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EXISTENCE AND UNIQUENESS OF INFINITE COMPONENTS IN GENERIC RIGIDITY PERCOLATION[1]

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We consider a percolation configuration on a general lattice in which edges are included independently with probability p. We study the rigidity properties of the resulting configuration, in the sense of generic rigidity in d dimensions. We give a mathematically rigorous treatment of the problem, starting with a definition of an infinite rigid component. We prove that, for a broad class of lattices, there exists an infinite rigid component for some p strictly below unity. For the particular case of two-dimensional rigidity on the two-dimensional triangular lattice, we prove first that the critical probability for rigidity percolation lies strictly above that for connectivity percolation and second that the infinite rigid component (when it exists) is unique for all but countably many values of p. We conjecture that this uniqueness in fact holds for all p. Some of our arguments could be applied to two-dimensional lattices in more generality.

1. Introduction. We consider a "percolation model"; that is, starting with an infinite graph having some regular structure (a "lattice"), we delete some of the edges at random, while retaining the others. In the case which we shall consider, individual edges are retained independently of each other, each with probability p. Questions relating to the connectivity properties of the resulting graph have been studied extensively (see [5], [6]). In particular, it is known that (under a wide range of conditions) there exists a critical probability p_c, satisfying $0 < p_c < 1$, such that for $p > p_c$ there is almost surely a unique infinite connected component, while for $p < p_c$ there is almost surely no infinite connected component. The most elegant proof of the uniqueness of the infinite connected component may be found in [2].

We shall consider a somewhat different problem. Given a percolation model, we regard the retained edges as solid bars, which are able to pivot freely at the vertices. Our aim is to study the rigidity properties of the resulting structure. The questions which arise have important applications in physics, and the subject has been approached in the physics literature by means of (partly) nonrigorous arguments and numerical simulations. See [8] for the results of an extensive and innovative simulation study, together with a summary of physical applications and references to previous studies. The basic application is to the behavior of materials, and in particular glasses; the retained edges represent chemical bonds between atoms. Our aim here is to approach the subject from a rigorous mathematical standpoint.

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One of the necessary steps in a mathematical treatment of the problem is to formulate a precise definition of rigidity. In particular, we shall see that the rigidity of a graph may depend on the particular way in which it is embedded in Euclidean space. However, it has been shown (see [3], [4]) that for "almost all" embeddings of a particular finite graph, the rigidity properties are identical, and this gives rise to a definition of "generic rigidity" for abstract graphs. We shall restrict our attention to generic rigidity. (This is also the approach taken in [8].) This restriction makes the concept of rigidity easier to deal with from a mathematical point of view, and it is also regarded as providing a realistic physical model of "glassy" materials. It should be noted that the subject of graph rigidity is of interest in itself from a combinatorial point of view, and we shall make some use of results from this area. The most important reference in this context is [4].

The main results presented here are as follows. Starting from the definition of generic rigidity for finite graphs, we formulate a definition of rigidity for infinite graphs, and use this to define rigidity percolation. We prove that for a broad class of lattices, the rigidity critical probability lies strictly below unity (Theorem 4.1). In the subsequent sections we restrict our attention to the particular case of two-dimensional generic rigidity percolation on the two-dimensional triangular lattice (as in [8]), although it is believed that our techniques could be applied to other lattices. For the triangular lattice we prove first, that the rigidity critical probability lies strictly above the connectivity critical probability (Theorem 7.1), and second, that for all p lying outside a particular (perhaps empty) countable set, the infinite rigid component (when it exists) is unique (Theorem 8.1). Our proof of the strict inequality of critical probabilities confirms some of the numerical findings of [8] (for example), where the rigidity critical probability is estimated to be 0.66020 ± 0.0003; the connectivity critical probability for the triangular lattice is known to be $2\sin(\pi/18) = 0.34730\cdots$. The uniqueness of the infinite rigid component is sometimes implicitly assumed in the physics literature, although it does not appear to have been mentioned explicitly, perhaps partly because there has not previously been any explicit definition of an infinite rigid component.

The paper is organized as follows. In Section 2 we present the concept of generic rigidity of finite graphs and give some standard results. In Section 3 we extend the definition of rigidity to infinite graphs and discuss rigidity of "lattices." In Section 4 we define rigidity percolation and derive some basic results which are valid for a wide range of lattices and in an arbitrary number of dimensions. The remaining sections are devoted to the study of two-dimensional rigidity percolation on the triangular latttice. In Section 5 we state and discuss our main results. In Section 6 we give some further results on rigidity which will be needed, and finally in Sections 7 and 8 we give proofs of the two main results.

2. Rigidity of finite graphs. In this section we define the concept of rigidity for an embedding of a graph in d dimensions, and describe how this may be used to define generic rigidity in d dimensions. This material will be treated fairly briefly. For a full account see, for example, [4]. See also Section 2 of [7] for a useful summary of generic rigidity.

Our approach to graph theory will be slightly unconventional, since it will be convenient to regard a graph simply as a set of edges rather than as a pair (V, E). For any set A, we write $\mathscr{P}(A)$ for the set of all subsets of A (the power set), and $A^{(r)}$ for the set of all subsets of A of size r (where $r \in \mathbb{N}$). Given an underlying set $\mathscr{V}$ (which we will regard as a set of *vertices*), we refer to $\mathscr{V}^{(2)}$ as the set of *edges* on $\mathscr{V}$. By a *graph* we mean any nonempty subset of $\mathscr{V}^{(2)}$, and by a *subgraph* we mean a nonempty subset of a graph. We define the *vertex set* of a graph E to be

$$V(E) = \bigcup_{e \in E} e,$$

and for a graph E and a set of vertices $U \subseteq V(E)$ we define

$$E(U) = \big\{\{x, y\} \in E\colon x, y \in U\big\}.$$

Given a graph E we write $\Delta_E(\cdot, \cdot)$ for graph-theoretic distance between pairs of vertices in $V(E)$. We shall sometimes say that a graph E *contains* a vertex x to mean that $x \in V(E)$.

Let E be a finite graph. An *embedding* of E in d dimensions is an injective mapping

$$r\colon V(E) \to \mathbb{R}^d.$$

A *framework* (E, r) is a graph together with an embedding.

A *motion* of a framework (E, r) is a differentiable family of embeddings $(r(t)\colon t \in [0, 1])$ of E (in some fixed number of dimensions) including r which preserves all edges lengths; that is, for every $\{x, y\} \in E$, $\|r(t)(x) - r(t)(y)\|$ is constant in t (where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^d$). A motion is a *rigid motion* if this holds for *every* pair $x, y \in V(E)$. A framework is *rigid* if all its motions are rigid motions.

The above definition of rigidity depends on the embedding r as well as on the graph E. However, we may define rigidity (in d dimensions) for an abstract graph via the concept of generic embeddings as follows. We say an embedding r is *generic* if the sequence of coordinates $(r(x)_i\colon x \in V(E),\ 1 \le i \le d)$ contains no repetitions, and the corresponding set $C = \{r(x)_i\colon x \in V(E), 1 \le i \le d\}$ is algebraically independent over the rationals [that is, any relation of the form $q_1(z_1^{\alpha_{1,1}} \cdots z_n^{\alpha_{1,n}}) + \cdots + q_m(z_1^{\alpha_{m,1}} \cdots z_n^{\alpha_{m,n}}) = 0$ where $q_1, \dots, q_m \in \mathbb{Q}$ and $\alpha_{1,1}, \alpha_{1,2}, \dots, \alpha_{m,n} \in \{0, 1, \dots\}$ which is satisfied by some $z_1, \dots, z_n \in C$ must be an "identity" satisfied by any $z_1, \dots, z_n \in \mathbb{R}$]. The idea of this definition is that a generic embedding cannot have any "special symmetries." In fact, the above condition is stronger than that which is actually required; for more details see [3] and [4]. Note that (with respect to Lebesgue measure on $\mathbb{R}^{d|V(E)|}$), almost all embeddings are generic.

The following result is essentially due to Gluck [3]. For further details see [4].

THEOREM 2.1. *For any given finite graph E and any fixed $d \geq 1$, either all generic embeddings of E are rigid, or all generic embeddings of E are not rigid.*

We say a graph is *generically rigid in d dimensions*, or simply *d-rigid*, if any (all) of its generic embeddings in d dimensions are rigid.

We shall now state, without proof, a number of standard results about d-rigidity. For proofs, see, for example, [4]. In all the immediately following propositions we assume that all the graphs mentioned are finite.

PROPOSITION 2.2. *A graph is* 1*-rigid if and only if it is connected.*

PROPOSITION 2.3. *For any $d > 1$, a d-rigid graph is $(d-1)$-rigid. (Thus, in particular, for any $d \geq 1$, a d-rigid graph is connected).*

PROPOSITION 2.4. *If A and B are d-rigid graphs and $|V(A) \cap V(B)| \geq d$, then $A \cup B$ is d-rigid.*

Proposition 2.4 expresses an important property of rigidity which we shall use repeatedly: if we join two d-rigid graphs together by identifying d vertices of one with d vertices of the other, the resulting graph is d-rigid. One possible proof depends on the fact that the only "small" isomorphism of $\mathbb{R}^d$ which fixes d generically embedded points is the identity. We shall also give a simple proof in the case $d = 2$ (Proposition 6.7) assuming the combinatorial characterization of 2-rigidity afforded by Laman's theorem (Theorem 6.3).

PROPOSITION 2.5. *If A is a d-rigid graph with $|V(A)| \geq d$ then $|A| \geq d|V(A)| - d(d+1)/2$.*

Proposition 2.5 will be used only in the proof of Proposition 3.2 concerning rigidity of general regular graphs. However, we shall make use of the similar but much stronger result of Laman's theorem (Theorem 6.3) in the case $d = 2$. The proof of the above proposition depends on the idea of "constraint-counting"; a set of $|V(A)|$ vertices has $d|V(A)|$ degrees of freedom, while a rigid body in d dimensions has $d(d+1)/2$, so at least $d|V(A)| - d(d+1)/2$ edge-constraints are required to induce rigidity.

3. Infinite rigidity, rigid components and lattices. In order to study rigidity percolation, we must extend our definition of rigidity to infinite graphs. There are several possible ways to do this, but we will adopt the following approach. Let A be any (possibly infinite) graph. We say A is *d-rigid* if every finite subgraph of A is a subgraph of some finite d-rigid subgraph of A (note that this is consistent with the existing definition in the case when A is finite).

