gamma(j,n)B(j,n)$$

satisfy (A.15), for some weights in question. Similarly, $A(j,k,n) \approx B(j,k,n)$ if the corresponding double sums [possibly weighted by some weights $\gamma(j,k,n)$] satisfy (A.15).

First,

$$\begin{aligned}\mathbf{E}(\log L_n)^2 &= \sum_{k=1}^n (\log^2 k)\,\mathbf{E}\,Z_{nk}^2 + 2\sum_{1\le j<k\le n}(\log j\log k)\,\mathbf{E}\big(Z_{nj}Z_{nk}\big)\\ &= \mathscr{S}_1 + \mathscr{S}_2.\end{aligned} \tag{A.16}$$

Notice that

$$\mathbf{E}\,Z_{nk}^2 = \mathbf{E}\big(Z_{nk}(Z_{nk}-1)\big) + \mathbf{E}\,Z_{nk}$$

and, analogously to $\mathbf{E}\log L_n$,

$$\sum_{j=1}^n \big(\log^2 j\big)\,\mathbf{E}\,Z_{nj} = n\sum_{j\ge 1} r(j)\log j + O\big(n^{1/2}\log n\big) \approx 0.$$

So, using (3.9),

$$\mathscr{S}_1 \approx \frac{1}{n^{n-1}}\sum_{j<n/2}\binom{n}{j}\binom{n-j}{j}\big(j^{j-1}\big)^2(n-2j)^{n-2j+1}\log^2 j.$$

Approximating the factorials, one can see that the terms with $j \ge n/3$ contribute at most $O\big(\log^2 n\big)$ to the value of the last sum. So,

$$\begin{aligned}\mathscr{S}_1 &\approx n^{3/2}\sum_{j<n/3} r^2(j)(n-2j)^{1/2}\big(1+O(j/n^2)\big)\\ &\approx n^2\sum_{j\ge 1} r^2(j).\end{aligned} \tag{A.17}$$

It remains to estimate sharply $\mathscr{S}_2$. According to (3.10),

$$\mathscr{S}_2 = \mathscr{S}_{21} + \mathscr{S}_{22},$$

$$\mathscr{S}_{21} := 2n^{-(n-1)} \sum_k \binom{n}{k} (n-k)^{n-k} k^{k-1} (\log k)(\mathbf{E} \log L_k) \tag{A.18}$$

$$- 2 \sum_k (\log^2 k)\, \mathbf{E}\, Z_{nk},$$

$$\mathscr{S}_{22} := \frac{n!}{n^{n-1}} \sum_{j \neq k} \frac{j^{j-1} \log j}{j!} \frac{k^{k-1} \log k}{k!} \frac{(n-k-j)^{n-k-j+1}}{(n-k-j)!}.$$

Begin with $\mathscr{S}_{21}$. As we have noticed, the second sum in the expression for $\mathscr{S}_{21}$ is ≈ 0. Consider the kth generic term in the first sum, denoting it $\mathscr{S}_{21}(k)$. Let $k \le n/2$. Using (A.1) for $n!$ and $(n-k)!$ and (A.14) for $\mathbf{E} \log L_k$, we estimate

$$\begin{aligned}
\mathscr{S}_{21}(k) &\approx 2n^{3/2} (n-k)^{-1/2} r(k)\, \mathbf{E} \log L_k \\
&= 2n^{3/2} \big(n^{-1/2} + (n-k)^{-1/2} - n^{-1/2}\big)\big(r_0(k) + \Delta(k)\big)\, \mathbf{E} \log L_k \\
&\approx 2nr_0(k)\, \mathbf{E} \log L_k + 2nr_0(k)\big((1-k/n)^{-1/2} - 1\big)\, \mathbf{E} \log L_k \\
&\approx 2nr_0(k)(kS - \sqrt{2\pi k}) + 2nr_0(k)\big((1-k/n)^{-1/2} - 1\big) kS \\
&= -2nk^{-1} \log k + 2Sn(2\pi)^{-1/2} k^{-1/2} (1-k/n)^{-1/2} \log k;
\end{aligned} \tag{A.19}$$

recall the definition of $r_0(k)$. For $k \ge n/2$, we apply (A.1) to $k!$ too, and see that this time

$$\begin{aligned}
\mathscr{S}_{21}(k) &\approx -2nk^{-1}(n-k)^{-1/2} \log k \\
&\qquad + 2Sn(2\pi)^{-1/2} k^{-1/2} (1-k/n)^{-1/2} \log k \\
&\approx 2Sn(2\pi)^{-1/2} k^{-1/2} (1-k/n)^{-1/2} \log k.
\end{aligned} \tag{A.20}$$

