The Annals of Applied Probability
1999, Vol. 9, No. 2, 529–566

COMPLETENESS OF SECURITIES MARKET MODELS—AN OPERATOR POINT OF VIEW[1]

BY ROBERT BÄTTIG

Cornell University

We propose a notion of market completeness which is invariant under change to an equivalent probability measure. Completeness means that an operator T acting on stopping time simple trading strategies has dense range in the weak* topology on bounded random variables. In our setup, the claims which can be approximated by attainable ones has codimension equal to the dimension of the kernel of the adjoint operator T^* acting on signed measures, which in most cases is equal to the "dimension of the space of martingale measures." From this viewpoint the example of Artzner and Heath is no longer paradoxical since all the dimensions are 1. We also illustrate how one can check for injectivity of T^* and hence for completeness in the case of price processes on a Brownian filtration (e.g., Black–Scholes, Heath–Jarrow–Morton) and price processes driven by a multivariate point process.

1. Introduction. Since the papers of Harrison and Kreps (1979) and Harrison and Pliska (1981), there has been much interest in the connection between notions of *no arbitrage* and *completeness* and the structure of the set of equivalent martingale measures. Roughly, the absence of arbitrage is characterized by the existence of an equivalent martingale measure, while completeness holds if and only if the equivalent martingale measure is unique. Precise results along these lines usually go under the names of the First and the Second Fundamental Theorem of Asset Pricing.

The question of completeness has typically been addressed by first fixing an equivalent martingale measure Q and then making the trading strategies [Harrison and Pliska (1981)], the topology used in the definition of completeness [Artzner and Heath (1995)] or the definition of a hedge [Ansel and Stricker (1994)] depend on Q. As was pointed out by Artzner and Heath (1995), such a measure-dependent notion of completeness is not generally sufficient to ensure uniqueness of the equivalent martingale measure or, more loosely speaking, is not sufficient to ensure unique pricing. In their example there are many martingale measures, and traders choosing different measures may not agree on which claims can be approximated by attainable ones. A trader will think that the market is complete if and only if she choose one of the two extremal equivalent martingale measures.

Received August 1997; revised September 1998.
[1] Supported in part by NSF Grant DMS-93-01070. Alfred P. Sloan Foundation Doctoral Dissertation Fellow, 1997–98.
AMS 1991 *subject classifications*. Primary 90A90; secondary 60H30.
Key words and phrases. Completeness of securities markets, equivalent martingale measures, Black–Scholes model, Heath–Jarrow–Morton model, weak* topology.

We propose instead a setup for a securities market model which is invariant under change to an equivalent probability measure and a notion of completeness that also has this property. This removes the logical dependence between the questions of no arbitrage and completeness, introduced by first fixing an equivalent martingale measure Q (in this connection see Section 6; the paragraph after the proof of Theorem 5 as well as Example 6.1 illustrate that the existence of an equivalent martingale measure may actually rule out the possibility of completeness). It also resolves the "paradox" of the Artzner–Heath example, since from our new viewpoint, the claims which can be approximated by attainable claims has codimension one, reflecting the fact that the "set of equivalent martingale measures has dimension 1."

As our title indicates, we approach the question of completeness by making use of operators. This approach to investigating completeness was first suggested by Jarrow and Madan (1997) and their paper inspired this work. In a different sense, operators were also used in Björk, Kabanov and Runggaldier (1996) and Björk, Di Masi, Kabanov and Runggaldier (1997). These authors associate operators to the jump-diffusion coefficients used in modeling the fundamental price processes.

We now describe in broad strokes our setup and the main results. All proofs as well as precise definitions and explanations not given here will appear in later sections. Given is a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t \in [0,1]}, P)$ satisfying the usual conditions, a set A of labels for assets (which may be infinite), and a family $\mathscr{Z} = \{(Z_t^\alpha)_{t \in [0,1]}\}_{\alpha \in A \cup \{\Delta\}}$ of fundamental price processes with $Z_t^\Delta \equiv 1$. In words, we assume that a deflation has been carried out, so that the riskless asset Δ is constant.

An agent is allowed to trade in a finite number of assets via self-financing, bounded, stopping time simple trading strategies that yield a bounded payoff at time 1. We denote by $\mathbb{Y}$ the space of all these trading strategies and by $\mathbb{C} = L^\infty(\mathscr{F}_1, P)$ the space of claims. A typical trading strategy is of the form $(x, (H^\alpha)_{\alpha \in A})$, where $x \in \mathbf{R}$ stands for the time 0 value of the portfolio and H^α is a bounded, stopping time simple process with H_t^α representing the holdings in asset α at time $t \in [0,1]$. Since an agent trades only in a finite number of assets simultaneously, all but finitely many H^α's are identically zero. Thus the linear operator $T\colon \mathbb{Y} \to \mathbb{C}$ given by

$$T(x, (H^\alpha)_{\alpha \in A}) = x + \sum_{\alpha \in A} \int_0^1 H_u^\alpha \, dZ_u^\alpha, \qquad (x, (H^\alpha)_{\alpha \in A}) \in \mathbb{Y}$$

is well defined and calculates the time 1 payoff resulting from a given trading strategy. Noe that the (stochastic) integrals make sense even if the Z^α's are not semimartingales, because the H^α's are stopping time simple and hence the integrals reduce to sums.

We let $\mathbb{M}$ denote the space of P-absolutely continuous signed measures on $\mathscr{F}_1$ and think of $\mu \in \mathbb{M}$ as an agent's personal way of assigning values to claims. She assigns the value $\int X \, d\mu$ to the claim $X \in \mathbb{C}$. An agent's valuation $\mu \in \mathbb{M}$ also gives her a way of measuring closeness of claims, in that the finite intersections of sets of the form $B(X, \varepsilon) = \{Y \in \mathbb{C} \mid |\int (X - Y)\, d\mu| < \varepsilon\}$, $X \in \mathbb{C}$ and $\varepsilon > 0$ are a basis for a topology τ^{μ} on $\mathbb{C}$. We endow $\mathbb{C}$ with the coarsest topology τ finer than all of the τ^{μ}, $\mu \in \mathbb{M}$. This topology is obviously agent–measure-independent. Loosely speaking, claims are close if any agent, regardless of her valuation $\mu \in \mathbb{M}$ agrees that they are close. Here $\mathbb{M}$ endowed with the total variation norm is a Banach space which is isomorphic to $L^1(\mathscr{F}_1, P)$ by the Radon–Nikodym theorem and hence its topological dual is $\mathbb{C} = L^{\infty}(\mathscr{F}_1, P)$. Furthermore, the topology τ on $\mathbb{C}$ can alternatively be described as the coarsest topology making all the elements of $\mathbb{M}$ continuous linear functionals on $\mathbb{C}$. Therefore, τ is what is frequently referred to as the weak* topology [see, e.g., Rudin (1991)] and we will also use this terminology.

If an agent wishes to trade according to $(x, (H^{\alpha})_{\alpha \in A}) \in \mathbb{Y}$, then she needs to put up an amount of money x at time 0. Thus $\pi_0(x, (H^{\alpha})_{\alpha \in A}) = x$ for $(x, (H^{\alpha})_{\alpha \in A}) \in \mathbb{Y}$ is the market's way of valuing trading strategies. On the other hand, an agent's personal valuation $\mu \in \mathbb{M}$ of claims induces a valuation $T^* \mu$ of trading strategies given by

$$(T^* \mu)(x, (H^{\alpha})_{\alpha \in A}) = \int T(x, (H^{\alpha})_{\alpha \in A})\, d\mu, \qquad (x, (H^{\alpha})_{\alpha \in A}) \in \mathbb{Y}.$$

We endow $\mathbb{Y}$ with the coarsest topology, making $\{T^*\mu\}_{\mu \in \mathbb{M}} \cup \{\pi_0\}$ continuous linear functionals on $\mathbb{Y}$. The topological dual of $\mathbb{Y}$ is denoted by $\mathbb{X}$ and we can regard T^* as a linear operator from $\mathbb{M}$ into $\mathbb{X}$. As will be seen later, T^* is in fact the adjoint operator of T. We summarize this pictorially as

$$\begin{aligned} \mathbb{X} &\xleftarrow{\quad T^* \quad} \mathbb{M}, \\ \mathbb{Y} &\xrightarrow{\quad T \quad} \mathbb{C}. \end{aligned}$$

Our setup is measure-independent in the sense that $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{C}$ and $\mathbb{M}$ are all invariant under change to an equivalent probability measure.

We let $\mathscr{A}_1$ denote the attainable claims and $\mathscr{A}_1^0$ the claims attainable at zero initial cost

$$\mathscr{A}_1 = \left\{ x + \sum_{\alpha \in A} \int_0^1 H_u^{\alpha}\, dZ_u^{\alpha} \,\middle|\, (x, (H^{\alpha})_{\alpha \in A}) \in \mathbb{Y} \right\} = \operatorname{Im} T,$$

$$\mathscr{A}_1^0 = \left\{ \sum_{\alpha \in A} \int_0^1 H_u^{\alpha}\, dZ_u^{\alpha} \,\middle|\, (0, (H^{\alpha})_{\alpha \in A}) \in \mathbb{Y} \right\}.$$

Let $\mathfrak{M}_{++}/\mathfrak{M}_{++}^{\mathrm{loc}}$ denote the P-equivalent martingale–local martingale measures for $\mathscr{Z}$. We are now ready to define completeness.

DEFINITION 1. The market is complete if $\mathscr{A}_1 = \operatorname{Im} T$ is dense in $\mathbb{C}$ with respect to the weak* topology.

Since the weak* topology as well as the space $\mathbb{Y}$ of trading strategies is agent–measure-independent, the same is true for our notion of completeness.

In referring to the measure-dependent notion of completeness used by Artzner and Heath (1995), we shall use the following terminology:

DEFINITION 2. For $Q \in \mathfrak{M}^{\text{loc}}_{++}$ we say that the market is Q-complete if $\mathscr{A}_1 = \text{Im}\, T$ is dense in $\mathbb{C}$ with respect to the $L^1(\mathscr{F}_1, Q)$ topology.

To be able to state our results, we need one more definition.

DEFINITION 3. The no arbitrage condition (NA) holds if $\mathscr{A}_1^0 \cap \mathbb{C}_+ = \{0\}$, where $\mathbb{C}_+$ denotes the r.v.'s in $\mathbb{C}$ which are P-a.s. nonnegative.

Since we consider only stopping time simple trading strategies and no closures of sets appear in Definition 3, NA is a mild no arbitrage condition. Stronger conditions are needed to obtain versions of the first fundamental theorem of asset pricing when the time set is infinite. See Dalang, Morton and Willinger (1990), Lakner (1993), Schachermayer (1994) and Delbaen and Schachermayer (1994). For our purposes NA is sufficient; it ensures that if $X \in \mathscr{A}_1$, then any two trading strategies resulting in the payoff X are valued the same by the market, that is, $X = T(x, (H^\alpha)_{\alpha \in A}) = T(\tilde{x}, (\tilde{H}^\alpha)_{\alpha \in A})$ implies $\pi_0(x, (H^\alpha)_{\alpha \in A}) = x = \tilde{x} = \pi_0(\tilde{x}, (\tilde{H}^\alpha)_{\alpha \in A})$. Thus, attainable claims are unambiguously priced by the initial investment required to attain them.

First, we give operator characterizations of completeness and Q-completeness. The numbering of conditions may seem strange. However, conditions (iii) and (iv) will appear a little later and the numbering has been chosen to be natural, given the relationship we will deduce between the various conditions. See Figure 1 for help in following the development.

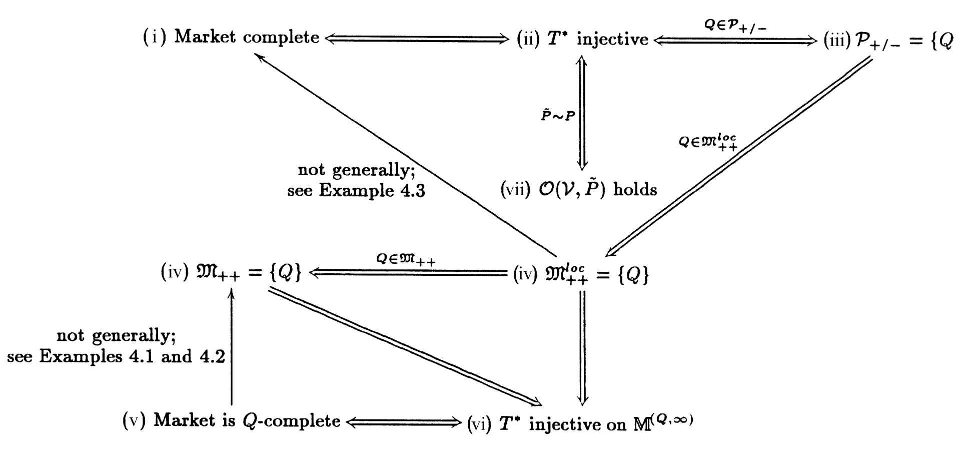

FIG. 1. *Except for the top row, it is assumed that NA holds and that the price processes are locally bounded. If $\mathscr{V}$ is finite or all the elements of $\mathscr{V}$ are processes with continuous sample paths, then all the conditions are equivalent.*

THEOREM 1. *The following are equivalent*:

(i) *The market is complete.*
(ii) *The operator* $T^*: \mathbb{M} \to \mathbb{X}$ *is injective.*

Let $Q \in \mathfrak{M}_{++}^{\mathrm{loc}}$. *The following are equivalent*:

(v) *The market is* Q-*complete.*
(vi) *The operator* T^* *restricted to* $\mathbb{M}^{(Q,\infty)}$ *is injective*, *where* $\mathbb{M}^{(Q,\infty)}$ *denotes the* (*signed*) *measures in* $\mathbb{M}$ *with bounded Radon–Nikodym derivative with respect to* Q.

REMARK 1. In particular, comparing (ii) and (vi) shows that our notion of completeness is stronger than Q-completeness.

Recall that $\pi_0(x,(H^\alpha)_{\alpha\in A}) = x$ for $(x,(H^\alpha)_{\alpha\in A}) \in \mathbb{Y}$ gives the market's valuation of trading strategies. We denote by $\mathscr{P}_{+/-}$ the signed measured whose induced valuation of trading strategies is consistent with the market, that is, $\mu \in \mathscr{P}_{+/-}$ if and only if $T^*\mu = \pi_0$. Finally, we led $\mathbb{M}_{++}$ denote the cone of P-equivalent positive measures in $\mathbb{M}$. The next result states that, under appropriate assumptions, the equivalent local martingale measures are simply the positive measures inducing a market consistent valuation of trading strategies.

PROPOSITION 1. *If NA holds* (*in particular if* $\mathfrak{M}_{++} \neq \varnothing$) *then* $\mathfrak{M}_{++}^{\mathrm{loc}} \subseteq \mathscr{P}_{+/-} \cap \mathbb{M}_{++}$. *If*, *furthermore*, *the elements of* $\mathscr{V}$ *are locally bounded*, *then* $\mathfrak{M}_{++}^{\mathrm{loc}} = \mathscr{P}_{+/-} \cap \mathbb{M}_{++}$.

The structure of the sets $\mathscr{P}_{+/-}$ and $\mathfrak{M}_{++}^{\mathrm{loc}}$ is closely connected to $\ker T^*$ and hence to completeness and Q-completeness:

THEOREM 2. *If* $Q \in \mathscr{P}_{+/-}$, *then* $\mathscr{P}_{+/-} = \{Q + \mu \,|\, \mu \in \ker T^*\}$. *Hence conditions* (i) *and* (ii) *of Theorem* 1 *are further equivalent to*:

(iii) $\mathscr{P}_{+/-} = \{Q\}$.

Assume that NA holds and that the elements of $\mathscr{V}$ *are locally bounded. If* $Q \in \mathfrak{M}_{++}^{\mathrm{loc}}$ *and we let* (iv) *denote the condition* $\mathfrak{M}_{++}^{\mathrm{loc}} = \{Q\}$ *then we have*

$$\text{(iii)} \Rightarrow \text{(iv)} \Rightarrow \text{(vi)}.$$

When $Q \in \mathfrak{M}_{++}$, (iv) *may be replaced by* (iv′) $\mathfrak{M}_{++} = \{Q\}$.
If $\mathscr{V}$ *is finite or if all the elements of* $\mathscr{V}$ *are processes with continuous sample paths then* (i)–(vi) *are all equivalent.*

REMARK 2. The Artzner–Heath example (see Examples 4.1 and 4.2 below) shows that (v), which by Theorem 1 is equivalent to (vi), is not generally sufficient for (iv) or (iv′) when $Q \in \mathfrak{M}_{++}$. In Example 4.3 we will see that (iv) does not generally imply (i), which is equivalent to (iii).

If Q is a probability measure, we let $\mathsf{A}^{\mathrm{loc}}(Q)$ denote the vector space of processes of Q-locally integrable variation. For $A \in \mathsf{A}^{\mathrm{loc}}(Q)$ we use A^p to denote the Q-compensator (or dual predictable projection) of A. We let $\mathscr{M}(Q)$ denote the space of uniformly integrable Q-martingales and $\mathscr{H}^p(Q)$, $1 \le p \le \infty$, the space of Q-martingales X_t for which $\|\sup_t |X_t|\,\|_{L^p(Q)} < \infty$. $\mathscr{M}^{\mathrm{loc}}(Q)$ and $\mathscr{H}^{p,\mathrm{loc}}(Q)$ consist of the processes which are locally in $\mathscr{M}(Q)$ and $\mathscr{H}^p(Q)$ $(1 \le p \le \infty)$, respectively. In particular, $\mathscr{M}^{\mathrm{loc}}(Q)$ is the space of local Q-martingales. We let $\mathsf{A}_0(Q) = \{X \in \mathsf{A}(Q) | X_0 = 0\}$ and $\mathscr{H}_0^p(Q), \mathscr{H}_0^{p,\mathrm{loc}}(Q)$, $\mathscr{M}_0(Q)$ and $\mathscr{M}_0^{\mathrm{loc}}(Q)$ are defined analogously. Finally, when no confusion can arise, we omit the dependence of these spaces on Q, writing, for example, A in place of $\mathsf{A}(Q)$.

We say that two adapted cadlag processes X, Y are orthogonal and we write $X \perp Y$ if $XY \in \mathscr{M}_0^{\mathrm{loc}}$. When X, Y are local martingales this coincides with the usual notion of orthogonality for local martingales. If $\mathscr{S}$ is a family of adapted cadlag processes, we write $X \perp \mathscr{S}$ to indicate that X is orthogonal to each element in $\mathscr{S}$. Of course, this notion of orthogonality is in reference to a particular probability measure. In Theorem 1 we saw that completeness was equivalent to injectivity of T^*. To make this a practically useful observation, we need a way of checking for injectivity of T^*. For this purpose we introduce the following orthogonality condition which also appears in Jacod (1979) as condition C_1 when $\mathscr{B} \subseteq \mathscr{M}_0^{\mathrm{loc}}$.

