

SEQUENTIAL SELECTION OF AN INCREASING SEQUENCE FROM A MULTIDIMENSIONAL RANDOM SAMPLE

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Let random points X_1, \dots, X_n be sampled in strict sequence from a continuous product distribution on Euclidean d -space. At the time X_j is observed it must be accepted or rejected. The subsequence of accepted points must increase in each coordinate. We show that the maximum expected length of a subsequence selected is asymptotic to $\gamma n^{1/(d+1)}$ and give the exact value of γ . This extends the $\sqrt{2n}$ result by Samuels and Steele for $d = 1$.

1. Introduction. Samuels and Steele [8] studied the following offspring of Ulam's problem on the longest increasing subsequence of a random permutation. Let X_1, \dots, X_n be independent random points sampled from a continuous distribution. The points are recognized successively, one at a time. As soon as the first j points become known, X_j must be selected or rejected. If a point is selected, it cannot be discarded later and if rejected, cannot be recalled. The points selected should make up an increasing subsequence of X_1, \dots, X_n . The problem is to determine a selection policy that yields a subsequence of maximum expected length and to find the value, say v_n , of this maximum. The total number of points and their distribution are assumed to be known and can be used in the decision rules.

Samuels and Steele constructed a threshold policy that approaches optimality as $n \rightarrow \infty$ and showed that $v_n \sim \sqrt{2n}$. Comparing this result with the well known $2\sqrt{n}$ asymptotics for the expected length of the *longest* increasing subsequence of X_1, \dots, X_n (see [7] and [10]) they interpreted the ratio $2:\sqrt{2}$ as the long-run advantage of a "prophet" with complete foresight of the sequence over an intelligent but nonclairvoyant individual who uses nonanticipating decision rules. A striking feature of this result is that the prophet performs better by only a finite factor. See [2] and [4] for alternative proofs.

In this paper, we study a multidimensional analogue of the Samuels–Steele problem. We consider independent random points X_1, \dots, X_n sampled from a continuous product distribution in the Euclidean d -space. For $i_1 < \dots < i_k$ we define the subsequence X_{i_1}, \dots, X_{i_k} to be increasing if it is increasing in each coordinate. This definition is related intrinsically with random partial orders and multivariate records as studied, for example, in [11], [6], [3] and [5]. The formulation of the selection problem and definition of v_n generalize directly to the multidimensional setting. Here is our central result:

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THEOREM 1. *As $n \rightarrow \infty$ we have*

$$(1) \quad v_n \sim \gamma n^{1/(d+1)} \quad \text{with } \gamma = \frac{d+1}{\{(d+1)!\}^{1/(d+1)}}.$$

From [1] we know that the expected length of the longest increasing subsequence in d dimensions is asymptotic to $\text{const} \cdot n^{1/(d+1)}$. Although for $d > 1$ the exact value of the constant is still unknown (some conjectures are found in [9], page 117), Bollobás and Winkler showed that for $d \rightarrow \infty$ the constant tends to $e = 2.718\dots$. The same limit holds for the factors in (1), thus we can conclude that the long-run advantage of the prophet becomes negligible as the dimension grows.

Our proof of (1) is based on asymptotic solution of the dynamic programming equation for an analogous selection problem related to the planar Poisson process. Above that, we construct a policy that achieves the asymptotic value (1). For the case of uniform distribution in the unit cube, this policy makes a selection each time a point hits a small simplex with vertex at the last point selected so far. The policy is stationary in the sense that the simplex depends only on n (this sequential method has strong points of similarity with the bottom-up chain construction found in [1]).

2. The optimal policy. We define the relations among vectors component-wise, for example, $x < y$ for $x = (x^{(1)}, \dots, x^{(d)})$ and $y = (y^{(1)}, \dots, y^{(d)})$ means $x^{(i)} < y^{(i)}$, $i = 1, \dots, d$. For $x < y$ we denote (x, y) the d -dimensional interval $\{z : x < z < y\}$ called a *box*. For the vectors with all zero and all unit components we use the notation $\mathbf{0}$ and $\mathbf{1}$, respectively.

Many properties of the partial order $<$ on Q hinge on the function

$$p(x) = \prod_{i=1}^d (1 - x^{(i)}), \quad x \in Q,$$

which is the Euclidean volume of $(x, \mathbf{1})$. The function $p(\cdot)$ is decreasing. If $p(x) = p(y)$, then two boxes $(x, \mathbf{1})$ and $(y, \mathbf{1})$ are isomorphic in the sense that there exists an affine one-to-one mapping between the boxes that preserves the measure and coordinatewise orders.