Combining (A.19), (A.20),

$$\begin{aligned}
\mathscr{S}_{21} &\approx -2n \sum_{k \le n/2} k^{-1} \log k + 2Sn(2\pi)^{-1/2} \sum_{k=1}^{n-1} k^{-1/2} (1-k/n)^{-1/2} \log k \\
&\approx -n \log^2 n + 2Sn(2\pi)^{-1/2} \int_0^n x^{-1/2} (1-x/n)^{-1/2} \log x \, dx \\
&= S(2\pi)^{1/2} n^{3/2} \log n + 8S(2\pi)^{-1/2} n^{3/2} \int_0^{\pi/2} \log \sin t \, dt - n \log^2 n.
\end{aligned} \tag{A.21}$$

Finally, we turn to the estimation of $\mathscr{S}_{22}$. First let $j + k \le n/2$. Consider $\mathscr{S}_{22}(j, k)$, the (j, k)th summand. Applying (A.1) to $(n-j-k)!$, we get

$$\begin{aligned}
\mathscr{S}_{22}(j, k) &\approx n^{3/2} (n-j-k)^{1/2} r(j) r(k) \\
&= n^2 r(j) r(k) - n^{3/2} \big(n^{1/2} - (n-j-k)^{1/2}\big) r(j) r(k).
\end{aligned}$$

Now

$$
\begin{aligned}
&n^2 \sum_{\substack{j+k\le n/2\\ j\ne k}} r(j)r(k)\\
&\quad = n^2 \sum_{j+k\le n/2} r(j)r(k) - n^2 \sum_{j\le n/4} r^2(j)\\
&\quad \approx n^2\Bigl(S^2 - \sum_{j\ge 1} r^2(j)\Bigr) - n^2 \sum_{j+k>n/2} r(j)r(k)\\
&\quad = n^2\Bigl(S^2 - \sum_{j\ge 1} r^2(j)\Bigr)\\
&\qquad - n^2\Biggl[2S\sum_{j>n/2} r(j) - \Bigl(\sum_{j>n/2} r(j)\Bigr)^2 + \sum_{\substack{j+k>n/2\\ j,\,k\le n/2}} r(j)r(k)\Biggr]\\
&\quad \approx n^2\Bigl(S^2 - \sum_{j\ge 1} r^2(j)\Bigr)\\
&\qquad - n^2\Biggl[2S\sum_{j>n/2} r_0(j) - \Bigl(\sum_{j>n/2} r_0(j)\Bigr)^2 + \sum_{\substack{j+k>n/2\\ j,\,k\le n/2}} r_0(j)r_0(k)\Biggr]\\
&\quad = n^2\Bigl(S^2 - \sum_{j\ge 1} r^2(j)\Bigr) - n^2 \sum_{m>n/2} \bigl(S_0(m) + 2(S-S_0)r_0(m)\bigr).
\end{aligned}
\tag{A.22}
$$

(Here we set

$$
S_0(m) := \sum_{j+k=m} r_0(j)r_0(k).) \tag{A.23}
$$

Next

$$
\begin{aligned}
&n^{3/2} \sum_{\substack{j+k\le n/2\\ j\ne k}} \bigl(n^{1/2} - (n-j-k)^{1/2}\bigr) r(j)r(k)\\
&\qquad \approx n^{3/2} \sum_{m\le n/2} \bigl(n^{1/2} - (n-m)^{1/2}\bigr) S(m),
\end{aligned}
\tag{A.24}
$$

where

$$
S(m) := \sum_{j+k=m} r(j)r(k).
$$

Furthermore

$$
\begin{aligned}
S(m) &\approx 2 \sum_{j \le m/2} r(j) r(m-j) \\
&= 2 r_0(m) \sum_{j \le m/2} r(j) + 2 \sum_{j \le m/2} r(j)\big(r_0(m-j) - r_0(m)\big) \\
&\quad + 2 \sum_{j \le m/2} r(j) \Delta(m-j), \\
&\approx 2 r_0(m) \sum_{j \le m/2} r(j) + 2 \sum_{j \le m/2} r_0(j)\big(r_0(m-j) - r_0(m)\big);
\end{aligned} \tag{A.25}
$$

indeed, the difference between the sum in (A.24) and the sum obtained if $S(m)$ is replaced by (A.25) is $\xi n + O\big(n^{1/2} \log n\big)$. So we have

$$
\begin{aligned}
S(m) &\approx S_0(m) + 2 r_0(m) \sum_{j \le m/2} \Delta(j) \\
&\approx S_0(m) + 2 r_0(m)(S - S_0).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&n^{3/2} \sum_{\substack{j+k \le n/2 \\ j \ne k}} \big(n^{1/2} - (n-j-k)^{1/2}\big) r(j) r(k) \\
&\quad \approx n^{3/2} \sum_{m \le n/2} \big(n^{1/2} - (n-m)^{1/2}\big)\big(S_0(m) + 2 r_0(m)(S - S_0)\big).
\end{aligned} \tag{A.26}
$$