META-PROPOSITION 3.1. *The statements of Propositions* 2.2, 2.3 *and* 2.4 *all hold if we allow the graphs concerned to be infinite.*

PROOF. In each case the result may be deduced easily from the above definition of rigidity for infinite graphs.

For example, to prove the "infinite version" of Proposition 2.4, suppose A and B are possibly infinite graphs satisfying the conditions of the proposition. Then if $E \subseteq A \cup B$ is a finite graph, we may find finite d-rigid graphs $A' \subseteq A$ and $B' \subseteq B$ such that $A' \supseteq A \cap E$, $B' \supseteq B \cap E$ and $|V(A') \cap V(B')| \geq d$. We now appeal to the original proposition. $\square$

For the sake of convenience, we shall henceforth use the original numbers 2.2, 2.3 and 2.4 to refer to the extensions of these propositions implied by Meta-proposition 3.1.

By a *d-rigid component* of a graph, we mean a maximal d-rigid subgraph. Note that by Proposition 2.4, if A and B are distinct rigid components of E, then $|V(A) \cap V(B)| < d$. Note also that in the case $d = 2$, this implies $A \cap B = \varnothing$. (Recall that A and B are sets of edges). Thus the 2-rigid components of E partition E.

In order for the phenomenon of rigidity percolation to be of any interest, we must work on an infinite graph which is itself d-rigid (for some $d > 1$). The following result gives some conditions under which this is *not* the case.

Given a graph E, for any subgraph $A \subseteq E$, we define

$$D(A) = \bigl\{\{x, y\} \in E\colon x \in V(A),\ \ y \notin V(A)\bigr\}.$$

Now define

$$S_n = \inf\bigl\{|D(A)|\colon A \subseteq E, |V(A)| = n\bigr\}.$$

(We interpret S_n as the "minimum surface of a sphere of size n" in E).

PROPOSITION 3.2. *Suppose E is an infinite regular graph of degree δ (that is, with every vertex having degree δ). If either*

(i) $\delta < 2d$,

or

(ii) $\delta \leq 2d$ *and* $S_n \to \infty$ *as* $n \to \infty$,

then E has no finite d-rigid subgraphs with vertex set larger than N for some N. Hence E has no infinite rigid subgraphs.

PROOF. Let A be a subgraph of E with $|V(A)| = n$. A simple counting argument shows that

$$|A| = \frac{\delta n - |D(A)|}{2}.$$

We may now appeal to Proposition 2.5. In case (i), the proposition implies that A is not rigid provided $n > \max\{d, d(d+1)/(2d-\delta)\}$. In case (ii), A is not rigid provided $n \geq d$ and n is large enough to ensure that $|D(A)| \geq d(d+1)$. $\square$

In particular, we note that the "square lattice" (usually written $\mathbb{L}^2$) is not 2-rigid, so it is of no interest to us. For this reason, we shall mainly study the (two-dimensional) "triangular lattice" $\mathbb{T}$, which we shall now define formally.

Define $V \subset \mathbb{R}^2$ by

$$V = \{a(1,0) + b(1/2, \sqrt{3}/2): a, b \in \mathbb{Z}\},$$

and define the origin $O = (0,0) \in V$. We use $+$ to denote vector space addition on V and $\|\cdot\|$ for Euclidean distance. We define the *triangular lattice* $\mathbb{T} \subset V^{(2)}$ by

$$\mathbb{T} = \{\{x, y\}: \|x - y\| = 1\},$$

and note that $V = V(\mathbb{T})$.

We shall make use of several subgraphs of $\mathbb{T}$ which for convenience we define here. We define the hexagons centered at O:

$$H(n) = \mathbb{T}(\{x: \Delta_{\mathbb{T}}(O, x) \le n\}),$$
$$\partial H(n) = \mathbb{T}(\{x: \Delta_{\mathbb{T}}(O, x) = n\}).$$

[Recall that $\mathbb{T}(X)$ denotes the set of edges of $\mathbb{T}$ with both vertices in the set X.] We shall also use the hexagons centred at an arbitrary vertex $x \in V(\mathbb{T})$, which we write $H(n)+x$ and $\partial H(n)+x$. Figure 1 is an illustration of a portion of the triangular lattice together with two examples of hexagons.

For the purposes of the next section, we shall also give a general definition of the term "lattice." We say a graph is a *lattice* if it is graph-theoretic isomorphic to a graph L with vertex set

$$V(L) = \mathbb{Z}^2 \times S,$$

where S is a nonempty, countable set (which may or may not be infinite), which satisfies

$$\{(x, s), (y, t)\} \in L \text{ if and only if } \{(x+z, s), (y+z, t)\} \in L \quad \text{for all } x, y, z \in \mathbb{Z}^2 \text{ and } s, t \in S$$

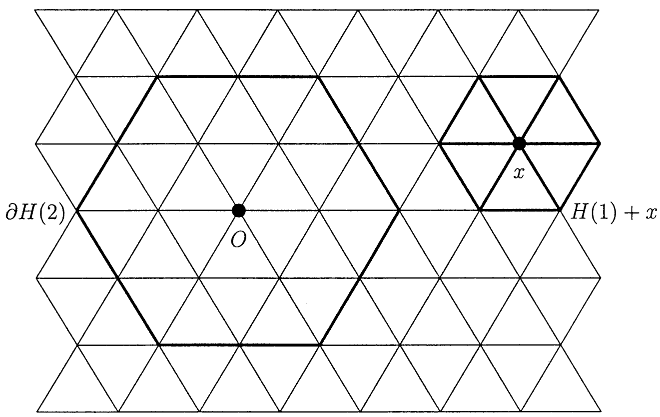

FIG. 1. *An illustration of a portion of the triangular lattice, together with two hexagons. Here x is the vertex specified by $a = 3$, $b = 1$, and the thickened subgraphs are $\partial H(2)$ and $H(1) + x$.*

(i.e., L is invariant under the natural translations of $\mathbb{Z}^2$). Note that we do not insist that a lattice is connected or place any restriction on numbers of edges. The fact that we allow S to be infinite allows lattices to have "dimension" greater than 2, so that, for example, the "cubic lattice" ($\mathbb{L}^3$) is a lattice.

We note that $\mathbb{T}$ is indeed a lattice. We note also that $\mathbb{T}$ is 2-rigid; this may be deduced by first using Proposition 2.4 to show that $H(n)$ is 2-rigid for any n.

4. Rigidity percolation. In this section we define the concept of rigidity percolation, and determine sufficient conditions for the critical probability to be strictly less than 1.

Let E be a countable infinite graph. We wish to consider a "random subgraph" K of E, in which any edge of E is included with probability p and distinct edges behave independently. To be precise, we define the sample space $\Omega\,(=\Omega_E) = \{0,1\}^E$, equipped with the product σ-field. For $p \in [0,1]$, we define P_p to be the product measure on Ω with parameter p. We define the random variable $K\colon \Omega_E \to \mathscr{P}(E)$ by $K(\omega) = \{e \in E\colon \omega(e) = 1\}$.

We define the event

$$\mathscr{R}^{(d)} = \bigl\{K \text{ has an infinite } d\text{-rigid component containing } O\bigr\},$$

and define the *d-rigidity percolation probability*

$$\rho^{(d)}(p) = P_p(\mathscr{R}^{(d)}).$$

REMARK. It is of course necessary to check that $\mathscr{R}^{(d)}$ is indeed a measurable event; we briefly describe one approach to this below. Similar arguments can be applied to all the events which we shall consider, and we shall therefore generally not mention questions of measurability. Write $E = \{e_1, e_2, \dots\}$, and define $E_n = \{e_1, \dots, e_n\}$. For natural numbers m and n, define the cylinder event

$$r_{m,n} = \bigl\{K \cap E_m \text{ has a } d\text{-rigid component containing } E_d \text{ and } e_n\bigr\}.$$

Then we have

$$\bigcup_m r_{m,n} = \bigl\{K \text{ has a } d\text{-rigid component containing } E_d \text{ and } e_n\bigr\},$$

and so

$$\Bigl\{\bigcup_m r_{m,n} \text{ occurs for infinitely many } n\Bigr\}$$
$$= \bigl\{K \text{ has an infinite } d\text{-rigid component containing } E_d\bigr\}.$$

But $\mathscr{R}^{(d)}$ is the union over all subgraphs of E of size d (such as E_d) containing O of the event that K has an infinite d-rigid component containing the subgraph. The required measurability follows.

We note that $\rho^{(d)}$ is a nondecreasing function, and define the *d-rigidity critical probability* $p_r^{(d)} = p_r^{(d)}(E)$ by

$$p_r^{(d)} = \sup\{p\colon \rho^{(d)}(p) = 0\}.$$

Since the concept of 1-rigidity is identical with that of connectivity (Proposition 2.2), we will normally write p_c for $p_r^{(1)}$. (This is the usual critical probability for connectivity percolation.)

By Proposition 2.3, for any particular graph we have

$$p_c \le p_r^{(d)} \quad \text{for any } d \ge 1$$

(and indeed $p_r^{(d)} \le p_r^{(d')}$ whenever $d \le d'$). Standard techniques from percolation theory may be used to show that for a broad family of graphs we have $0 < p_c$. The following result allows us to bound p_r away from 1.

THEOREM 4.1. *For any $d \ge 1$, if L is any d-rigid lattice then we have*

$$p_r^{(d)}(L) < 1.$$

The proof of this theorem depends on the following result of Liggett, Schonmann and Stacey [10]. The result concerns measures on the state space $\{0,1\}^{\mathbb{Z}^2}$ equipped with the product σ-field. We write $\sigma = (\sigma(x))_{x\in\mathbb{Z}^2}$ for a typical point in the sample space. For any $p \in [0,1]$ we write π_p for the product measure with parameter p. We say that a measure μ is *k-dependent* (for $k \ge 0$) if for any $A, B \subseteq \mathbb{Z}^2$ such that $\|x - y\| > k$ for all $x \in A$ and $y \in B$, the families $(\sigma(x))_{x\in A}$ and $(\sigma(x))_{x\in B}$ are independent of each other under μ. In words, μ has no dependence over distances greater than k.

THEOREM 4.2 (Liggett, Schonmann and Stacey). *For any $k > 0$ and $\alpha < 1$, there exists $\beta < 1$ such that for every k-dependent measure μ satisfying $\mu\{\sigma\colon \sigma(x) = 1\} \ge \beta$ for every $x \in \mathbb{Z}^2$, μ stochastically dominates π_α.*

A proof of Theorem 4.2 (in a much more general form) may be found in [10].

