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OPTIMAL INVESTMENT IN INCOMPLETE MARKETS WHEN WEALTH MAY BECOME NEGATIVE[1]

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This paper accompanies a previous one by D. Kramkov and the present author. While in [17] we considered utility functions $U : \mathbb{R}_+ \to \mathbb{R}$ satisfying the Inada conditions $U'(0) = \infty$ and $U'(\infty) = 0$, in the present paper we consider utility functions $U : \mathbb{R} \to \mathbb{R}$ which are finitely valued, for all $x \in \mathbb{R}$, and satisfy $U'(-\infty) = \infty$ and $U'(\infty) = 0$. A typical example of this situation is the exponential utility $U(x) = -e^{-x}$.

In the setting of [17] the following crucial condition on the asymptotic elasticity of U, as x tends to $+\infty$, was isolated: $\limsup_{x\to+\infty} \frac{xU'(x)}{U(x)} < 1$. This condition was found to be necessary and sufficient for the existence of the optimal investment as well as other key assertions of the related duality theory to hold true, if we allow for general semi-martingales to model a (not necessarily complete) financial market.

In the setting of the present paper this condition has to be accompanied by a similar condition on the asymptotic elasticity of U, as x tends to $-\infty$, namely, $\liminf_{x\to-\infty} \frac{xU'(x)}{U(x)} > 1$. If both conditions are satisfied — we then say that the utility function U has *reasonable asymptotic elasticity* — we prove an existence theorem for the optimal investment in a general locally bounded semi-martingale model of a financial market and for a utility function $U : \mathbb{R} \to \mathbb{R}$, which is finitely valued on all of $\mathbb{R}$; this theorem is parallel to the main result of [17]. We also give examples showing that the reasonable asymptotic elasticity of U also is a necessary condition for several key assertions of the theory to hold true.

1. Introduction. The present work accompanies the previous paper [17] by D. Kramkov and the author. For the motivation and history of the utility maximization as well as for references and notation we refer in the sequel to [17] without further notice.

In the present paper the setting differs from that of [17] in the following respect: we consider a utility function $U : \mathbb{R} \to \mathbb{R}$, which is defined and finitely valued everywhere on the real line; in addition we make the usual assumptions that U is smooth (i.e., continuously differentiable), increasing, strictly concave and s.t.

$$U'(\infty) := \lim_{x\to\infty} U'(x) = 0 \quad \text{and} \quad U'(-\infty) := \lim_{x\to-\infty} U'(x) = \infty. \tag{1}$$

As in [17] the financial market is modeled by a d-dimensional semi-martingale $S = ((S_t^i)_{1\le i\le d})_{0\le t\le T}$ describing the discounted price process of

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d risky traded assets. For the bond price process we let $B_t \equiv 1$. In the present paper we also assume that the semi-martingale S is locally bounded and this assumption will be crucial for the present methodology (see Remark 2.6 below).

DEFINITION 1.1. A probability measure $Q \sim P$ (resp. $Q \ll P$) is called an equivalent (resp. absolutely continuous) local martingale measure if S is a local martingale under Q.

The family of equivalent (resp. absolutely continuous) local martingale measure will be denoted by $\mathscr{M}^e(S)$ [resp. $\mathscr{M}^a(S)$]. We assume throughout this paper that

$$\mathscr{M}^e(S) \neq \emptyset. \tag{2}$$

Note that, under the present assumption that S is locally bounded, this definition coincides with definition 2.1 in [17]: indeed, it is easy to verify that a locally bounded semi-martingale S is a local martingale under Q iff Definition 2.1 of [17] is satisfied, that is, each positive process X of the form $X_t = X_0 + \int_0^t H_u dS_u$ with $X_0 \in \mathbb{R}_+$ is a local Q-martingale (compare [11] and [1]).

