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TREELIKE QUEUEING NETWORKS: ASYMPTOTIC STATIONARITY AND HEAVY TRAFFIC[1]

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This study establishes limiting distributions for customer waiting times and queue lengths in treelike networks with single-server nodes. The main result characterizes the limiting distributions when the network data (interarrival times, service times and routes) is "asymptotically stationary." This is a weak condition covering a variety of networks including standard ones where the network data is stationary, regenerative, Markovian, satisfies coupling, and so on. The dependencies in the network data may be customer centered or node centered. The proof is based on two preliminary results that are of interest by themselves. The first one justifies the existence of the waiting time and queue length processes on the entire time axis for any network whose service capacity has been adequate to handle all the customers as one looks back to the "beginning of time." This is a sample-path generalization of a result of Loynes for a queueing system with stationary data. The second preliminary result is a characterization of functionals of sequences that preserve the asymptotic stationarity property. This is somewhat analogous to continuous-mapping principles for weak convergence. We also present functional central limit theorems for the waiting time processes in a network when the partial sums of the network data obey a heavy-traffic functional limit property. The limiting waiting time sequence is a functional of a process that is typically a multivariate Brownian motion, or a process with stationary increments and long range dependence such as a fractional Brownian motion.

1. Introduction. A major issue for a stochastic network (e.g., computer, telecommunications or manufacturing network) is to characterize the limiting behavior of its queue lengths and the customer waiting times. There are extensive results in this regard for networks, such as Jackson networks, that can be analyzed by Markovian or coupling techniques. For other types of networks with intricate dependencies, little is known about the limiting behavior of waiting times.

The present study describes the limiting behavior of waiting times and queue lengths in treelike networks with single-server nodes and general dependencies on arrivals, routings and services. The dependencies are either

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based on the order of the arrivals to the network (customer centered) or on the order of arrivals to each node (node centered). In a treelike network, the subsequence of customers that enter a node is determined by the order in which customers enter the network and their routes, and the subsequence is not affected by the service times. This property, which is the key to our characterization of waiting times, is not satisfied in more general networks where service times affect the subsequence of arrivals to a node. How to model waiting times in such networks is an open problem.

A major root of this study is a classical result of Loynes (1962). Consider a service system that processes units (or customers) one at a time under a first-in–first-out discipline. The waiting times for the successive units satisfy the recursive equation $W_{n+1} = \max\{0, V_n - U_n + W_n\}$, $n \geq 0$, where V_n is the service time of unit n, the U_n is the time between the arrival of units n and $n+1$ and W_0 is arbitrary. Loynes showed that if (U_n, V_n) is a stationary, ergodic sequence with $EV_0 < EU_0$, then W_n converges in distribution to $\sup_{l \leq 0} \sum_{i=l}^{-1}(V_i - U_i)$. Moreover, the sequence of waiting times for a finite time horizon converges in distribution as the horizon length tends to infinity to a stationary sequence of waiting times. This limit is a sequence of waiting times for a "stationary version" of the system on the entire time axis.

Our results required a comparable result when the sequence (U_n, V_n) is not stationary. This led us to address the issue of finding minimal conditions on interarrival and services times under which the waiting times and queue lengths converge in distribution, and the limits are waiting times and queue lengths of a stationary version of the system. Our first result (Theorem 2) justifies the existence of a stationary version of a treelike network process on the entire time axis with finite waiting times under the natural condition that the service capacity has been adequate to handle all the customers as one looks back to the "beginning of time" (the cumulative interarrival times minus the cumulative service times of the last n units tends to infinity as $n \to \infty$). This result is based only on recursive dynamical system equations and does not require assumptions on the distribution, expectations or dependencies of the network data (i.e., routes, interarrival times and service times). Theorem 2 contains a Loynes-type result (Theorem 3) for the existence of a stationary treelike network process.

Our analysis uses the relatively new notion that a sequence of random elements $\{X_k\colon k \geq 0\}$ is *asymptotically stationary* with respect to convergence in distribution if $\{X_{k+n}\colon k \geq 0\}$ (the sequence shifted by n time units) converges in distribution to some sequence $\{\tilde{X}_k\colon k \geq 0\}$ as $n \to \infty$. The limiting sequence is necessarily stationary. This condition guarantees that $X_n \to_{\mathscr{D}} \tilde{X}_0$, as $n \to \infty$. In other words, to prove that a sequence converges in distribution, it suffices to show that it is asymptotically stationary. This is a very weak condition that is satisfied by most sequences that converge in distribution (e.g., Markov chains and many other sequences with less structure, such as those satisfying a coupling property). See Szczotka (1986) for a general discussion of asymptotic stationarity.

Our main result, Theorem 8, is for a treelike network whose data is asymptotically stationary. It says that if the service capacity is adequate and a few technical conditions hold, then the waiting times and queue lengths at the nodes are asymptotically stationary and hence have limiting distributions. Furthermore, the entire network process converges in distribution to a stationary version of the network. This theorem applies to two types of indexing schemes for the network information. The first one is *customer centered*, where Q_k^j, W_k^j are queue lengths and waiting times at node j seen by the kth unit to *enter the network*. The second type is *node centered*, where the index k on Q_k^j, W_k^j refers to the kth unit to *enter node j*. The proof of Theorem 8 uses Theorems 10 and 11 in Section 4, which characterize functionals of sequences that preserve the asymptotic stationarity property. These theorems play a role for asymptotic stationarity that is similar to the role of continuous-mapping principles for weak convergence of probability measures. They are useful for establishing asymptotic stationarity in a variety of contexts.

Theorem 8 for one node extends the results of Borovkov (1984, 1987) and Foss (1991) as well as those of Loynes (1962). The first two authors assume conditions on the data (U_n, V_n) that imply a coupling property, from which the limits follow. Coupling, however, is considerably stronger than asymptotic stationarity. In particular, Theorem 4.3.3 in Berbee (1979) says coupling is equivalent to *asymptotic stationarity in mean in total variation*, which implies asymptotic stationarity [Szczotka (1986)]. Here is another way of describing the difference between results based on coupling and our results based on asymptotic stationarity. Under coupling, one usually obtains a convergence in total variation of random elements $Z_n \to Z$, and then $\phi(Z_n)$ automatically converges in total variation and hence in distribution to $\phi(Z_n)$ for "any measurable function" ϕ. On the other hand, if Z_n is asymptotically stationary with limit Z, then $\phi(Z_n)$ does not automatically converge in distribution to $\phi(Z_n)$. One needs further criteria for such convergence, as we establish in Section 4.

Kelly and Szczotka (1990) present an analogue of Theorem 8 for a tandem network in which the system data is asymptotically stationary "in mean under total variation convergence." See Boxma (1979) and Kelly (1982) for examples when a customer requires identical service times at the nodes. In these strong-convergence settings, the interdeparture times automatically satisfy the same strong mode of convergence, and so they do not involve the extra analysis with asymptotically stationary functionals described in Section 4 that is needed for the weaker convergence in distribution. Other results on the stability of queue lengths in networks with stationary, node-centered system data are in Baccelli and Foss (1994). Dynamical system equations, like the classical one above for the waiting times, are studied in Borovkov (1984, 1987), Brandt, Franken and Lisek (1990) and Baccelli and Liu (1992). Further applications are in Afanas'eva (1987), Baccelli and Bremaud (1994), Baccelli and Foss (1994), Borovkov and Schassberger (1994), Foss (1991), Franken, Koenig, Arndt and Schmidt (1981), Kelly and Szczotka (1990), Kalashnikov and Rachev (1990), Konszantopolos and Walrand (1990) and Szczotka (1986).

The other main results of this study describe the limiting behavior of the treelike network in heavy traffic. There are no stationarity or asymptotic stationarity assumptions on the data. Section 5 contains multivariate functional central limit theorems for the waiting times at the nodes when the partial sums of the system data obey a functional limit property. We comment on how to obtain similar functional limit theorems for queue lengths. The limiting waiting time and queue length sequences are typically functionals of a multivariate Brownian motion or, more generally, a process with stationary increments such as a fractional Brownian motion. The latter processes are gaining attention because ATM teletraffic data appears to exhibit self-similar properties; see for instance Willinger, Taqqu, Leland and Wilson (1995).

Theorem 3 in Kelly and Szczotka (1990) is a related, but more restrictive, limit theorem for a system with stationary waiting times. Reiman (1984) and Peterson (1991) also proved heavy traffic results for queue lengths (but not waiting times) in open networks with Markovian-type assumptions on node operations and routes. Konstantopoulos and Lin (1995) proved similar results under node-centered service times whose normalized sums converge to fractional Brownian motion. The limiting behavior of the waiting times in these references under treelike routing can be characterized by our results. How to model similar waiting times under general routing is an open problem.

The rest of this study is organized as follows. Section 2 addresses the existence of nonstationary and stationary treelike networks. Section 3 covers limits of waiting times and queue lengths in symptotically stationary networks. Section 4 characterizes functionals that preserve the asymptotic stationarity property. Heavy traffic limit theorems are in Section 5. Finally, Sections 6–9 contain proofs of the main results.

2. Existence of nonstationary and stationary networks. This section contains preliminaries on the existence of a general treelike network on the entire time axis and the existence of a stationary version of it.

We shall consider a network shown in Figure 1 consisting of M nodes labeled $1, \dots, M$ that represent service stations or processing points (e.g., manufacturing work stations, computers, storage areas). Discrete units representing customers (parts, data packets, messages, etc.) move through the nodes where they are served (processed, stored temporarily, etc.). Each node serves the units one at a time on a first-come–first-served basis and there is unlimited space for units queueing for service. The network is in the form of a directed tree with a single root node, hereafter called node 1, and the possible routes of the units are all the root-to-leaf paths. That is, each unit enters the network at node 1 and proceeds along some root-to-leaf path (or branch) and then exits the network.

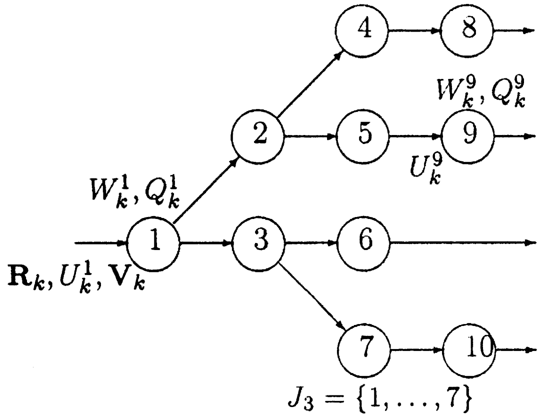

FIG. 1. *An open treelike network.*

Randomness may be present in the units' arrival times to node 1, their routes through the network and their service times at the nodes. Assuming the system has been operating since time $-\infty$, we let A_k^1 denote the arrival time to node 1 of unit k and adopt the standard conventions that k is in the set of all integers $\mathbb{Z}$,

$$
\begin{aligned}
&\cdots \leq A_{-1}^1 \leq A_0^1 \leq 0 < A_1^1 \leq A_2^1 \leq \cdots \quad \text{and} \\
&A_k^1 \to \pm\infty \text{ w.p.1 as } k \to \pm\infty.
\end{aligned}
\tag{1}
$$

The dependency in these arrival times is arbitrary: there may be multiple types of units, batch arrivals, arrivals depending the routing and services, and so on.