SKETCH PROOF OF THEOREM 4.1. We use a "two-dimensional block argument." Let L be a d-rigid lattice, and suppose $V(L) = S \times \mathbb{Z}^2$, as in the definition of a lattice in the previous section. Without loss of generality, we may assume that $|S| \ge d$, since we make S as large as desired by relabelling $V(L)$ as $(S \times \{0,\dots,m-1\}^2) \times (m\mathbb{Z})^2$ for any $m \ge 1$. Now let $T \subseteq S$ be a set of vertices of size d. Since L is rigid, we may find a finite rigid subgraph R of L such that $T \times \{(0,0),(1,0),(0,1)\} \subseteq V(R)$. Choose any such subgraph, and define R_x (where $x \in \mathbb{Z}^2$) to be a subgraph of L obtained by "translating R by x," that is, adding the vector x to the vector component of each vertex. Note that $R_0 = R$, and that R_x is graph-theoretic isomorphic to R.

The crucial point of this construction is that if $x, y \in \mathbb{Z}^2$ are "adjacent" in the sense that $\|x - y\| = 1$, then $|V(R_x) \cap V(R_y)| \ge d$. Since R is rigid, it follows by Proposition 2.4 that if $x_1, x_2, \dots$ is a sequence of distinct vectors in $\mathbb{Z}^2$ such that $\|x_{i+1} - x_i\| = 1$ for each i, then $\bigcup_i R_{x_i}$ is a rigid graph. We shall show that for p sufficiently close to 1, K contains an infinite graph of this type almost surely with respect to P_p.

Given $K \subseteq L$, define for each $x \in \mathbb{Z}^2$,

$$\sigma(x) = \begin{cases} 1, & \text{if } R_x \subseteq K, \\ 0, & \text{otherwise.} \end{cases}$$

We note that since R is finite, the law of σ induced by P_p must be k-dependent for some k. Also note that for each x we have $P_p(\sigma(x) = 1) = p^{|R|}$. We may now appeal to Theorem 4.2. For any $\alpha < 1$, we may choose p sufficiently large that the law of σ stochastically dominates the product measure π_α. Hence if we choose α to exceed the critical probability for site percolation on the square lattice in two dimensions (which is strictly less than 1), the result follows. □

In particular, we note that Theorem 4.1 may be applied to the triangular lattice to show that $p_r^{(2)}(\mathbb{T}) < 1$. It is a standard result of percolation theory that $p_c(\mathbb{T}) > 0$, so we also have $p_r^{(2)}(\mathbb{T}) > 0$.

5. Results for the triangular lattice. The remainder of this work will be devoted to the study of 2-rigidity percolation on the triangular lattice, $\mathbb{T}$. It is believed that our arguments could in principle be extended to deal with a large family of lattices, but that this would involve considerable difficulties of a rather technical graph-theoretic nature.

We have seen that the general results of the previous section imply the inequalities

$$0 < p_c(\mathbb{T}) \le p_r^{(2)}(\mathbb{T}) < 1,$$

so that there is a genuine rigidity phase transition, occurring at a critical probability greater than or equal to that for connectivity percolation.

Our first main result (Theorem 7.1) will be the strict inequality

$$p_c(\mathbb{T}) < p_r^{(2)}(\mathbb{T}).$$

Our approach to proving this is based on the general result of Aizenman and Grimmett [1], although we shall require a slight extension of the method used therein.

Our second main result (Theorem 8.1) concerns uniqueness of the infinite rigid component:

If p is such that $\rho^{(2)}(p) > 0$ and $\rho^{(2)}$ is either left-continuous or right-continuous at p, then

$$P_p(K \text{ has exactly one infinite 2-rigid component}) = 1.$$

Since $\rho^{(2)}$ is a nondecreasing function, this implies that the above displayed equation holds for all but countably many values of p in the interval

$\{p\colon \rho^{(2)}(p) > 0\}$. We shall also see that the result enables us to make some deductions about the continuity of $\rho^{(2)}$.

Our approach to proving this result is based on the method of Burton and Keane [2]. However, since we are dealing here with rigidity rather than connectivity percolation, we shall require a variety of additional techniques, and our final result is slightly weaker than the corresponding result in [2], in that we prove uniqueness only for all but countably many p. We briefly describe the reasons for this below.

In the physics literature, rigidity (in contrast with connectivity) is often described as a "long-range phenomenon." To see one example of what is meant by this, consider the effect of adding one extra edge to a graph. The effect on connectivity is simply to unite the two connected components of the two vertices into one connected component (if they were distinct initially). However, the effect on rigidity may be much more complicated. For example, consider starting with a square and adding one diagonal; initially each of the four edges forms a distinct 2-rigid component, but the addition of the extra edge unites all five edges into a single 2-rigid component. Thus it appears to be much more difficult to predict the effect of adding or removing edges on rigidity (as opposed to connectivity); perhaps the most useful partial information in this direction is provided by Proposition 6.9.

The method of Burton and Keane consists essentially of two steps. Assuming that the number of infinite components is infinite, one first deduces the existence of so-called "encounter points," which may be thought of as points where three "branches" of an infinite component meet. Second, a counting argument based on the "compatibility" of pairs of encounter points shows that this leads to a contradiction. For the case of 2-rigidity, one may define a natural analogue of an encounter point, which we shall later refer to as a "pretrifurcation"; roughly speaking, this is a zone where the removal of a small number of edges causes one infinite 2-rigid component to split into at least three infinite 2-rigid components. With the aid of Proposition 6.9, we may show that the existence of infinitely many infinite 2-rigid components implies the existence of pretrifurcations, although the argument is considerably more delicate than in the connectivity case. However, the conditions defining a pretrifurcation are not strong enough to allow the final counting argument to work, essentially because we do not have enough control over the effect which removing a few edges from near a pretrifurcation may have. This difficulty is overcome as follows. Using a further property of rigidity (Proposition 6.5), and Proposition 8.8, which is an extension of Russo's Formula (see [5]), we show that, given the continuity of $\rho^{(2)}$, the effect of removing edges near a pretrifurcation extends only over a finite range, and hence, with positive probability, the effect is *only* to split one infinite component into three. This establishes the existence of "trifurcations," whose definition is more restrictive, and whose properties are such that the counting argument may now be made to work.

In the section immediately following, we shall develop the tools which we will require to study 2-rigidity, and in the two final sections we shall prove the two main results for the triangular lattice.

6. Two-dimensional rigidity. In this section we shall study the properties of 2-rigid graphs in more detail. We shall do this first for the case of finite graphs, and then extend the results to infinite graphs. It will be convenient to use the rigidity closure operator, which is defined below. The material in this section is of a somewhat different nature from the rest of this work, being primarily combinatorial. The reader should note that the only information from this section which is logically necessary in what follows consists of the statements of Theorem 6.3, Propositions 6.4–6.9 and Meta-proposition 6.10. In particular, the proofs of these statements may be omitted on a first reading.

We write $\mathscr{F}(\mathscr{V})$ for the set of all finite graphs on $\mathscr{V}$. We define the two-dimensional rigidity closure operator $\langle\cdot\rangle$: $\mathscr{F}(\mathscr{V}) \to \mathscr{F}(\mathscr{V})$ as follows. Given $E \in \mathscr{F}(\mathscr{V})$, choose a generic embedding of E in two dimensions and let $\mathscr{M}$ be the collection of all motions of the resulting framework. We define $\langle E\rangle$ to be the set of all edges $\{x, y\} \in V(E)^{(2)}$ such that the "edge length" $\|r(t)(x)-r(t)(y)\|$ is constant for every motion $(r(t))$ in $\mathscr{M}$. It can be shown (see [4]) that, as we might expect in the light of Theorem 2.1, the set $\langle E\rangle$ so defined is not dependent on the choice of the generic embedding. By definition,

$$E \text{ is 2-rigid if and only if } \langle E\rangle = V(E)^{(2)}.$$

We remark that the closure operator is more commonly defined in a somewhat different but equivalent way in terms of infinitesimal rigidity. See [4] for details.

The following properties of $\langle\cdot\rangle$ may be deduced immediately from the definition.

LEMMA 6.1. *For any $A, B \in \mathscr{F}(\mathscr{V})$ we have the following*:

(i) $\langle A\rangle \supseteq A$;
(ii) *If* $A \subseteq B$ *then* $\langle A\rangle \subseteq \langle B\rangle$;
(iii) $\langle\langle A\rangle\rangle = \langle A\rangle$.

The following is an elementary consequence of the previous lemma.

COROLLARY 6.2. *For any $A, B \in \mathscr{F}(\mathscr{V})$ and $A \subseteq A' \subseteq \langle A\rangle$, we have $\langle A \cup B\rangle = \langle A' \cup B\rangle$.*

We note that all our observations so far could be applied equally to rigidity in any number of dimensions. Thus we could define a closure operator in the same way for rigidity in any number of dimensions, and Lemma 6.1 and Corollary 6.2 would still be valid. What makes the case of two dimensions special is the following purely combinatorial characterization of the closure operator, which is originally due (in a slightly different form) to Laman [9]. (See [4] for a treatment of the result in the form given here.) It is this which allows us to obtain simple proofs of the results which follow.

We say a graph $E \in \mathscr{F}(\mathscr{V})$ is *overconstrained* if $|E| > 2|V(E)| - 3$, and *balanced* if $|E| = 2|V(E)|-3$. We say E is *independent* if it has no overconstrained subgraph.

THEOREM 6.3 (From Laman). *The closure operator* $\langle\cdot\rangle$ *is characterized by*

$$\langle E\rangle = E \cup \big\{e\text{: there exists } F \subseteq E \text{ such that } F \text{ is independent but } F \cup \{e\} \text{ is not independent}\big\}.$$

In all that follows we shall restrict our attention to rigidity in two dimensions, and in the remainder of this section and those which follow we will use the term *rigid* to mean 2-rigid. Note that the results we give in this section subsume the general rigidity results of Propositions 2.2–2.5 in the two-dimensional case (as well as providing a considerable amount of further information), so that, for example, the statement of Proposition 6.7 is simply the specialization of Proposition 2.4 to the case $d = 2$. Thus all the results on two-dimensional rigidity which we shall require follow from Laman's characterization. In all the immediately following results, we assume that $A, B, E \in \mathscr{F}(\mathscr{V})$.

