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# LARGE DEVIATIONS OF PRODUCTS OF RANDOM TOPICAL OPERATORS

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A topical operator on $\mathbb{R}^d$ is one which is isotone and homogeneous. Let $\{A(n): n \geq 1\}$ be a sequence of i.i.d. random topical operators such that the projective radius of $A(n) \cdots A(1)$ is almost surely bounded for large $n$. If $\{x(n): n \geq 1\}$ is a sequence of vectors given by $x(n) = A(n) \cdots A(1)x_0$, for some fixed initial condition $x_0$, then the sequence $\{x(n)/n : n \geq 1\}$ satisfies a weak large deviation principle. As corollaries of this result we obtain large deviation principles for products of certain random aperiodic max-plus and min-plus matrix operators and for products of certain random aperiodic nonnegative matrix operators.

**1. Topical operators.** An operator $A$: $\mathbb{R}^d \to \mathbb{R}^d$ is *homogeneous* if it satisfies $A(x + a\mathbf{1}) = Ax + a\mathbf{1}$ for all $x \in \mathbb{R}^d$ and $a \in \mathbb{R}$, where $\mathbf{1}$ is the vector in $\mathbb{R}^d$ with all components equal to 1. An operator $A$ is *isotone* if it satisfies $Ax \leq Ay$ whenever $x \leq y$ (the order here and throughout this paper is the product order on $\mathbb{R}^d$). An operator which is both homogeneous and isotone is called *topical*. This terminology was introduced by Gunawardena and Keane (1995), who proposed the class of topical operators as a setting for the study of certain properties of discrete event systems. In this context, one considers recursive systems of equations of the form

$$x(n) = A(n)x(n-1), \qquad n = 1, 2, \ldots, \tag{1}$$

with the interpretation that $x(n) \in \mathbb{R}^d$ is a vector whose entries represent timing data: $x_i(n)$ is the time of the $n$th event of some type $i$, where $d$ is the number of types of events which may occur. The operators $A(n)$: $\mathbb{R}^d \to \mathbb{R}^d$ determine the delays and synchronization constraints present between events. Homogeneity of these operators reflects invariance of the system's dynamics under a shift in the origin of the time axis. Isotonicity of the $A(n)$'s implies that the system is monotonic, in the sense that if some events were to be artificially delayed, then all subsequent events would also be delayed, or at best they would occur no sooner than originally. For more information on topical operators and their application to discrete event systems, see Gunawardena and Keane (1995) and Gunawardena (1996) and the references therein.

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Well-known examples of topical operators include the max-plus and min-plus matrix operators, which are defined as follows: $A\colon \mathbb{R}^d \to \mathbb{R}^d$ is a *max-plus matrix operator* if it takes the form

$$(Ax)_i = \max_{j=1,\ldots,d} A_{ij} + x_j, \qquad i = 1, \ldots, d,$$

for every $x \in \mathbb{R}^d$, where $\{A_{ij} : i, j = 1, \ldots, d\}$ are elements of $\mathbb{R} \cup \{-\infty\}$. (We assume that each row of the matrix $\{A_{ij}\}$ has at least one entry different from $-\infty$, so that the image of $\mathbb{R}^d$ under $A$ is contained in $\mathbb{R}^d$.) A *min-plus matrix operator* is one which takes the form

$$(Ax)_i = \min_{j=1,\ldots,d} A_{ij} + x_j, \qquad i = 1, \ldots, d,$$

for each $x \in \mathbb{R}^d$, where now $\{A_{ij} : i, j = 1, \ldots, d\}$ are elements of $\mathbb{R} \cup \{+\infty\}$ (again with the caveat that each row of $\{A_{ij}\}$ has at least one finite entry). Matrix operators of these kinds arise in the theory of Markov decision processes and timed event graphs. A general reference is the book by Baccelli, Cohen, Olsder and Quadrat (1992). If we take a finite pointwise infimum of max-plus matrix operators, or a finite pointwise supremum of min-plus matrix operators, we obtain an operator which is again topical, known as a *min-max operator*. In the context of discrete event systems, min-max operators were introduced and studied by Olsder (1991) and Gunawardena (1994).

Another interesting class of topical operators can be constructed from the isotone linear operators on the positive cone $\mathbb{R}^d_+$, in the following way [Gunawardena (1996)]. Let exp: $\mathbb{R}^d \to \mathbb{R}^d_+$ be the componentwise exponential function and log: $\mathbb{R}^d_+ \to \mathbb{R}^d$ the componentwise logarithm: $\exp(x)_i := \exp(x_i)$ and $\log(x)_i := \log(x_i)$. If $A\colon \mathbb{R}^d_+ \to \mathbb{R}^d_+$ is isotone and satisfies $A(ax) = aAx$ for all $x \in \mathbb{R}^d$ and $a \in \mathbb{R}_+$, then the operator $\tilde{A}\colon \mathbb{R}^d \to \mathbb{R}^d$ defined by $\tilde{A}x := \log(A\exp(x))$ is topical. Note that $A$ might be, for example, a nonnegative matrix operator with at least one nonzero entry per row.

Our purpose in this paper is to study the large deviations of sequences $\{x(n) : n \geq 1\}$ which satisfy recursions of the form (1), in the case when $\{A(n) : n \geq 1\}$ is a random sequence of i.i.d. topical operators. The approach we will take requires an assumption that the $A(n)$'s satisfy a certain range condition, which we now state.

Let $\mathsf{t}$ and $\mathsf{b}$ denote the *top* and *bottom* functions on $\mathbb{R}^d$:

$$\mathsf{t}[x] := \max_i x_i, \qquad \mathsf{b}[x] := \min_i x_i.$$

Gunawardena and Keane (1995) showed that $A\colon \mathbb{R}^d \to \mathbb{R}^d$ is topical if and only if it is nonexpansive in $\mathsf{t}$:

$$\mathsf{t}[Ax - Ay] \leq \mathsf{t}[x - y] \qquad \forall x, y, \tag{2}$$

and if and only if it is noncontractive in $\mathsf{b}$:

$$\mathsf{b}[Ax - Ay] \geq \mathsf{b}[x - y] \qquad \forall x, y. \tag{3}$$

Together these inequalities imply that topical operators are nonexpansive in the $l_\infty$-norm on $\mathbb{R}^d$:

$$\|Ax - Ay\| \leq \|x - y\| \qquad \forall x, y,$$

where $\|x\| = \max_i |x_i| = \mathsf{t}[x] \vee (-\mathsf{b}[x])$. In fact, Crandall and Tartar (1980) showed that a homogeneous operator on $\mathbb{R}^d$ is isotone if and only if it is $l_\infty$ nonexpansive.