We denote the route that unit k takes through the network by the random vector $\mathbf{R}_k = (R_k^1, \ldots, R_k^M)$, where $R_k^j = 1$ or 0 according to whether or not unit k visits node j. For instance, $\mathbf{R}_{-8} = (1,1,0,1,0,0,0,1,0,0)$ means that unit -8 visits nodes $1, 2, 4, 8$, which is necessarily a root-to-leaf path. The routes may be determined by a variety of mechanisms and may depend on the arrival and service times. A standard example is that each route $\mathbf{R}_k$ is a realization of a one-node-at-a-time routing process in which unit k's route is a finite path of a Markov chain. The routes may also be determined by a unit's "type," where all the units of a given type follow the same deterministic (or random) route and the "type labels" on the units are generated by some random phenomenon. We make the innocuous assumption that, for each node j,

$$
\sum_{k=0}^{\pm\infty} R_k^j = \infty \qquad \text{w.p.1.} \tag{2}
$$

This ensures that an infinite stream of units visits j, and that, for each unit k, there are units before and after it that visit j. Denote the random subsequence of units that visit node j by

$$\cdots \le K^j_{-1} \le K^j_0 \le 0 < K^j_1 \le K^j_2 \le \cdots .$$

For instance, K^j_3 is the third unit after unit 0 that visits node j. We let J_s denote the set of nodes in the network that are *reachable in s steps or less*: a unit entering any $j \in J_s \setminus J_{s-1}$ will have visited $s-1$ nodes previously.

We denote the service times of unit k by the random vector $\mathbf{V}_k = (V^1_k, \dots, V^M_k)$, where V^j_k = service time of unit k at node j if $R^j_k = 1$ and $V^j_k = 0$ if $R^j_k = 0$. This service time does not include the time unit k waits in the queue at node j for its service. When unit k arrives at node j and there are no units there, its goes into service immediately; otherwise, it joins the queue and goes into service when the service time of the unit ahead of it ends. Upon finishing its service at node j, the unit proceeds immediately to the next node on its route, or exits the network if node j is a leaf node. The service times have arbitrary dependencies and they may depend on the service mechanism, the arrival times, the unit's type, or the unit's route. This includes the classic example that the service times at a node are independent with a common distribution depending on the node, or the dependency condition that a unit may require the same duration of service at each node it visits. At this point, we make no assumptions regarding the dependency among these variables; they will be made in the theorem statements.

Our interest is in describing the interarrival times U^j_k, waiting times W^j_k and quantities (or queue lengths) Q^j_k that unit k "sees" at node j. Namely, if unit k does visit node j ($R^j_k = 1$), then we have the following:

U^j_k = the time between the arrival of unit k at node j and the next unit that visits j.
W^j_k = the length of time unit k waits in the queue at node j for its service.
Q^j_k = the number of units at node j "just before" unit k arrives there (excluding unit k but including the units that might exit exactly when k arrives).

If unit k does not visit node j, then $U^j_k = W^j_k = Q^j_k = 0$. We will use the vector notation $\mathbf{U}_k = (U^1_k, \dots, U^M_k)$ and define $\mathbf{W}_k$ and $\mathbf{Q}_k$ similarly for each $k \in \mathbb{Z}$. In summary, the *basic data* and the *system variables* for this treelike service system are, respectively,

$$\boldsymbol{\xi}^\dagger = \left\{\boldsymbol{\xi}^\dagger_k = \left(\mathbf{R}_k, \mathbf{U}^1_k, \mathbf{V}_k\right): k \in \mathbb{Z}\right\}, \qquad \boldsymbol{\xi} = \left\{\boldsymbol{\xi}_k = (\mathbf{R}_k, \mathbf{U}_k, \mathbf{V}_k, \mathbf{W}_k, \mathbf{Q}_k): k \in \mathbb{Z}\right\}.$$

We sometimes consider the system on only the positive time axis with arrival times $0 \le A_0 \le A_1 \le \cdots$. The other system variables $U^j_k, V^j_k, \dots$ are defined as above only for $k \ge 0$. We call this the *positive-time system* to distinguish it from the system on the entire time axis. The preceding notation is *customer-centered* because the index k on the variables $Q^j_k, W^j_k, \dots$ refers to the kth unit to enter the network. For some applications, however, it is natural to use *node-centered* indices where the k refers to the kth unit to enter node j. We will present our main results with the customer-centered notation and point out how they also apply with node-centered indices.

We are now ready to begin our analysis. The following two results, which are the framework for our later analysis, show how the waiting times and queue lengths of the network are determined from the system data by standard recursive equations. First, consider the positive-time system. With no loss of generality, assume the network is initially empty so that the first customer to enter (unit K_1) has no wait. Otherwise, one would have to specify additional assumptions on the units initially in the system (where they are and their residual waiting or service times). Another needed random index is

$$\nu_k^j = \min\{l > k : R_l^j = 1\},$$

which is the *next unit following unit k that visits j*. We also let j^- denote the (unique) node in the tree preceding node j. Finally, we let $I(S)$ denote the indicator function that equals 1 or 0 according as the statement S is true or false.

LEMMA 1. *For the positive-time network, the interarrival times, waiting times and queue lengths are determined recursively as follows: for each* $s \geq 1$, $j \in J_s \setminus J_{s-1}$ *and* $k \geq K_1^j$,

$$U_k^1 \text{ is prespecified}, \tag{3}$$

$$U_k^j = R_k^j \left(\sum_{l=k}^{\nu_k^j - 1} U_l^{j^-} + V_{\nu_k^j}^{j^-} - V_k^{j^-} + W_{\nu_k^j}^{j^-} - W_k^{j^-} \right), \qquad j \geq 2, \tag{4}$$

$$W_k^j = R_k^j \max_{0 \leq l \leq k-1} \left[\sum_{i=k-l}^{k-1} \left(V_i^j - U_i^j \right) \right], \tag{5}$$

$$Q_k^j = R_k^j \sum_{l=1}^{k-1} I\left(R_{k-l}^j - 1 + W_{k-l}^j + V_{k-l}^j \geq \sum_{i=k-l}^{k-1} U_i^j \right). \tag{6}$$

For the proof, see Section 6.

We now consider the existence of the network process on the entire time axis. To define such a process, it suffices to find waiting times W_k^j that satisfy the dynamical system equation (31) derived in Section 6, which is

$$W_k^j = R_k^j \max\left\{0, V_\beta^j - U_\beta^j + W_\beta^j\right\}, \tag{7}$$

where β is the last unit before k to enter node j (this is $W_{n+1} = \max\{0, V_n - U_n + W_n\}$ for a single node). The other variables U_k^j and Q_k^j are then automatically determined by the system equations (4) and (6). Accordingly, we say that the network process *exists on the entire time axis* if there exist "finite" waiting times W_k^j and queue lengths that satisfy the dynamics (7) and (6) and that W_k^j is the minimal solution of (7) (for each j, if X_k^j is any other solution then $W_k^j \leq X_k^j$ w.p.1).

The following result says the network process exists under the mild condition that the service capacity has been adequate to handle all the customers as one looks back to the "beginning of time" (the cumulative interarrival times minus the cumulative service times of the last n units tends to infinity as $n \to \infty$).

THEOREM 2. *For the network process on the entire time axis, assume that, for each* j,

$$\sum_{l=k}^{0} \left(U_l^j - V_l^j\right) \to \infty \qquad w.p.1 \text{ as } k \to -\infty, \tag{8}$$

where U_k^j, *for* $k \in \mathbb{Z}$, *are determined recursively by* (4) *in terms of variables for the predecessor* j^- *obtained in previous iterations. Then the network process exists with interarrival times, waiting times and queue lengths determined recursively for each* $s \geq 1$, $j \in J_s \setminus J_{s-1}$ *and* $k \in \mathbb{Z}$ *by* (3), (4),

$$W_k^j = R_k^j \sup_{l \geq 0} \left[\sum_{i=k-l}^{k-1} \left(V_i^j - U_i^j\right) \right] \tag{9}$$

and (6) *with* $k - 1$ *in its first sum replaced by* ∞.

For the proof, see Section 6.

We now consider stationary systems. Recall that a sequence of random elements $\mathbf{X} = \{X_k : k \in \mathbb{Z}\}$ is *stationary* if $\mathbf{X} =_{\mathscr{D}} \theta X$, where θ is the usual shift operator ($\theta^n \mathbf{X} = \{X_{k+n} : k \in \mathbb{Z}\}$). This stationary sequence is *ergodic* if $P\{\mathbf{X} \in B\} = 0$ or 1 for each B such that $\{\mathbf{X} \in B\} = \{\theta \mathbf{X} \in B\}$. Similar definitions apply to sequences indexed by positive integers.

The following is the Loynes (1962) result for tandem nodes extended to treelike networks; the difference is the need here for the added conditioning on a unit's route. It says that if the data sequence $\xi^\dagger$ is stationary and ergodic and the expected service time is less than the expected interarrival time (10), then the entire system $\boldsymbol{\xi}$ is stationary. This result is a consequence of the preceding theorem. Here P_j and E_j denote the conditional probability and expectation of $\boldsymbol{\xi}^\dagger$ conditioned on the event that unit 0 enters node j. This P_j is also the *Palm probability* of $\boldsymbol{\xi}^\dagger$ conditioned on $R_0^j = 1$ [e.g., see Brandt, Franken and Lisek (1990) or Franken, Koenig, Arndt and Schmidt (1981)]. The assumption $P\{R_0^j = 1\} > 0$ ensures that units actually visit node j; otherwise, the node is superfluous.

THEOREM 3. *Suppose the system data* $\boldsymbol{\xi}^\dagger$ *is a stationary, ergodic sequence that satisfies* $P\{R_0^j = 1\} > 0$ *and*

$$EV_0^j < E\left(\sum_{k=0}^{K_1^j - 1} U_k^1 \right) < \infty, \qquad j = 1, \ldots, M. \tag{10}$$

Then the network process exists and the system variables $\boldsymbol{\xi}$ *are determined as in Theorem* 2. *Furthermore,* $\boldsymbol{\xi}$ *is stationary, ergodic and*

$$E_j\left(U_0^j\right) = E_j\left(\sum_{k=0}^{K_1^j - 1} U_k^1 \right), \qquad j = 1, \ldots, M. \tag{11}$$

For the proof, see Section 7.

The rest of this section contains further observations about stationary network processes. Does the stationarity of the system variables $\boldsymbol{\xi}_k$ (for all the units) also ensure the stationarity of the system variables for a substream of units that visit a certain node or a subset of nodes? No and yes. Here is an explanation.