Definition 4. Let $\tilde{P}$ be a P-equivalent probability measure. We say that the family of price processes $\mathscr{V}$ satisfies the orthogonality condition with respect to $\tilde{P}$ or more briefly that $\mathscr{O}(\mathscr{V}, \tilde{P})$ holds if $\{\xi \in \mathscr{H}^1(\tilde{P}) | \xi \perp \mathscr{V}\} = \{0\}$.

The next result will be useful in further characterizing injectivity of T^* and hence completeness in a number of settings (see Theorems 4 and 5 below):

Theorem 3. *Assume NA holds and that the elements of $\mathscr{V}$ are locally bounded. if $\tilde{P}$ is any P-equivalent probability measure then condition* (ii) *of Theorem* 1 *which states that the operator $T^*: \mathbb{M} \to \mathbb{X}$ is injective, is equivalent to*:

(vii) *$\mathscr{O}(\mathscr{V}, \tilde{P})$ holds.*

If $\tilde{P} = Q \in \mathfrak{M}_{++}^{\mathrm{loc}}$ then (vii) *takes the following form*:

(vii)′ $\{\xi \in \mathscr{M}_0(Q) | [\xi, Z^\alpha] \in \mathsf{A}^{\mathrm{loc}}(Q)$ *and* $[\xi, Z^\alpha]^p = 0\ \forall\, \alpha \in A\} = \{0\}$.

The results contained in Theorems 1–3 are graphically summarized in Figure 1.

Given a semimartingale X, let $\mathscr{E}(X)$ denote its *Doléans-Dade exponential*, that is, $Y_t = \mathscr{E}(X)_t$ is the unique solution to $dY_t = Y_{t-}\, dX_t$ with $Y_0 = 1$. In Section 5 we consider price processes which are positive continuous semimartingales on a (d-dimensional) Brownian filtration and hence are of the form

$$Z_t^\alpha = Z_0^\alpha \mathscr{E}(R^\alpha)_t,$$

where

$$R_t^\alpha = A_t^\alpha + \sum_{i=1}^{d} \int_0^t \sigma_u^\alpha(i)\, dB_{u\wedge 1}^i,$$

$\sigma^\alpha(i)$, $i = 1, \dots, d$, are predictable processes, the A^α's are continuous adapted finite variation processes with $A_0^\alpha = 0$, $\alpha \in A$ and $(B_t^i)_{i=1}^d$ is d-dimensional Brownian motion. R^α has the economic interpretation as the return process of asset α. One obvious type of model covered by this setup is a generalized Black–Scholes stock model with prices evolving as possibly time inhomogeneous diffusions. Another possibility is to have $\mathscr{Z} = \{(Z_t^T)_{t\in[0,1]}\}_{T\in\mathscr{T}} \cup \{1\}$, where $\mathscr{T} \subseteq [1, \infty)$ and Z^T is the deflated price of a bond with maturity $T \in \mathscr{T}$ as in Heath, Jarrow and Morton (1992). Since we are interested in completeness at time 1, the bonds under consideration clearly should have maturity after time 1, hence the restriction $\mathscr{T} \subseteq [1, \infty)$.

If the number of assets is finite, that is, $|A| < \infty$, we let Σ_t denote the predictable $|A| \times d$-matrix process whose row vectors are $(\sigma_t^\alpha(i))_{i=1}^d$, $\alpha \in A$. We call Σ_t the *volatility matrix* of the risky assets. Since there are only d "sources of randomness" coming from the Brownian motion, it is natural to expect that d "sufficiently independent" risky assets are needed for completeness. More precisely, we will use Theorem 3 to obtain Theorem 4.

THEOREM 4. *Assume $\mathfrak{M}_{++}^{\mathrm{loc}} \neq \emptyset$, that NA holds and that $|A| < \infty$. Let Σ_t be as in the preceding paragraph. Then the market is complete if and only if rank $\Sigma_t = d$ holds M^P-a.e.*

Theorem 4 is essentially known. Indeed by Theorem 3 the market is complete if and only if $|\mathscr{M}_{\mathrm{loc}}^{++}| = 1$, so the result follows from a slight modification of Théorème 6 of Ansel and Stricker (1992).

Price processes with jumps have been studied by various authors with different goals in mind. See, for example, Merton (1976), Mercurio and Runggaldier (1993) and Jeanblanc-Picqué and Pontier (1990). In Section 6 we will consider price processes which are driven by an E-valued multivariate point process μ with compensator $\nu(dt, dx) = K_t(dx)\, dt$. We will see that our assumptions on the price processes allow us to write them in the following form:

$$Z_t^\alpha = Z_0^\alpha \mathscr{E}(R^\alpha)_t,$$

where

$$R_t^\alpha = A_t^\alpha + \int_{[0,t]\times E} \sigma^\alpha(u, x)\{\mu(du, dx) - \nu(du, dx)\}$$

and the $\sigma^\alpha(t, x)$'s are bounded M_μ^P-a.e. Here σ^α, μ and ν may depend on ω, but as is customary, this dependence is suppressed. Again, R^α has the interpretation as the return process of asset α.

The points of E should be thought of as possible types of "shocks" which occur in the economy according to the multivariate point process μ and cause the price processes to adjust by jumping. If $|E| < \infty$ we may regard $\sigma^{\alpha}(t, x)$ as an $|E|$-dimensional vector for fixed ω and t, and we then write $(\sigma_t^{\alpha}(x))_{x \in E}$ in place of $\sigma^{\alpha}(t, x)$. If now $|A| + |E| < \infty$, we let Σ_t denote the predictable $|A| \times |E|$-matrix process whose row vectors are $(\sigma_t^{\alpha}(x))_{x \in E}$, $\alpha \in A$. Once again, we call Σ_t the volatility matrix of the risky assets. The situation is very much analogous to the Brownian setting. There are now $|E|$ "sources of randomness" corresponding to the different possible shocks. We will use the orthogonality condition in Theorem 3 to show that $|E|$ "sufficiently independent" risky assets are needed for completeness.

THEOREM 5. *Assume $\mathfrak{M}_{++}^{\mathrm{loc}} \neq \emptyset$, that NA holds and that $|E| + |A| < \infty$. Let Σ_t be as in the preceeding paragraph and assume that M^P-a.e. $K_t(\{x\}) > 0$ $\forall x \in E$. Then the market is complete if and only if rank $\Sigma_t = |E|$ holds M^P-a.e.*

This paper is organized as follows. In Section 2 we discuss the second fundamental theorem from an operator point of view for a securities market model on a finite probability space. This simple setting allows us to point out the fundamental connection between completeness, operators and equivalent martingale measures without having the technicalities of infinite-dimensional spaces obscure the basic ideas. Section 3 gives our general setup and establishes Proposition 1 and Theorems 1–3. Sections 4–6 are devoted to three types of examples. Section 4 contains the Artzner–Heath example as well as the other examples mentioned in Remark 2, which illustrate the problems that arise when there are an infinite number of discontinuous price processes. Sections 5 and 6 deal, respectively, with price processes on a Brownian filtration and price processes driven by a multivariate point process. There we use the orthogonality condition of Theorem 3 to establish the characterization of completeness given in Theorems 4 and 5.

2. Securities market models on a finite probability space. A complete analysis of finite models was provided by Taqqu and Willinger (1987). We briefly discuss the Second Fundamental Theorem of Asset Pricing in this simple context; here the basic ideas of our approach to completeness can be illustrated without having to deal with topological issues which are present in a more general setting.

We take as given a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t \in \Pi}, P)$, where $\Pi = \{0, 1, \dots, T\}$, $|\Omega| = M < \infty$, $\mathscr{F}_0 = \{\emptyset, \Omega\}$, $\mathscr{F}_T = \mathscr{F} = \mathscr{P}(\Omega)$ (power set of Ω) and $P(\omega) > 0$ $\forall \omega \in \Omega$. In addition to a riskless asset whose price process is identically 1, there are N risky assets whose prices are modelled by adapted processes $(Z_t^i)_{t \in \Pi}$ $(i = 1, \dots, N)$ and we set $(\mathbf{Z})_{t \in \Pi} = (Z_t^1, \dots, Z_t^N)_{t \in \Pi}$. For $t \in \Pi$, $\mathscr{F}_t$ is generated by a finite partition $\mathscr{P}^t = \{P_i^t\}_{i=1,\dots,n_t}$ of n_t nonempty disjoint subsets of Ω and for $t < T$ any element of $\mathscr{P}^t$ is the union of elements from $\mathscr{P}^{t+1}$. So in this case, the stochastic basis or filtered probability space can be pictured as a tree whose nodes at time t are given by $P_1^t, \dots, P_{n_t}^t$. One allows predictable, self-financing trading strategies and because of our simple setting there is no ambiguity as to the appropriate definitions of no arbitrage and completeness. The questions of no arbitrage, completeness, existence and uniqueness of an equivalent martingale measure (e.m.m.) may be investigated either globally, that is, for the tree as a whole or locally, that is, at a particular node. It is easily seen that no arbitrage, existence of an e.m.m. and uniqueness of an e.m.m. holds globally if and only if the corresponding property holds at each node. Also, when no arbitrage holds, completeness globally is equivalent to completeness at each node. In view of these remarks, there is no loss of generality in considering a one-time period model, that is, $T = 1$.

In the one-time period model, the space of trading strategy is $\mathbb{Y} = \{(x, \mathbf{H}) | x \in \mathbf{R},\ \mathbf{H} \in \mathbf{R}^N\}$. Here $\mathbf{H} = (H^1, \dots, H^N)$ gives the holdings in the N risky assets during the time interval $[0,1]$ and x denotes the time 0 value of the entire portfolio, so the holding in the riskless asset are $x - \mathbf{H} \cdot \mathbf{Z}_0$. With this convention, the time 1 value of the portfolio is given by $x + \mathbf{H}(\mathbf{Z}_1 - \mathbf{Z}_0)$. We let $\mathbb{C} = L^\infty(\Omega, \mathscr{P}(\Omega), P) \cong \mathbf{R}^M$, where "$\cong$" indicates that the two spaces are isomorphic as Banach spaces. We introduce the operator $T\colon \mathbb{Y} \to \mathbb{C}$ defined by $T(x, \mathbf{H}) = x + \mathbf{H}(\mathbf{Z}_1 - \mathbf{Z}_0)$ which calculates the time 1 value of a given strategy $(x, \mathbf{H}) \in \mathbb{Y}$ and completeness means that T is surjective. We let $\mathbb{M}, \mathbb{X}$ denote the topological duals of $\mathbb{C}$, $\mathbb{Y}$, respectively, and consider on $(\mathbb{Y}, \mathbb{X}), (\mathbb{C}, \mathbb{M})$ the natural bilinear forms. With the generalizations to come in mind, note that $\mathbb{M}$ can be thought of as the space of signed measures on $(\Omega, \mathscr{P}(\Omega))$ acting on $\mathbb{C}$ through integration; $\langle \mu, X \rangle = \int X \, d\mu$ for $\mu \in \mathbb{M}$ and $X \in \mathbb{C}$. We denote by $T^*\colon \mathbb{M} \to \mathbb{X}$ the adjoint operator of T and we have

$$
\begin{aligned}
\langle T^*\mu, (X, \mathbf{H}) \rangle &= \langle \mu, T(x, \mathbf{H}) \rangle \\
&= \int T(x, \mathbf{H}) \, d\mu = \int x + \mathbf{H}(\mathbf{Z}_1 - \mathbf{Z}_0) \, d\mu
\end{aligned}
\tag{2.1}
$$

for $\mu \in \mathbb{M}$ and $(x, \mathbf{H}) \in \mathbb{Y}$.

We let $\mathfrak{M}_{++}$ denote the set of equivalent martingale measures and $\mathbb{M}_{++}$ the set of strictly positive measures on $(\Omega, \mathscr{P}(\Omega))$. We assume that $\mathfrak{M}_{++} \neq \emptyset$, say $Q \in \mathfrak{M}_{++}$, which in this setting is equivalent to the no arbitrage assumption by the first fundamental theorem of asset pricing. From (2.1) we see that if $\pi_0 \in \mathbb{X}$ is defined by $\pi_0(x, \mathbf{H}) = x$ for $(x, \mathbf{H}) \in \mathbb{Y}$, then

$$
\mathfrak{M}_{++} = (T^*)^{-1}(\pi_0) \cap \mathbb{M}_{++} = \{Q + \mu | \mu \in \ker T^*\} \cap \mathbb{M}_{++}
$$

and hence

$$
\mathfrak{M}_{++} = \{Q\} \Leftrightarrow T^* \text{ is injective} \Leftrightarrow T \text{ is surjective},
$$

using the fact that $\mathbb{M}$ is finite-dimensional for the first equivalence and linear algebra for the second. Since, by definition, completeness means that T is surjective, we have recovered the classical version of the Second Fundamental Theorem of Asset Pricing, as follows.

THEOREM 6. *Assume no arbitrage holds, say $Q \in \mathfrak{M}_{++}$. Then the market is complete if and only if $\mathfrak{M}_{++} = \{Q\}$.*

3. General setup and results. Throughout this section, a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \in [0,1]}, P)$ satisfying the usual conditions, that is, the filtration is right-continuous and $\mathcal{F}_0$ contains all P-null sets, is fixed. We further assume that $\mathcal{F}_0$ contains only sets of P-measure zero or one. Also given is a family $\mathcal{Z} = \{(Z_t^\alpha)_{t \in [0,1]}\}_{\alpha \in A \cup \{\Delta\}}$ of adapted cadlag price processes on the time interval $[0, 1]$ with $Z_t^\Delta \equiv 1$. In other words, the Δ-asset plays the role of deflator and we assume that the deflation has been carried out. A is allowed to be infinite. Further properties of the elements of $\mathcal{Z}$ will be introduced later. For the moment, however, we do not even assume that the price processes are semimartingales. We can get away with this level of generality since we only consider stochastic integrals on stopping time simple integrands. Of course, if there is an equivalent local martingale measure for $\mathcal{Z}$ then the elements of $\mathcal{Z}$ are necessarily semimartingales [Jacod and Shiryaev (1987), Theorem 3.13, page 156]. For some purposes it is more convenient to work with the time set $\mathbf{R}_+$ and we therefore let $\mathcal{F}_t = \mathcal{F}_1$ and $Z_t^\alpha = Z_1^\alpha$ for $t \geq 1$. Since $\mathcal{F}_t = \mathcal{F}_1$ for $t \geq 1$, we will use $\mathbb{F}$ to denote both $(\mathcal{F}_t)_{t \in [0,1]}$ and $(\mathcal{F}_t)_{t \in \mathbf{R}_+}$. When we say that $(X_t)_{t \in [0,1]}$ is a local martingale we mean that X extended to $\mathbf{R}_+$ by $X_t = X_1$ for $t \geq$ is a local martingale on the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbf{R}_+}$.

A few comments about technicalities are in order. For basic stochastic calculus definitions we refer the reader to Jacod (1979) and Jacod and Shiryaev (1987). The measures we deal with are considered on the σ-algebra $\mathcal{F}_\infty = \bigvee_{t \in \mathbf{R}_+} \mathcal{F}_t = \mathcal{F}_1$. Given a process X and a stopping time τ, we write X^τ for the process stopped at time τ. When we write an equality between processes it is understood to hold up to a P-evanescent set. With these conventions in place, we note that our filtration has the property that $X = X^1$ for any martingale X and hence also for any local martingale. In other words, any local martingale on our filtration is constant after time 1. In particular, a martingale is automatically a uniformly integrable martingale, and, to check if a local martingale is of class D (or equivalently if it is a uniformly integrable martingale), it suffices to consider stopping times which are bounded by 1. Recall that if X is an adapted cadlag process which admits a terminal r.v. then a necessary and sufficient condition for X to be a uniformly integrable martingale is that $EX_\tau = EX_0$ for any stopping time τ. We will apply this fact repeatedly to processes that are constant after time 1, in which case it suffices to consider stopping times $\tau \leq 1$. Finally observe that since $\mathcal{F}_0$ is assumed to contain only sets of P-measure zero or one, X_0 is constant P-a.s. for any adapted process X and so $EX_0 = X_0$.

In the introduction we gave a nontechnical and economically motivated description of our setup. We now focus on the mathematical aspects and refer the reader to Grothendieck (1973) for functional analytic background material. We allow an agent to invest in the riskless asset plus a finite number of risky assets via self-financing, stopping time simple strategies [see (3.1) and (3.2) below] yielding bounded payoffs. This means that the agent can arbitrarily choose holdings in the riskless and the risky assets at time 0 and after time 0 chooses holdings in the risky assets, with the holdings in the riskless asset being determined by the self-financing condition. More precisely, let

$$\tilde{\mathbb{Y}} = \left\{ (x, (H^{\alpha})_{\alpha \in A}) | x \in \mathbf{R}, H_t^{\alpha} = \sum_{i=1}^{n_{\alpha}} h_{i-1}^{\alpha} 1_{(\tau_{i-1}^{\alpha}, \tau_i^{\alpha}]}(t) \right\}, \tag{3.1}$$

where $0 \le \tau_0^{\alpha} \le \cdots \le \tau_{n_{\alpha}}^{\alpha} \le 1$ are stopping times, $h_i^{\alpha} \in L^{\infty}(\mathscr{F}_{\tau_i^{\alpha}}, P)$ and $H^{\alpha} \equiv 0$ except for finitely many $\alpha \in A$. $(H^{\alpha})_{\alpha \in A}$ represents the holdings in the risky assets after time 0. Rather than specifying the holdings at time zero of the riskless and all the risky assets, we let x stand for the time 0 value of the entire portfolio. We denote by $L^0(\mathscr{F}_1)$ the vector space of $\mathscr{F}_1$-measurable r.v.'s modulo P-equivalent and define the operator $\tilde{T}$: $\tilde{\mathbb{Y}} \to L^0(\mathscr{F}_1)$ by

$$\tilde{T}(x, (H^{\alpha})_{\alpha \in A}) = x + \sum_{\alpha \in A} \int_0^1 H_u^{\alpha} \, dZ_u^{\alpha}.$$

Here the sum is finite and $\tilde{T}$ calculates the time 1 payoff resulting from the strategy $(x, (H^{\alpha})_{\alpha \in A}) \in \tilde{\mathbb{Y}}$. The space of claims will be $\mathbb{C} = L^{\infty}(\mathscr{F}_1, P)$, that is, the $\mathscr{F}_1$-measurable bounded r.v.'s modulo P-equivalence. To ensure that the payoffs are bounded, the space of trading strategies is taken to be

$$\mathbb{Y} = \tilde{\mathbb{Y}} \cap \tilde{T}^{-1}(L^{\infty}(\mathscr{F}_1, P)) \tag{3.2}$$

and we shall refer to the elements of $\mathbb{Y}$ as stopping time simple strategies yielding bounded payoffs. Finally, we denote by T: $\mathbb{Y} \to \mathbb{C}$ the restriction of $\tilde{T}$ to $\mathbb{Y}$.