PROPOSITION 6.4. *We have*

$$\langle E\rangle = \bigcup_{\substack{F\subseteq E:\\ F \text{ is rigid}}} V(F)^{(2)}.$$

PROOF. That the right-hand side is a subset of $\langle E\rangle$ follows from the definition of rigidity, as given above. For the reverse inclusion, suppose that $e \in \langle E\rangle$. If $e \in E$, then e is a member of the set on the right-hand side, since $\{e\}$ is a rigid graph. If not, we note that the graph F in the statement of Theorem 6.3 must have a balanced independent subgraph F' such that both vertices of e lie in $V(F')$. Hence, since F' is balanced and independent, Theorem 6.3 implies that $\langle F'\rangle = V(F')^{(2)}$; that is, F' is rigid. Hence the result follows. □

In words, Proposition 6.4 states that an edge lies in $\langle E\rangle$ if and only if both its vertices are contained in some rigid subgraph of E. It will be useful to bear in mind this interpretation of the closure operator in what follows. We remark that the statement corresponding to Proposition 6.4 in three or more dimensions is false (a counter-example in three dimensions may be constructed along the lines of the graph in Figure 1.7 of [4]).

PROPOSITION 6.5. *If* $s, t \in \mathscr{V}^{(2)} \setminus \langle E\rangle$ *and* $s \in \langle E \cup \{t\}\rangle$ *then* $t \in \langle E \cup \{s\}\rangle$.

PROOF. By Theorem 6.3, there must exist $F \subseteq E$ with $F \cup \{t\}$ independent and $F \cup \{s, t\}$ not independent. But then F is independent, and $F \cup \{s\}$ must be independent, for otherwise we would have $s \in \langle E\rangle$. Hence $t \in \langle F \cup \{s\}\rangle$. □

This is perhaps the least intuitive of our results on rigidity: if the addition of an edge t "locks" the two vertices of s by causing them to be contained in a rigid graph, then the addition of s has the same effect on the vertices of t. The result will be crucial to the proof of Theorem 8.1, since it will enable us to show that if, with positive probability, there are an infinite number of regions which are affected by a particular change near the origin ("hinges"), then, with positive probability, there are an infinite number of regions at which a change will affect the origin ("h-pivotal vertices"). It is this that allows us to deduce the existence of "trifurcations" from the existence of "pretrifurcations" and the continuity of $\rho^{(2)}$. (We remark that the statements of Lemma 6.1 and Proposition 6.5 together amount to the assertion that $\langle\cdot\rangle$ is a matroid closure operator. For more details, see [4].)

PROPOSITION 6.6. *If* $|V(A)\cap V(B)|\le 1$ *then* $\langle A\cup B\rangle \subseteq V(A)^{(2)}\cup V(B)^{(2)}$. *In particular, a rigid graph is connected.*

PROOF. Suppose on the contrary that there exist $x\in V(A)\setminus V(B)$ and $y\in V(B)\setminus V(A)$ such that $\{x,y\}\in\langle A\cup B\rangle$. Then there must exist disjoint graphs $C\subseteq A$ and $D\subseteq B$ with $C\cup D$ balanced and independent, and such that $x\in V(C)$ and $y\in V(D)$. Now since $C\cup D$ is balanced, we have

$$\begin{aligned}|C|+|D| &= |C\cup D|\\ &= 2|V(C\cup D)|-3\\ &\ge 2(|V(C)|+|V(D)|-1)-3\\ &= 2|V(C)|+2|V(D)|-5.\end{aligned}$$

But since C and D are not overconstrained we have

$$|C|\le 2|V(C)|-3$$

and

$$|D|\le 2|V(D)|-3,$$

giving a contradiction. □

PROPOSITION 6.7. *If* A *and* B *are rigid and* $|V(A)\cap V(B)|\ge 2$ *then* $A\cup B$ *is rigid.*

PROOF. We shall appeal to Corollary 6.2. It is sufficient to show that for any two vertices $x\in V(A)\setminus V(B)$ and $y\in V(B)\setminus V(A)$ we have $\{x,y\}\in\langle A\cup B\rangle$. Let u,v be two distinct vertices in $V(A)\cap V(B)$, and note that the edges $\{u,v\},\{x,u\},\{x,v\},\{y,u\},\{y,v\}$ all lie in $\langle A\cup B\rangle$. The result now follows from the observation that the graph consisting of these five edges is rigid. □

PROPOSITION 6.8. *Suppose* $x,y\in V(A)$ *and* $z\in\mathscr{V}\setminus V(A)$, *and let* $A'=A\cup\{\{x,z\},\{y,z\}\}$. *Then* A' *is rigid if and only if* A *is rigid.*

PROOF. An argument similar to the above shows that if A is rigid then A' is rigid. On the other hand, suppose A' is rigid but A is not rigid. Then there is some edge $\{u, v\} \in \langle A' \rangle \setminus \langle A \rangle$, where $u, v \in V(A)$. It is easily seen this implies the existence of some $C \subseteq A$ such that $C \cup \{\{x, z\}, \{y, z\}\}$ is independent and balanced and $u, v \in V(C)$. It follows that C is also independent and balanced, contradicting the assertion that $\{u, v\} \notin \langle A \rangle$. □

PROPOSITION 6.9. *If $A \cup B$ and B' are rigid, and $V(A) \cap V(B) \subseteq V(B')$, then $A \cup B'$ is rigid.*

PROOF. First observe that by Proposition 6.6 we must have $|V(A) \cap V(B)| \geq 2$.

We now construct a graph C as follows. Let x and y be two distinct vertices in $V(B')$, and let $V(B) \setminus V(A) = \{z_1, \dots z_k\}$. Then define

$$C = \big\{\{x, z_1\}, \{y, z_1\}, \{x, z_2\}, \{y, z_2\}, \dots, \{x, z_k\}, \{y, z_k\}\big\}.$$

Thus, by repeated application of Proposition 6.8 we deduce that $A \cup B'$ is rigid if and only if $A \cup B' \cup C$ is rigid, and also that $B' \cup C$ is rigid. But applying Corollary 6.2, we have

$$\begin{aligned} \langle A \cup B' \cup C \rangle &= \langle A \cup (V(B') \cup V(B))^{(2)} \rangle \quad \text{(since } B' \cup C \text{ is rigid)} \\ &= \langle V(A \cup B)^{(2)} \cup (V(B') \cup V(B))^{(2)} \rangle \quad \text{(since } A \cup B \text{ is rigid)} \\ &= (V(A) \cup V(B) \cup V(B'))^{(2)} \quad \text{(by Proposition 6.7).} \\ &= V(A \cup B' \cup C)^{(2)}. \end{aligned}$$

□

The statement of Proposition 6.9 expresses an intuitively plausible fact: if we remove some part B of a rigid graph $A \cup B$, then we can restore rigidity by adding any rigid graph B' which contains all the vertices of A from which edges have been removed.

As in Section 3, we may now extend our results to infinite graphs. Let E be any (possibly infinite) graph. We define $\langle E \rangle$ according to the statement of Proposition 6.4. We now have the following meta-proposition, whose proof is straightforward.

META-PROPOSITION 6.10. *The statements of Propositions* 6.5–6.9 *all hold if we allow the graphs concerned to be infinite.*

Again, for the sake of convenience, we shall use the original numbers to refer to the "infinite versions" of these propositions.

7. Strict inequality of critical points.

Our first main result about the triangular lattice is the following.

THEOREM 7.1. *We have the strict inequality*

$$p_c(\mathbb{T}) < p_r^{(2)}(\mathbb{T}).$$

Our approach to proving Theorem 7.1 is to translate the problem into a problem concerning only connectivity percolation, and then to solve this problem using the techniques of [1].

Let E be a graph. We say an edge $\{x, y\} \in E$ is a *bridge* of E if the following hold:

1. Each of x, y has degree exactly 2 in E.
2. There does not exist $z \in V(E)$ such that $\{x, z\}, \{y, z\} \in E$.

A bridge is illustrated in Figure 2. The idea of the definition is that removing bridges has no effect on the infinite rigid components of a graph, but does affect its connectivity properties.

LEMMA 7.2. *Let E be a finite graph and let $b \in E$ be a bridge. The only rigid subgraph of E which is not also a rigid subgraph of $E \setminus \{b\}$ is $\{b\}$ itself.*

PROOF. Suppose F is a rigid subgraph of E which is not a rigid subgraph of $E \setminus \{b\}$. Clearly we have $b \in F$. Further, provided $F \neq \{b\}$, by Proposition 6.6, F must also include the two edges e_1, e_2 of E adjacent to b, and also at least one further edge adjacent to each of e_1, e_2 (we say two edges are adjacent if they share a vertex). But now, by Proposition 6.6, $F \setminus \{b, e_2\}$ is not rigid, so by Proposition 6.8, F is not rigid, a contradiction. □

COROLLARY 7.3. *Let E be any graph, and let $B \subseteq E$ be a (possibly infinite) set of bridges of E. Then the only rigid subgraphs of E which are not also rigid subgraphs of $E \setminus B$ are singleton bridges $\{b\} \subseteq B$. In particular, the infinite rigid subgraphs of E are precisely the infinite rigid subgraphs of $E \setminus B$.*

PROOF. We prove the result first for finite rigid subgraphs of E. Since any such subgraph contains only finitely many bridges in B, we obtain this result simply by repeatedly applying the preceding lemma. The general result now follows from the definition of an infinite rigid graph. □

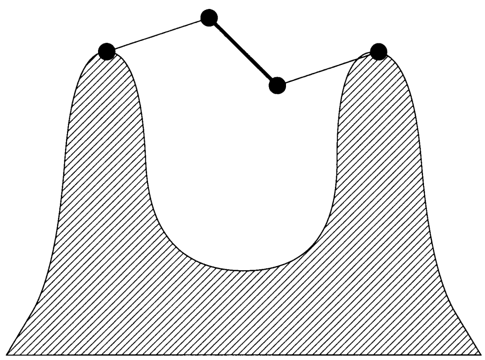

FIG. 2. *The thickened edge is an example of a bridge.*

The above result tells us that removing bridges from a graph has no effect on its infinite rigid components. The following result will show that removing a particular set of bridges *does* have an effect on the connectivity critical probability, which will enable us to deduce the main result, Theorem 7.1.