We will assume that the observations X_1, X_2, \dots are independent uniformly distributed random points in the unit cube $Q = (0, 1)^d$. (The more general case of an arbitrary continuous product distribution is easily reduced to this particular one by a monotone coordinatewise transformation.)

A *policy* is defined to be a collection of finite stopping times $\tau = (\tau_1, \tau_2, \dots)$ such that:

- (i) each τ_i is adapted to X_1, X_2, \dots .
- (ii) $\tau_1 < \tau_2 < \dots$.
- (iii) $X_{\tau_1} < X_{\tau_2} < \dots$.

We allow that some of the stopping times be undefined starting from some i , in which case the conditions are to be understood properly.

Here, the variables X_{τ_i} are interpreted as the points selected by τ . Condition (i) captures the intuitive idea that the policy is *nonanticipating* or *on-line*. That is to say, a decision about X_j depends solely on the first j observations and not on X_{j+1}, X_{j+2}, \dots . Condition (iii) requires that the selected sequence increase.

Let $L_\tau(X_1, \dots, X_n) = \#\{i : \tau_i \leq n\}$ be the number of points selected by τ from X_1, \dots, X_n . The chief quantity of interest is

$$v_n = \sup_{\tau} EL_\tau(X_1, \dots, X_n),$$

the maximum expected length of an increasing subsequence that can be achieved by nonanticipating decision policies.

An example of a policy is the *greedy* rule, which selects X_1 and then each consecutive X_j greater than predecessors X_1, \dots, X_{j-1} . The relevant τ_i and X_{τ_i} are called record times and record values, respectively. Counting the records shows that the expected number of selections by this policy is $\sum_{j=1}^n j^{-d}$, which for $d > 1$ remains bounded as $n \rightarrow \infty$. We shall see that one can do considerably better with more sophisticated policies.

We outline a dynamic programming approach to the optimization problem. Extending the last definition, introduce the value function

$$v_n(x) = \sup_{\tau \in \mathcal{P}(x)} EL_\tau(X_1, \dots, X_n), \quad x \in \mathcal{Q},$$

where

$$\mathcal{P}(x) = \{\tau = (\tau_1, \tau_2, \dots) : X_{\tau_i} > x\}$$

is the class of policies that do not accept points outside the interval $(x, \mathbf{1})$. Clearly, the value function is continuous, with $v_n(\mathbf{0}) = v_n$ and $v_n(\mathbf{1}) = 0$. Since $\mathcal{P}(x)$ becomes smaller as x increases, $v_n(x)$ is decreasing in x .

Consider policies from $\mathcal{P}(x)$ applied to $n + 1$ observations. A recurrence formula for the value function follows by conditioning on the value of X_1 . By the i.i.d. assumption, if X_1 is rejected, the maximal (conditional) expected length of a selected subsequence will be $v_n(x)$. Similarly, if $X_1 > x$ and it is accepted, then the maximal (conditional) expected length will be $1 + v_n(X_1)$. Noting that $X_1 \notin (x, \mathbf{1})$ with probability $1 - p(x)$, we have

$$v_{n+1} = (1 - p(x))v_n(x) + \int_{(x, \mathbf{1})} \max(v_n(x), 1 + v_n(y)) dy,$$

where dy stands for Lebesgue measure. Rearranging terms, we put this in a more suggestive form,

$$(2) \quad v_{n+1}(x) - v_n(x) = \int_{(x, \mathbf{1})} (1 + v_n(y) - v_n(x))^+ dy,$$

where $(\cdot)^+ = \max(\cdot, 0)$.

Using induction in n one can prove that an optimal policy $\tau^* = (\tau_1^*, \tau_2^*, \dots)$, which achieves v_n , is given by

$$\tau_1^* = \min\{j \leq n : v_{n-j}(X_j) + 1 \geq v_{n-j}(\mathbf{0})\}$$

$$\tau_{i+1}^* = \min\{j : \tau_i^* < j \leq n : X_j > X_{\tau_i^*}, \text{ and } v_{n-j}(X_j) + 1 \geq v_{n-j}(X_{\tau_i^*})\}.$$

The somewhat cumbersome stopping time notation belies the simplicity of the optimal policy, which amounts to the following rule: *Suppose at stage j we are to decide about observation y and x is the last point selected so far. Then y must be accepted iff*

$$(3) \quad y > x \quad \text{and} \quad v_{n-j}(y) + 1 \geq v_{n-j}(x)$$

(and rejected otherwise).