We let $\mathbb{M}$ denote the vector space of P-absolutely continuous signed measures on $(\Omega, \mathscr{F}_1)$. Then $(\mathbb{C}, \mathbb{M})$ forms a duality via the bilinear form $\langle \cdot , \cdot \rangle$: $\mathbb{C} \times \mathbb{M} \to \mathbf{R}$ given by $\langle X, \mu \rangle = \int X \, d\mu$ which is separated in the sense that $\langle X, \cdot \rangle \equiv 0$ only if $X = 0$ and $\langle \cdot , \mu \rangle \equiv 0$ only if $\mu = 0$ ($X \in \mathbb{C}$, $\mu \in \mathbb{M}$). Recall that $\pi_0(x, (H^{\alpha})_{\alpha \in A}) = x$, $(x, (H^{\alpha})_{\alpha \in A}) \in \mathbb{Y}$, is the market's way of valuing trading strategies. We let $\mathbb{X}$ denote the vector space generated by $\{\mu \circ T\}_{\mu \in \mathbb{M}} \cup \{\pi_0\}$. Since $\{\mu \circ T\}_{\mu \in \mathbb{M}} \cup \{\pi_0\}$ are linear functionals on $\mathbb{Y}$, $(\mathbb{Y}, \mathbb{X})$ forms a duality via the bilinear form $\langle \cdot , \cdot \rangle$: $\mathbb{Y} \times \mathbb{X} \to \mathbf{R}$ given by $\langle (x, (H^{\alpha})_{\alpha \in A}), \phi \rangle = \phi(x, (H^{\alpha})_{\alpha \in A})$ which is separated in $\mathbb{X}$ because $\langle \cdot , \phi \rangle \equiv 0$ only if $\phi = 0$ ($(x, (H^{\alpha})_{\alpha \in A}) \in \mathbb{Y}$, $\phi \in \mathbb{X}$).

The vector spaces $\mathbb{Y}$, $\mathbb{X}$, $\mathbb{C}$ and $\mathbb{M}$ become locally convex topological vector spaces when endowed with the weak topologies arising from the dualities described in the last paragraph. In particular, since $\mathbb{M}$ is isomorphic to $L^1(\mathscr{F}_1, P)$ by the Radon–Nikodym theorem, the topology on $\mathbb{C}$ is the weak* topology, viewing $\mathbb{C}$ as the topological dual of $L^1(\mathscr{F}_1, P)$. Also, it follows from Grothendieck [(1973), Proposition 24, page 80] that T: $\mathbb{Y} \to \mathbb{C}$ is weakly continuous and Corollary 2, page 81 of the same source yields the weakly continuous adjoint operator $T^*: \mathbb{M} \to \mathbb{X}$ explicitly given by

$$(T^* \mu)(x, (H^\alpha)_{\alpha \in A}) = \int T(x, (H^\alpha)_{\alpha \in A})\, d\mu, \qquad (x, (H^\alpha)_{\alpha \in A}) \in \mathbb{Y}.$$

In Section 4 we will want to discuss discrete-time models and the following argument shows that our continuous time setup still applies. Consider the time set $\Pi = \{t_0, t_1, t_2, \dots, t_{T-1}, t_T\}$ with $0 = t_0 < t_1 < t_2 < \cdots < t_{T-1} < t_T = 1$, the filtered probability space $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t \in \Pi}, P)$, and the family $\mathscr{V} = \{(Z^\alpha)_{t \in \Pi}\}_{\alpha \in A \cup \{\Delta\}}$ of finite-time price processes adapted to the filtration $\mathbb{F}$. Here one allows trading strategies which are $\mathbb{F}$-predictable, bounded and self-financing. The bounded claims attainable in the continuous-time model obtained by letting

$$\tilde{\mathscr{F}}_t = \begin{cases} \mathscr{F}_{t_i}, & t \in [t_i, t_{i+1}), \\ \mathscr{F}_1, & t = 1, \end{cases} \quad \text{and} \quad \tilde{Z}_t^\alpha = \begin{cases} Z_{t_i}^\alpha, & t \in [t_i, t_{i+1}), \\ Z_1^\alpha, & t = 1, \end{cases}$$

are the same as the bounded attainable claims in the finite-time model. In this way, finite-time models can always be embedded into continuous time.

We briefly recall some of the notation and definitions given in Section 1. The space of attainable claims is $\mathscr{A}_1 = \operatorname{Im} T$ and $\mathscr{A}_1^0$ the space of claims attainable at zero initial cost. Then $\mathfrak{M}_{++}/\mathfrak{M}_{++}^{\text{loc}}$ denote the P-equivalent martingale/local martingale measures for $\mathscr{V}$ and $\mathscr{P}_{+/-}$ the set of signed measured $\mu \in \mathbb{M}$ for which $T^*\mu = \pi_0$. Recall also that completeness means that $\mathscr{A}_1$ is weak* dense in $\mathbb{C}$, while for $Q \in \mathfrak{M}_{++}^{\text{loc}}$, Q-completeness means that $\mathscr{A}_1$ is dense in $\mathbb{C}$ with respect to the $L^1(\mathscr{F}_1, Q)$ topology. We also defined a weak no arbitrage condition (NA); $\mathscr{A}_1^0 \cap \mathbb{C}_+ = \{0\}$, where $\mathbb{C}_+ = \{X \in \mathbb{C} | X \geq 0$ (P-a.s.)$\}$. For a P-equivalent probability measure $\tilde{P}$ we say that the orthogonality condition $\mathscr{O}(\mathscr{V}, \tilde{P})$ holds if $\{\xi \in \mathscr{H}^1(\tilde{P}) | \xi \perp \mathscr{V}\} = \{0\}$. Since $\xi \in \mathscr{H}^1(\tilde{P})$ belongs to $\mathscr{H}_0^1(\tilde{P})$ if and only if $\xi \perp 1$, we have

$$\mathscr{O}(\mathscr{V}, \tilde{P}) \text{ holds} \Leftrightarrow \left\{\xi \in \mathscr{H}_0^1(\tilde{P}) | \xi \perp \{Z^\alpha\}_{\alpha \in A}\right\} = \{0\} \tag{3.3}$$

$$\Leftrightarrow \left\{\xi \in \mathscr{M}_0(\tilde{P}) | \xi \perp \{Z^\alpha\}_{\alpha \in A}\right\} = \{0\}, \tag{3.4}$$

using $\mathscr{M}_0(\tilde{P}) \subseteq \mathscr{H}_0^{1,\text{loc}}(\tilde{P})$ to get the last equivalence. Finally, we let $\mathscr{A}$ denote the space of value processes corresponding to the attainable claims

$$\mathscr{A} = \left\{ x + \sum_{\alpha \in A} \int_0^{\cdot} H_u^\alpha \, dZ_u^\alpha \,\middle|\, (x, (H^\alpha)_{\alpha \in A}) \in \mathbb{Y} \right\}.$$

We now prove Proposition 1 and Theorems 1–3.

PROOF OF THEOREM 1. From our setup we have that $(\mathbb{Y}, \mathbb{X})$ and $(\mathbb{C}, \mathbb{M})$ are dual systems separated in $\mathbb{X}$ and $\mathbb{M}$, respectively, and that $T: \mathbb{Y} \to \mathbb{C}$ is a weakly continuous linear operator. Proposition 26, page 82 of Grothendieck (1973) implies that $\operatorname{Im} T$ is weakly dense in $\mathbb{C}$ if and only if $T^*: \mathbb{M} \to \mathbb{X}$ is injective. Since the weak topology on $\mathbb{C}$ arising from the dual system $(\mathbb{C}, \mathbb{M})$ is precisely the weak* topology, the equivalence of (i) and (ii) follows.

For the equivalence of (v) and (vi), we just need to modify the argument of the last paragraph slightly. Note that $(\mathbb{C}, \mathbb{M}^{(Q,\infty)})$ forms a dual system via the bilinear form $\langle \cdot, \cdot \rangle$: $\mathbb{C} \times \mathbb{M}^{(Q,\infty)} \to \mathbf{R}$ given by $\langle X, \mu \rangle = \int X \, d\mu$ which is separated in $\mathbb{M}^{(Q,\infty)}$. Furthermore, T: $\mathbb{Y} \to \mathbb{C}$ is weakly continuous with respect to the dualities $(\mathbb{Y}, \mathbb{X})$ and $(\mathbb{C}, \mathbb{M}^{(Q,\infty)})$. Proposition 26, page 82 of Grothendieck (1973) now gives that $\operatorname{Im} T$ is weakly dense in $\mathbb{C}$ for the dualities $(\mathbb{Y}, \mathbb{X})$ and $(\mathbb{C}, \mathbb{M}^{(Q,\infty)})$ if and only if T^* restricted to $\mathbb{M}^{(Q,\infty)}$ is injective. By the corollary to Theorem 4, page 60 of Grothendieck (1973), this is in turn equivalent to $\operatorname{Im} T$ being dense in $\mathbb{C}$ with respect to the $L^1(\mathscr{F}_1, Q)$ topology because the $L^1(\mathscr{F}_1, Q)$ topology on $\mathbb{C}$ is consistent with the duality $(\mathbb{C}, \mathbb{M}^{(Q,\infty)})$ in the sense that the dual space of $\mathbb{C}$ when endowed with the $L^1(\mathscr{F}_1, Q)$ topology is $\mathbb{M}^{(Q,\infty)}$ [which is isomorphic to $L^\infty(\mathscr{F}_1, Q)$]. □

Before giving a proof of Proposition 1, we point out that the containment $\mathfrak{M}^{\text{loc}}_{++} \subseteq \mathscr{P}_{+/-} \cap \mathbb{M}_{++}$ may be strict if the local boundedness in Proposition 1 is dropped.

EXAMPLE 3.1. We consider the filtered probability space $(\Omega, \{\mathscr{F}_0, \mathscr{F}_1\}, P)$ where $\Omega = \mathbb{N}$, $\mathscr{F}_0 = \{\varnothing, \Omega\}$, $\mathscr{F}_1 = \mathscr{P}(\Omega)$ (the power set of Ω) and with $P(\omega) > 0$ $\forall\, \omega \in \Omega$. There is a single risky asset with price process $Z_0 \equiv 2$, $Z_1(i) = i$ for $i \geq 1$. In this case $\mathscr{A}_1$ consists only of constants and hence $\mathscr{P}_{++}$ is the set of all probability measures on $\mathbb{N}$ with support $\mathbb{N}$. It is clear that $\mathfrak{M}_{++} \neq \varnothing$ and hence NA holds. If Q is a probability measure on $\mathbb{N}$ with support $\mathbb{N}$ such that $Z_1 \notin L^1(\mathscr{F}_1, Q)$ then $Q \in \mathscr{P}_{++}$ but $Q \notin \mathfrak{M}^{\text{loc}}_{++} = \mathfrak{M}_{++}$.

The example is trivial in the sense that $\mathscr{A}_1$ consists only of constants. By introducing a second period, one can give an example to the same effect in which $\mathscr{A}_1$ consists of more than just constants.

PROOF OF PROPOSITION 1. First I claim that

$$\text{NA} \Rightarrow \text{the elements of } \mathscr{A} \text{ are bounded processes.} \tag{3.5}$$

Let $V \in \mathscr{A}$, say,

$$V_t = x + \sum_{\alpha \in A} \int_0^1 H_u^\alpha \, dZ_u^\alpha, \qquad (x, (H^\alpha)_{\alpha \in A}) \in \mathbb{Y}.$$

We assume that $|V_1| \leq M$ and show that

$$\sup_t |V_t| \leq M + 1, \quad P\text{-a.s.} \tag{3.6}$$

Consider the stopping times

$$\sigma^+ = \inf\{t | V_t > M + 1\} \wedge 1 \text{ and } \sigma^- = \inf\{t | V_t < -M - 1\} \wedge 1.$$

Then

$$V_1 = V_{\sigma^+} + \sum_{\alpha \in A} \int_0^1 H_u^\alpha 1_{(\sigma^+, 1]}(u) \, dZ_u^\alpha$$

and from the definition of σ^+ and the fact that $|V_1| \le M$ we deduce that

$$0 \le 1_{\{\sigma^+ < 1\}} \le \sum_{\alpha \in A} \int_0^1 - H_u^\alpha 1_{(\sigma^+, 1]}(u)\, dZ_u^\alpha. \tag{3.7}$$

The claim on the right-hand side is attainable with $(0, (-H_u^\alpha 1_{(\sigma^+, 1]}(u))_{\alpha \in A}) \in \mathbb{Y}$ and so NA implies that the inequalities in (3.7) are equalities. Hence $\sigma^+ = 1$ P-a.s. and similarly $\sigma^- = 1$ P-a.s. so that

$$\sigma^+ \wedge \sigma^- = \inf\{t \mid |V_t| > M + 1\} \wedge 1 = 1, \quad P\text{-a.s.},$$

which establishes (3.6) and hence (3.5).

It now follows that if NA holds, then

$$\mathscr{A} \subseteq \mathscr{H}^\infty(Q) \tag{3.8}$$

for any $Q \in \mathfrak{M}_{++}^{\mathrm{loc}}$. Hence

$$(T^* Q)(x, (H^\alpha)_{\alpha \in A}) = E_Q T(x, (H^\alpha)_{\alpha \in A}) = E_Q\left(x + \sum_{\alpha \in A} \int_0^1 H_u^\alpha \, dZ_u^\alpha \right)$$
$$= x = \pi_0(x, (H^\alpha)_{\alpha \in A})$$

for $(x, (H^\alpha)_{\alpha \in A}) \in \mathbb{Y}$ which shows that

$$\mathfrak{M}_{++}^{\mathrm{loc}} \subseteq \mathscr{P}_{+/-} \cap \mathbb{M}_{++}.$$

Conversely, let $Q \in \mathscr{P}_{+/-} \cap \mathbb{M}_{++}$ and note that $+\mathrm{Q}$ is a probability measure. Since we are working under the additional assumption that the elements of $\mathscr{V}$ are locally bounded, it suffices to show that for a fixed $\alpha^* \in A$ and any stopping time τ that $(Z^{\alpha^*})^\tau$ is bounded, $(A^{\alpha^*})^\tau$ is a Q-martingale. As pointed out at the beginning of this section, since Z^{α^*} is constant after time 1, this will follow if we show that

$$E_Q (Z^{\alpha^*})_\sigma^\tau = (Z^{\alpha^*})_0^\tau \quad \text{or equivalently} \quad E_Q\{(X^{\alpha^*})_\sigma^\tau - (Z^{\alpha^*})_0^\tau\} = 0$$

for any stopping time $\sigma \le 1$. Given such a σ and setting $H_u^\alpha \equiv 0$ if $\alpha \ne \alpha^*$ and $H_u^\alpha = 1_{u \le \tau \wedge \sigma}$ if $\alpha = \alpha^*$ we have $(0, (H^\alpha)_{\alpha \in A}) \in \mathbb{Y}$ and hence

$$E_Q\{(Z^{\alpha^*})_\sigma^\tau - (Z^{\alpha^*})_0^\tau\} = E_Q\{(Z^{\alpha^*})_{\tau \wedge \sigma} - (Z^{\alpha^*})_0\}$$
$$= E_Q(T(0, (H^\alpha)_{\alpha \in A}) = 0. \qquad \square$$

Let $\tilde{P}$ be any P-equivalent probability measure and for $\mu \in \mathbb{M}$ denote by $\varphi_{\tilde{P}}(\mu)_t$ the density process of μ with respect to $\tilde{P}$, that is, $\varphi_{\tilde{P}}(\mu)_t = E_{\tilde{P}}\{d\mu / d\tilde{P} \mid \mathscr{F}_t\}$. Then $\varphi_{\tilde{P}}$ defines a bijection between $\mathbb{M}$ and $\mathscr{M}(\tilde{P})$. The following lemma implies that $\ker T^*$ gets mapped onto $\{\xi \in \mathscr{M}_0(\tilde{P}) \mid \xi \perp \{Z^\alpha\}_{\alpha \in A}\}$ by $\varphi_{\tilde{P}}$ and plays a key role in the proofs of Theorems 2 and 3:

LEMMA 3.1. *Let* $\mu \in \mathbb{M}$ *with* $\mu(\Omega) = 0$. *Then* $\xi = \varphi_{\tilde{P}}(\mu) \in \mathscr{M}_0(\tilde{P})$ *and the following three conditions are equivalent*:

(i) $\xi \perp \{\mathrm{Z}^\alpha\}_{\alpha \in \mathrm{A}}$;
(ii) $\xi \perp \mathscr{A}$;
(iii) $\mu \in \ker T^*$.

PROOF OF LEMMA 3.1. It is clear that if $\mu \in \mathbb{M}$ with $\mu(\Omega) = 0$ then $\xi = \varphi_{\tilde{P}}(\mu) \in \mathscr{M}_0(\tilde{P})$. Now ξ is constant after time 1 and if η is any bounded adapted cadlag process which is also constant after time 1 then

$$(3.9)\quad \xi \perp \eta \Leftrightarrow \xi\eta \in \mathscr{M}_0(\tilde{P}) \Leftrightarrow E_{\tilde{P}}(\xi\eta)_T = 0 \qquad \forall \text{ stopping times } T \leq 1.$$

Since NA holds, the statement in (3.8) holds and implies that (3.9) applies to any $\eta \in \mathscr{A}$. This (for the first equivalence) and $E_{\tilde{P}}(\xi Y)_T = \int Y_T\, d\mu$ ($d\mu = \xi_T\, d\tilde{P}$ on $\mathscr{F}_T$) (for the second equivalence) yields

$$(3.10)\qquad \begin{aligned} &\text{(ii)} \Leftrightarrow E_{\tilde{P}}(\xi Y)_T = 0 \qquad \forall\, Y \in \mathscr{A} \text{ and stoppingtimes } T \leq 1,\\ &\Leftrightarrow \int Y_T\, d\mu = 0 \qquad \forall\, Y \in \mathscr{A} \text{ and stopping times } T \leq 1. \end{aligned}$$

Since $Y_T = Y_1^T$ and $Y^T \in \mathscr{A}$ whenever $Y \in \mathscr{A}$ and $T \leq 1$ is any stopping time, the condition in (3.10) is equivalent to

$$\int Y_1\, d\mu = 0, \qquad \forall\, Y \in \mathscr{A}$$

which is equivalent (iii).

It remains to check the equivalence of (i) and (ii). First we recall the following easily verified fact [He, Wang and Yan (1992), Theorem 7.38 and its proof, page 203]:

(3.11) Let $M \in \mathscr{M}(\tilde{P})$, T a stopping time and $g \in L^\infty(\mathscr{F}_T)$. Then $g(M - M^T) \in \mathscr{M}(\tilde{P})$.