Consider any subset of nodes $\mathscr{S} \subset \{1, \dots, M\}$. Let $K_k^{\mathscr{S}}$ denote the kth unit that visits $\mathscr{S}$ and define the indices k such that $\cdots \le K_{-1}^{\mathscr{S}} \le K_0^{\mathscr{S}} \le 0 < K_1^{\mathscr{S}} \le \cdots$. For instance, $K_3^{\mathscr{S}}$ is the third unit "after" unit 0 that visits $\mathscr{S}$. Then the $\mathscr{S}$-sector variables are $\boldsymbol{\xi}_k^{\mathscr{S}} = (\mathbf{R}_k^{\mathscr{S}}, \mathbf{U}_k^{\mathscr{S}}, \mathbf{V}_k^{\mathscr{S}}, \mathbf{W}_k^{\mathscr{S}}, \mathbf{Q}_k^{\mathscr{S}})$, $k \in \mathbb{Z}$, where $\mathbf{R}_k^{\mathscr{S}} = (R^j_{K_k^{\mathscr{S}}}\colon j \in \mathscr{S})$ and the rest of the vectors are defined similarly. Is $\boldsymbol{\xi}^{\mathscr{S}}$ a stationary sequence when the entire system $\boldsymbol{\xi}$ is? No. But it is stationary under the conditional probability that an entering unit visits $\mathscr{S}$.

THEOREM 4. *If the entire system $\boldsymbol{\xi}$ is stationary and ergodic, then the $\mathscr{S}$-sector $\boldsymbol{\xi}^{\mathscr{S}}$ is stationary and ergodic under the Palm probability of $\boldsymbol{\xi}$ conditioned that unit 0 visits $\mathscr{S}$.*

This result is an immediate consequence of the following well-known property of Palm probabilities applied to $\boldsymbol{\xi}_k^{\mathscr{S}} = \phi(\theta^{K_k^{\mathscr{S}}}\boldsymbol{\xi})$, where ϕ is a deterministic function.

LEMMA 5. *Suppose $\{X_k\colon k \in \mathbb{Z}\}$ is a stationary sequence and $\cdots \le K_{-1} \le K_0 \le 0 < K_1 \le \cdots$ are random indices such that $\theta^{K_k}\mathbf{X} \in B$ for some fixed set B. Then the sequence $\mathbf{Y} = \{\theta^{K_k}\mathbf{X}\colon k \in Z\}$ is stationary under the Palm probability P_0 of $\mathbf{X}$ conditioned that $K_0 = 0$. If, in addition, $\mathbf{X}$ is ergodic, then $\mathbf{Y}$ is also ergodic under P_0.*

We have been discussing the stationarity of the network system with respect to the sequence of units that enter it, that is, stationarity over shifts in the unit labels. How does this relate to the stationarity of the system in the continuous-time parameter t? Here are some insights on this.

The basic system data $\{(A_k^1, \mathbf{R}_k, \mathbf{V}_k)\colon k \in \mathbb{Z}\}$ is *stationary in continuous time* under its probability measure P if its distribution is invariant under any shift in the time axis; the distribution of $\{(A_k^1 - t, \mathbf{R}_k, \mathbf{V}_k)\colon k \in \mathbb{Z}\}$ is the same for each $t \in \mathbb{R}$. Ergodicity of the data in continuous time is also defined in the obvious way. In this setting, the system data is sometimes called a stationary, ergodic marked point process—the A_k^1's are points in time and $(\mathbf{R}^k, \mathbf{V}_k)$ is a "mark" associated with A_k^1. Let P^0 denote the Palm probability of the probability P of the data "conditioned" that a unit enters the network at node 1 at time 0.

THEOREM 6. *If the basic system data is stationary and ergodic in continuous time, then the sequence $\{(U_k^1, \mathbf{R}_k, \mathbf{V}_k)\colon k \in \mathbb{Z}\}$ is stationary and ergodic under P^0 (with respect to shifts in the labels k). If, in addition,* (10) *holds under the probability measure P^0, then the entire system sequence $\boldsymbol{\xi}$ is stationary and ergodic under P^0 (with respect to shifts in the labels k).*

The first assertion follows by a standard property of continuous-time Palm probabilities (similar to the discrete-time Lemma 5), and the second assertion follows by Theorem 3.

Next, let us consider the continuous-time processes $\mathbf{Q}(t) = (Q^1(t), \dots, Q^M(t))$ and $\mathbf{W}(t) = (W^1(t), \dots, W^M(t))$ of queue lengths and waiting times seen by the units that arrive at the nodes at or after time t. The process $(\mathbf{Q}, \mathbf{W}) = \{(\mathbf{Q}(t), \mathbf{W}(t)): t \in \mathbb{R}\}$ is *stationary in continuous time* under P if the distribution of $\theta^u(\mathbf{Q}, \mathbf{W}) = \{(\mathbf{Q}(t+u), \mathbf{W}(t+u)): t \in \mathbb{R}\}$ is independent of u.

THEOREM 7. *If the system data is stationary in continuous time, then* $(\mathbf{Q}, \mathbf{W})$ *is stationary in continuous time.*

For the proof, see Section 7.

3. Convergence of waiting times and queue lengths. We now consider nonstationary networks and address the following issues. Under what conditions do the queue length and waiting time vectors $\mathbf{Q}_k, \mathbf{W}_k$ for a nonstationary network converge in distribution as $k \to \infty$? If they do, are their limits equal in distribution to queue lengths and waiting times for a stationary version of the network process? We answer these questions by establishing that if the system data is asymptotically stationary and some technical conditions hold, then $\mathbf{Q}_k, \mathbf{W}_k$ are asymptotically stationary and hence converge in distribution. Furthermore, their limits are equal in distribution to the queue lengths and waiting times for a stationary network whose system data is equal in distribution to the limits of the original system data. This major result requires several preliminary results on asymptotic stationarity in Section 4.

We will use the following terminology. A sequence of random elements $\mathbf{X} = \{X_k: k \in \mathbb{Z}\}$ is *asymptotically stationary* (in the sense of weak convergence) if $\theta^n \mathbf{X} \to_{\mathscr{D}} \tilde{X}$ as $n \to \infty$ for some sequence $\tilde{\mathbf{X}}$. In particular, $X_n \to_{\mathscr{D}} \tilde{X}_0$ as $n \to \infty$. This limit $\tilde{\mathbf{X}}$ is stationary, since $\theta^n \mathbf{X} \to_{\mathscr{D}} \tilde{X}$ and $\theta^{n+1} X \to_{\mathscr{D}} \tilde{X}$ imply $\theta \tilde{\mathbf{X}} =_{\mathscr{D}} \tilde{\mathbf{X}}$. Hereafter we use a tilde over a sequence to denote that it is such a stationary limit. The preceding notions are defined similarly for one-sided sequences $\mathbf{X} = \{X_k: k \geq 0\}$. Szczotka (1986) discusses details of this (weak) asymptotic stationarity and applies it to obtain limiting distributions of general queueing systems. He also discusses the convergence of $\theta^n \mathbf{X}$ to $\tilde{\mathbf{X}}$ for five other stronger modes of convergence, including convergence in total variation and strong convergence in mean. The family of (weak) asymptotic stationary sequences is rather large; it contains most of the standard sequences that have limiting distributions, such as Markov chains, regenerative or semistationary sequences, periodic sequences and it even contains asymptotically stationary sequences in the five other convergence modes. Our results apply (with slight differences) to asymptotic stationarity under these other modes of convergence as well, but we will not give details on this.

We are now ready for our main result. It applies to the network on the entire time axis as well as the positive-time system, but we will state it only for the latter (more interesting) case. It says that if the system data $\boldsymbol{\xi}^{\dagger}$ is asymptotically stationary, the service capacity is adequate (10) and some technical continuous-mapping conditions hold (12)–(14), then the system $\boldsymbol{\xi}$ is asymptotically stationary.

THEOREM 8. *Suppose the system data* $\boldsymbol{\xi}^{\dagger}$ *is asymptotically stationary and its stationary limit* $\tilde{\boldsymbol{\xi}}^{\dagger} = (\tilde{\mathbf{R}}, \tilde{\mathbf{U}}^1, \tilde{\mathbf{V}})$ *is ergodic and satisfies the hypotheses of Theorem* 3. *In addition, assume that, for each* j *and* $\varepsilon > 0$,

$$\lim_{n' \to \infty} \limsup_{n \to \infty} P\left\{ \left| \sup_{n' \le l \le n-1} \sum_{k=n-l}^{n-1} \left(V_k^j - U_k^j \right) \right| > \varepsilon \right\} = 0, \tag{12}$$

$$P_j\left\{ \tilde{W}_0^j + \tilde{V}_0^j = \sum_{k=0}^{\tilde{K}_n^j - 1} \tilde{U}_k^j \right\} = 0 \quad \textit{for each } n \ge 1, \tag{13}$$

$$\lim_{n' \to \infty} \limsup_{n \to \infty} P\left\{ \begin{array}{c} R_{n-l}^j = R_n^j = 1,\ W_{n-l}^j + V_{n-l}^j \ge \Sigma_{i=1}^l U_{n-i}^j, \\ \textit{for some } l \in \{n'+1, \dots, n\} \end{array} \right\} = 0. \tag{14}$$

Then $\boldsymbol{\xi}$ *is asymptotically stationary and its stationary limit* $\tilde{\boldsymbol{\xi}}$ (*on the entire time axis*) *is ergodic. Moreover, the variables* $\tilde{\mathbf{U}}, \tilde{\mathbf{W}}, \tilde{\mathbf{Q}}$ *for the limit* $\tilde{\boldsymbol{\xi}}$ *are functions of the data* $\tilde{\boldsymbol{\xi}}^{\dagger}$ *as in Theorem* 2.

For the proof, see Section 8.

REMARKS. (a) From Theorem 11, it follows that (12) is also a necessary condition for the asymptotic stationarity of the waiting time sequence. Condition (14) plays a similar role for the queue length sequence.

(b) Conditions (13) and (14) are not needed in the proof of Theorem 8 to obtain the asymptotic stationarity of the waiting time vector; they are only needed for the asymptotic stationarity of the queue lengths. In other words, Theorem 8 holds without these conditions and without reference to queue lengths.

(c) Condition (13) is a natural condition that simply rules out the possibility of having arrivals and departures at the same time.