To find v_n and τ^* explicitly, we need to solve (2). The computations become difficult even for small n . For the first two values we have

$$v_1(x) = p(x), \quad v_2(x) = 2p(x) + (2^{-d} - 1)p^2(x).$$

These values suggest that $v_n(x)$ depends on $x \in Q$ only through $p(x)$. This is indeed true and can be derived by induction from (2). Alternatively, note that $v_n(x)$ is achieved by a policy in $\mathcal{P}(x)$ that pays no heed to observations falling outside $(x, \mathbf{1})$. Thus if this box is isomorphic to another box $(y, \mathbf{1})$, we can map such a policy to a policy from $\mathcal{P}(y)$, and keep the performance unchanged. The statement follows because two boxes are isomorphic exactly when the corresponding values of $p(\cdot)$ are equal.

The last remark sheds some light on the structure of the optimal policy. Introducing

$$D_n(x) = \{y \in (x, \mathbf{1}) : v_n(y) + 1 \geq v_n(x)\},$$

we can put the selection criterion (3) in the form

$$y \in D_{n-j}(x).$$

The volume of $D_n(x)$ depends on x through $p(x)$. Geometrically, $D_n(x)$ is obtained via intersecting the box $(x, \mathbf{1})$ by a hyperbolic hypersurface $p(\cdot) = \text{const}$ provided $p(x)$ is not too small [otherwise $D_n(x)$ coincides with the box].

3. A stationary policy. The form of the optimal policy suggests that we could approach optimality by a suitable choice of a family of decision sets $B_{n-j}(x)$ instead of the optimal $D_{n-j}(x)$. After a minute of reflection, one could guess that for $n - j$ large and $p(x)$ not too small, a good candidate for $B_{n-j}(x)$ would be a small simplex obtained by intersecting $(x, \mathbf{1})$ with a hyperplane parallel to the tangent space at x to the surface $p(\cdot) = p(x)$. A more delicate fact is that an asymptotically optimal policy can be obtained by taking a *single* simplex Σ (depending on n only) for all $B_{n-j}(x)$'s. An intuitive explanation for this phenomenon, which we will not justify here in detail, is that, for n large,

the optimal sequence concentrates near the main diagonal in Q and grows linearly with j (thus for j close to $n/2$, the last selected point is with high probability near the center of Q , and the number of points selected so far is close to half of the total length of the selected subsequence).

Consider a simplex $\Sigma = \{x \in Q : x^{(1)} + \dots + x^{(d)} < \delta\}$, where the side size δ is yet to be fixed. Define a policy $\sigma = (\sigma_1, \sigma_2, \dots)$ by setting

$$(4) \quad \begin{aligned} \sigma_1 &= \min\{j : X_j \in \Sigma\}, \\ \sigma_{i+1} &= \min\{j > \sigma_i : X_j - X_{\sigma_i} \in \Sigma\}. \end{aligned}$$

Set $Z_j = X_{\sigma_i}$ if σ_i is the largest of the stopping times that are less than or equal to j , or set $Z_j = \mathbf{0}$ if not. That is to say, Z_j is the last point selected from the first j points, or it is a zero vector if no selections have been made to instant j . Variable X_j is selected by σ if it hits $Z_{j-1} + \Sigma$ (and then $Z_j = X_j$), and rejected otherwise (in which case $Z_j = Z_{j-1}$).

Observe that the policy operates like the greedy rule as soon as the condition

$$(5) \quad Z_{j-1} + \Sigma \subset Q$$

is violated. If (5) holds, the point X_j is selected with probability $V = \delta^d/d!$, which is the volume of Σ . Conditionally on any value of Z_{j-1} satisfying (5) and given that $X_j \in Z_{j-1} + \Sigma$, the expectation of $Z_j - Z_{j-1}$ is equal to $m\mathbf{1}$, where $m = (d + 1)/\delta$ is the common value for all d coordinates of the center of gravity in Σ .