Consider an arbitrary $\alpha \in \mathscr{A}$ and let $T_n^\alpha \uparrow \infty$ a.s. be a sequence of stopping times such that $(Z^\alpha)^{T_n^\alpha}$ is bounded. For (i) $\Rightarrow$ (ii), we need to show that when (i) holds,

$$\xi \perp h\{(Z^\alpha)^\tau - (Z^\alpha)^\sigma\} \quad \text{for stopping times } \sigma \leq \tau \leq 1 \text{ and } h \in L^\infty(\mathscr{F}_\sigma),$$

which follows if we show

$$h 1_{\sigma \wedge T_n^\alpha = \sigma}\{\xi^{T_n^\alpha}(Z^\alpha)^{\tau \wedge T_n^\alpha} - \xi^{T_n^\alpha}(Z^\alpha)^{\sigma \wedge T_n^\alpha}\} \in \mathscr{M}_0(\tilde{P}) \qquad \forall\, n.$$

In view of (3.11) it is enough to show that

$$\xi^{T_n^\alpha}(Z^\alpha)^{\tau \wedge T_n^\alpha}, \xi^{T_n^\alpha}(Z^\alpha)^{\sigma \wedge T_n^\alpha} \in \mathscr{M}_0(\tilde{P}) \qquad \forall\, n,$$

since we can then take $M = \xi^{T_n^\alpha}(Z^\alpha)^{\tau \wedge T_n^\alpha} - \xi^{T_n^\alpha}(Z^\alpha)^{\sigma \wedge T_n^\alpha}$, $T = \sigma \wedge T_n^\alpha$ and $g = h 1_{T = \sigma}$ in (3.11). For any stopping time $T \leq 1$ we have

$$(3.12)\qquad \begin{aligned} E_{\tilde{P}}\left(\xi^{T_n^\alpha}(Z^\alpha)^{\tau \wedge T_n^\alpha}\right)_T &= E_{\tilde{P}}\, \xi_{T_n^\alpha \wedge T} Z^\alpha_{\tau \wedge T_n^\alpha \wedge T}\\ &= E_{\tilde{P}} E_{\tilde{P}}\left\{\xi_{T_n^\alpha \wedge T} Z^\alpha_{\tau \wedge T_n^\alpha \wedge T} | \mathscr{F}_{\tau \wedge T_n^\alpha \wedge T}\right\}\\ &= E_{\tilde{P}}\, \xi_{\tau \wedge T_n^\alpha \wedge T}(Z^\alpha)_{\tau \wedge T_n^\alpha \wedge T}\\ &= E_{\tilde{P}}\left(\xi^{T_n^\alpha}(Z^\alpha)^{T_n^\alpha}\right)_{T \wedge \tau}. \end{aligned}$$

If (i) holds then $\xi^{T_n^\alpha} \perp (Z^\alpha)^{T_n^\alpha}$ with $\xi^{T_n^\alpha} \in \mathscr{M}_0(\tilde{P})$ and $(Z^\alpha)^{T_n^\alpha}$ bounded and constant after time 1 so that (3.9) implies that the last expectation in (3.12) is zero. Therefore, (i) implies that

$$E_{\tilde{P}}\Big(\xi^{T_n^\alpha}(Z^\alpha)^{\tau\wedge T_n^\alpha}\Big)_T = 0 \qquad \forall \text{ stopping times } T \le 1,$$

which shows that $\xi^{T_n^\alpha}(Z^\alpha)^{\tau\wedge T_n^\alpha} \in \mathscr{M}_0(\tilde{P})$. One shows similarly that $\xi^{T_n^\alpha}(Z^\alpha)^{\sigma\wedge T_n^\alpha} \in \mathscr{M}_0(\tilde{P})$.

Finally,

$$(\xi Z^\alpha)^{T_n^\alpha} = \xi(Z^\alpha)^{T_n^\alpha} - (Z^\alpha)^{T_n^\alpha}\{\xi - \xi^{T_n^\alpha}\}$$

and if (ii) holds, then $\xi(Z^\alpha)^{T_n^\alpha} \in \mathscr{M}_0(\tilde{P})$ since $(Z^\alpha)^{T_n^\alpha} \in \mathscr{A}$ while $(Z^\alpha)^{T_n^\alpha}\{\xi - \xi^{T_n^\alpha}\} \in \mathscr{M}_0(\tilde{P})$ follows from (3.11). Hence $(\xi Z^\alpha)^{T_n^\alpha} \in \mathscr{M}_0(\tilde{P})\ \forall\ n$, that is, $\xi \perp Z^\alpha$ when (ii) holds and (ii) ⇒ (i) follows. □

Lemma 3.1 and (3.4) imply the equivalence of conditions (ii) and (vii) as asserted in Theorem 3, so its proof is almost complete:

PROOF OF THEOREM 3. It remains only to show that if $Q \in \mathfrak{M}_{++}^{\text{loc}}$ then (vii) takes the form (vii)′. Again it is convenient to use the characterization of (vii) contained in (3.4). If $\xi \in \mathscr{M}_0(Q)$ then

$$\xi \perp Z^\alpha \Leftrightarrow [\xi, Z^\alpha] \in \mathscr{M}_0^{\text{loc}}(Q) \Leftrightarrow [\xi, Z^\alpha] \in \mathscr{M}_0^{\text{loc}}(Q) \cap \mathsf{A}^{\text{loc}}(Q),$$

using Proposition 1.43 of Jacod [(1979), page 20] to obtain the last equivalence. Hence

$$\xi \perp Z^\alpha \Leftrightarrow [\xi, Z^\alpha] \in \mathsf{A}^{\text{loc}}(Q) \quad \text{and} \quad [\xi, Z^\alpha]^p = 0. \qquad \square$$

Finally, we establish Theorem 2.

PROOF OF THEOREM 2. Since by definition $\mathscr{P}_{+/-} = (T^*)^{-1}(\pi_0)$, it is clear that if $Q \in \mathscr{P}_{+/-}$, then $\mathscr{P}_{+/-} = \{Q + \mu | \mu \in \ker T^*\}$ and hence (iii) is equivalent to (ii) of Theorem 1. For the rest of the proof we assume that NA holds, that the elements of $\mathscr{V}$ are locally bounded and that $Q \in \mathfrak{M}_{++}^{\text{loc}}$. Then $\mathfrak{M}_{++}^{\text{loc}} = \mathscr{P}_{+/-} \cap \mathbb{M}_{++}$ by Proposition 1 and hence the implications (iii) ⇒ (iv) and (iii) ⇒ (iv)′ when $Q \in \mathfrak{M}_{++}$ are immediate.

Next we assume that (vi) fails and show that (iv) fails and that (iv)′ fails if $Q \in \mathfrak{M}_{++}$. Let $\mu \in (\ker T^* \setminus \{0\}) \cap \mathbb{M}^{(Q,\infty)}$ and multiplying by a constant, assume without loss of generality that $\|d\mu/dQ\|_\infty < 1/2$. Then $\tilde{Q} = Q + \mu$, that is, $d\tilde{Q} = (1 + d\mu/dQ)\,dQ$ defines a P-equivalent probability measure different from Q and we show that $\tilde{Q} \in \mathfrak{M}_{++}^{\text{loc}}$ and $\tilde{Q} \in \mathfrak{M}_{++}$ when $Q \in \mathfrak{M}_{++}$. Since $\mu \in \ker T^*$, Lemma 3.1 implies that $Z^\alpha \perp \varphi_Q(\mu)$, that is, $Z^\alpha\varphi_Q(\mu) \in \mathscr{M}_0^{\text{loc}}(Q)$, where $\varphi_Q(\mu)_t = E_Q\{d\mu/dQ | \mathscr{F}_t\}$ is the density process of μ with respect to Q. Also, $Z^\alpha \in \mathscr{M}^{\text{loc}}(Q)$ since $Q \in \mathfrak{M}_{++}^{\text{loc}}$ and so $Z^\alpha(1 + \varphi_Q(\mu)) \in \mathscr{M}^{\text{loc}}(Q)$. Equivalently, $Z^\alpha \in \mathscr{M}^{\text{loc}}(\tilde{Q})$ and therefore $\tilde{Q} \in \mathfrak{M}_{++}^{\text{loc}}$. Finally, observe that a family of r.v.'s is Q-uniformly integrable if and only if it is $\tilde{Q}$-uniformly integrable since $1/2 < 1 + \varphi_Q(\mu)_t < 3/2$. It follows that Z^α is of class D under Q if and only if it is of class D under $\tilde{Q}$. Now if $Q \in \mathfrak{M}_{++}$, then Z^α is of class D under Q hence under $\tilde{Q}$ which in turn implies that Z^{α} is a uniformly integrable $\tilde{Q}$-martingale and so $\tilde{Q} \in \mathfrak{M}_{++}$.

Since Theorem 3(vii) implies (ii), we complete the proof by showing that if $\mathscr{V}$ is finite or all the elements of $\mathscr{V}$ have continuous sample paths then the equivalent (Theorem 1) conditions (v) and (vi) imply (vii). From Lemma 3.1 we know that $\varphi_Q(\mu)$ defines a bijection between $\mathbb{M}$ and $\mathscr{M}(Q)$ and that ker T^* gets mapped onto $\{\xi \in \mathscr{M}_0(Q) | \xi \perp \{Z^{\alpha}\}_{\alpha \in A}\}$. It follows that ker $T^* \cap \mathbb{M}^{(Q,\infty)}$ gets mapped onto $\{\xi \in \mathscr{H}_0^{\infty}(Q) | \xi \perp \{Z^{\alpha}\}_{\alpha \in A}\}$. Hence

$$\text{(vi)} \Leftrightarrow \left\{\xi \in \mathscr{H}_0^{\infty}(Q) | \xi \perp \{Z^{\alpha}\}_{\alpha \in A}\right\} = \{0\} \tag{3.13}$$

$$\Leftrightarrow \left\{\xi \in \mathscr{H}_0^{\infty,\text{loc}}(Q) | \xi \perp \{Z^{\alpha}\}_{\alpha \in A}\right\} = \{0\}. \tag{3.14}$$

At this point one can appeal to Proposition 4.13, page 118 ($\mathscr{V}$ finite), to Proposition 4.67, page 146 and Corollary 4.12, page 117 (elements of $\mathscr{V}$ are continuous) of Jacod (1979) to get that the second statement in (3.13) is equivalent to (vii) when the elements of $\mathscr{V}$ are continuous or when the family $\mathscr{V}$ is finite. However, we now outline a proof, adapting the arguments of Jacod (1979) to the particularities of our setting.

Since $Q \in \mathfrak{M}_{++}^{\text{loc}}$ and NA holds, we know that $\mathscr{A} \subseteq \mathscr{H}^{\infty}(Q)$ by (3.8) from the proof of Proposition 1. First we establish that

$$\text{(v)} \Rightarrow \mathscr{A} \text{ is dense in } \mathscr{H}^1(Q) \tag{3.15}$$

and by Proposition 2.39 of Jacod [(1979), page 40] it suffices to show that $\mathscr{A}$ is dense in $\mathscr{H}^{\infty}(Q)$ with respect to the $\mathscr{H}^1(Q)$ topology. Consider then $X \in \mathscr{H}^{\infty}(Q)$ and using (v) find $X^n \in \mathscr{A}$ such that

$$\lim_{n \to \infty} \|X_1^n - X_1\|_{L^1(Q)} = 0. \tag{3.16}$$

By Doob's inequality,

$$Q\Big(\sup_{t \le 1} |X_t^n - X_t| \ge \lambda\Big) \le \frac{1}{\lambda} \|X_1^n - X_1\|_{L^1(Q)}, \qquad \lambda > 0$$

and so

$$\lim_{k \to \infty} \sup_{t \le 1} |X_t^{n_k} - X_t| = 0, \quad Q\text{-a.s.}, \tag{3.17}$$

for a subsequence $\{n_k\}$. Define stopping times

$$\tau^n = \inf\{t \,|\, |X_t^{n_k}| > n \text{ for some } k \ge 1\} \wedge 1$$

and note that

$$Q\text{-a.s. } \tau^n = 1 \quad \text{for } n \text{ large enough}$$

because of (3.17) and the fact that $X \in \mathscr{H}^{\infty}(Q)$. Hence

$$\lim_{n \to \infty} \sup_{t \le 1} |X_t^{\tau^n} - X_t| \to 0, \quad Q\text{-a.s.},$$

and since $X \in \mathscr{H}^{\infty}(Q)$, the dominated convergence theorem yields

$$\lim_{n \to \infty} E_Q \sup_{t \le 1} |X_t^{\tau^n} - X_t| = 0, \text{ that is, } \lim_{n \to \infty} \|X^{\tau^n} - X\|_{\mathscr{H}^1(Q)} = 0.$$

Noting that $(X^{n_k})^{\tau^n} \in \mathscr{A}$ for all k and n, it now suffices to show that for fixed n,

$$\lim_{k\to\infty} E_Q \sup_{t\le 1} \left| (X^{n_k})_t^{\tau^n} - X_t^{\tau^n} \right| = 0, \qquad \text{that is, } \lim_{k\to\infty} \left\| (X^{n_k})^{\tau^n} - X^{\tau^n} \right\|_{\mathscr{H}^1(Q)} = 0. \tag{3.18}$$

We have

$$\begin{aligned} E_Q \sup_{t\le 1} \left| (X^{n_k})_t^{\tau^n} - X_t^{\tau^n} \right| &\le E_Q \sup_{t<\tau^n} |X_t^{n_k} - X_t| + E_Q |X_{\tau^n}^{n_k} - X_{\tau^n}| \\ &\le E_Q \sup_{t<\tau^n} |X_t^{n_k} - X_t| + \|X_1^{n_k} - X_1\|_{L^1(Q)}, \end{aligned} \tag{3.19}$$

the last inequality following from the fact that a convex transformation of a martingale is a submartingale. Now

$$\sup_{t<\tau^n} |X_t^{n_k} - X_t| \le n + \|X\|_{\mathscr{H}^\infty(Q)} \qquad \forall\, k,$$

which, combined with (3.17), allows us to apply the dominated convergence theorem to obtain

$$\lim_{k\to\infty} E_Q \sup_{t<\tau^n} |X_t^{n_k} - X_t| = 0.$$

Also,

$$\lim_{k\to\infty} \|X_1^{n_k} - X_1\|_{L^1(Q)} = 0$$

by (3.16) and so letting $k \to \infty$ in (3.19) establishes (3.18) and hence (3.15).

If now the elements of $\mathscr{V}$ are all continuous, the same is true for the elements of $\mathscr{A}$ and when (vi) [equivalent to (v) by Theorem 1] holds, then (3.15) shows that $\mathscr{A}$ is dense in $\mathscr{H}^1(Q)$. Hence all elements of $\mathscr{H}^1(Q)$ are continuous and in particular locally bounded. Therefore,

$$\left\{ \xi \in \mathscr{H}_0^1(Q) \,|\, \xi \perp \{Z^\alpha\}_{\alpha\in A} \right\} \subseteq \left\{ \xi \in \mathscr{H}_0^{\infty,\log}(Q) \,|\, \xi \perp \{Z^\alpha\}_{\alpha\in A} \right\}$$

and in view of (3.14) and (3.3) we obtain that (vi) $\Rightarrow$ (vii).

Finally assume that the family $\mathscr{V}$ is finite, say $\mathscr{V} = \{1\} \cup \{Z^i | i = 1, \dots, N\}$. We show

$$\text{(v)} \Rightarrow \left\{ \xi \in \mathscr{H}_0^1(Q) \,|\, \xi \perp Z^i,\ i = 1, \dots, N \right\} = \{0\}$$

and the last statement is equivalent to (vii) by (3.3). We set $\mathbf{Z} = (Z^1, \dots, Z^N)$. Equation (3.15) tells us that when (v) holds, then $\xi \in \mathscr{H}_0^1(Q)$ may be written as

$$\xi_t = \lim_{n\to\infty} \int_0^t \sum_{i=1}^N H_u^i(n)\, dZ_u^i \quad \text{in } \mathscr{H}^1(Q)$$

for stopping time simple $H_u^i(n)$ ($i = 1, \dots, N$ and $n \in N$). It can be shown that

$$\xi_t = \lim_{n\to\infty} \int_0^t \sum_{i=1}^N H_u^i(n)\, dZ_u^i = \int_0^t \mathbf{H}_u\, d\mathbf{Z}_u \tag{3.20}$$

for a suitable N-dimensional predictable process $\mathbf{H}$; see Theorem 4.60, page 143 of Jacod (1979). Here $\int_0^t \mathbf{H}_u \, d\mathbf{Z}_u$ is the stochastic integral with respect to the multidimensional local Q-martingale $\mathbf{Z}$ as defined in Chapter 4, Section 4, of Jacod (1979) or in Jacod (1980). Using the associativity of the vector stochastic integral and (3.20) we have

$$\begin{aligned} &\left[\int_0^{\cdot} 1_{|\mathbf{H}|\le n}\, d\xi_u, \int_0^{\cdot} 1_{|\mathbf{H}|\le n}\, d\xi_u\right]_t \\ &\quad = \left[\int_0^{\cdot} 1_{|\mathbf{H}|\le n}\, d\xi_u, \int_0^{\cdot} (\mathbf{H}_u 1_{|\mathbf{H}|\le n})\, d\mathbf{Z}_u\right]_t \qquad (3.21)\\ &\quad = \int_0^t 1_{|\mathbf{H}|\le n}\, d\left[\xi, \int_0^{\cdot} (\mathbf{H}_u 1_{|\mathbf{H}|\le n})\, d\mathbf{Z}_u\right]_u. \end{aligned}$$

Now

$$\left[\xi, \int_0^{\cdot} (\mathbf{H}_u 1_{|\mathbf{H}|\le n})\, d\mathbf{Z}_u\right]_t = \int_0^t \left\{ \sum_{i=1}^N H_u^i 1_{|\mathbf{H}|\le n} \left(\frac{d[\xi, Z^i]}{dC}\right)_u \right\} dC_u,$$

where C is any increasing finite variation process with the property that $d[Z^i, Z^j]_j \ll dC_u$ for $i, j = 1, \dots, N$ [Jacod (1980), page 162]. However,

$$\int_0^t H_u^i 1_{|\mathbf{H}|\le n} \left(\frac{d[\xi, Z^i]}{C}\right)_u dC_u = \int_t^0 H_u^i 1_{|\mathbf{H}|\le n}\, d[\xi, Z^i]_u$$

and so (3.21) may be written as

$$\left[\int_0^{\cdot} 1_{|\mathbf{H}|\le n}\, d\xi_u, \int_0^{\cdot} 1_{|\mathbf{H}|\le n}\, d\xi_u\right]_t = \sum_{i=1}^N \int_0^t H_u^i 1_{|\mathbf{H}|\le n}\, d[\xi, Z^i]_u. \qquad (3.22)$$

Since $\xi \perp Z^i$, we have $[\xi, Z^i]_t \in \mathscr{M}_0^{\mathrm{loc}}(Q) \cap \mathsf{A}^{\mathrm{loc}}(Q)$ and the right-hand side of (3.22) is therefore in $\mathscr{M}^{\mathrm{loc}}(Q)$ by Corollary 1.44 of Jacod [(1979), page 20], which implies that

$$\int_0^{\cdot} 1_{|\mathbf{H}|\le n}\, d\xi_u = 0. \qquad (3.23)$$

On the other hand, the dominated convergence theorem for stochastic integrals [Jacod (1979), Proposition 2.73, page 56] implies that the left-hand side of (3.23) converges to ξ in $\mathscr{H}^1(Q)$ and so $\xi = 0$. □

4. Countably many price processes on a countable probability space. In this section we discuss the example of Artzner and Heath (1995), (Example 4.1), which shows that Q-completeness is not generally sufficient for uniqueness of the equivalent local martingale measure from our operator point of view. In the same setting we also give an Artzner–Heath type example in which the price processes are nonnegative (Example 4.2) and finally an example which shows that uniqueness of the equivalent local martingale measure is not generally sufficient (i.e., when there are an infinite number of discontinuous price processes) for our notion of completeness (Example 4.3).