Define the set

$$U = \{3a(1,0) + 2b(1/2, \sqrt{3}/2) \colon a, b \in \mathbb{Z}\} \subset V(\mathbb{T}).$$

We also define W to be the subgraph of $\mathbb{T}$ consisting of the eleven edges illustrated in Figure 3 and define the edges $b, e_1, e_2 \in W$ as in the figure. Let X be the graph $\{b, e_1, e_2\}$. For $u \in U$, we shall write $W + u$, $b + u$ and so on, for the "translated copies" of these graphs and edges (which also lie in $\mathbb{T}$). Note that for any distinct $u, v \in U$, the graphs $W + u$ and $W + v$ are disjoint.

Given the graph $K \subseteq \mathbb{T}$, we define $\psi(K) \subseteq \mathbb{T}$ as follows:

$$\psi(K) = K \Big\backslash \bigcup_{\substack{u \in U: \\ K \cap (W+u) = X+u}} \{b + u\}.$$

In words, we obtain $\psi(K)$ from K by deleting all edges of the form $b+u$ where the edges of $W + u$ present in K are precisely those of $X + u$ (so that $b + u$ is a bridge of K).

PROPOSITION 7.4 (After Aizenman and Grimmett). *There exists a nonempty interval* $(p_1, p_2) \subset [0, 1]$ *such that for* $p \in (p_1, p_2)$ *we have*

$$P_p(K \text{ has an infinite connected component containing } O) > 0,$$

but

$$P_p(\psi(K) \text{ has an infinite connected component containing } O) = 0.$$

This result is almost a special case of the result proved in [1]. However there are two essential differences.

1. Our result is for the triangular lattice, whereas [1] uses the "hypercubic lattice" ($\mathbb{L}^d$).

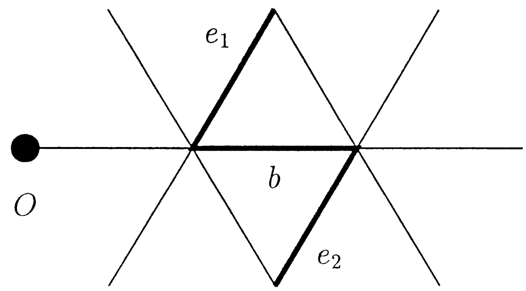

FIG. 3. *The subgraph W of $\mathbb{T}$ and some special edges of W.*

2. The general result in [1] is for "enhancements," that is, systematic alterations to a configuration which involve the addition of edges. Our alteration ψ might be called a disenhancement since it involves the removal of edges, and (to extend the terminology of [1] further), it is an "essential" disenhancement.

The proof of Proposition 7.4 is omitted; it presents no substantial difficulty and consists merely of repeating the steps used in [1], bearing in mind the above differences.

We are now in a position to prove the main result of this section.

PROOF OF THEOREM 7.1. Let p_1, p_2 be as in Proposition 7.4. Clearly we have $p_c \leq p_1$. Now consider $\psi(K)$. For $p \in (p_1, p_2)$, almost surely with respect to P_p, $\psi(K)$ has no infinite connected component and therefore no infinite rigid component. But since $\psi(K)$ is obtained from K by deleting bridges, by Corollary 7.3 this implies that K has no infinite rigid component. Hence $p_2 \leq p_r^{(2)}$, and the required inequality follows. $\square$

8. Uniqueness for almost all *p*. Throughout this section we shall work with the triangular lattice $\mathbb{T}$. Our main result is the following.

THEOREM 8.1. *If p is such that $\rho^{(2)}(p) > 0$ and $\rho^{(2)}$ is either left-continuous or right-continuous at p, then*

$$P_p(K \textit{ has exactly one infinite rigid component}) = 1.$$

Before proving this theorem, we shall make a few remarks on its consequences, and then outline the approach of the proof.

We say that "we have uniqueness at p" if p is such that the displayed equation above holds. Since $\rho^{(2)}$ is a nondecreasing function, it is an immediate corollary of Theorem 8.1 that we have uniqueness for all but countably many p in the interval $\{p: \rho^{(2)}(p) > 0\}$ (and hence for almost all such p with respect to Lebesgue measure). We conjecture that in fact we have uniqueness for all p in this interval.

It may be shown (by the same method as for connectivity percolation; see [5], pages 117–119, but replacing "a finite path" with "a finite rigid graph") that the following holds.

PROPOSITION 8.2. *If $p > p_r^{(2)}$ and we have uniqueness at p then $\rho^{(2)}$ is left-continuous at p.*

In particular, it follows from this and Theorem 8.1 that we have uniqueness at p ($> p_r^{(2)}$) if and only if $\rho^{(2)}$ is left-continuous at p. Also, if $\rho^{(2)}$ is left-discontinuous at p ($> p_r^{(2)}$), then it is also right-discontinuous at the same point.

We shall now give an outline of our approach to proving Theorem 8.1. Standard arguments can be used to show that for any given p, the number of infinite rigid components of K takes some value k with probability 1, and that $k \in \{0, 1, \infty\}$. Our task is therefore to rule out the possibility $k = \infty$. We shall show that this possibility leads to a contradiction, using an extension of the method in [2]. The main step is to prove the existence of "trifurcations," which play the same role as "encounter points" in the terminology of [2] (the term "trifurcation" is taken from [6]). We first prove the existence of "pretrifurcations," which have some of the required properties. To deduce the existence of trifurcations we use Proposition 6.5 to show that with positive probability, a pretrifurcation is "almost" a trifurcation when viewed on a sufficiently large scale. It is here that the continuity of $\rho^{(2)}$ is required, and it is this step which represents the most significant augmentation to the ideas in [2].

We shall make use of a number of special subgraphs and edges of $H(3)$. These are most conveniently defined by means of diagrams. Define Y to be the graph illustrated in Figure 4, and define the edges $f_1, f_2, f_3, z_0, \dots, z_8, w_2, w_2', w_4, w_4', w_6, w_6'$ of $H(3)$, and the vertices a_2, a_4, a_6 of $V(H(3))$ via Figure 5. Also, set $a_{i+1} = a_i, w_{i+1} = w_i, w_{i+1}' = w_i'$ for each $i = 2, 4, 6$, and define $Z = \{z_0, \dots, z_8\}$. Note in particular the form of the graph $Y \setminus Z$, which is illustrated in Figure 6. Finally, we define the following subgraphs of $H(3)$ which will be required in the proof of Lemma 8.9:

$$\begin{aligned}
X_1 &= (Y \setminus Z) \cup \{z_0, z_1\},\\
X_2 &= X_1 \cup \{z_2\},\\
&\ \vdots\\
X_7 &= X_6 \cup \{z_7\},\\
W_2 &= X_1 \cup \{w_2, w_2'\},\\
&\ \vdots\\
W_7 &= X_6 \cup \{w_7, w_7'\}.
\end{aligned}$$

As we shall see, the idea of this construction is that $X_1, \dots, X_7$ form a sequence of intermediate steps between $Y \setminus Z$ and Y, and that W_j, together with

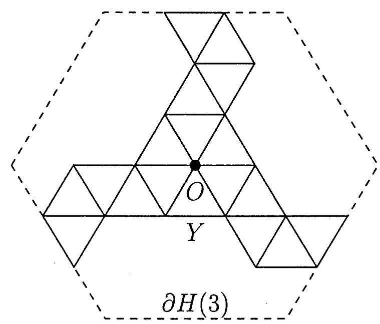

FIG. 4. *The subgraph Y of $H(3)$.*

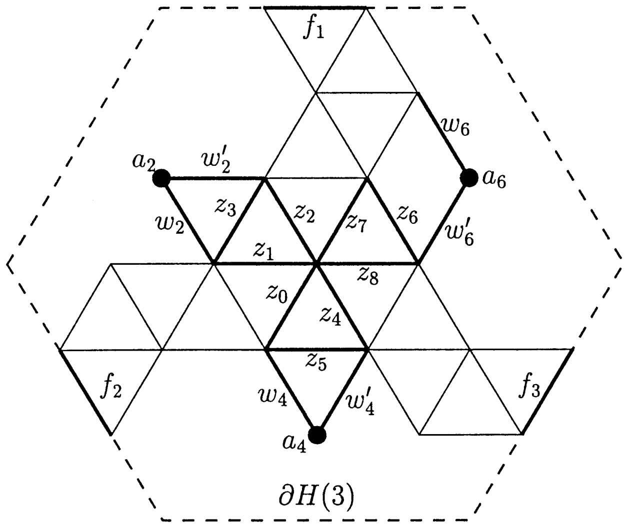

FIG. 5. *Some special edges and vertices of* $H(3)$.

the vertex a_j, forms a "test configuration" for testing the difference between X_{j-1} and X_j.

We shall make extensive use of the following result from percolation theory, whose proof is elementary.

LEMMA 8.3. *Let* $\mathscr{A}, \mathscr{B} \subseteq \mathscr{P}(\mathbb{T})$ *be two sets such that* $\{K \in \mathscr{A}\}$ *and* $\{K \in \mathscr{B}\}$ *are measurable events, and let* $S \subset \mathbb{T}$ *be finite. Suppose* $\zeta\colon \mathscr{P}(\mathbb{T}) \to \mathscr{P}(S)$ *is a function such that*

$$\text{if } K \in \mathscr{A} \text{ then } (K \setminus S) \cup \zeta(K) \in \mathscr{B}.$$

Then $P_p(K \in \mathscr{A}) > 0$ *implies* $P_p(K \in \mathscr{B}) > 0$.

In words, Lemma 8.3 states that if, starting from an event of positive probability, we can force a second event to occur by making changes to some fixed

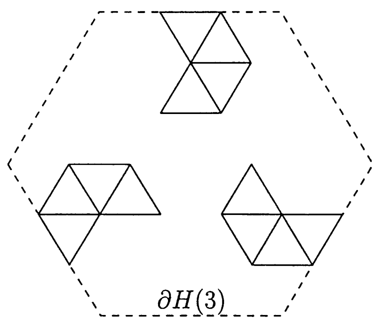

FIG. 6. *The subgraph* $Y \setminus Z$ *of* $H(3)$.

finite set of edges, then the second event also has positive probability. This assertion is closely related to the "finite energy condition" on measures which is used in [2].

We say that the origin is a *trifurcation* of K if the following three conditions hold:

1. $K \cap H(3) = Y$.
2. In the graph $K \setminus Z$ (whose intersection with $H(3)$ is $Y \setminus Z$; see Figure 6), each of the edges f_1, f_2, f_3 (see Figure 5) lies in an infinite rigid component, which we denote T_i ($i = 1, 2, 3$); and T_1, T_2, T_3 have pairwise disjoint vertex sets [that is, $V(T_i) \cap V(T_j) = \varnothing$ for each pair $i \neq j$].
3. $Z \cup T_1 \cup T_2 \cup T_3$ is a rigid component of K (in particular it is not contained in any strictly larger rigid subgraph of K).