Inequalities (2) and (3) also imply that topical operators are nonexpansive in the *projective semi-norm* $\|\cdot\|_{\mathsf{P}}$ defined by

$$\|x\|_{\mathsf{P}} = \mathsf{t}[x] - \mathsf{b}[x].$$

We define the *projective radius* of a topical operator $A$ to be the extended real number

$$\Pi[A] := \sup_{x \in \mathbb{R}^d} \|Ax\|_{\mathsf{P}}.$$

Note that the projective radius of a translation operator, for example, is $+\infty$. The interest in projective radius is that, if $A$ has finite projective radius, then there exist a vector $x \in \mathbb{R}^d$ and a scalar $c$ such that $Ax = x + c\mathbf{1}$ [Baccelli and Mairesse (1996)]. Such a vector is sometimes called a *generalized fixed point* of $A$. Finite projective radius is not, however, a necessary condition for the existence of a generalized fixed point. More details and references on the fixed-point properties of various types of topical operators can be found in Baccelli, Cohen, Olsder and Quadrat (1992) and Gaubert and Gunawardena (1998).

Turning to sequences of random operators, an important result is the following ergodic theorem.

THEOREM [Baccelli and Mairesse (1996)]. *Let $\{A(n) : n \geq 1\}$ be a stationary and ergodic sequence of random topical operators. If there is an integer $N$ and a real number $C$ such that*

$$\Pi[A(N) \cdots A(1)] \leq C,$$

*with positive probability, then there exists $\gamma \in \mathbb{R}$ such that*

$$\lim_{n\to\infty} \frac{1}{n} A(n) \cdots A(1) x_0 = \gamma \mathbf{1}$$

*almost surely, for every $x_0 \in \mathbb{R}^d$.*

In this paper we study deviations from the behavior described by this theorem, but under the following stronger assumption.

ASSUMPTION 1. (a) $\{A(n) : n \geq 1\}$ *is a random sequence of i.i.d. topical operators*; (b) *there exists an integer* $N$ *and a real number* $C$ *such that* $\Pi[A(N) \cdots A(1)] \leq C$ *almost surely*.

Our main result is that if this assumption holds for the sequence $\{A(n)\}$ and if $\{x(n) : n \geq 1\}$ is a sequence of vectors satisfying the recursive system (1) for some fixed initial condition, then the sequence $\{x(n)/n : n \geq 1\}$ satisfies a weak large deviation principle. The associated rate function is equal to $+\infty$ away from the line $x = a\mathbf{1}$, $a \in \mathbb{R}$, so that at this scale the system's behavior is effectively one-dimensional. This confinement is the consequence of part (b) of Assumption 1. We also present some results to characterize the rate function, but explicit calculations turn out to be difficult in all but trivial cases. It is well known that calculation of the Liapounov exponent $\gamma$ of the ergodic theorem is already a hard problem.

For the case of max-plus and min-plus matrix operators, our results extend previous work by Baccelli and Konstantopoulos (1991), Glasserman and Yao (1995) and Chang (1996).

To illustrate the significance of Assumption 1, we end this section with two simple examples from queuing theory, the first of which satisfies this assumption and the second of which violates it. Consider a closed cyclic queuing network consisting of two servers and a fixed number of customers. Customers leaving the first server immediately join the queue for the second and vice versa. At each server customers are served in their order of arrival in a work-conserving manner. For $n \geq 1$ let $\sigma_i(n)$ be the duration of the $n$th service at server $i$ and let $x_i(n)$ be the time at which this service is completed. If at time $x_i(n-1)$ there is a customer waiting in the queue for server $i$, then we will have

$$x_i(n) = x_i(n-1) + \sigma_i(n).$$

If there is no customer waiting in the queue, we will have

$$x_i(n) = x_j(n-1) + \sigma_i(n).$$

Hence $x(n) = A(n)x(n-1)$, where $A(n)$ is the max-plus matrix operator given by

$$A(n) \equiv \begin{bmatrix} \sigma_1(n) & \sigma_1(n) \\ \sigma_2(n) & \sigma_2(n) \end{bmatrix}.$$

The projective radius of $A(n) \cdots A(1)$ is

$$\Pi[A(n) \cdots A(1)] = \Pi[A(n)] = \sigma_1(n) \vee \sigma_2(n) - \sigma_1(n) \wedge \sigma_2(n).$$

Thus, if $\{\sigma_1(n) : n \geq 1\}$ and $\{\sigma_2(n) : n \geq 1\}$ are sequences of bounded i.i.d. random variables, Assumption 1 is satisfied. For extensions of this example to systems of several cyclic queues with various blocking mechanisms operating between them, see Mairesse (1997).

If $\sigma_1(n)$ and $\sigma_2(n)$ are not almost surely bounded, then Assumption 1 is violated, even though the conditions of Theorem 1 continue to hold. Consider, however, the example of the single server queue where new customers arrive from an external source, are served in their order of arrival and then depart from the system. Let $x_1(n)$ be the arrival time of the $n$th customer and let $x_2(n)$ be the time of his or her departure from the queue. Denote by $\sigma(n)$ the duration of service required by the $n$th customer, and by $\tau(n)$ the elapsed time between the arrival of this customer and his or her predecessor. Assuming that the server is nonidling, $x_1(n)$ and $x_2(n)$ satisfy the equations

$$x_1(n) = x_1(n-1) + \tau(n),$$
$$x_2(n) = [x_1(n-1) + \tau(n) + \sigma(n)] \vee [x_2(n-1) + \sigma(n)],$$

or $x(n) = A(n)x(n-1)$, where $A(n)$ is the max-plus matrix operator

$$A(n) \equiv \begin{bmatrix} \tau(n) & -\infty \\ \tau(n) + \sigma(n) & \sigma(n) \end{bmatrix}.$$

The projective radius of $A(n) \cdots A(1)$ is now equal to $+\infty$ for every $n$, so that neither Assumption 1 nor the conditions of Theorem 1 are satisfied. [The ergodic behavior of this system can, of course, be treated using methods other than those of Theorem 1; see Baccelli, Cohen, Olsder and Quadrat (1992).]