EXAMPLE 9 (Single service station). Consider Theorem 8 for a single service station whose interarrival and service time sequence $\{(U_k, V_k): k \ge 0\}$ is asymptotically stationary and its stationary limit $\{(\tilde{U}_k, \tilde{V}_k): k \in \mathbb{Z}\}$ is ergodic and satisfies $E\tilde{V}_0 < E\tilde{U}_0$. If (12) holds, then $\{(U_k, V_k, W_k): k \ge 0\}$ is asymptotically stationary and its stationary limit $\{(\tilde{U}_k, \tilde{V}_k, \tilde{W}_k): k \in \mathbb{Z}\}$ is ergodic, where

$$\tilde{W}_k = \sup_{l \ge 0} \left[\sum_{i=k-l}^{k-1} \left(\tilde{V}_i^j - \tilde{U}_i^j \right) \right], \qquad k \in \mathbb{Z}.$$

If, in addition, (14) is satisfied without the R's and j's and

$$P\left\{\tilde{W}_0 + \tilde{V}_0 = \sum_{k=0}^{n-1} \tilde{U}_k\right\} = 0, \qquad n \geq 1,$$

then $\{(U_k, V_k, W_k, Q_k)\colon k \geq 0\}$ is asymptotically stationary and its stationary limit $\{(\tilde{U}_k, \tilde{V}_k, \tilde{W}_k, \tilde{Q}_k)\colon k \in \mathbb{Z}\}$ is ergodic, where

$$\tilde{Q}_k = \sum_{l=1}^{\infty} I\left(\tilde{W}_{k-l} + \tilde{V}_{k-l} \geq \sum_{i=k-l}^{k-1} \tilde{U}_i\right), \qquad k \in \mathbb{Z}.$$

Results for node-centered indices. The treelike network with node-centered indices has system data and system variables

$$\boldsymbol{\xi}_k^{\dagger} = \left(\mathbf{D}_k, \mathbf{U}_k^1, \mathbf{V}_k\right) \quad \text{and} \quad \boldsymbol{\xi}_k = (\mathbf{D}_k, \mathbf{U}_k, \mathbf{V}_k, \mathbf{W}_k, \mathbf{Q}_k), \qquad k \in \mathbb{Z}.$$

The index k now refers to the kth unit served at node j (instead of the kth unit to enter the network), and D_k^j is such that the kth unit served at node j is the $(k + D_k^j)$th unit served at the previous node j^-, and $D_k^1 = 0$. This "difference" vector $\mathbf{D}_k$, which takes the place of the routing vector $\mathbf{R}_k$, is the vehicle for relating the customer indices at the nodes. In particular, the kth unit to be served at node j has the index $\gamma_k(i, j)$ at any preceding node i, where these indices are defined by $\gamma_k(j, j) = k$ and the backward recursion

$$\gamma_k(i^-, j) = \gamma_k(i, j) + D^i_{\gamma_k(i, j)}$$

for each i on the branch from node 1 to j.

The information in this node-centered system is simply a reindexing of the information of the customer-centered system (somewhat like a randomly indexed subsequence X_{ν_k} being related to the sequence X_k). Consequently, all the preceding results and those in the next section apply to this setting. In particular, by obvious relabeling of information (requiring no analysis), Lemma 1 and Theorems 2–8 hold with the following minor changes.

1. The R_k^j's equal 1.
2. Expression (5) is replaced by

$$U_k^j = \sum_{l=D_k^j}^{D_{k+1}^j} U_{k+l}^{j^-} + V_{k+1+D_{k+1}^j}^{j^-} - V_{k+D_k^j}^{j^-} + W_{k+1+D_{k+1}^j}^{j^-} - W_{k+D_k^j}^{j^-}.$$

3. The sum $\sum_{k=0}^{K_1^j-1}$ in Theorem 3 is replaced by $\sum_{k=\gamma_0(1,j)}^{-1}$ and a similar replacement is done in expressions (19) and (20) in Theorem 8.

4. Functionals that preserve asymptotic stationarity. The continuous-mapping principle is a useful tool for establishing weak convergence of probability measures [Billingsley (1968)]. This section contains similar principles for establishing asymptotic stationarity. They are of interest by themselves and are the basis for proving Theorem 8.

Throughout this section, we assume that $\mathbf{X} = \{X_k: k \in Z\}$ is an asymptotically stationary sequence of random elements in a Polish space E and its limit is $\tilde{\mathbf{X}}$. We consider the sequence

$$Y_k = \phi_k(\theta^k \mathbf{X}), \qquad k \in \mathbb{Z},$$

where ϕ_k is a measurable function from $E^{\mathbb{Z}}$ to a Polish space E'. We say that $\mathbf{Y}$ is an *asymptotically stationary functional* of $\mathbf{X}$ if

$$(\theta^n \mathbf{X}, \theta^n \mathbf{Y}) \to_{\mathscr{D}} (\tilde{\mathbf{X}}, \tilde{\mathbf{Y}}) \quad \text{and} \quad \tilde{Y}_k = \phi(\theta^k \tilde{\mathbf{X}}), \qquad k \in \mathbb{Z},$$

where ϕ is a measurable function from $E^{\mathbb{Z}}$ to E'. This says that $(\mathbf{X}, \mathbf{Y})$ is asymptotically stationary and $\tilde{\mathbf{Y}}$ is a stationary functional of $\tilde{\mathbf{X}}$. The next two theorems give criteria for establishing the asymptotic stationarity of this $\mathbf{Y}$.

THEOREM 10. *Suppose* $\tilde{\mathbf{X}} \in \mathscr{C}_\phi$ *w.p.*1, *where*

$$\mathscr{C}_\phi = \{\mathbf{x}: \phi_n(\mathbf{x}_n) \to \phi(\mathbf{x}) \text{ for any } \mathbf{x}_n \to \mathbf{x}\},$$

which is the set of points at which ϕ_n *converges "continuously" to* ϕ. *Then* $\mathbf{Y}$ *is an asymptotically stationary functional of* $\mathbf{X}$.

PROOF. Define $\psi_n(\mathbf{x}) = (\mathbf{x}, \{\phi_{k+n}(\theta^k \mathbf{x}): k \in \mathbb{Z}\})$ and

$$\mathscr{C}_\psi = \left\{\mathbf{x}: \psi_n(\mathbf{x}_n) \to \psi(\mathbf{x}) \equiv \left(\mathbf{x}, \{\phi(\theta^k \mathbf{x}): k \in \mathbb{Z}\}\right) \text{ for any } \mathbf{x}_n \to \mathbf{x}\right\}.$$

A little thought shows $\mathscr{C}_\psi = \mathscr{C}_\phi$, and so $\tilde{\mathbf{X}} \in \mathscr{C}_\psi$ w.p.1. Then by the "generalized" continuous mapping principle [Theorem 5.5 in Billingsley (1968)], we have $(\theta^n \mathbf{X}, \theta^n \mathbf{Y}) = \psi_n(\theta^n \mathbf{X}) \to_{\mathscr{D}} \psi(\tilde{\mathbf{X}}) = (\tilde{\mathbf{X}}, \tilde{\mathbf{Y}})$. $\square$

The next result is a more general criterion for asymptotic stationary functionals. Here d and d' are metrics on the respective spaces E and E'.

THEOREM 11. *Suppose*

$$\phi_n(\tilde{\mathbf{X}}) \xrightarrow{P} \phi(\tilde{\mathbf{X}}) \quad \text{as } n \to \infty, \tag{15}$$

$$P\{\tilde{\mathbf{X}} \text{ is in the continuity set of } \phi_k\} = 1, \qquad k \in \mathbb{Z}. \tag{16}$$

Then $\mathbf{Y}$ *is an asymptotic stationary functional of* $\mathbf{X}$ *if and only if*, *for any* $\varepsilon > 0$,

$$\lim_{n' \to \infty} \limsup_{n \to \infty} P\{d'(\phi_{n'}(\theta^n \mathbf{X}), \phi_n(\theta^n \mathbf{X})) > \varepsilon\} = 0. \tag{17}$$

For the proof, see Section 8.

The preceding discussion of asymptotically stationarity for "two-sided" sequences also applies to "one-sided" sequences $\mathbf{X} = \{X_k: k \geq 0\}$. Here the limit of $\theta^n \mathbf{X} = (X_n, X_{n+1}, \dots)$ is a one-sided stationary sequence $\tilde{\mathbf{X}}$. This $\tilde{\mathbf{X}}$ can be extended as usual to a two-sided stationary sequence, which we also denote by $\tilde{\mathbf{X}}$.

5. Treelike networks in heavy traffic. This section characterizes the asymptotic behavior of the treelike network in heavy traffic. The results are multivariate functional central limit theorems for waiting times. One can obtain analogous limit theorems for the joint convergence of queue lengths and waiting times using the approach in Serfozo, Szczotka and Topolski (1994), which also covers a novel relationship between the two sequences. But we will confine the following discussion only to waiting times.

Consider the network on the positive time axis. Recall that K_k^j is the label of the kth unit that *enters node* j or unit k's *local rank* at j. We change notation slightly in this section and let $U_k^j, V_k^j, W_k^j, \dots$ denote the system variables associated with unit K_k^j (e.g., W_k^j is now the waiting time of the kth entry to j instead of the kth entry to 1 as before; this W_k^j is $W_{K_k^j}^j$ in the old notation). This new notation eliminates those system variables for a node that were previously set to zero whenever units did not visit the node.

We shall consider the network on the positive time axis under the assumption that it also depends on another parameter n that indirectly represents the underlying "traffic intensity." The intensity increases as n increases, which causes the waiting times and queue lengths to increase. For simplicity, the parameter n is suppressed from the system variables. Our aim is to describe the asymptotic behavior of the waiting times W_k^j as both k and n tend to ∞.

Our analysis involves the weak convergence of random elements in the space $D = [0, \infty)$ of real-valued functions on $[0, \infty)$ that are right-continuous with left-hand limits, and D is endowed with the Skorohod topology. We will characterize the convergence of the waiting times via the waiting time processes

$$W_n^j(t) = b_n^{-1} W_{[a_n t]}^j, \qquad t \geq 0,$$

where $[a]$ is the integer part of a and a_n, b_n are constants that converge to infinity. These waiting times are *node-oriented* waiting times in that the subscript k (or $[a_n t]$) refers to the kth unit to enter node j. Later we consider the slightly different *route-oriented* waiting times where k refers to the kth unit that traverses a certain route.

The following result describes the convergence of the waiting time processes in terms of the convergence of the processes

$$X_n^j(t) = b_n^{-1} \left(\sum_{k=1}^{[a_n t]} V_k^j - A_{K_{[a_n t]}^j}^1 \right), \qquad \tau_n^j(t) = a_n^{-1} \sum_{k=1}^{[a_n t]} R_k^j, \qquad t \geq 0.$$

The X_n^j represents the difference in partial sums of the service times at node j and interarrival times at node 1 for those units that visit j, and τ_n^j is the normalized number of visits of units to node j. The "heavy traffic" condition for the network is implicit in assumption (18) on the convergence of the X_n^j's. We will use the mapping h from D to D defined by

$$h(x)(t) = \sup_{s \leq t} (x(t) - x(s)), \qquad t \geq 0.$$

Also, $x \circ y(t) = x(y(t))$ is the *composition* of x and y in D and, if $x \in D$ has nondecreasing paths, its *inverse* is $\hat{x}(t) = \inf\{s\colon x(s) > t\}$. Finally, we say that a random element Z of D has *stochastically continuous paths* if $P\{Z(t-) = Z(t)\} = 1$, $t \in \mathbb{R}_+$.

THEOREM 12. *Suppose*

$$\{(X_n^j, \tau_n^j)\colon 1 \le j \le M\} \to_{\mathscr{D}} \{(X^j, \tau^j)\colon 1 \le j \le M\} \tag{18}$$

$$b_n^{-1} V^j_{[a_n \cdot]} \to_{\mathscr{D}} 0, \qquad 1 \le j \le M, \tag{19}$$

where each X^j has stochastically continuous paths and each τ^j has continuous strictly increasing paths. Then

$$\{(X_n^j, W_n^j, \tau_n^j)\colon 1 \le j \le M\} \to_{\mathscr{D}} \{(X^j, W^j, \tau^j)\colon 1 \le j \le M\}, \tag{20}$$

where the W^j are defined recursively by

$$W^j = h\Bigl(X^j - \sum_{i \in B_{j^-}} W^i \circ \tau^i \circ \hat{\tau}^j\Bigr). \tag{21}$$

and B_{j^-} is the branch of nodes from 1 *to* j^- (*when* $j = 1$, *the summation term is* 0). *Also, each W^j has stochastically continuous paths.*

For the proof, see Section 9.