After this rather harmless task of fixing the proper definition of $\mathscr{M}^e(S)$ and $\mathscr{M}^a(S)$ we now pass to a more delicate issue, namely the concept of admissible trading strategies which is appropriate in the present context. Recall the subsequent definition from [7] which essentially is the same concept as used in [17]:

DEFINITION 1.2. A predictable S-integrable process H is an *admissible trading strategy* if the stochastic integral $(H \cdot S)_t = \int_0^t H_u dS_u$ is uniformly bounded from below.

For $x \in \mathbb{R}$, we denote by $\mathcal{X}^b(x)$ the set of processes

$$X_t = x + (H \cdot S)_t, \qquad 0 \le t \le T, \tag{3}$$

where H runs through the admissible trading strategies.

We have used the super-script b to indicate that the processes in $\mathcal{X}^b(x)$ are uniformly bounded from below. However, in the present context of maximizing expected utility for a utility function $U(x)$ which is finitely valued, for all $x \in \mathbb{R}$, it is natural to consider processes $(X_t)_{0 \le t \le T}$ such that X_t is not necessarily uniformly bounded from below, if one wants to have a chance to find the maximizer of the utility maximization problem (6) below.

We adopt the following concept.

DEFINITION 1.3. For $x \in \mathbb{R}$, define the set $\mathscr{C}_U^b(x)$ by

$$\begin{aligned} \mathscr{C}_U^b(x) = \{ G_T \in L^0(\Omega, \mathscr{F}_T, P) : G_T \le X_T \\ \text{for some } X \in \mathcal{X}^b(x) \text{ and } E[|U(G_T)|] < \infty \}, \end{aligned} \tag{4}$$

and let $\mathscr{C}_U(x)$ denote the set

$$\mathscr{C}_U(x) = \{F_T \in L^0(\Omega, \mathscr{F}_T, P; \mathbb{R} \cup \{\infty\}) : U(F_T) \text{ is in the } L^1(P)\text{-closure of } \{U(G_T) : G_T \in \mathscr{C}_U^b(x)\}\}. \tag{5}$$

Let us interpret the above concept: $\mathscr{C}_U^b(x)$ consists of all random variables G_T such that $U(G_T)$ is P-integrable (so that the expected utility may be defined) and such that G_T is dominated by some final wealth X_T which may be achieved by an economic agent with initial endowment x and a finite credit line, by trading on the stock S. Note that we don't impose integrability conditions on G_T but only on $U(G_T)$. [Recall that $L^0(\Omega, \mathscr{F}_T, P)$ denotes the set of all (equivalence classes of) $\mathscr{F}_T$-measurable $\mathbb{R}$-valued random variables.]

In the next step we enlarge the set $C_U^b(x)$ by considering the closure of the random variables $U(G_T)$, where the closure is taken with respect to the norm of $L^1(\Omega, \mathscr{F}_T, P)$. As U defines a bijection between $\mathbb{R}$ and $\mathbb{R}$, in the case when $U(\infty) = \infty$, and a bijection between $\mathbb{R} \cup \{+\infty\}$ and $]-\infty, U(\infty)]$, in the case when $U(\infty) < \infty$, we can write these random variables as $U(F_T)$, where F_T are $\mathscr{F}_T$-measurable random variables, possibly assuming the value $+\infty$ in the case $U(\infty) < \infty$.

Speaking in economic terms, $\mathscr{C}_U(x)$ describes all random variables F_T modeling (possibly infinite) wealth at time T such that the *utility* $U(F_T)$ may be approximated by the *utility* $U(G_T)$, where G_T ranges through the set of random variables dominated by X_T, for some X in $\mathfrak{X}^b(x)$, with respect to the norm of $L^1(P)$; this norm is natural as our optimization criterion (6) below pertains to maximizing *expected* utility.

The subscript T in the notation F_T pertains to the $\mathscr{F}_T$-measurability of this random variable and the fact that it describes a quantity related to time T. But the reader should note that F_T was *not* defined as the terminal random variable of some process $(F_t)_{0 \le t \le T}$ which in turn should be given by some stochastic integral on the process S (or as a random variable dominated by such an object).