**2. Large deviations.** Let $\{A(n) : n \geq 1\}$ be a sequence of random topical operators on $\mathbb{R}^d$. Given a fixed initial vector $x_0 \in \mathbb{R}^d$, we let $\{x(n; x_0) : n \geq 1\}$ be the sequence defined by

$$x(n; x_0) := A(n)A(n-1) \cdots A(1)x_0, \qquad n = 1, 2, \dots.$$

It will be convenient to let $A(l, m)$ stand for the product of the $A(n)$'s from $n = l + 1$ up to $n = m$, where $m > l \geq 0$:

$$A(l, m) := A(m)A(m-1) \cdots A(l+1).$$

We shall also use $x(l, m)$ to denote the vector $A(l, m)x_0$.

With the assumption that $x_0$ is a fixed, rather than random, initial condition, the nonexpansive property of the $A(n)$'s ensures that the large deviations of the sequence $\{x(n; x_0)/n : n \geq 1\}$ are in fact independent of $x_0$. If $y_0$ is another fixed initial condition, then

$$\|x(n; x_0) - x(n; y_0)\| = \|A(0, n)x_0 - A(0, n)y_0\| \leq \|x_0 - y_0\|,$$

implying that, for any $\varepsilon > 0$,

$$\mathbb{P}\big(\|x(n; x_0) - x(n; y_0)\| > n\varepsilon\big) = 0$$

for $n$ large enough. The sequences $\{x(n; x_0)/n\}$ and $\{x(n; y_0)/n\}$ are therefore exponentially equivalent [Dembo and Zeitouni (1998), Chapter 4], so that one satisfies a large deviation principle if and only if the other does, and with the same rate function. We set $x_0$ equal to the zero vector $\mathbf{0}$ and suppress the dependence of $x(n; x_0)$ on $x_0$ henceforth.

Let $\mathbb{M}_n$ be the law of $x(n)/n$ and let $\overline{m}$ and $\underline{m}$ be set functions defined on the Borel subsets of $\mathbb{R}^d$ by

$$\overline{m}[B] := \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{M}_n[B],$$

$$\underline{m}[B] := \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{M}_n[B].$$

The *upper* and *lower deviation functions* $\overline{\mu}$ and $\underline{\mu}$ associated with the sequence $\{\mathbb{M}_n : n \geq 1\}$ are the maps from $\mathbb{R}^d$ into $[-\infty, 0]$ given by

$$\overline{\mu}(x) := \inf_{G \ni x} \overline{m}[G],$$

$$\underline{\mu}(x) := \inf_{G \ni x} \underline{m}[G],$$

where the infima on the right-hand sides are taken over all open sets $G$ containing the point $x$. As both $\overline{m}$ and $\underline{m}$ are increasing set functions, these infima may in fact be taken over any base of Borel neighborhoods of $x$. The properties of $\overline{\mu}$ and $\underline{\mu}$ are discussed in the review of Lewis and Pfister (1995). They are upper semicontinuous and for all open sets $G$ satisfy

$$\overline{m}[G] \geq \sup_{x\in G} \overline{\mu}(x),$$

$$\underline{m}[G] \geq \sup_{x\in G} \underline{\mu}(x).$$

In addition, $\overline{\mu}$ satisfies

$$\overline{m}[K] \leq \sup_{x\in K} \overline{\mu}(x)$$

for all compact sets $K$.

The sequence $\{\mathbb{M}_n : n \geq 1\}$ satisfies a *weak large deviation principle* with rate function $l$ if and only if $l$ is lower semicontinuous and the inequalities

$$\overline{m}[K] \leq -\inf_{x\in K} l(x),$$

$$\underline{m}[G] \geq -\inf_{x\in G} l(x)$$

hold for all compact sets $K$ and open sets $G$. A neccessary and sufficient condition for the weak l.d.p. to hold with rate function $l$ is that $\overline{\mu}$ and $\underline{\mu}$ should coincide and be equal to $-l$ throughout $\mathbb{R}^d$ [Lewis and Pfister (1995)]. The sequence $\{\mathbb{M}_n : n \geq 1\}$ satisfies a *large deviation principle* with rate function $l$ if and only if it satisfies a weak l.d.p. in which the upper bound for compact sets extends to all closed subsets $F$ of $\mathbb{R}^d$:

$$\overline{m}[F] \leq -\inf_{x\in F} l(x).$$

Lemma 1 below establishes that, if the $A(n)$'s are i.i.d., then $\overline{\mu}(x) = \underline{\mu}(x)$ on the line $x = a\mathbf{1}$, $a \in \mathbb{R}$. The argument is based on the following observation: if $A_1, A_2$ are any pair of topical operators and $a, b$ are any pair of real numbers, then

$$\begin{aligned} &\|A_1 A_2 \cdot \mathbf{0} - (a+b)\mathbf{1}\| \\ &\quad \le \|A_1 A_2 \cdot \mathbf{0} - A_1 \cdot a\mathbf{1}\| + \|A_1 \cdot a\mathbf{1} - (a+b)\mathbf{1}\| \\ &\quad \le \|A_2 \cdot \mathbf{0} - a\mathbf{1}\| + \|A_1 \cdot \mathbf{0} - b\mathbf{1}\|. \end{aligned} \tag{4}$$