Note that assumption (18) does not specify the form of the limiting process $(X^1, \dots, X^M)$. This process may be a Brownian motion when the system has short range dependencies, or it might be a fractional Brownian motion or other process with stationary increments when the system has long range dependencies. The latter types of limits are becoming of interest in modeling ATM networks; see for instance Konstantopoulos and Lin (1995) and Konstantopoulos and Walrand (1990).

The waiting times in Theorem 12 were node indexed in that they were for the kth entries into the nodes. We now consider route-indexed waiting times indexed by the customers that travel certain routes. Let $\mathscr{L}$ denote the set of leaf nodes in the network (the last nodes where the units exit the network). For each $l \in \mathscr{L}$, the kth unit that traverses the branch of nodes B_l from 1 to l is unit K_k^l and its local rank at node $j \in B_l$ is $\gamma_k \equiv \sum_{\nu=1}^{K_k^l} R_\nu^j$. Then the waiting time of this unit at node j is $W^j_{\gamma_k}$. We also consider the waiting times of an arbitrary unit entering the network. Let L_k denote the last node (a leaf of the tree) that the kth unit entering the network visits. This kth unit has the local rank $\gamma_k' = \sum_{k'=1}^{k} R^j_{k'}$ at node $j \in B_{L_k}$ and its waiting time at j is $W^j_{\gamma_k'}$.

COROLLARY 13. (i) *If the hypotheses of Theorem* 12 *hold*, *then the normalized waiting times of the units traversing the branches satisfy*

$$\Bigl\{\bigl(b_n^{-1} W^j_{\gamma_{[a_n \cdot]}}\colon j \in B_l\bigr)\colon l \in \mathscr{L}\Bigr\} \to_{\mathscr{D}} \Bigl\{\bigl(W^j \circ \tau^j \circ \hat{\tau}^l\colon j \in B_l\bigr)\colon l \in \mathscr{L}\Bigr\}, \tag{22}$$

where W^j are given by (21). *Furthermore, the normalized sojourn times of the* *units traversing the branches satisfy*

$$\left\{b_n^{-1}\sum_{j\in B_l}\left(W^j_{\gamma_{[a_n\cdot]}}+V^j_{[a_n\cdot]}\right):l\in\mathscr{L}\right\}\to_{\mathscr{D}}\left\{\sum_{j\in B_l}W^j\circ\tau^j\circ\hat{\tau}^l:l\in\mathscr{L}\right\}. \tag{23}$$

(ii) *If* (19) *holds and*, *for any* $\nu_n\to\infty$,

$$\left(L_{\nu_n},\left\{\left(X_n^j,\tau_n^j\right):1\le j\le M\right\}\right)\to_{\mathscr{D}}\left(L,\left\{\left(X^j,\tau^j\right):1\le j\le M\right\}\right) \quad \text{in } \{1,\dots,M\}\times D^{2l}, \tag{24}$$

then the normalized waiting times for the units entering the network satisfy

$$\left\{b_n^{-1}W^j_{\gamma'_{[a_n\cdot]}}:j\in B_{L_{[a_n\cdot]}}\right\}\to_{\mathscr{D}}\left\{W^j\circ\tau^j:j\in B_L\right\}. \tag{25}$$

For the proof, see Section 9.

The preceding results apply to a variety of contexts in which the network data satisfies the heavy-traffic functional limit property (18). An elementary illustration is below. First, a preliminary observation. Suppose $\mathbf{Y}_{n,k}=(Y^j_{n,k}: j\in J)$, $k\ge 1$, are independent identically distributed random vectors that satisfy

$$\sup_n E|Y^j_{n,1}-\mu_n^j|^{2+\varepsilon}<\infty, \qquad \lim_{n\to\infty}\operatorname{Cov}\left(Y^i_{n,1},Y^j_{n,1}\right)=\sigma_{ij},$$

where $\mu_n^j=EY^j_{n,1}$, $\sigma_{ij}>0$ and $\varepsilon>0$. Then, as a variation of a result in Prokhorov (1956), the vector process $n^{-1/2}\sum_{k=1}^{[nt]}(\mathbf{Y}_{n,k}-\boldsymbol{\mu}_n)$ converges in distribution to a Wiener process $\mathscr{W}=(\mathscr{W}^j: j\in J)$ with covariance matrix σ_{ij}. We say that such a sequence $\mathbf{Y}_{n,k}$, $k\ge 1$, satisfies a *FCLT*($\mathscr{W}$) if it satisfies the preceding conditions.

EXAMPLE 14 (Route-dependent services). Suppose the network data satisfies the following conditions. Here the heavy-traffic parameter n is not suppressed.

(A1) The "arrival-rate process" $\tau_n^j(t)=n^{-1}\sum_{k=1}^{[nt]}R^j_{n,k}$ converges in distribution as $n\to\infty$ to the function $\lambda_j t$, where λ_j is a positive constant.

(A2) The interarrival times $\{U^1_{n,k}: k\ge 1\}$ satisfy a FCLT($\mathscr{W}^0$) and they are independent of the service times.

(A3) The service times $\{V^{jl}_{n,k}: j\in B_l,\ l\in\mathscr{L}\}$ satisfy a FCLT($\mathscr{W}$), where $V^{jl}_{n,k}$ denotes the service time of the kth arrival to node j that exits the network from node l.

(A4) The limit $c_j=\lim_{n\to\infty}n^{1/2}[\lambda_j^{-1}EU^1_{n,1}-\sum_{l\in\mathscr{L}}\lambda_l^{-1}EV^{jl}_{n,1}]$ exists.

Under these conditions, the waiting time processes $(n^{-1/2}W^j_{n,[nt]}: 1\le j\le m)$ converge in distribution to the processes $(W^j: 1\le j\le m)$ that are determined recursively by $W^j=h(X^j-\sum_{j\in B_{j^-}}W^i(\lambda_i\lambda_j^{-1}\cdot))$, where

$$X^j(t)=\sum_{l\in\mathscr{L}}\mathscr{W}^{jl}\left(\lambda_l^{-1}t\right)-\mathscr{W}^0\left(\lambda_j^{-1}t\right)-c_jt, \tag{26}$$

and the Weiner processes $\mathscr{W}^{jl}$ are independent of $\mathscr{W}^0$. This is an application of Theorem 12 and its proof is in Section 9. This result readily extends to stationary service times instead of independent ones as in [Serfozo, Szczotka and Topolski (1994)] or to other types of dependencies on the data.

6. Proofs of expressions for system variables. This section contains the proofs of expressions (4)–(6) for the system variables as functions of the data for the positive-time system (Lemma 1) and for the system on the entire time axis (Theorem 2).

PROOF OF LEMMA 1. Consider a fixed state s and a node $j \in J_s \setminus J_{s-1}$ that is reachable in exactly s steps. If $R_k^j = 0$, then (4)–(6) are automatically true. Therefore we only consider the case $R_k^j = 1$. To formulate the interarrival times, let $A_{K_1^j}^j$ denote the time at which the first unit enters node j and define

$$A_k^j = A_{K_1^j}^j + \sum_{i=K_1^j}^{k-1} U_i^j, \qquad k \ge K_1^j, \tag{27}$$

which is the arrival time of unit k to node j. Also note that

$$A_{\nu_k^j}^{j^-} - A_k^{j^-} = \sum_{l=k}^{\nu_k^j - 1} U_l^{j^-}. \tag{28}$$

Now, the arrival time A_k^j for $j \ge 2$ is equal to its departure time from node j^-, and so

$$A_k^j = A_k^{j^-} + W_k^{j^-} + V_k^{j^-}. \tag{29}$$

In light of this, it is clear that, if $R_k^j = 1$, then

$$U_k^j = R_k^j \left(A_{\nu_k^j}^j - A_k^j \right). \tag{30}$$

Applying (29) to both of these arrival times and using (28) yields (4). Next, note that when $R_k^j = 1$, by the definition of waiting times and $A_k^j = A_\beta^j + U_\beta^j$ from (29), we have

$$\begin{aligned} W_k^j &= R_k^j \max\{0, (\text{departure time from } j \text{ of unit } \beta) - A_k^j\} \\ &= R_k^j \max\{0, V_\beta^j - U_\beta^j + W_\beta^j\}, \end{aligned} \tag{31}$$

where $\beta = \max\{l < k : R_l^j = 1\}$ is the last unit before k to enter node j. Using (31) in an induction argument proves (5). Finally, by the definition of queue lengths, we have

$$Q_k^j = \sum_{l=1}^{k-1} I\left(R_{k-l}^j = 1,\, A_{k-l}^j + W_{k-l}^j + V_{k-l}^j \ge A_k^j \right).$$

Here unit $k - l$ is counted if it enters j and departs at or after A_k^j. Applying (27) to A_k^j yields (6). $\square$

The following is the proof of the existence of the network process on the entire time axis.

PROOF OF THEOREM 2. The proof will be by induction on the stage parameter s. The arguments for $s = 1$ and a general s are essentially the same, and so we present the argument only for the latter case. Accordingly, assume the assertion is true for some $s - 1 \geq 1$ and that $J_s \setminus J_{s-1}$ is not empty (otherwise the induction is already complete; recall that $s \leq M$). Similarly to the positive-time network, expression (4) determines U_k^j from previously defined variables, which exist under the induction hypothesis. Next, note that W_k^j defined by (9) is finite for each k because of assumption (8). If $R_k^j = 0$, then (9) is true. If $R_k^j = 1$, then W_k^J given by (9) must satisfy the system dynamics (31). But this is true since

$$\begin{aligned} W_k^j &= R_k^j \max\left\{0, V_\beta^j - U_\beta^j + \sup_{l \geq 0}\left[\sum_{i=\beta-l}^{\beta-1}\left(V_i^j - U_i^j\right)\right]\right\} \\ &= R_k^j \max\left\{0, V_\beta^j - U_\beta^j + W_\beta^j\right\}. \end{aligned}$$

Thus, the waiting times are finite and are determined by (9) for $j \in J_s \setminus J_{s-1}$. Furthermore, arguing as in the proof of Lemma 1 of Loynes (1962), one can show that W_k^j is the minimal solution of (31).

Similarly to the positive-time network, expression (6) with $k - 1$ in the first sum replaced by ∞ defines the queue lengths Q_k^j, and they will be finite if, for each k,

$$\zeta \equiv \limsup_{r \to -\infty}\left(W_r^j + V_r^j - \sum_{i=r}^{k-1} U_i^j\right) = -\infty \qquad \text{w.p.1.} \tag{32}$$

Using the representation above for W_r^j, and letting $S_r = \sum_{i=r}^{k-1}(V_i^j - U_i^j)$, we have

$$\begin{aligned} \zeta &= \limsup_{r \to -\infty}\left\{R_r^j \sup_{l \geq 0}\left[\sum_{i=r-l}^{r-1}\left(V_i^j - U_i^j\right)\right] + V_r^j - \sum_{i=r}^{k-1} U_i^j\right\} \\ &\leq \limsup_{r \to -\infty}\ \sup_{l \geq 0}\left[S_{r-l} - \sum_{i=r+1}^{k-1} V_i^j\right] \leq \limsup_{r \to -\infty}\ \sup_{l \leq r} S_l \\ &= \limsup_{r \to -\infty} S_r = -\infty \qquad \text{w.p.1.} \end{aligned}$$

The last line follows because $\sup_{l \leq r} S_l$ is decreasing as $r \to -\infty$ and $S_r \to -\infty$ w.p.1. by the hypothesis (8). This proves (32), which completes the induction step for s. □

7. Proofs of stationarity results. This section contains the proofs of Theorems 3 and 7 for stationary systems.

We will use the property that certain functions of stationary ergodic processes are also stationary and ergodic. This hereditary property is an immediate consequence of the definitions of stationarity and ergodicity.