Note that conditions (1) and (2) imply that $Z \cup T_1 \cup T_2 \cup T_3$ is a rigid graph, so that the only way in which condition (3) can fail is for $Z \cup T_1 \cup T_2 \cup T_3$ to be contained in a strictly larger rigid graph. It is precisely to rule out this possibility that we require the assumption about the continuity of $\rho^{(2)}$ in the statement of Theorem 8.1. A trifurcation is illustrated in Figure 7; after the removal of the nine edges of Z from the center of the picture, the shaded regions will form parts of three distinct rigid components.

We say that the origin is a *pretrifurcation* if conditions (1) and (2) hold. We say $x \in V$ is a (pre)trifurcation if the corresponding statements hold for the translated graph $K - x$.

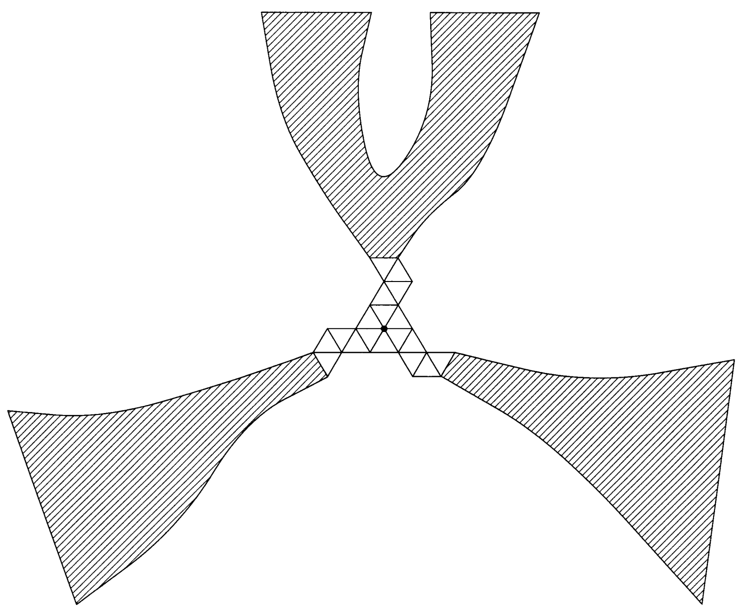

FIG. 7. *An illustration of the event $\{O$ is a trifurcation$\}$.*

PROPOSITION 8.4. *Suppose* p *is such that*

$$P_p(K \textit{ has infinitely many infinite rigid components}) = 1.$$

Then we have

$$P_p(O \textit{ is a pretrifurcation of } K) > 0.$$

In proving this result we shall make use of the following lemmas, 8.5 and 8.6, whose validity is "obvious," although full proofs would be somewhat lengthy. A sketch proof of Lemma 8.5 is given, and an example of the construction involved in Lemma 8.6 is illustrated in Figure 10.

By an *interval* of $\partial H(m)$ we mean a nonempty proper subset of $\partial H(m)$ which forms a connected graph. We say two intervals are *separate* if they have no common vertex.

LEMMA 8.5. *Let* D_1, D_2, D_3 *be infinite connected subgraphs of* $\mathbb{T} \setminus H(m)$ *with pairwise disjoint vertex sets. Write* $J_i = V(D_i) \cap V(\partial H(m))$, *and suppose* $|J_i| \geq 2$ *for each* i. *Then there exist pairwise separate intervals* I_1, I_2, I_3 *of* $\partial H(m)$ *with* $J_i \subseteq V(I_i)$.

SKETCH OF PROOF. For any collection of distinct vertices $v_1, \dots, v_r \in V(\partial H(m))$, we write

$$v_1 \curvearrowright v_2 \curvearrowright \cdots \curvearrowright v_r$$

for the assertion that the vertices are encountered in the order $v_1, v_2, \dots, v_r$ when $\partial H(m)$ is traversed in the clockwise direction. (Thus, $v_1 \curvearrowright \cdots \curvearrowright v_r$ is equivalent to $v_r \curvearrowright v_1 \curvearrowright \cdots \curvearrowright v_{r-1}$, and so on. Note that for any *two* distinct vertices v_1 and v_2 we always have $v_1 \curvearrowright v_2$ and $v_2 \curvearrowright v_1$). Also, for $u, v \in V(\partial H(m))$, we write $[u, v]$ for the interval of $\partial H(m)$ whose "anticlockwise end" is u and whose "clockwise end" is v. (See Figure 8.)

Note that for $j, j' \in J_1$ and $k, k' \in J_2 \cup J_3$, we cannot have $j \curvearrowright k \curvearrowright j' \curvearrowright k'$. This is because there is an infinite path in $D_2 \cup D_3$ starting from each of k, k',

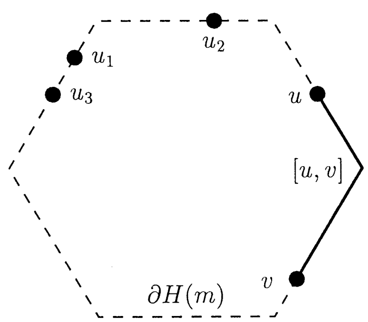

FIG. 8. *An illustration of the assertion* $u_1 \curvearrowright u_2 \curvearrowright u_3$ *and of the interval* $[u, v]$.

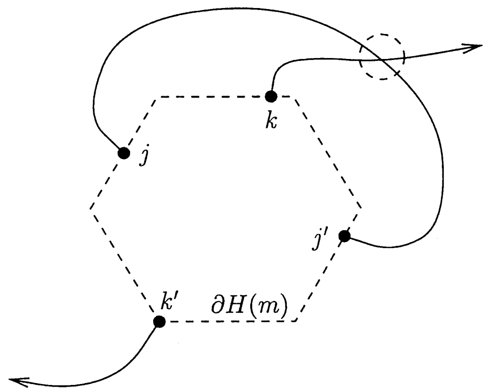

FIG. 9. *An illustration of intersecting paths in D_1 and $D_2 \cup D_3$ leading to a contradiction.*

and there is a path in D_1 from j to j' which cannot intersect either of these infinite paths. (See Figure 9.)

Now write $J_1 = \{j_1, \dots, j_r\}$ where $j_1 \curvearrowright \cdots \curvearrowright j_r$. By the above observation, there must exist some pair $(j_s, j_{s'})$ where $s' \equiv s + 1 \pmod r$, such that

$$J_2 \cup J_3 \subseteq V([j_s, j_{s'}]).$$

Without loss of generality we may take $(j_s, j_{s'}) = (j_r, j_1)$. Now define $I_1 = [j_1, j_r]$, and define I_2 and I_3 similarly.

Clearly $J_i \subseteq V(I_i)$, and it is easily verified that I_1, I_2, I_3 are pairwise separate. □

LEMMA 8.6. *There exists some fixed R such that if $m \geq R$ and I_1, I_2, I_3 are pairwise separate intervals of $\partial H(m)$, then there exist rigid graphs $A_1, A_2, A_3 \subseteq (H(m) \setminus H(3)) \cup \partial H(3)$ with pairwise disjoint vertex sets, such that, after relabelling indices if necessary, we have $A_i \cap \partial H(m) = I_i$ and $A_i \cap \partial H(3) = \{f_i\}$ for $i = 1, 2, 3$.*

The proof of Lemma 8.6 is omitted. See Figure 10 for an example of the construction involved. That the graphs A_1, A_2, A_3 are rigid may be deduced from Proposition 6.8.

Some of the steps in the following proof are illustrated in Figures 11 and 12.

PROOF OF PROPOSITION 8.4. Suppose the condition of the proposition holds. Define the event

$$F_N = \{\text{at least three infinite rigid components of } K \text{ intersect } H(N)\}.$$

Choose N large enough that $P_p(F_N) \geq 1/2$.

Suppose F_N occurs, and let C_1, C_2, C_3 be the "first" three such components (with respect to some suitable ordering). Since C_1, C_2, C_3 are distinct rigid

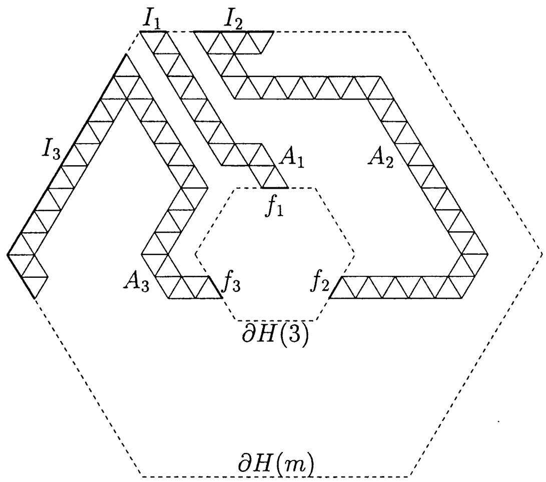

FIG. 10. *An example of the construction of the graphs* A_1, A_2, A_3.

components of a graph, Proposition 6.7 implies that any pair have at most one vertex in common, so we can take M sufficiently large that

$$P_p\bigl(C_1, C_2, C_3 \text{ have pairwise no common vertices outside } V(H(M-1)) \mid F_N\bigr) \geq 1/2.$$

Take $M' = \max\{M, R\}$ where R is as in Lemma 8.6, and let

$$G = \bigl\{F_N \text{ occurs and } C_1, C_2, C_3 \text{ have pairwise no common vertices outside } V(H(M'-1))\bigr\},$$

so that $P_p(G) \geq 1/4$. (See Figure 11.)

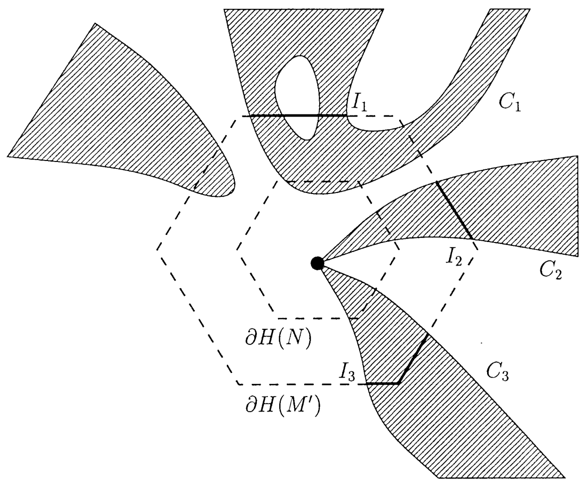

FIG. 11. *An illustration of the graphs* C_1, C_2, C_3.