Let $B_r(a\mathbf{1})$ denote the $l_\infty$-ball of radius $r$ centered at $a\mathbf{1}$. Since $x(n+m) = A(n, n+m)A(0,n) \cdot \mathbf{0}$, the inequality (4) implies that the sequence $\{\mathbb{M}_n[B_r(a\mathbf{1})]\}$ is supermultiplicative:

$$\begin{aligned} &\mathbb{M}_{n+m}[B_r(a\mathbf{1})] \\ &\quad = \mathbb{P}\big(\|x(n+m) - (n+m)a\mathbf{1}\| < (n+m)r\big) \\ &\quad \ge \mathbb{P}\big(\|x(n) - na\mathbf{1}\| < nr,\ \|x(n, n+m) - ma\mathbf{1}\| < mr\big) \\ &\quad = \mathbb{M}_n[B_r(a\mathbf{1})]\mathbb{M}_m[B_r(a\mathbf{1})]. \end{aligned}$$

A variant of the standard subadditivity lemma [see Lanford (1973) and Lewis, Pfister and Sullivan (1994)] may now be used to show that $\overline{\mu}(a\mathbf{1}) = \underline{\mu}(a\mathbf{1})$, and also that the resulting function $a \longmapsto \overline{\mu}(a\mathbf{1}) = \underline{\mu}(a\mathbf{1})$ is concave.

LEMMA 1. *Under part* (a) *of Assumption* 1, $\underline{\mu}(a\mathbf{1}) = \overline{\mu}(a\mathbf{1})$ *for each* $a \in \mathbb{R}$.

PROOF. Fix $a \in \mathbb{R}$ and put $n = ps + q$, where $s > 0$, $p > 0$ and $0 \le q < s$. We have from (4) that

$$\|x(n) - na\mathbf{1}\| \le \|x(ps, ps+q) - qa\mathbf{1}\| + \|x(0, ps) - psa\mathbf{1}\|,$$

and, continuing the expansion,

$$\|x(n) - na\mathbf{1}\| \le \|x(ps, ps+q) - qa\mathbf{1}\| + \sum_{k=1}^{p} \big\|x\big((k-1)s, ks\big) - sa\mathbf{1}\big\|. \tag{5}$$

Let $z_s(k)$ denote the contribution coming from the $k$th block of size $s$:

$$z_s(k) := x\big((k-1)s, ks\big).$$

Then $\{z_s(k) : k \ge 1\}$ is a sequence of i.i.d. random variables and the law of $z_s(1)$ is $\mathbb{M}_s$. It follows from (5) that, for each $\varepsilon > 0$,

$$\begin{aligned} &\mathbb{P}\big(\|x(n) - na\mathbf{1}\| < nr\big) \\ &\quad \ge \mathbb{P}\left(\|x(ps, ps+q) - qa\mathbf{1}\| < n\varepsilon,\ \sum_{k=1}^{p} \|z_s(k) - sa\mathbf{1}\| < ps(r-\varepsilon)\right) \\ &\quad \ge \mathbb{P}\big(\|x(q) - qa\mathbf{1}\| < n\varepsilon\big)\big[\mathbb{P}\big(\|z_s(1) - sa\mathbf{1}\| < s(r-\varepsilon)\big)\big]^p, \end{aligned}$$

and therefore

$$(6)\qquad \begin{aligned}&\frac{1}{n}\log \mathbb{M}_n[B_r(a\mathbf{1})]\\ &\qquad \ge \frac{1}{n}\log \mathbb{P}\big(\|x(q)-qa\mathbf{1}\|<n\varepsilon\big)+\frac{p}{n}\log \mathbb{M}_s[B_{r-\varepsilon}(a\mathbf{1})].\end{aligned}$$

Now, since any finite collection of probability measures on $\mathbb{R}^d$ is tight, we can find a compact set $K\subset\mathbb{R}^d$ such that $x(q)$ falls in $K$ with positive probability for each $q=0,\dots,s-1$. Define

$$\alpha_s := \min_{0\le q<s}\mathbb{P}\big(x(q)\in K\big).$$

Then $\alpha_s>0$, and there exists $M<\infty$ such that, for all $n\ge M$ and each $q=0,\dots,s-1$,

$$\mathbb{P}\big(\|x(q)-qa\mathbf{1}\|<n\varepsilon\big)\ge \mathbb{P}\big(x(q)\in K\big)\ge\alpha_s.$$

Returning to inequality (6), this yields

$$\begin{aligned}\frac{1}{n}\log \mathbb{M}_n[B_r(a\mathbf{1})] &\ge \frac{1}{n}\log\alpha_s+\frac{p}{n}\log \mathbb{M}_s[B_{r-\varepsilon}(a\mathbf{1})]\\ &\ge \frac{1}{n}\log\alpha_s+\frac{1}{s}\log \mathbb{M}_s[B_{r-\varepsilon}(a\mathbf{1})],\end{aligned}$$

with $\log\alpha_s>-\infty$. Taking first the lim inf in $n$ and then the lim sup in $s$, we obtain

$$\liminf_{n\to\infty}\frac{1}{n}\log \mathbb{M}_n[B_r(a\mathbf{1})]\ge \limsup_{s\to\infty}\frac{1}{s}\log \mathbb{M}_s[B_{r-\varepsilon}(a\mathbf{1})],$$

or $\underline{m}[B_r(a\mathbf{1})]\ge\overline{m}[B_{r-\varepsilon}(a\mathbf{1})]$. The statement of the lemma follows on taking infima over $r>\varepsilon$, giving

$$\inf_{r>\varepsilon}\underline{m}[B_r(a\mathbf{1})]\ge\overline{\mu}(a\mathbf{1}),$$

and then over $\varepsilon>0$, to get $\underline{\mu}(a\mathbf{1})\ge\overline{\mu}(a\mathbf{1})$. □

LEMMA 2. *The map $a\longmapsto\underline{\mu}(a\mathbf{1})=\overline{\mu}(a\mathbf{1})$ resulting from Lemma* 1 *is concave.*

PROOF. For $a_1,a_2\in\mathbb{R}$, we have from (4) that

$$\|x(2n)-n(a_1+a_2)\mathbf{1}\|\le\|x(n)-na_1\mathbf{1}\|+\|x(n,2n)-na_2\mathbf{1}\|,$$

which implies

$$\mathbb{M}_{2n}\big[B_r\big((a_1\mathbf{1}+a_2\mathbf{1})/2\big)\big]\ge \mathbb{M}_n[B_r(a_1\mathbf{1})]\,\mathbb{M}_n[B_r(a_2\mathbf{1})].$$