LEMMA 15. *Suppose* $\{X_k: k \in \mathbb{Z}\}$ *is a stationary sequence and* $Y_k = \phi(\theta^k \mathbf{X})$, $k \in \mathbb{Z}$, *where* ϕ *is a measurable function. Then* $\{(X_k, Y_k): k \in \mathbb{Z}\}$ *is stationary, and it is also ergodic when* $\mathbf{X}$ *is.*

We will call a sequence $\mathbf{Y}$ as in Lemma 15 a *stationary functional of* $\mathbf{X}$. This concept is transitive in the following sense.

LEMMA 16. *If* $\mathbf{Y}$ *is a stationary functional of* $\mathbf{X}$ *and* $\mathbf{Z}$ *is a stationary functional of* $(\mathbf{X}, \mathbf{Y})$, *then* $\mathbf{Z}$ *is a stationary functional of* $\mathbf{X}$.

PROOF. Suppose ϕ, ψ are the measurable functions such that $Y_k = \phi(\theta^k \mathbf{X})$, $Z_k = \psi(\theta^k(\mathbf{X}, \mathbf{Y}))$. Then $Z_k = f(\theta^k \mathbf{X})$, where $f(\mathbf{x}) = \psi(\mathbf{x}, \{\phi(\theta^k \mathbf{x})\})$ is clearly measurable. This proves the assertion. □

We are now ready to prove the Loynes-type result on the existence of stationary networks.

PROOF OF THEOREM 3. For each stage $s \geq 1$, define $\mathbf{U}(s) = (\mathbf{U}^j: j \in J_s)$, and define $\mathbf{V}(s), \mathbf{W}(s), \mathbf{Q}(s)$ similarly. Keep in mind that s is not a time index —Theorems 7 and 12 use $\mathbf{W}(t)$ to denote waiting time processes where t is a time parameter. To prove Theorem 3, it suffices by Lemma 15 to show that $\mathbf{U}(s), \mathbf{W}(s), \mathbf{Q}(s)$ are finite stationary functionals of the data $\boldsymbol{\xi}^\dagger$ and (11) holds for each $j \in J_s$. We will prove this statement by induction on s. Since the argument for $s = 1$ is essentially the same as that for a general s, we will present only the latter (more interesting) case.

Accordingly, assume the induction hypothesis is true for some $s - 1 \geq 1$. Suppose $J_s \setminus J_{s-1}$ is not empty; otherwise the induction is already complete. It follows from (4) that we can write

$$\mathbf{U}_k(s) = f\big(\theta^k(\mathbf{R}, \mathbf{U}(s-1), \mathbf{V}(s-1), \mathbf{W}(s-1))\big), \tag{33}$$

where $f = (f_j: j \in J_s)$, $k_1^j(\mathbf{r}) = \min\{l > 0: r_l^j = 1\}$ and

$$f_j(\mathbf{r}, \mathbf{u}, \mathbf{v}, \mathbf{w}) = \begin{cases} u_0^1, & \text{if } j = 1, \\ r_0^j \left(\displaystyle\sum_{l=0}^{k_1^j - 1} u_l^{j^-} + v_{k_1^j(r)}^{j^-} - v_0^{j^-} + w_{k_1^j(r)}^{j^-} - w_0^{j^-} \right), & \text{if } j \in J_s \setminus \{1\}. \end{cases} \tag{34}$$

Here $\mathbf{r}, \mathbf{u}, \mathbf{v}, \mathbf{w}$ are in the space of realizations of $(\mathbf{R}, \mathbf{U}(s-1), \mathbf{V}(s-1), \mathbf{W}(s-1))$. This random vector is a stationary functional of $\boldsymbol{\xi}^\dagger$ by the induction hypothesis, and an easy check shows that f is a measurable function. Thus, Lemmas 15 and 16 ensure that $\mathbf{U}(s)$ is a stationary functional of $\boldsymbol{\xi}^\dagger$. Also, $\mathbf{U}(s)$ is clearly finite since each term on the right side of (33) is finite.

Next, observe that, similarly to (33), we can write

$$\mathbf{W}_k(s) = g\big(\theta^k(\mathbf{R}, \mathbf{U}(s), \mathbf{V}(s))\big), \qquad \mathbf{Q}_k(s) = h\big(\theta^k(\mathbf{R}, \mathbf{U}(s), \mathbf{V}(s), \mathbf{W}(s))\big),$$

where $g = (g_j\colon j \in J_s)$, $h = (h_j\colon j \in J_s)$ and g_j and h_j are the functions defined on the right sides of equations (5) and (6), respectively. Clearly g and h are measurable. Then Lemma 15 ensures that $\mathbf{W}(s)$ is a stationary functional of $(\mathbf{R}, \mathbf{U}(s), \mathbf{V}(s))$. The latter is a stationary functional of $\boldsymbol{\xi}^{\dagger}$, and so Lemma 16 ensures that $\mathbf{W}(s)$ is a stationary functional of $\boldsymbol{\xi}^{\dagger}$. Similarly, $\mathbf{Q}(s)$ is a stationary functional of $\boldsymbol{\xi}^{\dagger}$. The finiteness of $\mathbf{W}(s)$ and $\mathbf{Q}(s)$ will be established shortly.

We now prove (11) for $j \in J_s$. Consider a fixed j and let $\overline{U}_i^j, \overline{V}_k^j, \overline{W}_i^j$ denote interarrival, service and waiting times of unit K_k^j (the k-unit that actually visits node j). Then from (4),

$$\overline{U}_k^j = \overline{U}_k^{j^-} + \overline{V}_{k+1}^{j^-} - \overline{V}_k^{j^-} + \overline{W}_{k+1}^{j^-} - \overline{W}_k^{j^-}. \tag{35}$$

Iterating this on j^- and all the other nodes back to node 1, it follows that

$$\overline{U}_k^j = \overline{U}_k^1 + \sum_{l \in B_{j^-}} \left(\overline{V}_{k+1}^l - \overline{V}_k^l + \overline{W}_{k+1}^l - \overline{W}_k^l \right),$$

where B_{j^-} is the set of nodes comprising the 1-to-j^- branch. Summing the last equation on k, we get

$$\sum_{k=1}^{n} \overline{U}_k^j = \sum_{k=1}^{n} \overline{U}_k^1 + \zeta_n, \tag{36}$$

where

$$\zeta_n = \sum_{l \in B_{j^-}} \left(\overline{V}_{n+1}^l - \overline{V}_1^l + \overline{W}_{n+1}^l - \overline{W}_1^l \right).$$

Since $\{(\mathbf{R}_k, \mathbf{U}_k(s), \mathbf{V}_k(s), \mathbf{W}_k(s))\colon k \in \mathbb{Z}\}$ is stationary and ergodic, it follows by Lemma 5 that $\{(\overline{\mathbf{U}}_i(s), \overline{\mathbf{V}}_i(s), \overline{\mathbf{W}}_i(s))\colon i \in \mathbb{Z}\}$ is stationary and ergodic under the Palm probability P_j. Therefore, by the ergodic theorem for Palm probabilities, we have the following convergences w.p.1 under P_j;

$$n^{-1} \sum_{i=1}^{n} \overline{U}_i^j \to E_j\left(\overline{U}_0^j\right), \qquad n^{-1} \zeta_n \to 0,$$

$$n^{-1} \left(\sum_{i=1}^{n} \overline{U}_i^1 + \zeta_n \right) \to E_j\left(\overline{U}_0^1\right).$$

Now because of the equality (36), the preceding first and third limits must be equal, which proves (11) (here $K_0^j = 0$ and $\overline{U}_0^j = U_0^j$ w.p.1 under P_j).

It remains to prove that $\mathbf{W}(s)$ and $\mathbf{Q}(s)$ are finite. They will be finite by Theorem 2, provided that (8) holds for $j \in J_s$. Now, the ergodic theorem under P_j and conditions (10), (11) imply that

$$n^{-1} \sum_{i=-n}^{0} \left(\overline{V}_i^j - \overline{U}_i^j \right) \to E_j\left(\overline{V}_0^j\right) - E_j\left(\overline{U}_0^j\right) \le 0 \quad \text{w.p.1 under } P_j.$$

This limit is also true under P; see for instance, Theorem A1.3.4 of Brandt, Franken and Lisek (1990). Thus (8) follows for $j \in J_s$. $\square$

The following is the proof that the queue length and waiting time processes are stationary in continuous time if the data is.

PROOF OF THEOREM 7. First note that the quantity in the system at time t is

$$Q^j(t) = \sum_{k=1}^{\infty} Q_k^j I\left(A_{k-1}^j < t \le A_k^j\right) \tag{37}$$

(only one of the terms in the sum is not zero). A short proof shows that $A_k^j = A_k^1 + S_k^j$, where $S_k^j = h_j(\theta^k(\mathbf{R}, \mathbf{U}, \mathbf{V}))$ for a measurable function h_j. Then we can write

$$Q^j(t+u) = \sum_k Q_k^j I\left(A_k^1 - u + S_k^j \le t < A_{k+1}^1 - u + S_{k+1}^j\right).$$

A similar expression holds for the waiting time $W^j(y+u)$. From these representations and the continuous-time stationarity of the system data, it follows that the distribution of $\theta^u(\mathbf{Q}, \mathbf{W})$ is independent of u. Thus $(\mathbf{Q}, \mathbf{W})$ is stationary in continuous time. $\square$

8. Proofs of asymptotic stationarity results. We begin with the proof that the network is asymptotically stationary if its data is.

PROOF OF THEOREM 8. For each stage $s \ge 1$, define the one-sided interarrival vector $\mathbf{U}(s) = \{(U_k^j \colon j \in J_s) \colon k \ge 0\}$, and define the service, waiting and queue length vectors $\mathbf{V}(s), \mathbf{W}(s), \mathbf{Q}(s)$ similarly (keep in mind that s is not a time parameter). We will show by induction that $(\mathbf{R}, \mathbf{U}(s), \mathbf{V}(s), \mathbf{W}(s), \mathbf{Q}(s))$ is asymptotically stationary for each s. Again, the argument for $s = 1$ is similar to that for general s, and so we present only the latter case. To this end, assume the induction hypothesis for $s - 1 \ge 1$ and assume $J_s \setminus J_{s-1}$ is not empty. We will prove the induction statement for s by establishing the following properties:

(i) $\mathbf{U}(s)$ is an asymptotically stationary functional of $(\mathbf{R}, \mathbf{U}(s-1), \mathbf{V}(s-1), \mathbf{W}(s-1))$;

(ii) $\mathbf{W}(s)$ is an asymptotically stationary functional of $(\mathbf{R}, \mathbf{U}(s), \mathbf{V}(s))$;

(iii) $\mathbf{Q}(s)$ is an asymptotically stationary functional of $(\mathbf{R}, \mathbf{U}(s), \mathbf{V}(s), \mathbf{W}(s))$.