Loosely speaking, for δ small and n large, the selection process is governed by the law of large numbers. The expected number of selections cannot exceed nV . On the other hand, the mean number of selections as long as (5) holds will be at most $1/m$. Since one quantity is increasing in δ while the other is decreasing, the maximum value of the minimum of these two quantities is attained when they are equal. Equating nV and $1/m$ yields

$$(6) \quad \delta = \left(\frac{(d + 1)!}{n} \right)^{1/(d+1)}.$$

The next theorem gives tight bounds on the performance of this policy.

THEOREM 2. *The mean number of points selected by the stationary policy (4) with the simplex side size (6) satisfies*

$$\gamma n^{1/(d+1)}(1 - R(n)) < L_\sigma(X_1, \dots, X_n) < \gamma n^{1/(d+1)},$$

where $R(n) = O(n^{-1/(2d+2)})$ as $n \rightarrow \infty$.

PROOF. The idea it to compare $\{Z_j\}$ with the partial sums process

$$S_0 = \mathbf{0}, \quad S_j = S_{j-1} + Y_j,$$

which has i.i.d. increments $Y_j = X_j \mathbf{1}_{\{X_j \in \Sigma\}}$.

It is easily seen that

$$(7) \quad \begin{aligned} P(Y_j \neq \mathbf{0}) &= V, & E(Y_j^{(k)}) &= mV, \\ \text{Var}(Y_j^{(k)}) &< 2\delta^2 V, & k &= 1, \dots, d, \end{aligned}$$

and $ES_n = \mathbf{1}$ by the choice of δ .

The initial piece of Z_1, Z_2, \dots taken as long as (5) holds has the same distribution as $S_1, S_2, \dots, S_{\rho \wedge n}$, where $\rho = \min\{j: S_{j-1} + \Sigma \notin Q\}$ and $\wedge = \min$. We will speak of a *jump* at index j each time $Y_j \neq \mathbf{0}$. The upper bound on $L_\sigma(X_1, \dots, X_n)$ follows by noting that the acceptance domain is a subset of a translate of Σ and

$$EL_\sigma(X_1, \dots, X_n) < E\left(\sum_{j=1}^n 1_{\{Y_j \neq \mathbf{0}\}}\right) = nV.$$

For the lower bound, note that the expected number of selections by σ is not less than the expected number of jumps in $S_1, S_2, \dots, S_{\rho \wedge n}$. Applying Wald's identity to stopping time $\rho \wedge n$, we have for the number of jumps

$$E\left(\sum_{j \leq n \wedge \rho} 1_{\{Y_j \neq \mathbf{0}\}}\right) = V E(\rho \wedge n) = V(n - E(n - \rho)^+).$$

To estimate the expectation in the brackets, observe that $\rho = \rho_1 \wedge \dots \wedge \rho_d$, where $\rho_k = \min\{j: S_j^{(k)} > 1 - \delta\}$. Therefore

$$(8) \quad E(n - \rho)^+ \leq E\left(\max_{k \leq d} (n - \rho_k)^+\right) \leq E\left(\sum_{k=1}^d (n - \rho_k)^+\right) = dE(n - \rho_1)^+.$$

Since ρ_1 is a stopping time, by the independence we have

$$\begin{aligned} E(S_n^{(1)} - S_{\rho_1}^{(1)})^+ &= E\left(\sum_{i=1}^n (S_n^{(1)} - S_{n-i}^{(1)}) 1_{\{n-\rho_1=i\}}\right) \\ &= \sum_{i=1}^n iP(n - \rho_1 = i)E(Y_1^{(1)}) = mV \cdot E(n - \rho_1)^+. \end{aligned}$$

Plugging this into (8) and using (7) gives

$$E(n - \rho)^+ \leq \frac{d}{mV} E(S_n^{(1)} - S_{\rho_1}^{(1)})^+.$$

Using the Cauchy-Schwarz inequality and $ES_n^{(1)} = 1$ we compute

$$\begin{aligned} E(S_n^{(1)} - S_{\rho_1}^{(1)})^+ &\leq E(S_n^{(1)} - 1 + \delta)^+ \\ &< (E(S_n^{(1)} - 1 + \delta)^2)^{1/2} = (\text{Var}(S_n^{(1)}) + \delta^2)^{1/2}. \end{aligned}$$

Now from (7) we have $\text{Var}(S_n^{(1)}) = n \text{Var}(Y_1^{(1)}) = O(n\delta^2 V)$, whence

$$E(n - \rho)^+ = O(n^{1-1/(2d+2)})$$

and the lower bound follows. \square

While the asymptotic optimality of σ among all policies will follow from our main result, there is a variational explanation why σ cannot be improved by another stationary policy: Of all bodies in the positive orthant and with given volume, a standard simplex minimizes the largest coordinate of the center of gravity.