Therefore,

$$\overline{m}\big[B_r\big((a_1\mathbf{1}+a_2\mathbf{1})/2\big)\big] \ge \liminf_{n\to\infty} \frac{1}{2n}\log \mathbb{M}_{2n}\big[B_r\big((a_1\mathbf{1}+a_2\mathbf{1})/2\big)\big]$$
$$\ge \frac{1}{2}\,\underline{m}[B_r(a_1\mathbf{1})] + \frac{1}{2}\,\underline{m}[B_r(a_2\mathbf{1})],$$

and, taking infima over $r > 0$, we get

$$\overline{\mu}\big((a_1\mathbf{1}+a_2\mathbf{1})/2\big) \ge \frac{1}{2}\,\underline{\mu}(a_1\mathbf{1}) + \frac{1}{2}\,\underline{\mu}(a_2\mathbf{1}).$$

This inequality may be extended to cover all convex combinations $\lambda a_1\mathbf{1} + (1-\lambda)a_2\mathbf{1}$, where $\lambda$ is a dyadic rational in [0, 1], by iterating the above argument. The concavity of the map $a \mapsto \underline{\mu}(a\mathbf{1}) = \overline{\mu}(a\mathbf{1})$ then follows from the fact that it is upper semicontinuous. $\square$

Lemma 1 is enough to establish a weak large deviation principle if both parts of Assumption 1 are satisfied.

THEOREM 3. *Let both parts of Assumption* 1 *hold. The sequence* $\{\mathbb{M}_n : n \ge 1\}$ *satisfies a weak large deviation principle with a convex rate function* $l$ *which is equal to* $+\infty$ *on the set* $\{x : \|x\|_{\mathsf{P}} > 0\}$.

PROOF. For $\varepsilon > 0$ and $n \ge N$,

$$\mathbb{P}\big(\|x(n)\|_{\mathsf{P}} > n\varepsilon\big) = \mathbb{P}\big(\|A(n-N,n)x(n-N)\|_{\mathsf{P}} > n\varepsilon\big)$$
$$\le \mathbb{P}\big(\Pi[A(n-N,n)] > n\varepsilon\big)$$
$$= \mathbb{P}\big(\Pi[A(0,N)] > n\varepsilon\big),$$

which is 0 for $n$ sufficiently large. Therefore $\underline{\mu}(x) = \overline{\mu}(x) = -\infty$ for each $x$ with $\|x\|_{\mathsf{P}} > 0$. Combining this with Lemma 1, we have $\underline{\mu} = \overline{\mu}$ everywhere, and the resulting rate function $l = -\overline{\mu} = -\underline{\mu}$ is convex by Lemma 2. $\square$

It is not difficult to construct examples which violate Assumption 1 and where the l.d.p. cannot hold with a rate function of the form appearing in Theorem 3. Consider again the cyclic queuing example of Section 1. Here each $A(n)$ is a max-plus matrix operator of the form

$$A(n) \equiv \begin{bmatrix} \sigma_1(n) & \sigma_1(n) \\ \sigma_2(n) & \sigma_2(n) \end{bmatrix}.$$

Assume that $\{\sigma_1(n) : n \ge 1\}$ and $\{\sigma_2(n) : n \ge 1\}$ are sequences of i.i.d. exponentially distributed random variables. Then

$$\|A(n)x\|_{\mathsf{P}} = \sigma_1(n) \vee \sigma_2(n) - \sigma_1(n) \wedge \sigma_2(n)$$

for any $x$, so that the projective radius of each $A(n)$ is almost surely finite but not almost surely bounded. We find that

$$\mathbb{P}\big(\|x(n)\|_{\mathsf{P}} > n\varepsilon\big) = \mathbb{P}\big(\sigma_1(n) \vee \sigma_2(n) - \sigma_1(n) \wedge \sigma_2(n) > n\varepsilon\big),$$

which does not vanish on an exponential scale as $n \to \infty$. Systems which do not exhibit bounded projective radius need not, therefore, be confined to the line $x = a\mathbf{1}$, $a \in \mathbb{R}$.

The next two lemmas are directed toward proving that the rate function $l$ of Theorem 3 is the convex dual of the scaled cumulant generating function $\lambda$ of the sequence $\{x(n)\}$. For $\theta \in \mathbb{R}^d$, let $\{\mathbb{M}_n^\theta\}$ be the sequence of measures defined by

$$\mathbb{M}_n^\theta[B] := \int_B e^{n\langle\theta,x\rangle}\mathbb{M}_n[dx].$$

Theorem 3 implies that $\{\mathbb{M}_n^\theta : n \geq 1\}$ satisfies a weak large deviation principle with rate function $l^\theta$ given by $l^\theta(x) = l(x) - \langle\theta, x\rangle$. Let $\lambda_n$ be the cumulant generating function of $x(n)$ (automatically proper, convex and l.s.c.):

$$\lambda_n(\theta) := \log \mathbb{M}_n^\theta[\mathbb{R}^d] = \log \mathbb{E}e^{\langle\theta,x(n)\rangle}.$$

Lemma 4 shows that the limit $\lambda(\theta) := \lim_{n\to\infty} \lambda_n(\theta)/n$ exists, and Lemma 5 shows that $\lambda$ is the convex dual of $l$.

Lemma 4 also gives two expressions for $\lambda$ which may be of use in approximating it. To state these, we define $\psi_n \colon \mathbb{R}^2 \to \mathbb{R}$ to be the map

$$\psi_n(\phi_1, \phi_2) := \log \mathbb{E}\exp\big(\phi_1 \mathsf{t}[x(n)] + \phi_2 \mathsf{b}[x(n)]\big).$$

For $\theta \in \mathbb{R}^d$ we let $\theta_+$ represent the sum of the positive components of $\theta$ and $\theta_-$ the sum of the negative components.