To prove (i), first note that, similarly to (33) for the two-sided vectors, we can write

$$U_k(s) = f\left(\theta^k(\mathbf{R}, \mathbf{U}(s-1), \mathbf{V}(s-1), \mathbf{W}(s-1))\right).$$

A straightforward proof shows that the function $f = (f_j \colon j \in J_s)$ is continuous. By the induction hypothesis, we know that $(\mathbf{R}, \mathbf{U}(s-1), \mathbf{V}(s-1), \mathbf{W}(s-1))$ is asymptotically stationary. Then to prove statement (i), it suffices by Theorem 10 to show that the limit $(\tilde{\mathbf{R}}, \tilde{\mathbf{U}}(s-1), \tilde{\mathbf{V}}(s-1), \tilde{\mathbf{W}}(s-1))$ is in the domain of f w.p.1. We need only check that $\tilde{K}_1^j < \infty$ w.p.1 for each j, which is equivalent to $\sum_{k=0}^{\infty} \tilde{R}_k^j = \infty$ w.p.1. However, this is true since this sum is the limit in distribution as $n \to \infty$ of $\sum_{k=0}^{\infty} R_{n+k}^j$ which equals ∞ w.p.1. by assumption (2).

Next consider statement (ii). From (5), we have $\mathbf{W}_k(s) = \phi_k(\theta^k \eta)$, where $\phi_k = (\phi_k^j\colon j \in J_s)$,

$$\phi_k^j(\mathbf{r}, \mathbf{u}, \mathbf{v}) = r_0^j \max_{0 \le l \le k-1} \left[\sum_{i=-l}^{-1} (v_i^j - u_i^j) \right], \tag{38}$$

$$\boldsymbol{\eta}_k = \begin{cases} (\mathbf{R}_0, 0, 0), & \text{if } k < 0, \\ (\mathbf{R}_k, \mathbf{U}_k(s), \mathbf{V}_k(s)), & \text{if } k \ge 0. \end{cases}$$

By the induction hypothesis, $(\mathbf{R}, \mathbf{U}(s), \mathbf{V}(s))$ is asymptotically stationary. Then clearly $\boldsymbol{\eta}$ is also asymptotically stationary and its limit is $\tilde{\boldsymbol{\eta}} = (\tilde{\mathbf{R}}, \tilde{\mathbf{U}}(s), \tilde{\mathbf{V}}(s))$.

To prove statement (ii), it suffices to show that $\mathbf{W}(s)$ is an asymptotically stationary functional of $\boldsymbol{\eta}$. To this end, we will apply Theorem 11. Define $\phi^j(r, u, v)$ by the right side of (38) with $k = \infty$. Note that $\phi(\tilde{\boldsymbol{\eta}})$ equals $\tilde{\mathbf{W}}_0(s) = (\tilde{W}_0^j\colon j \in J_s)$. By Theorem 2, the $\phi(\tilde{\boldsymbol{\eta}})$ will be finite if (8) holds for the limit $\tilde{\boldsymbol{\xi}}$. But this follows by the ergodic theorem and the assumption that $\tilde{\boldsymbol{\xi}}$ satisfies (10). It is also clear that $\phi_k(\tilde{\boldsymbol{\eta}}) \to \phi(\tilde{\boldsymbol{\eta}})$ w.p.1 as $k \to \infty$. Next, note that each ϕ_k is continuous, and so $\tilde{\boldsymbol{\eta}}$ is in the continuity set of $\boldsymbol{\phi}_k$ w.p.1. Finally, by the definition of ϕ_k, the condition

$$\lim_{n' \to \infty} \limsup_{n \to \infty} P\{ |\phi_{n'}(\theta^n \eta) - \phi_n(\theta^n \eta)| > \varepsilon \} = 0$$

is implied by assumption (12). Thus, by Theorem 11, $\mathbf{W}(s)$ is an asymptotically stationary functional of η.

We now prove statement (iii) by a similar argument. From (6), we know that $\mathbf{Q}_k(s) = \boldsymbol{\psi}_k(\theta^k \boldsymbol{\zeta})$, where $\boldsymbol{\psi}_k = (\psi_k^j\colon j \in J_s)$,

$$\zeta_k = \begin{cases} (\mathbf{R}_k, \mathbf{U}_k(s), \mathbf{V}_k(s), \mathbf{W}_k(s)), & \text{if } k \ge 0, \\ (R_0, 0, 0, 0,), & \text{if } k < 0, \end{cases}$$

$$\psi_k^j(\mathbf{r}, \mathbf{u}, \mathbf{v}, \mathbf{w}) = r_0^j \sum_{l=1}^{k-1} I\left(r_{-l}^j - 1 + w_{-l}^j + v_{-l}^j \ge \sum_{i=1}^{l} u_{-i}^j \right). \tag{39}$$

By the induction hypothesis, it follows that $\boldsymbol{\zeta}$ is asymptotically stationary with limit $\tilde{\boldsymbol{\zeta}} = (\tilde{\mathbf{R}}, \tilde{\mathbf{U}}(s), \tilde{\mathbf{V}}(s), \tilde{\mathbf{W}}(s))$. Define $\psi^j(\mathbf{r}, \mathbf{u}, \mathbf{v}, \mathbf{w})$ by the right side of (39) with $k = \infty$ and let $\psi = (\psi^j\colon j \in J_s)$. The $\psi(\tilde{\boldsymbol{\zeta}})$ is the two-sided $\tilde{\mathbf{Q}}_0(s) = (\tilde{Q}_0^j\colon j \in J_s)$, which is finite by Theorem 2 since assumption (8) holds for the limit $\tilde{\boldsymbol{\xi}}$. Clearly, $\psi_k(\tilde{\zeta}) \to \psi(\tilde{\zeta})$ w.p.1 as $k \to \infty$. Next, note that

$$\text{discontinuity set of } \psi_k \subset \bigcup_{j \in J_s} \bigcup_{l=1}^{k-1} D_{j_l}, \tag{40}$$

where

$$D_{j_l} = \left\{(\mathbf{r}, \mathbf{u}, \mathbf{v}, \mathbf{w}) \colon r^j_{-l} - 1 + w^j_{-l} + v^j_{-l} = \sum_{i=1}^{l} u^j_{-i}\right\}.$$

Since $\tilde{\boldsymbol{\zeta}}$ is stationary, the assumption (13) and Lemma 5 ensure $P\{\tilde{\boldsymbol{\zeta}} \in D_{j_l}\} = 0$ for each j, l. This and (40) imply that $\tilde{\boldsymbol{\zeta}}$ is in the continuity set of $\boldsymbol{\psi}_k$ w.p.1. Next, note that by the definition of ψ_k, it follows that

$$\left\{\left|\boldsymbol{\psi}_{n'}(\theta^n\boldsymbol{\zeta}) - \psi_n(\theta^n\boldsymbol{\zeta})\right| > \varepsilon\right\} = \bigcup_{j \in J_s} B^j(n', n),$$

where

$$B^j(n', n) = \left\{\tilde{R}^j_n = 1,\ \tilde{R}^j_{n-k} - 1 + \tilde{W}^j_{n-k} + \tilde{V}^j_{n-k} \geq \sum_{l=1}^{k} \tilde{U}^j_{n-l} \quad \text{for some } k \in \{n'+1, \dots, n\}\right\}, \tag{41}$$

which is the event in (14). This and (14) imply

$$\begin{aligned}\lim_{n' \to \infty} \limsup_{n \to \infty} P&\left\{\left|\boldsymbol{\psi}_{n'}(\theta^n\boldsymbol{\zeta}) - \psi_n(0^n\zeta)\right| > \varepsilon\right\}\\ &\leq \sum_{j \in J_s} \lim_{n' \to \infty} \limsup_{n \to \infty} P\left\{B^j(n', n)\right\} = 0.\end{aligned}$$

Thus, by Theorem 11, $\mathbf{Q}(s)$ is an asymptotically stationary functional of $\boldsymbol{\zeta}$, which proves statement (iii). This completes the proof of Theorem 8; the last assertion of Theorem 8 follows from Theorem 3. □

The following is the proof of the key continuous-mapping principle for asymptotic stationarity.

PROOF OF THEOREM 11. Because E and $E^{\mathbb{Z}}$ are Polish spaces, the convergence $(\theta^n\mathbf{X}, \theta^n\mathbf{Y}) \to_{\mathscr{D}} (\tilde{\mathbf{X}}, \tilde{\mathbf{Y}})$, is equivalent to the finite-dimensional vector convergence $h_n(\theta^n\mathbf{X}) \to h(\tilde{\mathbf{X}})$, for each I, where

$$\begin{aligned} h_n(\mathbf{x}) &= \left(\left(x_k, \phi_{n+k}(\theta^k\mathbf{x})\right) \colon k \in I\right),\\ h(\mathbf{x}) &= \left(\left(x_k, \phi(\theta^k\mathbf{x})\right) \colon k \in I\right)\end{aligned} \tag{42}$$

and I is an interval. Then the assertion to be proved is that

$$h_n(\theta^n\mathbf{X}) \to_{\mathscr{D}} h(\tilde{\mathbf{X}}) \quad \text{for each } I, \text{ if and only if (17) holds.}$$

First suppose that (17) holds. Let ρ denote the Prokhorov metric for distributions of random elements. Then

$$\begin{aligned}\rho\left(h_n(\theta^n\mathbf{X}), h(\tilde{\mathbf{X}})\right) &\leq \rho\left(h_n(\theta^n\mathbf{X}), h_{n'}(\theta^n\mathbf{X})\right)\\ &\quad + \rho\left(h_{n'}(\theta^n\mathbf{X}), h_{n'}(\tilde{\mathbf{X}})\right) + \rho\left(h_{n'}(\tilde{\mathbf{X}}), h(\tilde{\mathbf{X}})\right).\end{aligned}$$

Now the last two terms converge to 0 as n and n' tend to ∞ by (16), (15) and the asymptotic stationarity of $\mathbf{X}$. Also, using the metric α for convergence in probability [see, for instance, Dudley (1968)], we have

$$\rho\big(h_n(\theta^n \mathbf{X}), h_{n'}(\theta^n \mathbf{X})\big) \le \alpha\big(h_n(\theta^n \mathbf{X}), h_{n'}(\theta^n \mathbf{X})\big) = \alpha\big(\phi_n(\theta^n \mathbf{X}), \phi_{n'}(\theta^n \mathbf{X})\big).$$

These observations and (17) yield

$$\lim_{n\to\infty} \rho\big(h_n(\theta^n \mathbf{X}), h(\tilde{\mathbf{X}})\big) \le \lim_{n'\to\infty} \limsup_{n\to\infty} \alpha\big((\phi_{n'}(\theta^n \mathbf{X}), \phi_n(\theta^n \mathbf{X}))\big) = 0.$$

Thus $h_n(\theta^n \mathbf{X}) \to_{\mathscr{D}} h(\tilde{\mathbf{X}})$ for each I.