Combining (A.22) and (A.26) yields

$$
\begin{aligned}
&\sum_{\substack{j+k \le n/2 \\ j \ne k}} \mathscr{S}_{22}(j,k) \\
&\quad = n^2 \bigg(S^2 - \sum_{j \ge 1} r^2(j)\bigg) - n^2 \sum_{m > n/2} \big(S_0(m) + 2(S - S_0) r_0(m)\big) \\
&\qquad - n^{3/2} \sum_{m \le n/2} \big(n^{1/2} - (n-m)^{1/2}\big)\big(S_0(m) + 2 r_0(m)(S - S_0)\big).
\end{aligned} \tag{A.27}
$$

For $j + k > n/2$ we apply (A.1) to whichever of j, k exceeds $n/4$, and in a similar fashion obtain

$$
\begin{aligned}
&\sum_{\substack{j+k > n/2 \\ j \ne k}} \mathscr{S}_{22}(j,k) \\
&\quad \approx n^{3/2} \sum_{n/2 < m \le n} (n-m)^{1/2}\big(S_0(m) + 2(S - S_0) r_0(m)\big).
\end{aligned} \tag{A.28}
$$

Putting together (A.27) and (A.28), we conclude that

$$\begin{aligned}\mathscr{S}_{22} &\approx n^2\Big(S^2 - \sum_{j\ge 1} r^2(j)\Big)\\ &\quad - n^{3/2}\sum_{m=1}^{n-1}\big(n^{1/2}-(n-m)^{1/2}\big)\Big(S_0(m)+2(S-S_0)r_0(m)\Big)\\ &\quad - n^2\sum_{m\ge n}\big(S_0(m)+2(S-S_0)r_0(m)\big).\end{aligned} \tag{A.29}$$

To use this formula, we need a sharp asymptotic expansion for $S_0(m)$ defined in (A.23). It is immediate that $S_0(m)$ is asymptotic to $2S_0r_0(m)$, S_0 being $\sum_j r_0(j)$. A careful study shows that more precisely,

$$S_0(m) \approx 2S_0r_0(m) + \frac{A}{2\pi}m^{-2}\log^2 m + \frac{B}{2\pi}m^{-2}\log m$$

(with the first omitted term asymptotic to m^{-2}), where

$$\begin{aligned}A &= -5 + \int_0^1 \big(y^{-3/2}-1\big)\big((1-y)^{-3/2}-1\big)\,dy,\\ B &= -8 + 2\int_0^1 y^{-3/2}\big((1-y)^{-3/2}-1\big)\log y\,dy.\end{aligned}$$

To compute the first integral, we substituted $y = \sin^2 t$ and found, after several by parts integrations, that the corresponding antiderivative equals

$$I(t) = \sin^2 t + 2\sin^{-1} t + \frac{4\sin^2 t - 2\sin t - 2}{\cos t \sin t}. \tag{A.30}$$

Now $I(0+) = -2$, $I(\pi/2-) = 3$, hence the integral equals 5, and $A = 0$ (!). To evaluate the second integral, we write it as

$$\int_0^1 (y^{-3/2}-1)((1-y)^{-3/2}-1)\log y\,dy + \int_0^1 \big(y^{-3/2}-1\big)\log(1-y)\,dy,$$

and compute the first integral here via the same substitution and integration by parts, using (A.30), and the second integral, by expanding $\log(1-y)$ at $y=0$. The corresponding values are $-4\pi+3+4\log 2$ and $1-4\log 2$. So their sum is $4-4\pi$, and $B=-8\pi$.

Therefore in (A.29),

$$S_0(m) + 2(S-S_0)r_0(m) \approx 2S(2\pi m^3)^{-1/2}\log m - 4m^{-2}\log m.$$

We plug this approximation into (A.29) and estimate the resulting sums, like the expressions $\mathbf{E}\log L_n$ and $\mathscr{S}_{21}$ earlier:

$$\begin{aligned}\mathscr{S}_{22} \approx n^2\biggl(S^2 - \sum_{j\ge 1} r^2(j)\biggr)\\ - S(2\pi)^{1/2}n^{3/2}\log n - 8S(2\pi)^{-1/2}\\ \times\biggl(\frac{\pi}{2} + \int_0^{\pi/2}\log\sin t\,dt\biggr)n^{3/2}\\ + n\log^2 n.\end{aligned} \tag{A.31}$$

Invoking (A.17), (A.21) and (A.31) enables us to conclude that

$$\mathbf{E}(\log L_n)^2 \approx n^2S^2 - 2S(2\pi)^{1/2}n^{3/2}.$$

Thus [see (A.14)],

$$\begin{aligned}\operatorname{Var} L_n &= \mathbf{E}(\log L_n)^2 - \mathbf{E}^2\log L_n\\ &= bn\log n + \xi_1 n + \xi_2 n^{1/2}\log^2 n + O\bigl(n^{1/2}\log n\bigr).\end{aligned}$$

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