4. A Poisson process problem. We introduce next a Poisson analogue of the fixed- n sequential selection problem and argue that both problems are asymptotically similar.

Let \mathcal{N} be a homogeneous Poisson point process on $[0, \infty) \times Q$, with Lebesgue measure as intensity. Let $(T_1, X_1), (T_2, X_2), \dots$ be the atoms of \mathcal{N} labelled by increasing the time component, $T_1 < T_2 < \dots$. Think of X_j as a point observed at time T_j . It is well known that X_1, X_2, \dots are i.i.d. uniform in Q , $(T_1, T_2 - T_1, T_3 - T_2, \dots)$ are i.i.d. standard exponential and both sequences are independent. Consider stopping times adapted to $(X_1, T_1), (X_2, T_2), \dots$ and define a policy to be a sequence of stopping times $\tau = (\tau_1, \tau_2, \dots)$ satisfying

- (i') each τ_i is adapted to $(X_1, T_1), (X_2, T_2), \dots$,

along with (ii) and (iii) of Section 2.

We denote by \mathcal{N}_t the restriction of \mathcal{N} onto $[0, t] \times Q$. With a slight abuse of notation, we write $L_\tau(\mathcal{N}_t) = \#\{i : T_{\tau_i} \leq t\}$ for the number of Poisson points selected by τ and write $\mathcal{P}(x)$ for the family of policies that do not accept points outside $(x, \mathbf{1})$. Let

$$u(t) = \sup_{\tau} EL_\tau(\mathcal{N}_t)$$

be the maximum expected length for horizon t .

What makes the Poisson model so attractive is a stronger invariance property: for any two “time–space” $(d + 1)$ -dimensional intervals of the same volume there exists an isomorphism that respects both the measure and the coordinatewise order. The supremum of $EL_\tau(\mathcal{N}_t)$ over $\mathcal{P}(x)$ is achieved within the subclass of policies measurable with respect to the restriction of \mathcal{N} onto $[0, t] \times (x, \mathbf{1})$, as follows by the independence properties of the Poisson process. Mapping the box $[0, t] \times (x, \mathbf{1})$ on $[0, p(x)t] \times Q$, we see that the supremum is equal to $u(tp(x))$. That the value function depends on a single parameter makes things much easier [as compared with two parameters n and $p(x)$ in the fixed- n problem].

To derive a dynamic programming equation on $u(\cdot)$, consider the Poisson problem with horizon $t + \varepsilon$. With probability $e^{-\varepsilon} = 1 - \varepsilon + o(\varepsilon)$ there are no arrivals until instant ε , thus no selection is made. Otherwise, with probability $\varepsilon + o(\varepsilon)$ there is a single arrival and a decision based on the observed X_1 should be made. Therefore

$$u(t + \varepsilon) = (1 - \varepsilon)u(t) + \varepsilon \int_Q \max(u(tp(y)) + 1, u(t)) dy + o(\varepsilon).$$

Rearranging terms and taking limits yields an integrodifferential equation

$$(9) \quad u'(t) = \int_Q (u(tp(y)) + 1 - u(t))^+ dy,$$

which should be complemented by the initial condition $u(0) = 0$.

It does not seem possible to find a closed form solution to (9). One of the difficulties that arises is that $u(\cdot)$ has singularities (the minimal singular point is where the value 1 is taken). In what follows we will find the asymptotics for large t .

THEOREM 3. *All solutions to (9) satisfy*

$$u(t) \sim \gamma t^{1/(d+1)}$$

as $t \rightarrow \infty$, whatever the initial value $u(0)$.

Let $u(\cdot)$ be a solution of (9). Our proof hinges on properties of the functionals

$$J_t(f) = \int_Q (f(tp(y)) + 1 - u(t))^+ dy,$$

parameterized by $t \geq 0$ and defined for continuously differentiable functions on $[0, t]$. First, note that J_t is monotone in the sense that $f(s) \leq g(s)$ for $s \leq t$ implies $J_t(f) \leq J_t(g)$. Second, $J_t(f + \text{const}) = J_t(f)$ for any constant. In these terms, the equation is written as

$$u'(t) = J_t(u), \quad t \geq 0,$$

and any function $u(\cdot) + \text{const}$ is a solution too.