LEMMA 4. *Under Assumption* 1, $\lambda(\theta)$ *exists for each* $\theta \in \mathbb{R}^d$ *and is given by*

$$\lambda(\theta) = \sup_{n\geq 1} \frac{1}{n}\psi_n(\theta_-, \theta_+) \quad \textit{and} \quad \lambda(\theta) = \inf_{n\geq 1} \frac{1}{n}\psi_n(\theta_+, \theta_-).$$

PROOF. For $n, m \geq 1$,

$$\begin{aligned} x(n+m) &= A(n, n+m)A(0, n)\mathbf{0} \\ &\leq A(n, n+m)\big(\mathsf{t}[A(0, n)\mathbf{0}]\mathbf{1}\big) = A(n, n+m)\mathbf{0} + \mathsf{t}[A(0, n)\mathbf{0}]\mathbf{1}, \end{aligned}$$

and, similarly,

$$x(n+m) \geq A(n, n+m)\mathbf{0} + \mathsf{b}[A(0, n)\mathbf{0}]\mathbf{1}.$$

These yield the inequalities

$$\mathsf{t}[x(n+m)] \leq \mathsf{t}[x(n, n+m)] + \mathsf{t}[x(n)], \tag{7}$$

$$\mathsf{b}[x(n+m)] \geq \mathsf{b}[x(n, n+m)] + \mathsf{b}[x(n)], \tag{8}$$

which together imply that the sequence $\{\psi_n(\theta_-, \theta_+) : n \geq 1\}$ is superadditive:

$$\psi_{n+m}(\theta_-, \theta_+) \geq \psi_n(\theta_-, \theta_+) + \psi_m(\theta_-, \theta_+)$$

for all $n, m \geq 1$. Now $\psi_n$ is the cumulant generating function of the pair of random variables $(\mathsf{t}[x(n)], \mathsf{b}[x(n)])$. As these random variables are real valued, $\psi_n$ cannot take the value $-\infty$. It follows that the limit

$$\lim_{n\to\infty} \frac{1}{n}\psi_n(\theta_-, \theta_+) = \sup_{n\geq 1} \frac{1}{n}\psi_n(\theta_-, \theta_+)$$

exists for all $\theta$.

Clearly, $\lambda_n(\theta) \geq \psi_n(\theta_-, \theta_+)$ for all $n$. But for $n \geq N$, $x(n)$ satisfies $\mathsf{t}[x(n)] - \mathsf{b}[x(n)] \leq C$, so that $\mathsf{t}[x(n)] - x_i(n) \leq C$ and $x_i(n) - \mathsf{b}[x(n)] \leq C$ for each $i$. Therefore

$$\langle \theta, x(n)\rangle \leq \theta_+ \mathsf{b}[x(n)] + \theta_+ C + \theta_- \mathsf{t}[x(n)] - \theta_- C,$$

implying that

$$\lambda_n(\theta) \leq \psi_n(\theta_-, \theta_+) + \|\theta\|_1 C$$

and

$$\lim_{n\to\infty} \frac{1}{n}\lambda_n(\theta) = \lim_{n\to\infty} \frac{1}{n}\psi_n(\theta_-, \theta_+).$$

To prove the second identity for $\lambda$, we first note that the argument just given also establishes that

$$\psi_n(\theta_+, \theta_-) - \psi_n(\theta_-, \theta_+) \leq 2C\|\theta_1\|$$

for all $n \geq N$. Therefore $\psi_n(\theta_+, \theta_-)/n$ converges to $\lambda(\theta)$ as $n \to \infty$. Furthermore, inequalities (7) and (8) imply that $\{\psi_n(\theta_+, \theta_-)\}$ is a subadditive sequence: for all $n, m \geq 1$,

$$\psi_{n+m}(\theta_+, \theta_-) \leq \psi_n(\theta_+, \theta_-) + \psi_m(\theta_+, \theta_-);$$

hence

$$\lim_{n\to\infty} \frac{1}{n}\psi_n(\theta_+, \theta_-) = \inf_{n\geq 1} \frac{1}{n}\psi_n(\theta_+, \theta_-). \qquad \square$$

Recall that the convex dual of $l$ is the function $l^*\colon \mathbb{R}^d \to [-\infty, +\infty]$ given by

$$l^*(\theta) := \sup_{x\in\mathbb{R}^d} \{\langle \theta, x\rangle - l(x)\}.$$

LEMMA 5. *Under Assumption* 1, $\lambda$ *is the convex dual of* $l$.

PROOF. The sequence of measures $\{\mathbb{M}_n^\theta\}$ satisfies a weak large deviation principle with rate function $l(x) - \langle\theta, x\rangle$. Since $\mathbb{R}^d$ is an open set, the large deviation lower bound gives us

$$\lambda(\theta) = \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{M}_n^\theta[\mathbb{R}^d] \geq -\inf_x \bigl\{l(x) - \langle\theta, x\rangle\bigr\},$$

or $\lambda(\theta) \geq l^*(\theta)$. To prove the opposite inequality, let $z_m(n)$ be the random variable $x((n-1)m, nm)$ and let $y_m(n)$ denote the sum

$$y_m(n) := \frac{1}{nm} \sum_{k=1}^{n} z_m(k).$$

For fixed $m$, the sequence $\{z_m(k) : k \geq 1\}$ is i.i.d. Let $\mathbb{L}_n^{(m)}$ be the law of $y_m(n)$; by Cramér's theorem, the sequence $\{\mathbb{L}_n^{(m)}, n \geq 1\}$ satisfies a large deviation principle with convex rate function $l_m$ equal to

$$l_m(x) = \sup_\theta \bigl\{\langle\theta, x\rangle - \lambda_m(\theta/m)\bigr\}.$$

Since both $l_m$ and $\lambda_m$ are proper convex l.s.c. functions, we also have

$$\lambda_m(\theta) = \sup_x \bigl\{\langle m\theta, x\rangle - l_m(x)\bigr\}.$$

Now for $n, m \geq N$ the two inequalities (7) and (8) imply that

$$\|x(n+m) - x(n, n+m) - x(n)\| \leq C.$$

Setting $n = pm$ with $m \geq N$ and applying this result repeatedly, we have

$$\|x(n) - ny_m(p)\| = \left\| x(n) - \sum_{k=1}^{p} z_m(k) \right\| \leq pC.$$

Therefore

$$\begin{aligned} \mathbb{M}_n[B_{r+C/m}(x)] &= \mathbb{P}\bigl(\|x(n) - nx\| < nr + pC\bigr) \\ &\geq \mathbb{P}\bigl(\|ny_m(p) - nx\| < nr\bigr) \\ &= \mathbb{L}_p^{(m)}[B_r(x)]. \end{aligned}$$