We work with a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_0, \mathscr{F}_1\}, P)$ with Ω countable, $\mathscr{F}_0 = \{\varnothing, \Omega\}$, $\mathscr{F}_1 = \mathscr{F} = \mathscr{P}(\Omega)$ (the power set of Ω) and with $P(\omega) > 0$ $\forall$ $\omega \in \Omega$. For $D \subseteq \Omega$ we denote by 1_D the r.v. which is one on the set D and zero otherwise; $1_D(\omega) = 1$ if $\omega \in D$ and $1_D(\omega) = 0$ if $\omega \notin D$. When $D = \{\omega'\}$ we abuse notation slightly and write $1_{\omega'}$ instead of $1_{\{\omega'\}}$. We consider a family $\mathscr{Z} = \{(Z_t^\alpha)_{t \in (0,1)}\}_{\alpha \in A} \cup \{1\}$ of bounded price processes, where A is countable and since the price processes are bounded we have $\mathfrak{M}_{++}^{\text{loc}} = \mathfrak{M}_{++}$. Here

$$\mathbb{Y} = \left\{ \left(x, (H^\alpha)_{\alpha \in A} \right) \in \mathbf{R} \times \mathbf{R}^A | (H^\alpha)_{\alpha \in A} \text{ has finite support} \right\};$$

$$\mathbb{C} = \ell^\infty(\Omega) = \{ f \colon \Omega \to \mathbf{R} | f \text{ is bounded} \};$$

$$\mathbb{M} = \ell^1(\Omega) = \left\{ f \colon \Omega \to \mathbf{R} | \sum_{\omega \in \Omega} |f(\omega)| < \infty \right\}$$

and $\mathbb{X}$ is the topological dual of $\mathbb{Y}$ where $\mathbb{Y}$ is topologized as in Section 3. Also, for $(x, (H^\alpha)_{\alpha \in A}) \in \mathbb{Y}$ and $\mu \in \mathbb{M}$ we have

$$T\left(x, (H^\alpha)_{\alpha \in A} \right) = x + \sum_{\alpha \in A} H^\alpha \left(Z_1^\alpha - Z_0^\alpha \right),$$

$$\left(T^*(\mu) \right)\left(x, (H^\alpha)_{\alpha \in A} \right) = \mu(\Omega) + \sum_{\alpha \in A} H^\alpha \int \left(Z_1^\alpha - Z_0^\alpha \right) d\mu.$$

The vector spaces $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{C}$ and $\mathbb{M}$ are infinite-dimensional here and we can point out some problems which arise in this context.

EXAMPLE 4.1. This example is due to Artzner and Heath (1995) and we discuss it here from our operator point of view. Let $p \in (0,1)$, $q \in (p, 1)$, set $c = p/(1-p) + q/(1-q)$ and take

$$\Omega = \mathbb{Z} \setminus \{0\};$$

$$A = \mathbb{Z};$$

$$Z_0^i \equiv 1 \quad \text{for } i \in \mathbb{Z};$$

$$Z_1^0(j) = \frac{c}{(p+q)} \left(1_{-1}(j) + 1_1(j) \right);$$

$$Z_1^i(j) = \frac{c\left(q^{i+1} - p^{i+1}\right)}{(pq)^i (q-p)} 1_i(j) + \frac{c\left(p^i - q^i\right)}{(pq)^i (q-p)} 1_{i+1}(j) \quad \text{for } i \in \mathbb{N}, j \in \mathbb{Z} \setminus \{0\};$$

$$Z_1^i(j) = Z_1^{-i}(-j) \quad \text{for } -i \in \mathbb{N},\ j \in \mathbb{Z} \setminus P\{0\}.$$

Here we can explicitly find $\ker T^*$. Indeed, a signed measure μ on $\mathbb{Z} \setminus \{0\}$ is in $\ker T^*$ if and only if $\mu(\Omega) = 0$ and $\int Z_1^i - Z_0^i \, d\mu = 0$, $i \in \mathbb{Z}$. This implies that $\int Z_1^i \, d\mu = 0$, $i \in \mathbb{Z}$ and using the fact that $Z_1^i(j) = Z_1^{-i}(-j)$ we see that μ must solve the following equations:

$$Z_1^0(-1)\mu(-1) + Z_1^0(1)\mu(1) = 0;$$

$$Z_1^i(i)\mu(i) + Z_1^i(i+1)\mu(i+1) = 0, \qquad i \geq 1;$$

$$Z_1^i(i)\mu(-i) + Z_1^i(i+1)\mu(-i-1) = 0, \qquad i \geq 1.$$

On the other hand, any μ satisfying these equations automatically has total mass zero and so $\mu \in \ker T^*$. Note that $\mu^*(i) = -\mu^*(-i) = q^i - p^i$, $i \geq 1$ defines a signed measure solving the above equations and that any other such measure is a scalar multiple of μ^*. Hence

$$\ker T^* = \{\gamma\mu^* | \gamma \in \mathbf{R}\}$$

and by Theorem 1 we do not have completeness. Next observe that $P_0(i) = c^{-1}(p^i 1_{i>0} + q^{-i} 1_{i<0})$ is an element of $\mathfrak{M}_{++}$ and so by Proposition 1 and Theorem 2,

$$\mathfrak{M}_{++} = \mathscr{P}_{+/-} \cap \mathbb{M}_{++} = \{P_0 + \mu | \mu \in \ker T^*\} \cap \mathbb{M}_{++} = \{P_0 + \gamma\mu^* | \gamma \in \Gamma\},$$

where Γ consists of all γ for which the measure $P_0 + \gamma\mu^*$ is strictly positive. We have $\gamma \in \Gamma$ if and only if $c^{-1}/\{1 - (q/p)^i\} < \gamma < c^{-1}/\{1 - (p/q)^i\}$ for $i \geq 1$ and since $c^{-1}/\{1 - (q/p)^i\} \uparrow 0$ and $c^{-1}/\{1 - (p/q)^i\} \downarrow c^{-1}$ we see that $\Gamma = [0, c^{-1}]$. Therefore,

$$\mathfrak{M}_{++} = [P_0, P_1],$$

where P_1 is the measure we get by taking $\gamma = c^{-1}$.

If now $Q \in \mathfrak{M}_{++}$, say $Q = \varepsilon P_0 + (1 - \varepsilon)P_1$ with $\varepsilon \in [, 1]$, then we know from Theorem 1 that Q-completeness is equivalent to injectivity of T^* on $\mathbb{M}^{(Q,\infty)}$. Since $\ker T^* = \{\gamma\mu^* | \gamma \in \mathbf{R}\}$, this is equivalent to saying that

$$\frac{d\mu^*}{dQ}(i) = \frac{(q^i - p^i)1_{i>0} + (p^{-i} - q^{-i})1_{i<0}}{\varepsilon\big(p^i 1_{i>0} + q^{-i}1_{i<0}\big) + (1 - \varepsilon)\big(q^i 1_{i>0} + p^{-i} 1_{i<0}\big)}, \quad i \in \mathbb{Z}\setminus\{0\}$$

is unbounded, which is the case if and only if $\varepsilon = 0$ or 1. In other words, Q-completeness holds if and only if Q is extremal, that is, $Q = P_0$ or P_1. This fact is true in general; see Chapter XI of Jacod (1979).

To summarize, we do not have completeness since $\ker T^*$ is one-dimensional, that is, the space of attainable claims has codimension 1. If, however, one is interested in Q-completeness, then one requires only that $\ker T^*$ not contain any nonzero measures having bounded Radon–Nikodyn derivative with respect to Q and this example illustrates that this may hold for multiple $Q \in \mathfrak{M}_{++}$.

We now consider a slightly different setup and give two more examples. As before, $p \in (0, 1)$ and $q \in (p, 1)$, but now

$$\Omega = \mathbb{N} \cup \{0\};$$
$$A = \mathbb{N};$$
$$Z_0^i \equiv 1 \quad \text{for } i \in \mathbb{N};$$
$$Z_1^i(j) = z(i,i)1_i(j) + z(i, i+1)1_{i+1}(j) \quad \text{for } i \in \mathbb{N}, j \in \mathbb{N} \cup \{0\}.$$

We will take $z(i,i), z(i, i+1) > 0$, $i \geq 1$ and hence the price processes are nonnegative (and bounded). Let $P_0(i) = p^i(1-p)$, $i \in \mathbb{N} \cup \{0\}$ and note that $P_0 \in \mathfrak{M}_{++}$ if and only if

$$z(i,i)p^i(1-p) + z(i,i+1)p^{i+1}(1-p) = 1, \qquad i \geq 1. \tag{4.1}$$

Also, a signed measure μ on $\mathbb{N} \cup \{0\}$ is in $\ker T^*$ if and only if

$$(4.2) \quad \mu(\Omega) = 0 \quad \text{and} \quad z(i,i)\mu(i) + z(i,i+1)\mu(i+1) = 0, \qquad i \geq 1.$$

If μ is any signed measure on $\mathbb{N} \cup \{0\}$ with $\mu(\Omega) = 0$ and $\mu(i-1)\mu(i) < 0$, $i \geq 1$, then (4.1) and (4.2) can be solved uniquely for $z(i,i), z(i,i+1) > 0$. Hence we have nonnegative price processes, $P_0 \in \mathfrak{M}_{++}$ and

$$\ker T^* = \{\gamma\mu | \gamma \in \mathbf{R}\}.$$

Thus the market is incomplete and

$$\mathfrak{M}_{++} = \{P_0 + \gamma\mu | \gamma \in \Gamma\},$$

where Γ consists of all γ for which the measure $P_0 + \gamma\mu$ is strictly positive. The next two examples simply correspond to particular choices of the signed measure μ and we then figure out what $\mathfrak{M}_{++}$ looks like.

EXAMPLE 4.2. This is an Artzner–Heath-type example with nonnegative price processes. We take $\mu(2i) = q^{2i}$, $\mu(2i-1) = -ap^{2i-1}2i/(2i+1)$, $i \geq 1$ and $\mu(0) = -(q^2/(1-q^2) - a\sum_{i \geq 1} p^{2i-1}2i/(2i+1))$ with $a > 0$ chosen large enough to make $\mu(0) > 0$. Then $\gamma \in \Gamma$ if and only if $\gamma > (p-1)/\mu(0)$ and $p^{2i-1}(1-p)/\{ap^{2i-1}2i/(2i+1)\} > \gamma > -p^{2i}(1-p)/q^{2i}$, $i \geq 1$. It follows that $\Gamma = [0, (1-p)/a]$ and therefore,

$$\mathfrak{M}_{++} = [P_0, P_1],$$

where P_1 corresponds to choosing $\gamma = (1-p)/a$. Thus we have an Artzner–Heath-type example with nonnegative price processes; the market is incomplete, but is P_i-complete since $d\mu/dP_i$ is unbounded for $i = 0, 1$.

EXAMPLE 4.3. If we take $\mu(2i) = q^{2i}$, $\mu(2i-1) = -q^{2i-1}$, $i \geq 1$ and $\mu(0) = (q - q^2)/(1-q^2)$, then $\gamma \in \Gamma$ if and only if $\gamma > (p-1)/\mu(0)$ and $(1-p)(p/q)^{2i-1} > \gamma > (p-1)(p/q)^{2i}$, $i \geq 1$. Since $(p/q)^i \to 0$ as $i \to \infty$, it follows that $\Gamma = \{0\}$ and therefore

$$\mathfrak{M}_{++} = \{P_0\}.$$

In other words, there is a unique equivalent martingale measure because as soon as we add a nonzero multiple of μ to P_0 we end up with a nonpositive measure. However, the market is incomplete.

The purpose of the final two sections is to illustrate the usefulness of the orthogonality condition appearing in Theorem 3 by using it to establish the characterizations of completeness given in Theorems 4 and 5.

5. Positive price processes on a Brownian filtration. In this section we use various results from continuous sample path stochastic calculus which can be found in Revuz and Yor (1991). We take as given a probability space $(\Omega, \mathscr{G}, P)$ with a d-dimensional P-Brownian motion $\mathbf{B}_t = (B_t^1, \dots, B_t^d)$ on it and we let $\mathbb{F} = (\mathscr{F}_t)_{t \in [0,1]}$ be the smallest right-continuous P-complete filtration to which $(\mathbf{B}_t)_{t \in [0,1]}$ is adapted. We also set $\mathscr{F} = \mathscr{F}_1$ and we then have the filtered probability space $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t \in [0,1]}, P)$ satisfying all the properties we required in Section 3. Here the price processes are $\mathscr{Z} = \{(Z_t^\alpha)_{t\in[0,1]}\}_{\alpha\in A} \cup \{1\}$, where the Z^α's are strictly positive semimartingales with continuous sample paths.

As was pointed out in Section 3, $\mathbb{F}$ may be considered as a filtration on the time set $\mathbf{R}_+$ by letting $\mathscr{F}_t = \mathscr{F}_1$ for $t \geq 1$ and we then let $Z_t^\alpha = Z_1^\alpha$ for $t \geq 1$. Recall also that a process $(X_t)_{t\in[0,1]}$ is a local martingale if it is a local martingale with respect to $\mathbb{F} = (\mathscr{F}_t)_{t\in\mathbf{R}_+}$ when extended to a process on $\mathbf{R}_+$ by letting $X_t = X_1$ for $t \geq 1$. Equivalently, $(X_t)_{t\in[0,1]}$ is a local martingale if there exists a sequence of stopping times $T_n \uparrow 1$ P-a.s. such that $P(T_n \geq 1) \to 1$. Throughout this section we shall think of $\mathbb{F}$ as a filtration on $\mathbf{R}_+$ and of the Z^α's as processes on $\mathbf{R}_+$. We let $\mathscr{P}$ denote the predictable σ-algebra on $\Omega \times \mathbf{R}_+$ corresponding to $\mathbb{F}$ and consider on $\mathscr{P}$ the Doléans measure M^P associated to the Brownian motion stopped at time 1, that is, $M^P(A) = E_P \int_0^\infty 1_A(\omega, s) 1_{s\leq 1}\, ds = E_P \int_0^1 1_A(\omega, s)\, ds$ for $A \in \mathscr{P}$. We denote by $L^2(M^P)$ the L^2-space on the measure space $(\Omega \times \mathbf{R}_+, \mathscr{P}, M^P)$ and we let $L^0(M^P)$ and $L^\infty(M^P)$ denote, respectively, all $\mathscr{P}$-measurable functions and all bounded $\mathscr{P}$-measurable functions modulo M^P-equivalence. Finally let $L^{2,\mathrm{loc}}(M^P)$ denote the set

$$\left\{H \in L^0(M^P) | \exists \text{ stopping times } T_n \uparrow \infty,\ P\text{-a.s. such that } H_u 1_{u\leq T_n} \in L^2(M^P)\right\}$$

and note that

$$L^{2,\mathrm{loc}}(M^P) = \left\{H \in L^0(M^P) \,\middle|\, \int_0^1 H_u^2\, du < \infty,\ P\text{-a.s.}\right\}.$$

Hence if Q is a P-equivalent probability measure on $\mathscr{F} = \bigvee_{t\in\mathbf{R}_+} \mathscr{F}_t = \mathscr{F}_1$ and we define M^Q analogously to M^P, then $L^{2,\mathrm{loc}}(M^P) = L^{2,\mathrm{loc}}(M^Q)$ and so we shall simply write $L^{2,\mathrm{loc}}$.

First we determine what the positive, continuous semimartingales and local martingales on $\mathbb{F}$ look like. If R is a continuous semimartingale we let $\mathscr{E}(R)_t$ denote the Doléans-Dade exponential of R, that is, $\mathscr{E}(R)_t = \exp(R_t - R_0 - \langle R\rangle_t/2)$. Alternatively, $\mathscr{E}(R)_t$ is the unique solution to $dY_t = Y_t\, dR_t$ with $Y_0 = 1$ from which we see that $\mathscr{E}(R) \in \mathscr{M}^{\mathrm{loc}}(P)$ if and only if $R \in \mathscr{M}^{\mathrm{loc}}(P)$. Observe that if X is a positive, continuous semimartingale then $X_t = X_0 \mathscr{E}(R^X)_t$, where $R_t^X = \int_0^t 1/X_u\, dX_u$ is also a continuous semimartingale, hence decomposes as $R_t^X = A_t^X + M_t^X$ for a continuous adapted finite variation process A^X with $A_0^X = 0$ and $M^X \in \mathscr{M}_0^{\mathrm{loc}}(P)$. Since M^X is constant after time 1 it is also a local martingale on the natural filtration generated by $(\mathbf{B}_t)_{t\in\mathbf{R}_+}$ and from the Brownian representation theorem [Revuz and Yor (1991), Theorem 3.5, page 188], one easily deduces that M^X is of the form $M_t^X = \sum_{i=1}^d \int_0^t \sigma_u^X(i)\, dB_{u\wedge 1}^i$, where $\sigma^X(i) \in L^{2,\mathrm{loc}}$ $(i = 1, \ldots, d)$. To summarize our findings, an arbitrary positive, continuous semimartingale on $\mathbb{F}$ is of the form $X_t = X_0 \mathscr{E}(A^X + \sum_{i=1}^d \int_0^\cdot \sigma_u^X(i)\, dB_{u\wedge 1}^i)_t$, where A^X is a continuous adapted finite variation process with $A_0^X = 0$ and $\sigma^X(i) \in L^{2,\mathrm{loc}}$ $(i = 1, \ldots, d)$. Furthermore, $X \in \mathscr{M}^{\mathrm{loc}}(P)$ if and only if $A^X \equiv 0$. In particular, the price processes Z^α are of the form

$$Z_t^\alpha = Z_0^\alpha \mathscr{E}(R^\alpha)_t, \tag{5.1}$$

where

$$R_t^\alpha = A_t^\alpha + \sum_{i=1}^d \int_0^t \sigma_u^\alpha(i)\, dB_{u\wedge 1}^i, \tag{5.2}$$

$\sigma^\alpha(i) \in \mathrm{L}^{2,\mathrm{loc}}$ $(i = 1,\dots,d)$ and the A^α's are continuous adapted finite variation processes with $A_0^\alpha = 0$, $\alpha \in A$. The process R^α has the economic interpretation as the return on asset α.