Conversely, assume the last convergence statement holds. Then using the Skorohod w.p.1 representation of convergence in distribution, one can show [as with Theorem 1 in Szczotka (1990)] that there exist $\mathbf{X}_n^*, \mathbf{X}^*$ on a common probability space such that $\mathbf{X}_n^* =_{\mathscr{D}} \theta^n \mathbf{X}$, $\mathbf{X}^* =_{\mathscr{D}} \tilde{\mathbf{X}}$ and

$$h_n(\mathbf{X}_n^*) \to h(\mathbf{X}^*) \quad \text{w.p.1 as } n \to \infty. \tag{43}$$

Let $\bar{d}$ be the metric on $E^I \times E'^I$ defined by

$$\bar{d}((\mathbf{x}, \mathbf{y}), (\tilde{\mathbf{x}}, \tilde{\mathbf{y}})) = \sum_{k \in l} \big[d(x_k, \tilde{x}_k) + d'(y_k, \tilde{y}_k)\big].$$

Now, applying (43),

$$\begin{aligned} P\big\{d'(\phi_{n'}(\theta^n \mathbf{X}), \phi_n(\theta^n \mathbf{X})) > \varepsilon\big\} &= P\big\{\bar{d}(h_{n'}(\mathbf{X}_n^*), h_n(\mathbf{X}_n^*)) > \varepsilon\big\} \\ &\le P\big\{\bar{d}(h_{n'}(\mathbf{X}_n^*), h(\mathbf{X}^*)) > \varepsilon/2\big\} \\ &\quad + P\big\{\bar{d}(h(\mathbf{X}^*), h_n(\mathbf{X}_n^*)) > \varepsilon/2\big\} \\ &\to P\big\{\bar{d}(h_{n'}(\mathbf{X}^*), h(\mathbf{X}^*)) > \varepsilon/2\big\} \quad \text{as } n \to \infty. \end{aligned}$$

Assumption (15) ensures that the last term also converges to 0 as $n' \to \infty$. These observations prove (17). □

9. Proofs of heavy-traffic results.

The following proofs concern the heavy-traffic limiting behavior of the waiting time process.

PROOF OF THEOREM 12. We begin by deriving a convenient representation for the waiting time process. By (5), we have

$$\begin{aligned} W_n^j(t) &= b_n^{-1} \max_{0 \le k \le [a_n t]-1} \left[\sum_{\nu=[a_n t]-k}^{[a_n t]-1} \big(V_\nu^j - U_\nu^j\big)\right] \\ &= \max_{1 \le k \le [a_n t]} \big(\bar{X}_n^j(t) - \bar{X}_n^j(k/a_n)\big) = \sup_{s \le t} \big(\bar{X}_n^j(t) - \bar{X}_n^j(s)\big), \end{aligned}$$

where

$$\bar{X}_n^j(t) = b_n^{-1} \sum_{\nu=1}^{[a_n t]-1} \big(V_\nu^j - U_\nu^j\big).$$

That is,

$$W_n^j(t) = h\big(\bar{X}_n^j\big)(t). \tag{44}$$

Next, we express $\bar{X}_n^j$ in terms of X_n. Note that by the definition of arrival times,

$$(45)\qquad \sum_{\nu=1}^{k-1} U_\nu^j = A_k^j - A_1^j, \quad \text{and} \quad A_k^j = A_{K_k^j}^1 + \sum_{i \in B_{j^-}} \left(W_{\gamma_k}^i + V_{\gamma_k}^i\right),$$

where $\gamma_k \equiv \gamma_k(i,j) \equiv \sum_{\nu=1}^{K_k^j} R_\nu^i$, which is the local rank at node i of unit K_k^j (the kth unit that enters j). The last sum in (45) is the time that unit K_k^j spends at nodes in B_{j^-}. From these expressions,

$$\sum_{\nu=1}^{k-1} \left(V_\nu^j - U_\nu^j\right) = \left[\sum_{\nu=1}^{k} V_\nu^j - A_{K_k^j}^1\right] - V_k^j + A_1^j - \sum_{i \in B_{j^-}} \left(W_{\gamma_k}^i + V_{\gamma_k}^i\right).$$

Then the stochastic process version of this equality (where k is replaced by $[a_n t]$) is

$$(46)\qquad \begin{aligned} \bar{X}_n^j(t) &= X_n^j(t) - \sum_{i \in B_{j^-}} W_n^i\left(a_n^{-1}\gamma_{[a_n t]}\right) - b_n^{-1} V_{[a_n t]}^j \\ &\quad + b_n^{-1} A_1^j - b_n^{-1} \sum_{i \in B_{j^-}} V_{\gamma_{[a_n t]}}^i. \end{aligned}$$

Finally, note that

$$(47)\qquad a_n^{-1}\gamma_{[a_n t]} = \tau_n^i \circ \hat{\tau}_n^j(t) - a_n^{-1},$$

since $R_{K_k^j}^i = 1$ and

$$(48)\qquad \hat{\tau}_n^j(t) = \inf\left\{s: \sum_{\nu=1}^{[a_n s]} R_\nu^j > a_n t\right\} = a_n^{-1} K_{[a_n t]+1}^j.$$

The limiting behavior of the preceding random processes, which is the issue before us, is established via the following weak convergence properties concerning h and the composition, addition and inverse mappings; see Whitt (1980).

(i) If $Z_n \to_{\mathscr{D}} Z$ in D and Z has stochastically continuous paths, then $h(Z_n) \to_{\mathscr{D}} h(Z)$ and $h(Z)$ also has stochastically continuous paths.
(ii) If $(Y_n, Z_n) \to_{\mathscr{D}} (Y, Z)$ in D^2 and the discontinuity sets of the processes Y and Z are disjoint w.p.1 (which is true if one has stochastically continuous paths), then $Y_n + Z_n \to_{\mathscr{D}} Y + Z$.
(iii) If $(Z_n, \tau_n) \to_{\mathscr{D}} (Z, \tau)$ in D^2, where τ_n has nondecreasing paths, τ has continuous nondecreasing paths and Z has stochastically continuous paths, then $Z_n \circ \tau_n \to_{\mathscr{D}} Z \circ \tau$ and $Z \circ \tau$ also has stochastically continuous paths.
(iv) If $\tau_n \to_{\mathscr{D}} \tau$ in D, where τ_n has nondecreasing paths and τ has continuous nondecreasing paths, then $\hat{\tau}_n \to_{\mathscr{D}} \hat{\tau}$ in D.

We are now ready to establish the asserted convergence statements. Recall that the set of nodes in the network reachable within s steps is J_s. We will prove by induction on s that

$$(49)\qquad \left\{\left(X_n^j, W_n^j, \tau_n^j, \hat{\tau}_n^j\right): j \in J_s\right\} \to_{\mathscr{D}} \left\{\left(X^j, W^j, \tau^j, \hat{\tau}^j\right): j \in J_s\right\}, \qquad s \ge 1,$$

and each W^j has stochastically continuous paths. Consider this convergence for $s = 1$, which is only for $j = 1$. From (44) and (46), we have $W_n^1 = h(X_n^1 - b_n^{-1}V_{[a_n \cdot]}^1 + b_n^{-1}A_1^1)$, since B_{1^-} is the empty set. Also, under the hypotheses, $(X_n^1, b_n^{-1}V_{[a_n \cdot]}^1, b_n^{-1}A_1^1) \to_{\mathscr{D}} (X^1, 0, 0)$. These facts along with the hypotheses and preceding weak convergence properties justify (49) for $j = 1$.

Next, suppose (49) is true for some s and consider any $j \in J_{s+1} \setminus J_s$, assuming this set is nonempty—otherwise the induction is complete. Recall the representation (44), which is $W_n^j(t) = h(\overline{X}_n^j)(t)$, where $\overline{X}_n^j$ is given by (46). Applying the induction hypothesis and the other assumptions to (46) and using the preceding weak convergence properties, it follows that

$$\overline{X}_n^j \to_{\mathscr{D}} \overline{X}^j \equiv X^j - \sum_{i \in B_{j^-}} W^i \circ \tau^i \circ \hat{\tau}^j,$$

[the last three terms in (46) and the last term in (47) converge in distribution to zero], and $\overline{X}^j$ has stochastically continuous paths. Furthermore, by a similar argument for joint convergence,

$$\begin{aligned} &\left\{X_n^i, W_n^i, X_n^j, \overline{X}_n^j \colon i \in J_s, j \in J_{s+1} \setminus J_s\right\} \\ &\qquad \to_{\mathscr{D}} \left\{X^i, W^i, X^j, \overline{X}^j \colon i \in J_s, j \in J_{s+1} \setminus J_s\right\}. \end{aligned}$$

This along with $W_n^j(t) = h(\overline{X}_n^j)(t)$ and the weak convergence property of h yields (49) for $s + 1$. This completes the induction and the proof of Theorem 12. □

Proof of Corollary 13. Similarly to (46) and (47), we have

$$b_n^{-1} W^j_{\gamma_{[a_n t]}} = W_n^j\left(\tau_n^j \circ \hat{\tau}_n^l(t) - a_n^{-1}\right), \qquad b_n^{-1} W^j_{\gamma'_{[a_n t]}} = W_n^j \circ \tau_n^j(t).$$

Using these representations and the result (20) in weak convergence arguments like the proof of Theorem 12 yields (22) and (23). Analogously, (25) follows since one can show, under assumptions (19) and (24), that $(L_{[a_n t]}, (X_n^j, \tau_n^j)\colon 1 \le j \le M\})$ converges jointly in distribution with $\{W_n^j\colon 1 \le j \le M\}$. □

Proof of Example 14. Consider the process

$$\begin{aligned} X_n^j(t) &= n^{-1/2}\left(\sum_{k=1}^{[nt]} V_{n,k}^j - A^1_{K^j_{n,[nt]}}\right) \\ &= \sum_{l \in \mathscr{L}} n^{-1/2} \sum_{k=1}^{K^l_{n,[nt]}} \left(V_{n,k}^{jl} - EV_{n,1}^{jl}\right) - n^{-1/2} \sum_{k=1}^{K^j_{n,[nt]}} \left(U_{n,k}^1 - EU_{n,1}^1\right) \\ &\quad - n^{1/2}\left[n^{-1}K^j_{n,[nt]} EU^1_{n,1} - \sum_{l \in \mathscr{L}} n^{-1} K^l_{n,[nt]} EV^{jl}_{n,1}\right]. \end{aligned} \tag{50}$$

By property (iv) above, (48) and assumption (A1), it follows that the process $n^{-1}K^j_{n,[nt]}$ converges in distribution to the function $\lambda_j^{-1}t$. This and assumption (A4) ensure that the last line in (50) converges in distribution to $c_j t$.

Then applying assumptions (A2) and (A3) to (50) and using property (iii) and the convergence of $n^{-1}K^j_{n,[nt]}$, it follows that $(X^j_n: 1 \leq j \leq m) \to_{\mathscr{D}} (X^j: 1 \leq j \leq m)$, which are defined by (26). This and assumption (A1) yield the convergence (18). Also, assumption (A3) yields (19). Thus the assertion in Example 14 follows by Theorem 12. □

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