Taking logs, dividing by $n$ and letting $p \to \infty$, this becomes

$$\overline{m}[B_{r+C/m}(x)] \geq \liminf_{p\to\infty} \frac{1}{pm} \log \mathbb{L}_p^{(m)}[B_r(x)],$$

and, taking infima over $r > 0$, we get

$$\inf_{r > C/m} \overline{m}[B_r(x)] \geq -\frac{1}{m}\, l_m(x).$$

Now fix $r' = C'/m > C/m$ and let $\overline{B}_{r'}(x)$ be the closed $l_\infty$-ball of radius $r'$ centered at $x$. Applying the large deviation upper bound for the set $\overline{B}_{r'}(x)$ produces

$$- \inf_{y \in \overline{B}_{r'}(x)} l(y) \geq \overline{m}[B_{r'}(x)] \geq -\frac{1}{m} l_m(x).$$

Hence

$$\begin{aligned}\sup_{y \in \overline{B}_{r'}(x)} \{\langle \theta, y \rangle - l(y)\} &\geq \langle \theta, x \rangle - \frac{1}{m}\|\theta\| C' - \inf_{y \in \overline{B}_{r'}(x)} l(y) \\ &\geq \langle \theta, x \rangle - \frac{1}{m}\big(\|\theta\| C' + l_m(x)\big),\end{aligned}$$

and, taking the supremum over $x$ on both sides,

$$l^*(\theta) \geq \frac{1}{m}\big(\lambda_m(\theta) - \|\theta\| C'\big).$$

The upper bound $\lambda(\theta) \leq l^*(\theta)$ is now obtained by letting $m \to \infty$. $\square$

If $\lambda$ is finite in a neighborhood of the origin, then the sequence $\{\mathbb{M}_n : n \geq 1\}$ is *exponentially tight*: there exists a sequence of compact sets $\{K_n : n \geq 1\}$ such that

$$\limsup_{n \to \infty} \overline{m}[\mathbb{R}^d \setminus K_n] = -\infty.$$

Under exponential tightness the weak l.d.p. for $\{\mathbb{M}_n : n \geq 1\}$ extends to a full l.d.p. [see Lewis and Pfister (1995) or Dembo and Zeitouni (1998), Chapter 1].

THEOREM 6. *Let Assumption* 1 *hold. The rate function $l$ of Theorem* 3 *is the convex dual of $\lambda$. If $\lambda$ is finite in a neighborhood of the origin, then the sequence $\{\mathbb{M}_n\}$ satisfies a large deviation principle with rate function $l$.*

PROOF. Recall that $\lambda$ exists by Lemma 4, and by Lemma 5 it is the convex dual of $l$. Since $l$ is a proper convex l.s.c. function, it follows that $l = l^{**} = \lambda^*$. If $\lambda$ is finite in a neighborhood of the origin, then the sequence $\{\mathbb{M}_n\}$ is exponentially tight and the l.d.p. follows. $\square$

Note that since $\lambda(\theta) \leq \psi_n(\theta_+, \theta_-)/n$ for all $n$ one needs only that $\psi_n(\theta_+, \theta_-)$ be finite in a neighborhood of the origin, for any $n$, in order to establish the l.d.p.

**3. Matrix operators.** From the results of the last section, we may deduce the l.d.p. for certain classes of the matrix operators introduced in Section 1.

LEMMA 7. *A max-plus matrix operator has finite projective radius if and only if each of its columns has all entries finite or all entries equal to $-\infty$. Similarly, a min-plus operator has finite projective radius if and only if each of its columns has all entries finite or all entries equal to $+\infty$.*

PROOF. Suppose that $A$ is a max-plus matrix operator with all matrix entries finite and let $x$ be any vector in $\mathbb{R}^d$. For a given value of $i \in [1, d]$, let $J(i)$ be the value of $j$ which maximizes $A_{ij} + x_j$. Then

$$\mathsf{t}[Ax] = \max_{i,j} A_{ij} + x_j = \max_i A_{iJ(i)} + x_{J(i)}$$

and

$$\mathsf{b}[Ax] = \min_k \max_l A_{kl} + x_l \geq \min_k A_{kJ(i)} + x_{J(i)}.$$

Therefore

$$\|Ax\|_{\mathsf{P}} \leq \max_i \bigl(A_{iJ(i)} - \min_k A_{kJ(i)}\bigr) \leq \max_{i,j} A_{ij} - \min_{k,l} A_{kl}.$$

This gives a finite upper bound on $\|Ax\|_{\mathsf{P}}$, independent of $x$. Next, if one or more columns of $A$ are identically equal to $-\infty$, then the projective radius of $A$ is equal to that of the matrix obtained by deleting these columns. If the remaining entries are all finite, then so is $\Pi[A]$. (Recall that we assume each row of $A$ has at least one finite entry, so that $A$ cannot be identically equal to $-\infty$.)

Now suppose that for some column $j$ we have $A_{ij}$ finite and $A_{kj} = -\infty$. Then, for any $x$,

$$\|Ax\|_{\mathsf{P}} \geq (Ax)_i - (Ax)_k \geq A_{ij} + x_j - \max_{l \neq j}(A_{kl} + x_l),$$

and since $x_j$ can be made arbitrarily large it follows that $\Pi[A] = +\infty$.

The proof for min-plus matrices is similar. $\square$

In particular, if $A$ is an *aperiodic* matrix operator, then there exists $N < \infty$ such that $A^N$ has all entries finite, and therefore finite projective radius. Turning to random sequences of matrix operators, we say that $\{A(n) : n \geq 1\}$ *has fixed structure* if, for each $i$, $j$, $A_{ij}(n)$ is equal to $-\infty$ for all $n$ with probability 1 or 0.