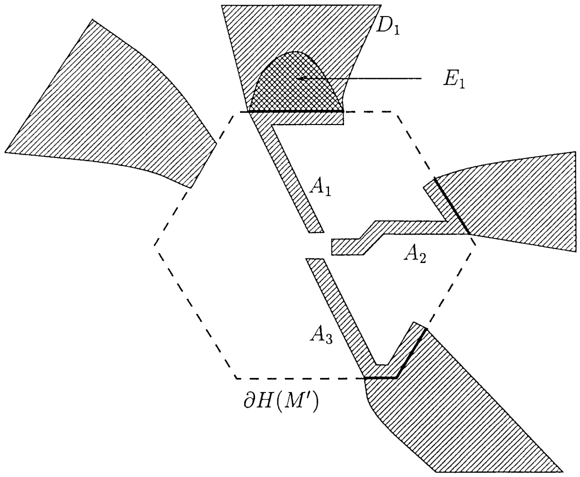

FIG. 12. *An illustration of some of the graphs used in the proof of Proposition* 8.4.

We shall use Lemma 8.3. We show that for any K such that G occurs, we can define $\zeta(K) \subseteq H(M')$ so that O is a pretrifurcation of $(K \setminus H(M')) \cup \zeta(K)$, and the result will follow.

Suppose G occurs. For each i, $C_i \setminus H(M')$ must have an infinite connected component, hence let D_i be the "first" such component. By Proposition 6.6, since C_i is rigid we must have $|V(D_i) \cap V(\partial H(M'))| \geq 2$. By Lemma 8.5, there must exist separate intervals I_1, I_2, I_3 of $\partial H(M')$ such that $V(D_i) \cap V(\partial H(M')) \subseteq V(I_i)$ for $i = 1, 2, 3$. Take I_1, I_2, I_3 to be minimal such intervals, so that their end vertices lie in the respective $V(D_i)$.

Now let A_1, A_2, A_3 be as in Lemma 8.6 with $m = M'$, and put $\zeta(K) = Y \cup A_1 \cup A_2 \cup A_3$. We claim that O is a pretrifurcation of $(K \setminus H(M')) \cup \zeta(K)$.

First observe that by Proposition 6.9, $A_i \cup D_i$ is rigid. We must also show that the rigid components of the graph $(K \setminus H(M')) \cup A_1 \cup A_2 \cup A_3$ (which we shall refer to as L) containing each $A_i \cup D_i$ are distinct and have pairwise disjoint vertex sets, from which the result will follow. Let P_i be a finite path in D_i connecting the two end vertices of I_i, and let E_i be the set of edges of K in the finite region enclosed by P_i and I_i. (See Figure 12.) We claim that the rigid component of L containing $A_i \cap D_i$ is contained in $A_i \cup C_i \cup E_i$. Indeed suppose that $A_i \cup C \cup E \cup Q \subseteq L$ is rigid, where $C \subseteq C_i \setminus E_i$, $E \subseteq E_i$ and $Q \subseteq K \setminus (C_i \cup E_i)$. Then by Proposition 6.9, since $V(E) \cap V(A_i \cup C \cup Q) \subseteq V(A_i \cup D_i)$, we deduce that $A_i \cup C \cup (A_i \cup D_i) \cup Q$ is rigid, but this is simply $A_i \cup C \cup D_i \cup Q$. Further, since $V(A_i) \cap V(C \cap D_i \cap Q) \subseteq V(C_i)$, we deduce that $C_i \cup C \cup D_i \cup Q$ is rigid, but this is simply $C_i \cup Q$. Hence (since C_i is a rigid component of K), $Q = \varnothing$.

The result now follows by Propositions 6.6 and 6.8. $\square$

Our aim is now to prove the following.

PROPOSITION 8.7. *Suppose p is such that*

$$P_p(K \textit{ has infinitely many infinite rigid components}) = 1$$

and $\rho^{(2)}$ is either left-continuous or right-continuous at p. Then we have

$$P_p(O \textit{ is a trifurcation of } K) > 0.$$

The proof depends on the following lemmas, 8.8 and 8.9.

We start with a definition. Let $\mathscr{A} \subseteq \mathscr{P}(\mathbb{T})$ be such that $\{K \in A\}$ is an *increasing* measurable event, so that if $K \subseteq K'$ then $K \in \mathscr{A}$ implies $K' \in \mathscr{A}$. Given K, we say that a vertex x is *h-pivotal* to $\mathscr{A}$ (or to the event $\{K \in \mathscr{A}\}$, when the latter is more convenient) if

$$K \cup (H(1) + x) \in \mathscr{A} \quad \text{and} \quad K \setminus (H(1) + x) \notin \mathscr{A}.$$

We shall write $N_{\mathscr{A}}$ (or $N_{\{K \in \mathscr{A}\}}$) for the (random) number of vertices h-pivotal to $\mathscr{A}$.

LEMMA 8.8. *Let $\mathscr{A} \subseteq \mathscr{P}(\mathbb{T})$ be such that $\{K \in \mathscr{A}\}$ is an increasing event. For any $p < 1$ we have*

$$\lim_{\delta \downarrow 0} \big(P_{p+\delta}(\mathscr{A}) - P_p(\mathscr{A})\big) \geq P_p(N_{\mathscr{A}} = \infty)$$

and

$$\lim_{\delta \downarrow 0} \big(P_p(\mathscr{A}) - P_{p-\delta}(\mathscr{A})\big) \geq P_p(N_{\mathscr{A}} = \infty).$$

Suppose O is a pretrifurcation of K. We say a vertex x is a *hinge* if x lies in fewer rigid components of K than of $K \setminus Z$. See Figure 13 for an illustration of one way in which hinges can occur; the five shaded regions lie in a single rigid component, but when the edges of Z are removed, each will lie in a distinct rigid component, so each of the three marked vertices will lie in two distinct rigid components. The idea of this definition is that it is the existence of hinges which may prevent a pretrifurcation from being a trifurcation. This is essentially because any rigid component is connected, so given any vertex contained in the same rigid component as the origin, this rigid component must contain a path from the vertex to the origin. If the chosen vertex is not contained in $Z \cup T_1 \cup T_2 \cup T_3$, then this path must contain a hinge. We shall make this argument precise in the proof of Proposition 8.7. We shall also show that, under the continuity assumption of Proposition 8.7, the number of hinges must be finite, and we may therefore "remove" all hinges (with positive probability) by taking a hexagon large enough to contain all of them and appealing to Lemma 8.3. We shall show that the number of hinges is finite by observing that if x is a hinge then, by altering the configuration near the origin (and shifting coordinates), we may make x h-pivotal to $\mathscr{R}^{(2)}$ and then using Lemma 8.8.

LEMMA 8.9. *Suppose*

$$P_p(O \textit{ is a pretrifurcation of } K \textit{ and there are infinitely many hinges}) > 0.$$

Then $P_p(N_{\mathscr{R}^{(2)}} = \infty) > 0$.

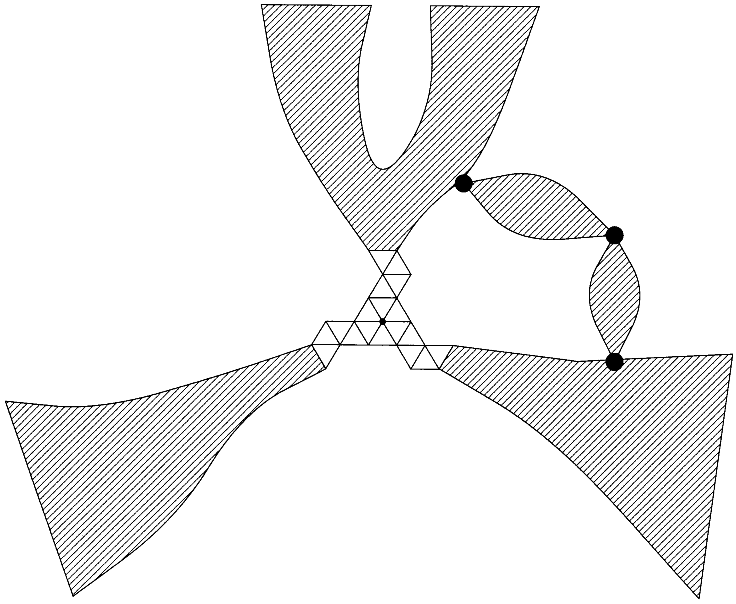

FIG. 13. *The three marked vertices are examples of hinges.*

PROOF OF LEMMA 8.8. We shall prove only the first inequality, the proof of the second being similar. We introduce the usual coupling of P_p and $P_{p+\delta}$: let $(\eta_e)_{e\in\mathbb{T}}$ be a collection of independent Uniform[0, 1] random variables and set $K^{(p)} = \{e\colon \eta_e < p\}$. Then for all $\delta > 0$ we have

$$\begin{aligned} P_{p+\delta}(K \in \mathscr{A}) - P_p(K \in \mathscr{A}) &= P(K^{(p+\delta)} \in \mathscr{A},\ K^{(p)} \notin \mathscr{A}) \\ &\geq P(F), \end{aligned}$$

say, where

$$\begin{aligned} F = \bigl\{&\text{there exists } x \in V \ h\text{-pivotal to } \mathscr{A} \text{ for } K^{(p)} \text{ such that} \\ &\qquad\qquad p \leq \eta_e \leq p+\delta \text{ for all } e \in H(1)+x\bigr\}. \end{aligned}$$

But we have

$$P(F) \geq P(F \mid N_{\mathscr{A}} = \infty \text{ for } K^{(p)})P_p(N_{\mathscr{A}} = \infty).$$

Now whenever $N_{\mathscr{A}} = \infty$, there is an infinite set of h-pivotal vertices S such that for any $x, y \in S$ with $x \neq y$, the graphs $H(1) + x$ and $H(1) + y$ are disjoint. It follows that $P_p(F \mid N_{\mathscr{A}} = \infty) = 1$, completing the proof. □

PROOF OF LEMMA 8.9. Suppose O is a pretrifurcation of K. Define the graphs $K_0 \subset \cdots \subset K_8$ by

$$\begin{aligned} &K_0 = K \setminus Z, \\ &K_8 = K, \\ &K_j = (K \setminus H(3)) \cup X_j \ (j = 1, \ldots, 7). \end{aligned}$$

We say $x \in V(\mathbb{T})$ is a *j-hinge* if x lies in fewer rigid components of K_j than of K_{j-1}. Thus if x is a hinge, it must be a j-hinge for some j. It is also clear that 1-hinges and 8-hinges cannot exist (by Proposition 6.8 and Corollary 6.2). Hence if there are infinitely many hinges, there must be infinitely many j-hinges for some $2 \le j \le 7$. Thus choose j to be the smallest value such that

$$P_p\big(O \text{ is a pre-trifurcation of } K \text{ and there are infinitely many } j\text{-hinges}\big) > 0.$$

We define $\hat{K} = (K \setminus H(3)) \cup W_j$ and also recall the definition of the vertex a_j. The required result follows from the observation (which we justify below) that any j-hinge outside $V(H(3))$ is h-pivotal to the event $\{a_j$ lies in an infinite rigid component$\}$ in $\hat{K}$. Hence, whenever the above event occurs, there are infinitely many such vertices, so the result follows by Lemma 8.3 and the fact that $N_{\mathscr{B}(2)}$ is translation-invariant.