Consider next a P-equivalent probability measure Q on $\mathscr{F} = \bigvee_{t\in \mathbf{R}_+} \mathscr{F}_t = \mathscr{F}_1$. The corresponding density process Z^Q is a strictly positive element of $\mathscr{M}(P)$ with $Z_0^Q = 1$ and so the last paragraph implies that it is of the form $Z_t^Q = \mathscr{E}(L^Q)_t$, where

$$L_t^Q = \sum_{i=1}^d \int_0^1 \sigma_u^Q(i)\, dB_{u\wedge 1}^i \tag{5.3}$$

for $\sigma^Q(i) \in \mathrm{L}^{2,\mathrm{loc}}$ $(i = 1,\dots,d)$ and in particular

$$dQ = \mathscr{E}(L^Q)_1\, dP.$$

Let

$$W_t^i = B_{t\wedge 1}^i - \langle B^i, L^Q\rangle_{t\wedge 1}$$

for $i = 1,\dots,d$. It readily follows from Revuz and Yor [(1991), Theorem 1.12, page 306], that $\mathbf{W}_t = (W_t^i)_{i=1}^d$ is $(\mathbb{F}, Q)$-BM stopped at time 1 and (5.2) may be written as

$$R_t^\alpha = A_t^\alpha + \sum_{i=1}^d \int_0^{t\wedge 1} \sigma_u^\alpha(i)\, d\langle B^i, L^Q\rangle_u + \sum_{i=1}^d \int_0^t \sigma_u^\alpha(i)\, dW_u^i. \tag{5.4}$$

We deduce from here that L^Q corresponding to $Q \in \mathfrak{M}_{++}^{\mathrm{loc}}$ satisfies the equation

$$A_t^\alpha + \sum_{i=1}^d \int_0^{t\wedge 1} \sigma_u^\alpha(i)\, d\langle B^i, L^Q\rangle_u \equiv 0$$

and hence (5.4) becomes

$$R_t^\alpha = \sum_{i=1}^d \int_0^t \sigma_u^\alpha(i)\, dW_u^i. \tag{5.5}$$

In the present setting, the orthogonality condition (vii)′ of Theorem 3 can be expressed in terms of the coefficients σ^α used in modelling the Z^α's to obtain the following lemma.

LEMMA 5.1. *Assume $\mathfrak{M}_{++}^{\mathrm{loc}} \neq \varnothing$ and that NA holds. Then the market is complete if and only if*

$$\left\{ (\gamma(i))_{i=1}^d \in \left(L^\infty(M^P)\right)^d \,\middle|\, \sum_{i=1}^d \gamma_u(i)\sigma_u^\alpha(i) = 0,\ M^p\text{-a.e. } \forall\, \alpha \in A \right\} = \{0\}.$$

PROOF OF LEMMA 5.1. We show that

$$\left\{(\gamma(i))_{i=1}^{d} \in \left(L^{\infty}(M^{P})\right)^{d} \,\middle|\, \sum_{i=1}^{d} \gamma_u(i)\,\sigma_u^{\alpha} = 0,\ M^{P}\text{-a.e. } \forall\, \alpha \in A\right\} = \{0\}$$

is equivalent to (vii)′ of Theorem 3. Let $Q \in \mathfrak{M}_{++}^{\text{loc}}$ and let L^Q be as in (5.3). In particular, L^Q is continuous and $dP = \mathscr{E}(-L^Q + \langle L^Q\rangle)_1\, dQ$. Hence

$$\eta \in \mathscr{M}^{\text{loc}}(P) \Leftrightarrow \eta\mathscr{E}(-L^Q + \langle L^Q\rangle) \in \mathscr{M}^{\text{loc}}(Q)$$

and so

$$\mathscr{M}^{\text{loc}}(Q) = \left\{\eta\mathscr{E}(-L^Q + \langle L^Q\rangle) | \eta \in \mathscr{M}^{\text{loc}}(P)\right\}. \tag{5.6}$$

It follows from (5.3) that $\mathscr{E}(-L^Q + \langle L^Q\rangle)$ is continuous and since all the elements of $\mathscr{M}^{\text{loc}}(P)$ are also continuous (having an integral representation) we conclude from (5.6) that all the elements of $\mathscr{M}^{\text{loc}}(Q)$ are continuous. Hence if $\xi \in \mathscr{M}_0^{\text{loc}}(Q)$ we can apply Girsanov's theorem [Revuz and Yor (1991), Theorem 1.7, page 305] to get that

$$\mathscr{E} - \langle \xi, -L^Q + \langle L^Q\rangle\rangle \in \mathscr{M}_0^{\text{loc}}(P).$$

Using the Brownian representation theorem [Revuz and Yor (1991), Theorem 3.5, page 188] we can then find $\sigma^{\xi}(i) \in L^{2,\text{loc}}$ $(i = 1,\dots,d)$ such that

$$\begin{aligned}\xi_t - \langle \xi, -L^Q + \langle L^Q\rangle\rangle_t &= \sum_{i=1}^{d} \int_0^t \sigma_u^{\xi}\, dB_{u\wedge 1}^{i}\\ &= \sum_{i=1}^{d} \int_0^t \sigma_u^{\xi}(i)\, dW_u^{i} + \sum_{i=1}^{d} \int_0^t \sigma_u^{\xi}(i)\, d\langle B^i, L^Q\rangle_u.\end{aligned}$$

Hence

$$\xi_t - \sum_{i=1}^{d} \int_0^t \sigma_u^{\xi}(i)\, dW_u^{i} = \langle \xi, -L^Q + \langle L^Q\rangle\rangle_t + \sum_{i=1}^{d} \int_0^t \sigma_u^{\xi}(i)\, d\langle B^i, L^Q\rangle_u$$

and from here we see that $\xi_t - \sum_{i=1}^{d} \int_0^t \sigma_u^{\xi}(i)\, dW_u^{i}$ is a predictable (continuous in fact) finite variation process in $\mathscr{M}_0^{\text{loc}}(Q)$ and so is identically zero. Therefore

$$\xi_t = \sum_{i=1}^{d} \int_0^t \sigma_u^{\xi}(i)\, dW_u^{i}. \tag{5.7}$$

To summarize, we have shown that all the elements of $\mathscr{M}_0^{\text{loc}}(Q)$ are continuous and have a predictable integral representation with respect to $\mathbf{W} = (W^i)_{i=1}^{d}$. It should be noted that $\mathscr{M}_0^{\text{loc}}(Q)$ refers to the filtration $\mathbb{F}$ and this filtration can be strictly larger than the natural filtration generated by $\mathbf{W} = (W^i)_{i=1}^{d}$.

Since all the elements of $\mathscr{M}_0^{\text{loc}}(Q)$ are continuous, we have $\mathscr{M}_0^{\text{loc}}(Q) \subseteq \mathscr{H}_0^{2,\text{loc}}(Q)$ and it follows that (vii)′ of Theorem 3 is equivalent to

$$\left\{\xi \in \mathscr{H}_0^{2}(Q) \,\middle|\, [\xi, Z^{\alpha}] \in \mathsf{A}^{\text{loc}}(Q) \text{ and } [\xi, Z^{\alpha}]^{p} = 0\ \forall\, \alpha \in A\right\} = \{0\}. \tag{5.8}$$

From (5.7) we obtain

$$(5.9)\quad \mathscr{H}_0^2(Q) = \left\{ \sum_{i=1}^{d} \int_0^t \sigma_u^{\xi}(i)\, dW_u^i \,\middle|\, \sigma_u^{\xi}(i) \in L^2(M^Q)\ (i = 1, \dots, d) \right\}.$$

Also,

$$(5.10)\qquad \xi = 0 \quad \Leftrightarrow \quad \sigma_u^{\xi}(i) = 0, \qquad i = 1, \dots, d.$$

Consider now $\xi \in \mathscr{H}_0^2(Q)$, say, $\xi_t = \sum_{i=1}^d \int_0^t \sigma_u^{\xi}(i)\, dW_u^i$. Since ξ is continuous, the same is true for $[\xi, Z^{\alpha}]$. Hence $[\xi, Z^{\alpha}] \in \mathsf{A}^{\mathrm{loc}}(Q)$ and using (5.1) and (5.5) we get

$$\begin{aligned}
[\xi, Z^{\alpha}]_t^p &= \langle \xi, Z^{\alpha} \rangle_t \\
&= \left\langle \sum_{i=1}^{d} \int_0^{\cdot} \sigma_u^{\xi}(i)\, dW_u^i, Z_0^{\alpha} + \int_0^{\cdot} Z_u^{\alpha}\, dR^{\alpha} \right\rangle_t \\
&= \left\langle \sum_{i=1}^{d} \int_0^{\cdot} \sigma_u^{\xi}(i)\, dW_u^i, \int_0^{\cdot} Z_u^{\alpha} \left(\sum_{i=1}^{d} \sigma_u^{\alpha}(i) \right) dW_u^i \right\rangle_t \\
&= \int_0^t Z_u^{\alpha} \left(\sum_{i=1}^{d} \sigma_u^{\xi}(i)\, \sigma_u^{\alpha}(i) \right) du.
\end{aligned}$$

Therefore,

$$(5.11)\quad \begin{aligned}
[\xi, Z^{\alpha}]_t^p = 0 \quad &\Leftrightarrow \quad Z_t^{\alpha} \left(\sum_{i=1}^{d} \sigma_t^{\xi}(i)\, \sigma_t^{\alpha}(i) \right) = 0, \quad M^Q\text{-a.e.} \\
&\Leftrightarrow \quad \sum_{i=1}^{d} \sigma_t^{\xi}(i)\, \sigma_t^{\alpha}(i) = 0, \quad M^Q\text{-a.e.}
\end{aligned}$$

From (5.9), (5.10) and (5.11) it follows that the condition in (5.8) which we know to be equivalent to completeness holds if and only if

$$(5.12)\quad \begin{aligned}
&\left\{ \left(\sigma^{\xi}(i)\right)_{i=1}^d \in \left(L^2(M^Q)\right)^d \,\middle|\, \sum_{i=1}^{d} \sigma_u^{\xi}(i)\, \sigma_u^{\alpha}(i) = 0, \right. \\
&\qquad\qquad \left. M^Q\text{-a.e. } \forall\, \alpha \in A \right\} = \{0\}.
\end{aligned}$$

Replacing $(\sigma^{\xi}(i))_{i=1}^d \in (L^2(M^Q))^d \setminus \{0\}$ by

$$\left(\frac{\sigma^{\xi}(i)}{\left\| \left(\sigma^{\xi}(i)\right)_{i=1}^d \right\|} 1_{\|(\sigma^{\xi}(i))_{i=1}^d\| > 0} \right)_{i=1}^{d} \in \left(L^{\infty}(M^Q)\right)^d \setminus \{0\},$$

we see that the condition in (5.12) is equivalent to

$$\left\{ \left(\sigma^{\xi}(i)\right)_{i=1}^d \in \left(L^{\infty}(M^Q)\right)^d \,\middle|\, \sum_{i=1}^{d} \sigma_u^{\xi}(i)\, \sigma_u^{\alpha}(i) = 0,\ M^Q\text{-a.e. } \forall\, \alpha \in A \right\} = \{0\}$$

or finally to

$$\left\{ \left(\sigma^{\xi}(i) \right)_{i=1}^{d} \in \left(L^{\infty}(M^P) \right)^d \;\middle|\; \sum_{i=1}^{d} \sigma_u^{\xi}(i) \sigma_u^{\alpha}(i) = 0,\ M^P\text{-a.e. } \forall\, \alpha \in A \right\} = \{0\}$$

since $M^P \sim M^Q$. □

Finally, suppose that $|A| < \infty$ and let Σ_t denote the predictable $|A| \times d$-matrix process whose row vectors are $(\sigma_t^{\alpha}(i))_{i=1}^{d}$, $\alpha \in A$. We will sometimes write $\Sigma(\omega, t)$ in place of Σ_t if it is necessary to emphasize the dependence of Σ on ω and t. Here the condition of Lemma 5.1 takes the form

$$\left\{ (\gamma(i))_{i=1}^{d} \in \left(L^{\infty}(M^P) \right)^d \;\middle|\; \Sigma_t (\gamma_t(i))_{i=1}^{d} = 0 \left(\text{in} \left(L^0(M^P) \right)^{|A|} \right) \right\} = \{0\}. \tag{5.13}$$

Clearly this should be equivalent to "rank $\Sigma_t = d$ M^P-a.e." That is the content of Theorem 4 which we now prove.

Proof of Theorem 4. If the rank condition holds then for M^P-a.e. $(\omega, u) \in \Omega \times \mathbf{R}_+$, $\Sigma^t(\omega, u)$: $\mathbf{R}^{|A|} \to \mathbf{R}^d$ is surjective or equivalently $\Sigma(\omega, \mathrm{Iu})$: $\mathbf{R}^d \to \mathbf{R}^{|A|}$ is injective. But then the condition in (5.13), and hence completeness, holds. We now suppose that the rank condition fails, that is, rank $\Sigma_t = d$ M^P-a.e. is false. Intuitively, one can then choose on a set of positive M^P-measure unit vectors which are orthogonal to the row vectors of Σ_t and we simply have to make sure that the vectors can be chosen so as to "piece together" to a predictable ($\mathbf{R}^d$-valued) process to conclude that (5.13) fails. To this end we use the section theorem from the general theory of stochastic processes. To apply it, we define

$$D = \left\{ (\omega, u, \mathbf{x}) \in \Omega \times \mathbf{R}_+ \times \mathbf{R}^d \;\middle|\; \|\mathbf{x}\| = 1 \text{ and } \mathbf{x} \in \left[\operatorname{Im}\left(\Sigma^t(\omega, u) \right) \right]^{\perp} \right\}.$$

Let Π_i: $\mathbf{R}^d \to \mathbf{R}$ denote the projection onto the ith coordinate and that Π_i can be considered as a $\mathscr{P} \times \mathscr{B}(\mathbf{R}^d)$-measurable function ($i = 1, \dots, d$). Since $\sigma^{\alpha}(i)$ is predictable, it can also be considered as a $\mathscr{P} \times \mathscr{B}(\mathbf{R}^d)$-measurable function ($i = 1, \dots, d$; $\alpha \in A$). Writing

$$\begin{aligned} D = {} & \left\{ (\omega, u, \mathbf{x}) \in \Omega \times \mathbf{R}_+ \times \mathbf{R}^d \;\middle|\; \sum_{i=1}^{d} (\Pi_i(\mathbf{x}))^2 = 1 \right\} \\ & \cap \left\{ \bigcap_{\alpha \in A} \left\{ (\omega, u, \mathbf{x}) \in \Omega \times \mathbf{R}_+ \times \mathbf{R}^d \;\middle|\; \sum_{i=1}^{d} \Pi_i(\mathbf{x}) \sigma^{\alpha}(i)(\omega, u) = 0 \right\} \right\}, \end{aligned}$$

we see that $D \in \mathscr{P} \times \mathscr{B}(\mathbf{R}^d)$. If we let Π: $\Omega \times \mathbf{R}_+ \times \mathbf{R}^d \to \Omega \times \mathbf{R}_+$ denote the projection which "forgets" the $\mathbf{R}^d$-coordinate then we have

$$\Pi(D) = \left\{ (\omega, u) \in \Omega \times \mathbf{R}_+ \;\middle|\; \text{rank } \Sigma(\omega, u) < d \right\}.$$

The section theorem [Dellacherie and Meyer (1975), Section 84, pages 219 and 220, Vol. A] implies that $\Pi(D)$ belongs to the (universal) completion of $\mathscr{P}$ and since the rank condition fails, $M^P(\Pi(D)) > 0$. The section theorem further gives the existence of a $\mathscr{P}$-measurable map ζ taking values in $\mathbf{R}^d \cup \{\infty\}$ with the property that

$$M^P(\zeta \in \mathbf{R}^d) = M^P(\Pi(D)) > 0$$

and

$$(\omega, u, \zeta(\omega, u)) \in D \quad \text{whenever } \zeta(\omega, u) \in \mathbf{R}^d.$$

It follows that $\zeta 1_{\zeta \in \mathbf{R}^d}$ is a nonzero element of

$$\left\{ (\gamma(i))_{i=1}^d \in \left(L^\infty(M^P)\right)^d \middle| \sum_{i=1}^d \gamma_t(i) \sigma_t^\alpha(i) = 0,\ M^P\text{-a.e. } \forall\, \alpha \in A \right\}$$
$$= \left\{ (\gamma(i))_{i=1}^d \in \left(L^\infty(M^P)\right)^d \middle| \Sigma_t(\gamma_t(i))_{i=1}^d = 0 \left(\text{in } \left(L^0(M^P)\right)^{|A|}\right) \right\}.$$

Hence the condition in (5.13) fails and we do not have completeness. □

REMARK 5.1. If $Q \sim P$ then $M^Q \sim M^P$ and consequently the statements of Lemma 5.1 and Theorem 4 are invariant under change to an equivalent probability measure.

6. Price processes driven by a multivariate point process. The basic reference furnishing the probability background for this section is Jacod and Shiryaev (1987). We take as given a probability space $(\Omega, \mathscr{G}, P)$ and a Polish space $(E, \mathscr{E})$. For the most part E will actually be finite and $\mathscr{E}$ the power set of E. First we need a definition.

DEFINITION 5. An E-valued multivariate process μ is a family of positive measures on the space $(\mathbf{R}_+ \times E, \mathscr{B}(\mathbf{R}_+) \times \mathscr{E})$ indexed by $\omega \in \Omega$ and satisfying:

(i) $\mu(\omega; A) \in \mathbb{N} \cup \{\infty\}$, $A \in \mathscr{B}(\mathbf{R}_+) \times \mathscr{E}$;
(ii) $\mu(\omega; \{t\} \times E) \le 1$, $t \in \mathbf{R}_+$;
(iii) $\mu(\omega; \{0\} \times E) = 0$;
(iv) $\mu(\omega; [0, t] \times E) < \infty$, $t \in \mathbf{R}_+$.

For $n \in \mathbb{N}$, define $T_n(\omega) = \inf\{t \mid \mu(\omega; [0, t] \times E) \ge n\}$. Then $T_n \uparrow \infty$ and $T_n < T_{n+1}$ on $\{T_n < \infty\}$. Let $D_\mu = \{(\omega, t) \mid \mu(\omega; \{t\} \times E) = 1\} = \{(\omega, t) \mid \exists\ n \text{ s.t. } t = T_n(\omega)\}$ and note that there exist $X_n\colon \Omega \to E$ such that

$$\mu(\omega; dt, dx) = \sum_n 1_{T_n(\omega) < \infty} \delta_{(T_n(\omega), X_n(\omega))}(dt, dx).$$

We let $\mathbb{G} = (\mathscr{G}_t)_{t \in \mathbf{R}_+}$ be the natural filtration generated by μ, that is, $\mathscr{G}_t = \mathscr{N}^P \vee \sigma(\mu([0, s] \times B) | s \le t,\ B \in \mathscr{E})$, where $\mathscr{N}^P$ denotes the null sets of P. Considered on this filtration, μ is a multivariate point process in the sense of Jacod and Shiryaev (1987) and their assumption 1.25, page 135, holds.