ASSUMPTION 2. *The sequence $\{A(n) : n \geq 1\}$ is a random sequence of i.i.d. aperiodic max-plus matrix operators, with fixed structure. In addition, the finite components of $A(1)$ take values in a bounded subset of $\mathbb{R}$.*

As in Section 2, $\mathbb{M}_n$ denotes the law of $x(n)/n$, where

$$x(n) = A(n) \cdots A(1)x_0, \qquad n = 1, 2, \ldots,$$

with $x_0$ fixed.

THEOREM 8. *Let Assumption 2 hold. Then the sequence $\{\mathbb{M}_n\}$ satisfies a large deviation principle with convex rate function $I$ equal to $\lambda^*$.*

PROOF. Let $N < \infty$ be such that the matrix $(A(1))^N$ has all entries finite. Note that, when taking the product of matrices $A(1)$ and $A(2)$, the positions of the finite entries in $A(2)A(1)$ depend only on which entries of $A(1)$ and $A(2)$ are finite (and not on the values of these entries; this is similar to the situation with the zeros of products of positive matrices under the standard algebra). Since each matrix $A(k)$ has its finite entries in the same positions (due to the fixed structure assumption), it follows that $A(0, N) = A(N) \cdots A(1)$ has all entries finite, and therefore finite projective radius. Under the second part of Assumption 2, the entries of $A(0, N)$ actually take values in a bounded subset of $\mathbb{R}$, implying that $\Pi[A(0, N)]$ is almost surely bounded. It follows from Theorems 3 and 6 that the sequence $\{\mathbb{M}_n\}$ satisfies a weak l.d.p. with rate function $\lambda^*$. Assumption 2 also implies that, for each $n$, $x(n)$ is a bounded random variable, so that the functions $\psi_n$ of Lemma 4 are finite throughout $\mathbb{R}^2$. Therefore so is $\lambda(\theta)$ and the l.d.p. holds for $\{\mathbb{M}_n\}$. $\square$

The l.d.p. for a certain class of nonnegative matrix operators on the positive cone $\mathbb{R}_+^d$ can be proved along similar lines. Recall that if $A\colon \mathbb{R}_+^d \to \mathbb{R}_+^d$ is a nonnegative matrix operator having at least one nonzero entry per row, then $\tilde{A}\colon \mathbb{R}^d \to \mathbb{R}^d$ is the topical operator defined by $\tilde{A}x := \log(A\exp(x))$.

LEMMA 9. *Let $A\colon \mathbb{R}_+^d \to \mathbb{R}_+^d$ be a nonnegative matrix operator, with at least one nonzero entry per row. Then $\tilde{A}$ has finite projective radius if and only if each column of $A$ has all entries greater than* 0 *or all entries equal to* 0.

PROOF. Assume that $A$ has no matrix entries equal to 0 and let $x$ be any vector in $\mathbb{R}^d$. For a given value of $i$, let $J(i)$ be the value of $j$ which maximizes $A_{ij}e^{x_j}$. Then

$$\mathsf{t}[\tilde{A}x] = \max_i \log\left(\sum_j A_{ij}e^{x_j}\right) \le \max_i \log\big(dA_{iJ(i)}e^{x_{J(i)}}\big)$$

and

$$\mathsf{b}[\tilde{A}x] = \min_k \log\left(\sum_l A_{kl}e^{x_l}\right) \ge \min_k \log\big(A_{kJ(i)}e^{x_{J(i)}}\big).$$

Therefore

$$\begin{aligned}\|\tilde{A}x\|_{\mathsf{P}} &\le \max_i\big(\log(dA_{iJ(i)}) - \min_k \log A_{kJ(i)}\big)\\ &\le \log d + \max_{i,j}\log A_{ij} - \min_{k,l}\log A_{kl},\end{aligned}$$

a finite upper bound which is independent of $x$. As in the max-plus matrix case, one can now observe that if $A$ has all entries of one column equal to 0, then the projective radius of $\tilde{A}$ is equal to that of the operator obtained by excluding this column. This proves the "if" part of the lemma.

On the other hand, if column $j$ of $A$ has a nonzero entry $A_{ij}$ and a zero entry $A_{kj}$, then, for any $x \in \mathbb{R}^d$,

$$\|\tilde{A}x\|_{\mathsf{P}} \geq \log A_{ij} + x_j - \log\Biggl(\sum_{l \neq j} A_{il} e^{x_l}\Biggr).$$

Since $x_j$ can be made arbitrarily large, it follows that $\Pi[\tilde{A}] = +\infty$. $\square$

The assumption analogous to Assumption 2 is therefore the following. A *fixed structure* sequence of random nonnegative matrices $\{A(n)\}$ will be one in which, for each $i$, $j$, $A_{ij}(n)$ is 0 for all $n$ with probability either 1 or 0.

ASSUMPTION 3. *The sequence $\{A(n) : n \geq 1\}$ is a random sequence of i.i.d. nonnegative aperiodic matrix operators, with fixed structure. In addition, the nonzero entries of $A(1)$ take values in a compact subset of the positive real line.*

We continue to let $x(n)$ denote the vector $A(n) \cdots A(1)x_0$, for a fixed $x_0 \in \mathbb{R}_+^d$, but we now take $\mathbb{M}_n$ to be the law of $(\log x(n))/n$.

THEOREM 10. *Under Assumption 3, the sequence $\{\mathbb{M}_n : n \geq 1\}$ satisfies a large deviation principle with rate function $I$ equal to $\lambda^*$, where $\lambda \colon \mathbb{R}^d \to \mathbb{R}$ is given by*

$$\lambda(\theta) = \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E} \prod_{i=1}^{d} \bigl(x_i(n)\bigr)^{\theta_i}.$$

PROOF. Letting $\tilde{x}(n) = \log x(n)$, we find that $\tilde{x}(n)$ satisfies

$$\tilde{x}(n) = \tilde{A}(n) \cdots \tilde{A}(1)\tilde{x}_0.$$

The remainder of the proof parallels the proof of Theorem 8 and is omitted. $\square$

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