For positive α and t consider the function

$$h(s) = \alpha s^{1/(d+1)} - \alpha t^{1/(d+1)} + u(t), \quad 0 \leq s \leq t,$$

where the constant terms are selected so that $h(t) = u(t)$. The derivative h' at $s = t$ is increasing and, by the monotonicity, $J_t(h)$ is decreasing in α .

LEMMA 4. $J_t h_\alpha \sim ((d + 1)/\alpha)^d ((d + 1)!)^{-1} t^{-d/(d+1)}$ as $t \rightarrow \infty$.

PROOF. We have, by the change of variable,

$$\begin{aligned} J_t(h_\alpha) &= \int_Q (\alpha(tp(x))^{1/(d+1)} - \alpha t^{1/(d+1)} + 1)^+ dx \\ &= \int_0^1 (\alpha(t(1 - \xi))^{1/(d+1)} - \alpha t^{1/(d+1)} + 1)^+ db(\xi), \end{aligned}$$

where $b(\xi)$ is the volume of $\{x \in Q : p(x) \geq 1 - \xi\}$. Using induction in d one can show that

$$b(\xi) = 1 - (1 - \xi) \sum_{i=0}^{d-1} |\log(1 - \xi)|^i / i!$$

from which

$$db/d\xi = |\log(1 - \xi)|^{d-1}/(d - 1)!.$$

The integrand is positive for $\xi < \xi_0$, where

$$\xi_0 \sim (d + 1)\alpha^{-1}t^{-1/(d+1)}.$$

The statement follows by taking the leading term of the Taylor expansion for small ξ and integrating over $[0, \xi_0]$. \square

PROOF OF THEOREM 3. Note that γ is a single solution to

$$\alpha = \left(\frac{d + 1}{\alpha}\right)^d ((d + 1)!)^{-1},$$

which is the only value of α that gives the match $J_t(h) \sim h'(t)$. It follows from the lemma and by monotonicity that, for t sufficiently large,

$$(10) \quad J_t(h) < h'(t) \quad \text{for } \alpha > \gamma,$$

$$(11) \quad J_t(h) > h'(t) \quad \text{for } \alpha < \gamma.$$

We claim that $u(t) - \alpha t^{1/(d+1)}$ is bounded from above, provided $\alpha > \gamma$. Indeed, assume this is unbounded. For $C > 0$, define $t(C)$ as the minimal point where the function $u(t) - \alpha t^{1/(d+1)}$ crosses the level C . We can select C arbitrarily large and so that the derivative is strictly positive at $t(C)$. For such $t = t(C)$ we have

$$u'(t) > h'_\alpha(t) \quad \text{and} \quad u(t) = \alpha t^{1/(d+1)} + C,$$

while for $s < t$,

$$u(s) \leq \alpha s^{1/(d+1)} + C = h(s).$$

Again by monotonicity and from (9),

$$u'(t) = J_t(u) \leq J_t(h_\alpha),$$

whence $h'_\alpha(t) < J_t h_\alpha$, which is a contradiction with (10), because letting $C \rightarrow \infty$, we have $t(C) \rightarrow \infty$.

A similar argument involving (11) proves that $u(t) - \alpha t^{1/(d+1)}$ is bounded from below for $\alpha < \gamma$. The theorem follows. \square

Now we can prove Theorem 1.

PROOF OF THEOREM 1. In view of Theorems 2 and 3, all that we need to show is that the value in the Poisson problem with $t = n$ yields an appropriate asymptotic upper bound for v_n .

Indeed, let $N(t)$ be the number of Poisson atoms in $[0, t] \times Q$. For small $\varepsilon > 0$ set $t = n(1 + \varepsilon)$ and apply the optimal- n policy to the first n Poisson arrivals, paying attention only to the X_j 's. Given that N is larger than n , this

policy selects, on the average, v_n points. Because the policy is only suboptimal in the Poisson problem, we have

$$v_n P(N \geq n) < u(t).$$

For $n \rightarrow \infty$, this probability goes to 1. Thus setting $\varepsilon \rightarrow 0$ and using Theorem 2, we establish $v_n \sim u(n)$ and (1) follows. \square

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