Since our time interval is $[0, 1]$, we work with the filtration $\mathbb{F} = (\mathscr{F}_t)_{t \in [0,1]}$ defined by $\mathscr{F}_t = \mathscr{G}_t$ for $t \le 1$ and we consider it as a filtration on the time interval $\mathbf{R}_+$ by setting $\mathscr{F}_t = \mathscr{F}_1$ for $t \ge 1$. The filtration $\mathbb{G}$ is right continuous [see Remark 1.31, page 136 of Jacod and Shiryaev (1987)] and P-complete, hence the same is true for the filtration $\mathbb{F}$. We shall refer to $\mathbb{F}$ as the filtration generated by the multivariate point process μ stopped at time 1. Unless explicitly mentioned otherwise, the term local martingale will refer to the filtration $\mathbb{F}$. We set $\mathscr{F} = \bigvee_{t \in \mathbf{R}_+} \mathscr{F}_t = \mathscr{F}_1$. We let $\mathscr{P}$ and $\mathscr{P}(\mathbb{G})$ denote the predictable σ-algebra on $\Omega \times \mathbf{R}_+$ corresponding to the filtrations $\mathbb{F}$ and $\mathbb{G}$. On $\Omega \times \mathbf{R}_+ \times E$ we consider the σ-algebras $\tilde{\mathscr{P}} = \mathscr{P} \times \mathscr{E}$ and $\tilde{\mathscr{P}}(\mathbb{G}) = \mathscr{P}(\mathbb{G}) \times \mathscr{E}$. Clearly $\mathscr{P} \subseteq \mathscr{P}(\mathbb{G})$ and $\tilde{\mathscr{P}} \subseteq \tilde{\mathscr{P}}(\mathbb{G})$ and on $\tilde{\mathscr{P}}(\mathbb{G})$ the Doléans measure M_μ^P of μ is defined by $M_\mu^P = E_P \int_{[0,\infty)\times E} 1_A(\omega, u, x)\mu(\omega; du, dx)$, $A \in \tilde{\mathscr{P}}(\mathbb{G})$. We let ν denote the P-compensator (or dual predictable projection) of the optional and $\tilde{\mathscr{P}}(\mathbb{G})$-$\sigma$-finite random measure μ and M_ν^P the associated Doléans measure. Random measures and their Doléans measures are defined on $\tilde{\mathscr{P}}(\mathbb{G})$, although we will consider them only as measures on $\tilde{\mathscr{P}} \subseteq \tilde{\mathscr{P}}(\mathbb{G})$ and representations of measures which follow below implicitly refer to the σ-algebra $\tilde{\mathscr{P}}$. If no confusion can arise, we henceforth suppress the dependence of processes, $\tilde{\mathscr{P}}(\mathbb{G})$-measurable functions and random measures on ω.

So far we have a general filtered probability space $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t \in \mathbf{R}_+}, P)$ with the filtration generated by the multivariate point process μ stopped at time 1. We consider a family of price processes $\mathscr{V} = \{(Z_t^\alpha)_{t \in [0,1]}\}_{\alpha \in A} \cup \{1\}$ and the following assumptions are made:

1. The Z^α's defined by $Z_t^\alpha = Z_1^\alpha$ for $t \geq 1$ are semimartingales on the filtration $\mathbb{F}$. Furthermore, having the interpretation of price processes, they are assumed to be positive. More precisely, P-a.s. $Z_t^\alpha > 0$ and $Z_{t-}^\alpha > 0$ for all $t \in \mathbf{R}_+$. As in Section 5, if R is a semimartingale then $\mathscr{E}(R)_t$ denotes the Doléans-Dade exponential, that is, the unique solution to $dY_t = Y_{t-}\,dt$ with $Y_0 = 1$, but now R is not continuous. Because of the positivity of Z^α, one may write $Z_t^\alpha = Z_0^\alpha \mathscr{E}(R^\alpha)_t$, where $R_t^\alpha = \int_0^t 1/Z_{u-}^\alpha\, dZ_u^\alpha$ has the interpretation as the return process on asset α.
2. $(\Delta R^\alpha)_t$ $(= (\Delta Z^\alpha)_t / Z_{t-}^\alpha)$ is bounded, say, by $C^\alpha \in \mathbf{R}$. In particular, R^α is a special semimartingale which means that the finite variation part in the semimartingale decomposition can be chosen (uniquely) as a predictable process. We let $R_t^\alpha = A_t^\alpha + M_t^\alpha$, where $M^\alpha \in \mathscr{M}_0^{\mathrm{loc}}(P)$ and A^α is predictable of finite variation with $A_0^\alpha = 0$, be the canonical decomposition.
3. The drifts A^α of R^α have bounded jumps, say, $|(\Delta A^\alpha)_t| \leq D^\alpha$ for $D^\alpha \in \mathbf{R}$.
4. The compensator ν of μ is continuous in the sense that $\nu(dt \times E) \ll dt$, P-a.s.

In assumption (1), if $\mathfrak{M}_{++}^{\mathrm{loc}} \neq \emptyset$, then $Z_{t-}^\alpha > 0$ follows automatically from $Z_t^\alpha > 0$. Assumption (2) states that for any given asset, the price jump as a percentage of the prejump price is bounded. Note that the strict positivity of Z^α already implies that $(\Delta R^\alpha)_t > -1$. Assumption (2) also ensures that the Z^α are locally bounded. Assumption (4) implies that we can find a $\mathscr{P}$-measurable function ϕ such that $\nu(dt \times E) = \phi_t\, dt$ and since $(E, \mathscr{E})$ is Polish (in particular, Blackwell), we can disintegrate [Jacod and Shiryaev (1987), 1.2, page 65] ν as $\nu(dt, dx) = \alpha(t; dx)\nu(dt \times E)$, where $\alpha(t; dx)$ is a transition kernel from $(\Omega \times \mathbf{R}_+, \mathscr{P})$ into $(E, \mathscr{E})$. Setting $K_t(dx) = \alpha(t; dx)\phi_t$, we can write $\nu(dt, dx) = K_t(dx)\,dt$. This is the representation of ν which will be the most useful for our purposes. Finally, if the local martingale parts M^α of R^α are locally square integrable and one assumes the existence of an equivalent local martingale measure with locally square integrable density process, then the drifts A^α are in fact absolutely continuous so that (3) automatically holds.

As was pointed out in Section 3, the filtration $\mathbb{F}$ (being constant after time 1) has the property that all local martingales are constant after time 1. Hence the M^α's are also $(\mathbb{G}, P)$-local martingales and we easily deduce from Theorem 4.37 of Jacod and Shiryaev [(1987), page 177], that there exist $\tilde{\mathscr{P}}$-measurable σ^α such that

$$M_t^\alpha = \int_{[0,t]\times E} \sigma^\alpha(u, x)\{\mu(du, dx) - \nu(du, dx)\}$$

for

$$\int_{[0,t]\times E} |\sigma^\alpha(u, x)|\mu(du, dx) \in \mathsf{A}^{\mathrm{loc}}(P)$$

or equivalently

$$\int_{[0,t]\times E} |\sigma^\alpha(u, x)|\nu(du, dx) \in \mathsf{A}^{\mathrm{loc}}(P).$$

The continuity assumption (4) on ν implies that $\Delta M^\alpha(\omega, t) \neq 0$ only if $(\omega, t) \in D_\mu$, that is, $t = T_n(\omega)$ for some n in which case it equals $\sigma^\alpha(\omega, T_n(\omega), X_n(\omega))$. Since

$$|(\Delta M^\alpha)_t| \le |(\Delta A^\alpha)_t| + |(\Delta R^\alpha)_t| \le C^\alpha + D^\alpha,$$

it follows that $|\sigma^\alpha(\omega, t, x)| \le C^\alpha + D^\alpha$ M_μ^P-a.e. In summary, our price processes are of the form

$$Z_t^\alpha = Z_0^\alpha \mathscr{E}(R^\alpha)_t,$$

where

$$R_t^\alpha = A_t^\alpha + \int_{[0,t]\times E} \sigma^\alpha(u, x)\{\mu(du, dx) - \nu(du, dx)\} \tag{6.1}$$

and

$$|\sigma^\alpha(\omega, t, x)| \le C^\alpha + D^\alpha, \quad M_\mu^P\text{-a.e.} \tag{6.2}$$

Let now Q be a probability measure on $\mathscr{F} = \bigvee_{t\in\mathbf{R}_+}\mathscr{F}_t = \mathscr{F}_1$ which is equivalent to P and let Z^Q denote the associated density process. We let Y^Q be the M_μ^P-a.e. unique $\tilde{\mathscr{P}}$-measurable function such that

$$E_P \int_{[0,\infty)\times E} W(u, x) Y^Q(u, z)\mu(du, dx) = E_P \int_{[0,\infty)\times E} W(u, x) \frac{Z_u^Q}{Z_{u-}^Q}\mu(du, dx)$$

for all nonnegative $\tilde{\mathscr{P}}$-measurable functions W. Then $Y^Q > 0$ M_μ^P-a.e. since Z^Q is a strictly positive element of $\mathscr{M}(P)$ and Girsanov's theorem for random measures [see, e.g., Theorem 3.17, page 157 of Jacod and Shiryaev (1987)] tells us that

$$\nu^Q(dt, dx) = Y^Q(t, x)\nu(dt, dx) = K_t^Q(dx)\,dt, \tag{6.3}$$

where $K_t^Q(dx) = Y^Q(t, x)K_t(dx)$, is a version of the Q-compensator of μ. Next note that $Q \in \mathfrak{M}_{++}^{\mathrm{loc}}$ if and only if $R^\alpha \in \mathscr{M}^{\mathrm{loc}}(Q)$ for all $\alpha \in A$ and if $Q \in \mathfrak{M}_{++}^{\mathrm{loc}}$ then $\int_{[0,t]\times E} |\sigma^\alpha(u, x)|\, \mu(du, dx) \in \mathsf{A}^{\mathrm{loc}}(Q)$ in view of (6.2) which allows us to obtain

$$\begin{aligned} R_t^\alpha - \int_{[0,t]\times E} & \sigma^\alpha(u, x)\{\mu(du, dx) - \nu^Q(du, dx)\} \\ &= A_t^\alpha + \int_{[0,t]\times E} \sigma^\alpha(u, x)\{Y^Q(u, x) - 1\}\nu(du, dx) \end{aligned}$$

from (6.1). But then $A_t^\alpha + \int_{[0,t]\times E} \sigma^\alpha(u, x)\{Y^Q(u, x) - 1\}\nu(du, dx)$ is a predictable element of $\mathscr{M}_0^{\mathrm{loc}}(Q)$, hence identically zero. Thus for $Q \in \mathfrak{M}_{++}^{\mathrm{loc}}$, Y^Q satisfies

$$A_t^\alpha = \int_{[0,t]\times E} \sigma^\alpha(u, x)\{1 - Y^Q(u, x)\}\nu(du, dx) \tag{6.4}$$

and under Q we have the representation

$$R_t^\alpha = \int_{[0,t]\times E} \sigma^\alpha(u, x)\{\mu(du, dx) - \nu^Q(du, dx)\}. \tag{6.5}$$

As in Section 5, we let M^P (no subscript) denote the Doléans measure on $\mathscr{P}$ associated with Brownian motion stopped at time 1, that is, $M^P(A) = E_P \int_0^1 1_A(\omega, u)\, du$, $A \in \mathscr{P}$ and recall that the Q-compensator ν^Q of μ is given by (6.3). The orthogonality condition, which by Theorem 3 is equivalent to completeness, can be used here to obtain the following characterization of completeness.

LEMMA 6.1. *Assume that NA holds and the* $\mathfrak{M}_{++}^{\mathrm{loc}} \neq \varnothing$, *say*, $Q \in \mathfrak{M}_{++}^{\mathrm{loc}}$. *Then the market is complete if and only if*

$$\left\{\gamma \in \mathscr{T} \,\middle|\, \int_E \gamma(t, x)\sigma^\alpha(t, x)K_t^Q(dx) = 0,\ M^P\text{-a.e. } \forall\, \alpha \in A\right\} = \{0\},$$

where $\mathscr{T} = \{\gamma \mid \gamma(t, x)$ *is* $\tilde{\mathscr{P}}$*-measurable s.t.* $\int_{[0,t]\times E} |\gamma(t, x)|\, \mu(dt, dx) \in \mathsf{A}^{\mathrm{loc}}(Q)\}$.

PROOF OF LEMMA 6.1. As we pointed out after listing the assumptions on the price processes, the Z^α's are locally bounded and so with the hypotheses of the Lemma, Theorem 3 and a localization argument show that completeness is equivalent to

$$\left\{\xi \in \mathscr{M}_0^{\mathrm{loc}}(Q) \,\middle|\, [\xi, Z^\alpha] \in \mathsf{A}^{\mathrm{loc}}(Q) \text{ and } [\xi, Z^\alpha]^p = 0\ \forall\, \alpha \in A\right\} = \{0\}. \tag{6.6}$$

By Theorem 4.37 of Jacod and Shiryaev [(1987), page 177] we have

$$\mathscr{M}_0^{\text{loc}}(Q) = \left\{ \int_{[0,\cdot]\times E} \gamma(u,x)\{\mu(du,dx) - \nu^Q(du,dx)\} \,\middle|\, \gamma \in \mathscr{T} \right\}. \tag{6.7}$$

If $\xi \in \mathscr{M}_0^{\text{loc}}(Q)$, say,

$$\xi_t = \int_{[0,t]\times E} \gamma^{\xi}(u,x)\{\mu(du,dx) - \nu^Q(du,dx)\} \tag{6.8}$$

for $\gamma^{\xi} \in \mathscr{T}$, then

$$[\xi, Z^{\alpha}]_t = \sum_{u\le t} (\Delta\xi)_u (\Delta Z^{\alpha})_u = \sum_{u \le t} (\Delta \xi)_u (\Delta R^{\alpha})_u Z^{\alpha}_{u-}.$$

In view of (6.5) and (6.8) and since $\nu^Q(dt \times E) \ll dt$ P-a.s., $\Delta\xi(\omega, u)$ and $\Delta R^{\alpha}(\omega, u)$ are nonzero only if $(\omega, u) \in D_{\mu}$, that is, $u = T_n(\omega)$ for some n in which case they are respectively equal to $\gamma^{\xi}(\omega, T_n(\omega), X_n(\omega))$ and $\sigma^{\alpha}(\omega, T_n(\omega), X_n(\omega))$. Thus

$$\begin{aligned}[\xi, Z^{\alpha}]_t &= \sum_{T_n \le t} \gamma^{\xi}(T_n, X_n)\sigma^{\alpha}(T_n, X_n) Z^{\alpha}_{T_n -} \\ &= \int_{[0,t]\times E} \gamma^{\xi}(u,x)\sigma^{\alpha}(u,x) Z^{\alpha}_{u-}\,\mu(du,dx)\end{aligned}$$

and this process is in $\mathsf{A}^{\text{loc}}(Q)$ because of (6.2), the fact that Z^{α}_{u-} is locally bounded and $\gamma^{\xi} \in \mathscr{T}$. Therefore

$$\begin{aligned}[\xi, Z^{\alpha}]^p_t &= \int_{[0,t]\times E} \gamma^{\xi}(u,x)\sigma^{\alpha}(u,x) Z^{\alpha}_{u-}\,\nu^Q(du,dx) \\ &= \int_0^t \int_E \gamma^{\xi}(u,x)\sigma^{\alpha}(u,x) Z^{\alpha}_{u-} K^Q_u(dx)\,du\end{aligned}$$

and so

$$\begin{aligned}[\xi, Z^{\alpha}]^p = 0 \quad &\Leftrightarrow \quad \int_E \gamma^{\xi}(u,x)\sigma^{\alpha}(u,x) Z^{\alpha}_{u-} K^Q_u(dx) \\ &\qquad = 0, \quad M^P\text{-a.e.} \\ &\Leftrightarrow \quad \int_E \gamma^{\xi}(u,x)\sigma^{\alpha}(u,x) K^Q_u(dx) \\ &\qquad = 0, \quad M^P\text{-a.e.}\end{aligned} \tag{6.9}$$

using $Z^{\alpha}_{u-} > 0$ for the last equivalence. Finally,

$$\begin{aligned}\xi = 0 \quad &\Leftrightarrow \quad [\xi, \xi]_{\infty} = \sum_n \left(\gamma^{\xi}(T_n, X_n)\right)^2 = 0, \quad P\text{-a.s.} \\ &\Leftrightarrow \quad \gamma^{\xi} = 0, \quad M^P_{\mu}\text{-a.e.}\end{aligned} \tag{6.10}$$

and it follows from (6.7), (6.9) and (6.10) that the statement in (6.6), and hence completeness, is equivalent to

$$\left\{ \gamma \in \mathscr{T} \,\middle|\, \int_E \gamma(u,x)\sigma^{\alpha}(u,x) K^Q_u(dx) = 0,\ M^P\text{-a.e. } \forall\, \alpha \in A \right\} = \{0\}. \qquad \square$$

Finally we suppose that $|A| + |E| < \infty$. In this case, if γ is a $\tilde{\mathscr{P}}$-measurable function then for fixed ω and t, γ may be viewed as an $|E|$-dimensional vector. When we want to emphasize this view of γ we will write $(\gamma_t(x))_{x \in E}$, as always suppressing the dependence on ω. We let Σ_t denote the predictable $|A| \times |E|$-matrix process whose row vectors are $(\sigma_t{}^{\alpha})(x))_{x \in E}$, $\alpha \in A$. Then Σ_t may be interpreted as the volatility matrix of the risky assets. Theorem 5 asserts that completeness in this setting is equivalent to a rank condition on Σ_t and we now supply the proof.

PROOF OF THEOREM 5. Let $Q \in \mathfrak{M}^{\mathrm{loc}}_{++}$ and from (6.3) we have $K_t^Q(\{x\}) = Y_t^Q(x)K_t(\{x\})$. Now $Y_t^Q > 0$ M_μ^P-a.e. is equivalent to $Y_t^Q > 0$ M_ν^P-a.e. The continuity assumption on ν and the fact that M^P-a.e. $K_t(\{x\}) > 0$ $\forall$ $x \in E$ imply that M^P-a.e. $Y_t^Q(x) > 0$ $\forall$ $x \in E$, so we also have that M^P-a.e. $K_t^Q(\{x\}) > 0$ $\forall$ $x \in E$. Since $|A| + |E| < \infty$, the condition of Lemma 6.1 takes the form

$$\left\{\gamma \in \mathscr{T} \,\middle|\, \sum_{x \in E} \gamma_t(x)\sigma_t{}^{\alpha}(x)K_t^Q(\{x\}) = 0 \ M^P\text{-a.e.}\ \forall\, \alpha \in A\right\} = \{0\}.$$

If $\gamma \in \mathscr{T} \setminus \{0\}$ satisfies

$$\sum_{x \in E} \gamma_t(x)\sigma_t{}^{\alpha}(x)K_t^Q(\{x\}) = 0, \quad M^P\text{-a.e.}\ \forall\, \alpha \in A,$$

then

$$(\tilde{\gamma}_t(x))_{x \in E} = \left(\frac{\gamma_t(x)K_t^Q(\{x\})1_{\sum_{x \in E}|\gamma_t(x)K_t^Q(\{x\})| > 0}}{\sum_{x \in E}|\gamma_t(x)K_t^Q(\{x\})|}\right)_{x \in E}$$

is an element of $(L^\infty(\mathscr{P}, M^P))^{|E|}$ and $\sum_{x \in E}\tilde{\gamma}_t(x)\sigma_t{}^{\alpha}(x) = 0$ M^P-a.e. $\forall$ $\alpha \in A$. Furthermore, $\tilde{\gamma} \neq 0$ since $\gamma \neq 0$ and M^P-a.e. $K_t^Q(\{x\}) > 0$ $\forall$ $x \in E$.