A separate argument is required for each j. For the sake of brevity, we give the argument only for the case $j = 2$, the other cases being similar.

Suppose $x \in V(\mathbb{T}) \setminus V(H(3))$ is a 2-hinge. The relevant subgraphs of $H(3)$ are illustrated in Figure 14. The edges of X_1 are indicated by thin solid lines, while X_2 consists of these edges together with the dashed edge $\{O, b\}$, and W_2 consists of the edges of X_1 together with the two thickened edges. Recall that the graphs K_1, K_2 and $\hat{K}$ differ only on $H(3)$, and we have $K_1 \cap H(3) = X_1$, $K_2 \cap H(3) = X_2$, $\hat{K} \cap H(3) = W_2$. The vertices O and a_2 are also marked, and we define the additional vertices b and c via the figure. Now, x must have two neighbors u and v in K such that $\{x, u\}$, $\{x, v\}$ lie in distinct rigid components of K_1 but not of K_2. Consider the edge $\{u, v\}$, which is not necessarily an edge of $\mathbb{T}$. Consider also the edge $\{O, b\}$ which is present in K_2 but not in K_1. We have $\{u, v\}, \{O, b\} \notin \langle K_1 \rangle$, and $\{u, v\} \in \langle K_1 \cup \{\{O, b\}\}\rangle$, so by Proposition 6.5, $\{O, b\} \in \langle K_1 \cup \{\{u, v\}\}\rangle$. Hence in $K_1 \cup \{\{u, v\}\}$, there must be some rigid subgraph containing O and b, and we claim that it must also contain c. Indeed, any rigid subgraph containing O must include both of the two edges incident to O in K_1, otherwise we would have a contradiction to Proposition 6.6. It follows by Proposition 6.8 that a_2 lies in an infinite rigid

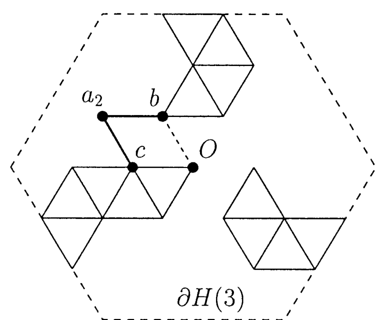

FIG. 14. *An illustration of the case $j = 2$.*

component of $\hat{K} \cup \{\{u, v\}\}$ but not of $\hat{K}$. Hence since $\{u, v\} \in \langle H(1) + x \rangle$, a_2 lies in an infinite rigid component of $\hat{K} \cup (H(1)+x)$ but not of $\hat{K} \setminus (H(1)+x)$; that is, x is h-pivotal to $\{a_2$ lies in an infinite rigid component$\}$ in $\hat{K}$, as required. $\square$

PROOF OF PROPOSITION 8.7. It follows from the two previous lemmas that if $\rho^{(2)}$ is either left-continuous or right-continuous at p, then

$$P_p\big(O \text{ is a pretrifurcation of } K \text{ and there are infinitely many hinges}\big) = 0.$$

Therefore

$$P_p\big(O \text{ is a pretrifurcation of } K \text{ and there are finitely many hinges}\big) > 0.$$

Hence for some fixed $S < \infty$ we have

$$P_p\big(O \text{ is a pretrifurcation of } K \text{ and there are no hinges outside } V(H(S-1))\big) > 0.$$

We now appeal to Lemma 8.3. Suppose the above event occurs and let T_1, T_2, T_3 be the three rigid components of $K \setminus Z$ containing f_1, f_2, f_3. Define $\zeta(K) = H(S) \cap (Z \cup T_1 \cup T_2 \cup T_3)$. Define $K' = (K \setminus H(S)) \cup \zeta(K)$. We shall show that O is a trifurcation of K'.

Clearly T_1, T_2, T_3 are rigid components of $K' \setminus Z$, since they are rigid components of $K \setminus Z$ and we have $K' \setminus Z \subseteq K \setminus Z$.

Suppose the rigid component of K' containing $H(1)$ is strictly larger than $Z \cup T_1 \cup T_2 \cup T_3$. Since the rigid component is connected, it must contain some edge $\{x, y\} \notin Z \cup T_1 \cup T_2 \cup T_3$ with $x \in V(T_i)$ (some i). But $\{x, y\} \notin H(S)$ [since $K' \cap H(S) = Z \cup T_1 \cup T_2 \cup T_3$], and $\{x, y\}$ must also lie in the rigid component of K containing $H(1)$ (since we have $K \supseteq K'$). We claim that x is a hinge, which gives a contradiction since x does not lie in $V(H(S-1))$. To see that x is a hinge, note that in $K \setminus Z$, x is contained in the rigid component T_i and also in another distinct rigid component, T' say, which includes the edge $\{x, y\}$. In K, the graphs T_i and T' both form part of a single rigid component. Hence, since the addition of Z cannot make any rigid component smaller, we deduce that x lies in fewer rigid components of K than of $K \setminus Z$. $\square$

The final ingredient is the following result about "compatibility" of trifurcations. See Figure 15 for an illustration of the proof of the lemma.

PROPOSITION 8.10. *Suppose s, t are distinct trifurcations of K which are contained in the same rigid component of K. Write $K_{(s)} = K \setminus (Z+s)$, $K_{(t)} = K \setminus (Z+t)$ and $K_{(s)(t)} = K \setminus ((Z+t) \cup (Z+t))$, and write S_1, S_2, S_3 for the three rigid components of $K_{(s)}$ containing f_1+s, f_2+s, f_3+s, respectively; define similarly T_1, T_2, T_3 for $K_{(t)}$.*

Then, after relabelling indices if necessary, we have

$$S_2 \cup S_3 \subseteq T_1.$$

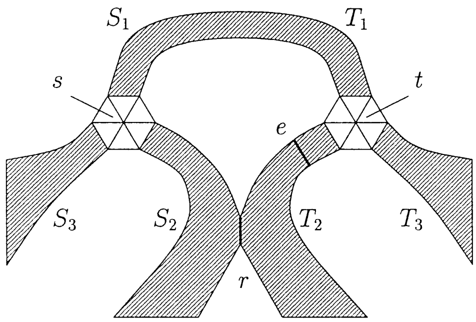

FIG. 15. *An illustration of the proof of "compatibility" of trifurcations.*

PROOF. Due to condition (1) of a trifurcation, it is clear that $s + H(1)$ and $t + H(1)$ must be disjoint, so after relabelling if necessary, we may assume $H(1)+s \subseteq T_1$ and $H(1)+t \subseteq S_1$. We have $S_2 \cup S_3 \subseteq T_1 \cup T_2 \cup T_3$, so we must show that $S_i \cap T_j = \varnothing$ for each pair $i, j \in \{2, 3\}$. By symmetry it is sufficient to show that $S_2 \cap T_2 = \varnothing$.

Suppose $r \in S_2 \cap T_2$. Write $e = f_2 + t$ (see Figure 15). Then since $r \in S_2$ and $e \in S_1$, r and e lie in distinct rigid components of $K_{(s)}$, hence they lie in distinct rigid components of $K_{(s)(t)}$. But r and e lie in the same rigid component, T_2, of $K_{(t)}$, so we must have $Z + s \subseteq T_2$ also, a contradiction since $H(1)+s \subseteq T_1$. $\square$

We now follow [2]. Let Z be a finite set with $|Z| \geq 3$. A *partition* of Z is a set $P = \{P_1, P_2, P_3\}$ of disjoint nonempty subsets of Z with $P_1 \cup P_2 \cup P_3 = Z$.

LEMMA 8.11 (Burton and Keane). *If $\mathbb{P}$ is a set of partitions of Z such that for any $P, Q \in \mathbb{P}$, after relabelling indices if necessary we have*

$$P_2 \cup P_3 \subseteq Q_1,$$

then

$$|\mathbb{P}| \leq |Z| - 2.$$

See [2] for a proof of Lemma 8.11.

PROOF OF THEOREM 8.1. All that is now required is to repeat the steps of [2]. Let N be the number of infinite rigid components of K. By ergodicity, $P_p(N = k) = 1$ for some constant k. Also, if $2 \leq k < \infty$ then it can be shown using Lemma 8.3 that $P_p(N < k) > 0$, a contradiction. Also $k = 0$ is impossible since we are given that $\rho^{(2)}(p) > 0$. Hence $k \in \{1, \infty\}$.

Suppose $k = \infty$. Consider $H(n)$, and let $F = \{\{x, y\} \in H(n),\ x \in V(\partial H(n))\}$. Let $T = \{x \in V(H(n-1))\colon x$ is a trifurcation of $K\}$. Let $U \subseteq T$ be the set of trifurcations in T belonging to a particular rigid component C of K, and let $Z = F \cap C$. Then each trifurcation in U induces a partition of Z, and by Lemma 8.10, the condition of Lemma 8.11 is satisfied, hence

$|U| \le |Z| - 2$. Applying this to each rigid component containing trifurcations in T and summing, we may deduce that $|T| \le |F|$. However, Proposition 8.7 implies that $E(|T|) \ge \varepsilon n^2$ for some constant $\varepsilon > 0$, and we have $|F| \le cn$ for some constant c, so we obtain a contradiction for large n. □

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