On the other hand, if $(\tilde{\gamma}_t(x))_{x \in E} \in (L^\infty(\mathscr{P}, M^P))^{|E|} \setminus \{0\}$ is such that

$$\sum_{x \in E} \tilde{\gamma}_t(x)\sigma_t{}^{\alpha}(x) = 0, \quad M^P\text{-a.e.}\ \forall\, \alpha \in A,$$

then

$$(\gamma_t)_{x \in E} = \left(\frac{\tilde{\gamma}_t(x)}{K_t^Q(\{x\})}\right)_{x \in E}$$

is well defined since M^P-a.e. $K_t^Q(\{x\}) > 0$ $\forall$ $x \in E$. Also, $\gamma \neq 0$ and

$$\sum_{x \in E} \gamma_t(x)\sigma_t{}^{\alpha}(x)K_t^Q(\{x\}) = 0, \quad M^P\text{-a.e.}\ \forall\, \alpha \in A.$$

Finally, γ is $\tilde{\mathscr{P}}$-measurable and

$$\int_{[0,t] \times E} |\gamma(u, x)| \nu^Q(du, dx) = \int_0^t \sum_{x \in E} \left|\frac{\tilde{\gamma}_u(x)}{K_u^Q(\{x\})}\right| K_u^Q(\{x\})\, du \leq \sum_{x \in E} \|\tilde{\gamma}(x)\| t$$

where $\|\cdot\|$ refers to the $L^\infty(\mathscr{P}, M^P)$-norm and we conclude that $\gamma \in \mathscr{T}$.

Hence we have shown that in the present setting, the condition of Lemma 6.1 which characterizes completeness is equivalent to

$$\left\{\gamma \in \left(L^\infty(\mathscr{P}, M^P)\right)^{|E|} \,\middle|\, \sum_{x \in E} \gamma_t(x)\sigma_t{}^{\alpha}(x) = 0,\ M^P\text{-a.e.}\ \forall\, \alpha \in A\right\} = \{0\}$$

or in linear algebra notation to

$$\left\{\gamma \in \left(L^{\infty}(\mathscr{P}, M^P)\right)^{|E|} \,\middle|\, \Sigma_t(\gamma_t(x))_{x\in E} = 0 \left(\text{in } \left(L^0(\mathscr{P}, M^P)\right)^{|A|}\right)\right\} = \{0\}.$$

The last condition is seen to be equivalent to the rank condition on Σ_t by arguing as in the proof of Theorem 4. □

We continue to assume that $|A| + |E| < \infty$ and M^P-a.e. $K_t(\{x\}) > 0$ $\forall$ $x \in E$. We consider returns R^α of the form $R_t^\alpha = \int_{[0,t]\times E} \sigma^\alpha(u, x)\mu(du, dx)$, that is, $A_t^\alpha = \int_{[0,t]\times E} \sigma^\alpha(u, x)\nu(du, dx)$. If we assume $\mathfrak{M}_{++} \neq \emptyset$, say, $Q \in \mathfrak{M}_{++}$, then (6.4) here takes the form

$$\int_{[0,t]\times E} \sigma^\alpha(u, x) Y^Q(u, x)\,\nu(du, dx) = 0 \qquad \forall\, \alpha \in A$$

or equivalently

$$\int_0^t \sum_{x\in E} \sigma_u^{\alpha}(x) K_u^Q(\{x\})\, du = 0 \qquad \forall\, \alpha \in A,$$

which holds if and only if

$$\sum_{x\in E} \sigma_t^{\alpha}(x) K_t^Q(\{x\}) = 0 \qquad \forall\, \alpha \in A\; M^P\text{-a.e.} \tag{6.11}$$

In linear algebra notation (6.11) may be written as

$$\Sigma_t\left(K_t^Q(\{x\})\right)_{x\in E} = 0 \quad \left(\text{in } \left(L^0(\mathscr{P}, M^P)\right)^{|A|}\right) \tag{6.12}$$

In the proof of Theorem 5 we saw that M^P-a.e. $K_t^Q(\{x\}) > 0$ $\forall$ $x \in E$ and so (6.12) shows that the existence of an equivalent martingale measure implies that the rank condition on Σ_t fails and hence by Thoerem 5 we cannot have completeness.

We conclude with the simple but important case where μ is a homogeneous Poisson random measure, that is, a random measure $\mu(\omega; dt, dx)$ with compensator ν of the form $\nu(dt, dx) = F(dx)\lambda_t\, dt$, where $F(dx)$ is independent of ω and t and $\lambda_t > 0$ is independent of ω and x. We can alternatively describe μ by counting processes $(N_t^x)_{x\in E}$ defined by $N_t^x = \mu([0,t]\times\{x\})$. Then N_t^x are independent Poisson processes of intensity $EN_t^x = E\mu([0,t]\times\{x\}) = f(\{x\})\int_0^t \lambda_u\, du$. Here N_t^x counts the number of events of type x which have occurred by time t. The condition on $K_t(dx)$ appearing in Theorem 5 here amounts to $F(\{x\}) > 0$ $\forall$ $x \in E$, that is, only events having positive probability of occurring are listed in E.

EXAMPLE 6.1. We consider an economy in which one of two events may occur at random times. More precisely, with notation as in the last paragraph, we take $E = \{1, 2\}$, $\lambda_u \equiv \lambda$ and $F(\{1\}) = p$, $F(\{2\}) = 1 - p$ for $p \in (0, 1)$. We then have the independent Poisson processes N_t^1 and N_t^2 with respective intensities $p\lambda t$ and $(1-p)\lambda t$. There are two stocks $(i = 1, 2)$ with price processes Z_t^i growing at a fixed rate g_i and responding to events 1 and 2 by fixed percentage price jumps of $\sigma_i(1) > -1$ and $\sigma_i(2) > -1$, respectively. Hence assuming without loss of generality that $Z_0^i = 1$ we have

$$Z_t^i = \exp(g_i t)(1 + \sigma_i(1))^{N_t^1}(1 + \sigma_i(2))^{N_t^2}$$

or

$$Z_t^i = \mathscr{E}(R^i)_t \quad \text{where } R_t^i = g_i t + \sigma_i(1) N_t^1 + \sigma_i(2) N_t^2.$$

The canonical decomposition of R_t^i is therefore given by $R_t^i = A_t^i + M_t^i$ with

$$A_t^i = \{g_i + \sigma_i(1) p\lambda + \sigma_i(2)(1-p)\lambda\} t$$

and

$$M_t^i = \sigma_i(1)\{N_t^1 - p\lambda t\} + \sigma_i(2)\{N_t^2 - (1-p)\lambda t\}.$$

Here the matrix process Σ appearing in Theorem 5 is independent of ω and t and is given by

$$\Sigma = \begin{bmatrix} \sigma_1(1) & \sigma_1(2) \\ \sigma_2(1) & \sigma_2(2) \end{bmatrix}$$

In this case and after simplifying, (6.4) says that if $Q \in \mathfrak{M}_{++}^{\text{loc}}$ then

$$\begin{bmatrix} \sigma_1(1) & \sigma_1(2) \\ \sigma_2(1) & \sigma_2(2) \end{bmatrix} \begin{pmatrix} p\lambda Y_t^Q(1) \\ (1-p)\lambda Y_t^Q(2) \end{pmatrix} = \begin{pmatrix} -g_1 \\ -g_2 \end{pmatrix}, \quad M^P\text{-a.e.} \tag{6.13}$$

First we consider the case when the stock prices stay unchanged unless one of the two events occurs, in other words the growth rates are zero and we show that:

(a) The existence of an equivalent martingale measure implies that completeness does not hold.

(b) If the rank condition holds then there are arbitrage opportunities and the market is complete.

(a) simply illustrates more concretely the point made earlier [see (6.12)]. Since $g_1 = g_2 = 0$ here, (6.13) shows that if $Q \in \mathfrak{M}_{++}^{\text{loc}}$ then the rank condition fails because $Y_t^Q(1), Y_t^Q(2) > 0$ M^P-a.e. and (a) now follows from Thoerem 5.

As for (b), suppose that the rank condition holds. If $Y \in L^\infty(\mathscr{F}_1)$, then

$$\lim_{n\to\infty} E\{Y \mid \mathscr{F}_{T_n \wedge 1}\} = Y \quad \text{boundedly a.s.,}$$

where T_n denotes the time of the nth jump, and so completeness follows if we show

$$L^\infty(\mathscr{F}_{T_n \wedge 1}) \subseteq \overline{\mathscr{A}} \qquad \forall\, n \text{ (the bar denotes the weak* closure)}. \tag{6.14}$$

We prove only the case $n = 1$ since a straightforward induction yields it for arbitrary n. Consider the increasing family of σ-algebras $\mathscr{G}^k$, $k \geq 1$, defined by

$$\mathscr{G}^k = \sigma\Big(\{X_1 = i\} \cap \{T_1 \in [l/2^k, (l+1)/2^k)\} \Big| i = 1, 2; l = 0, \dots, 2^k - 1\Big),$$

where $X_n\colon \Omega \to E$ indicates which type of jump occurs at time T_n; in other words, $\mu(dt, dx) = \sum_n \delta_{(T_n, X_n)}(dt, dx)$. Since $\mathscr{G}^k \uparrow \mathscr{F}_{T_1 \wedge 1}$,

$$\lim_{n\to\infty} E\{Y|\mathscr{G}^k\} = Y \quad \text{boundedly a.s.}$$

whenever $Y \in L^\infty(\mathscr{F}_{T_1 \wedge 1})$ and so (6.14) will hold for $n = 1$ if we show

$$L^\infty(\mathscr{G}^k) \subseteq \mathscr{A} \qquad \forall\, k.$$

To this end it is enough to check that

$$1_{\{X_i = i, T_1 < l/2^k\}} \subseteq \mathscr{A}, \qquad i = 1, 2;\ l = 1, \dots, 2^k.$$

Note that this will also establish that there are arbitrage opportunities; we will have $1_{\{T_1 < 1\}} \in \mathscr{A}$ and the strategy which generates this payoff must have zero initial investment. The rank condition on Σ implies that

$$\begin{bmatrix} 1 & \sigma_1(1) & \sigma_2(1) \\ 1 & \sigma_1(2) & \sigma_2(2) \\ 1 & 0 & 0 \end{bmatrix}$$

has rank 3 and hence we can find (x, H^1, H^2) such that

$$\begin{bmatrix} 1 & \sigma_1(1) & \sigma_2(1) \\ 1 & \sigma_1(2) & \sigma_2(2) \\ 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ H^1 \\ H^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Then the strategy with initial investment x and respective holdings of $H^1 1_{u \le T_1 \wedge l/2^k}$ and $H^2 1_{u \le T_1 \wedge l/2^k}$ in assets 1 and 2 yields time 1 payoff of $1_{\{X_1 = 1,\, T_1 < 1/2^k\}}$. Similarly, $1_{\{X_1 = 2,\, T_1 < l/2^k\}} \subseteq \mathscr{A}$.

Finally we leave the zero growth rate setting and show that if Σ satisfies the rank condition, that is, $\sigma_1(1)\sigma_2(2) - \sigma_2(1)\sigma_1(2) \neq 0$ and

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{bmatrix} \sigma_1(1) & \sigma_1(2) \\ \sigma_2(1) & \sigma_2(2) \end{bmatrix}^{-1} \begin{pmatrix} -g_1 \\ -g_2 \end{pmatrix}$$

has strictly positive components, then there exists an equivalent martingale measure and the market is complete. It suffices to show the existence of an equivalent martingale measure because in view of the rank condition completeness then holds by Theorem 5.

With the given hypotheses we can define the positive process

$$\xi_t = \exp\left(\lambda t + \left(\log\frac{a_1}{p\lambda}\right)N_t^1 - a_1 t + \left(\log\frac{a_2}{(1-p)\lambda}\right)N_t^2 - a_2 t\right).$$

We must now merely show that ξ_t, $Z_t^1\xi_t$ and $Z_1^2\xi_t$ are (genuine) P-martingales since $dQ = \xi_1\, dP$ then defines an e.m.m. This follows from the lemma.

LEMMA 6.2. *Let c, α_1, α_2 be constants with $\alpha_1, \alpha_2 > 0$. Then $\xi_t = \exp(ct + (\log \alpha_1)N_t^1 + (\log \alpha_2)N_t^2)$ is a (positive) P-martingale if and only if $c = \lambda - \lambda p\alpha_1 - \lambda(1-p)\alpha_2$.*

PROOF. Using the independence of N_t^1 and N_t^2, we get

$$E\xi_t = \exp\big(\{c - \lambda + \lambda p\alpha_1 + \lambda(1-p)\alpha_2\}t\big)$$

from which the forward implication is immediate. Suppose conversely that $c = \lambda - \lambda p\alpha_1 - \lambda(1-p)\alpha_2$. It is enough to show that ξ_t is a local P-martingale, since, being positive, it is then a P-supermartingale and having constant expectation it must be a genuine P-martingale. After some cancellations, Itô's lemma [Jacod and Shiryaev (1987); see, e.g., Theorem 4.57, page 57] yields

$$\xi_t = 1 + \int_0^t \xi_{u-} c\,du + \int_0^t \xi_{u-}\{(\alpha_1 - 1)\,dN_u^1 + (\alpha_2 - 1)\,dN_u^2\}.$$

Compensating the Poisson processes and using $c = \lambda - \lambda p\alpha_1 - \lambda(1-p)\alpha_2$ gives the desired conclusion. □

REMARK 6.1. Equation (6.4) suggests the following strategy for finding $Q \in \mathfrak{M}_{++}^{\mathrm{loc}}$. Solve (6.4) for a positive, $\tilde{\mathscr{P}}$-measurable Y and then find a P-equivalent probability Q such that $Y^Q = Y$. The question of constructing such a Q is addressed in Jacod (1975). The Q of the last paragraph could be thought of as the Q for which $Y_t^Q(1) = a_1/(\lambda p)$ and $Y_t^Q(2) = a_2/(\lambda(1-p))$. Hence $\nu^Q(dt, dx) = Y^Q(t,x)\nu(dt,dx) = Y^Q(t,x)F(dx)\lambda\,dt$ is deterministic here so that μ is actually a Poisson random measure under Q [Jacod and Shiryaev (1987), Theorem 4.8, page 104].

Acknowledgments. I thank Rick Durrett for all his mathematical help and for being an incredibly supportive and caring advisor throughout my time at Cornell. His enthusiasm and sense of humor made our weekly meetings very enjoyable. I am very grateful to Bob Jarrow for getting me started on this project and for his valuable help and guidance along the way.

REFERENCES

ANSEL, J.-P. and STRICKER, C. (1992). Lois de martingales, densitiés, et décomposition de Follmer–Schweizer, *Ann. Inst. H. Poincaré Probab. Statist.* **28** 375–392.

ANSEL, J.-P. and STRICKER, C. (1994). Couverture des actifs contingents et prix maximum. *Ann. Inst. H. Poincaré Probab. Statist.* **30** 303–315.

ARTZNER, P. and HEATH, D. (1995). Approximate completeness with multiple martingale measures. *Math. Finance* **5** 1–11.

BJÖRK, T., DI MASI, G., KABANOV, Y. and RUNGGALDIER, W. (1997). Towards a general theory of bond markets. *Finance and Stochastics*. To appear.

BJÖRK, T., KABANOV, Y. and RUNGGALDIER, W. (1996). Bond market structure in the presence of marked point processes. Preprint.

DALANG, R. C., MORTON, A. and WILLINGER, W. (1990). Equivalent martingale measures and no-arbitrage in stochastic securities market models. *Stochastics Stochastic Rep.* **29** 185–201.

DELBAEN, F. and SCHACHERMAYER, W. (1994). A general version of the fundamental theorem of asset pricing. *Math. Ann.* **300** 463–520.

DELLACHERIE, C. and MEYER, P.-A. (1975). *Probabilités et Potentiel*, **1**–**3**. Hermann, Paris.

GROTHENDIECK, A. (1973). *Topological Vector Spaces*. Gordon and Breanch, Montreux.

HARRISON, J. M. and KREPS, D. M. (1979). Martingales and arbitrage in multiperiod securities markets. *J. Econom. Theory* **20** 381–408.

HARRISON, J. M. and PLISKA, S. R. (1981). Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Process. Appl.* **11** 215–260.

HE, S.-W., WANG, J.-G. and YAN, J.-A. (1992). *Semimartingale Theory and Stochastic Calculus*. Science Press, Beijing.

HEATH, D., JARROW, R. and MORTON, A. (1992). Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation. *Econometrica* **60** 77–105.

JACOD, J. (1975). Multivariate point processes: predictable projection, Radon–Nikodym derivatives, representation of martingales. *Z. Wahrsch. Verw. Gebiete* **31** 235–253.

JACOD, J. (1979). *Calcul Stochastique et Problèmes de Martingales*. *Lecture Notes in Math*. **714**. Springer, Berlin.

JACOD, J. (1980). Integrales stochastiques par rapport a une semimartingale vectorielle et changments de filtration. *Séminaire de Probabilité XIV*. *Lecture Notes in Math*. **784** 161–172. Springer, Berlin.

JACOD, J. and SHIRYAEV, A. N. (1987). *Limit Theorems for Stochastic Processes*. Springer, Berlin.

JARROW, R. A. and MADAN, D. B. (1997). Hedging contingent claims on semimartingales. *Finance and Stochastics*. To appear.

JEANBLANC-PICQUÉ, M. and PONTIER, M. (1990). Optimal portfolio for a small investor in a market model with discontinuous prices. *Appl. Math. Optim.* **22** 287–310.

LAKNER, P. (1993). Martingale measures for a class of right-continuous processes. *Math. Finance* **3** 43–53.

MERCURIO, F. and RUNGGALDIER, W. J. (1993). Option pricing for jump diffusions: approximations and their interpretation. *Math. Finance* **3** 191–200.

MERTON, R. C. (1976). Option pricing when underlying stock returns are discontinuous. *J. Financial Econom.* **3** 125–144.

REVUZ, D. and YOR, M. (1991). *Continuous Martingales and Brownian Motion*. Springer, Berlin.

RUDIN, W. (1991). *Functional Analysis*, 2nd ed. McGraw-Hill, New York.

SCHACHERMAYER, W. (1994). Martingale measures for discrete-time processes with infinite horizon. *Math. Finance* **4** 25–55.

TAQQU, M. S. and WILLINGER, W. (1987). The analysis of finite security markets using martingales. *Adv. Appl. Probab.* **19** 1–25.