The rationale behind this approach is the following: we believe that the natural domain for the utility maximization problem (6) below should be chosen to be some closure of the set of terminal values X_T resulting from processes X in $\mathfrak{X}^b(x)$: indeed, economic considerations suggest that one only should allow quantities which may be approximated (in some sense to be specified) by the situation describing economic agents with finite credit lines, which precisely is the idea behind the definition of $\mathfrak{X}^b(x)$.

The set $\mathscr{C}_U(x)$ is the largest conceivable set obeying this criterion [it is the closure of $\mathscr{C}_U^b(x)$ with respect to the weakest conceivable topology if we are interested in expected utility]. Of course, we could impose additional restrictions to make this set smaller, such as requiring that the random variables F_T are of the form $F_T = x + (H \cdot S)_T$ for some "reasonable" integrand H.

We deliberately don't do this at the present stage, but rather formulate our optimization problem over the "big" set $\mathscr{C}_U(x)$. It will turn out that–under appropriate assumptions–the optimal solution $\widehat{F}_T(x) \in \mathscr{C}_U(x)$ to the optimization problem (6) will *automatically* be the terminal value of an integral on the process S, which, a posteriori, gives a more satisfactory solution than restricting a priori the possible domain of optimization.

We now have prepared the ingredients for the definition of the optimization problem studied in this paper:

$$\{E[U(F_T)], F_T \in \mathscr{C}_U(x)\} \longrightarrow \max ! \tag{6}$$

We still observe that it follows from the definition of $\mathscr{C}_U(x)$ that we have the equality

$$\sup_{F_T \in \mathscr{C}_U(x)} E[U(F_T)] = \sup_{G_T \in \mathscr{C}_U^b(x)} E[U(G_T)]. \tag{7}$$

As in [17] we denote by u the value function

$$u(x) = \sup_{F_T \in \mathscr{C}_U(x)} E[U(F_T)], \tag{8}$$

which now is defined on the entire real line $\mathbb{R}$. Throughout the paper we assume

$$u(x) < U(\infty) := \lim_{x \to \infty} U(x) \quad \text{for some } x \in \mathbb{R}, \tag{9}$$

to exclude trivial cases (see Remark 3.7 below for a thorough discussion of this assumption). Noting that a convex combination of admissible integrands is an admissible integrand we deduce from (7) that u is a concave function on $\mathbb{R}$; hence assumption (9) readily implies that $u(x)$ is finitely-valued for each $x \in \mathbb{R}$. [For the fact that $u(x) > -\infty$, for all $x \in \mathbb{R}$, simply note that $B_t \equiv 1$ implies that $u(x) \geq U(x) > -\infty$.]

We note in passing that, under assumption (9), for $X \in \mathfrak{X}^b(x)$, we automatically have that $E[|U(X_T)|] < \infty$, which would allow to simplify the definition (4) of $\mathscr{C}_U^b(x)$: by requiring that $G_T \leq X_T$, for some $X \in \mathfrak{X}^b(x)$ and G_T is uniformly bounded from below we automatically have $E[|U(G_T)|] < \infty$. This shows in particular that, under assumption (9), the set $\mathscr{C}_U^b(x)$ does not depend on U.

We now turn to the central notion of this paper, which is the counterpart of the concept of the asymptotic elasticity $AE(U) = \limsup_{x \to \infty} \frac{xU'(x)}{U(x)}$ as defined in [17], where $+\infty$ now is replaced by $-\infty$:

DEFINITION 1.4. For a utility function $U : \mathbb{R} \to \mathbb{R}$ the *asymptotic elasticity at* $-\infty$ is defined as

$$AE_{-\infty}(U) = \liminf_{x \to -\infty} \frac{x\,U'(x)}{U(x)}. \tag{10}$$

To keep in line with the above notation we shall write $AE_{+\infty}(U)$ for the quantity $\limsup_{x \to +\infty} \frac{x\,U'(x)}{U(x)}$, which was denoted by $AE(U)$ in [17].

One easily checks that under assumption (1) the asymptotic elasticity at $-\infty$ is a well-defined number in $[1, \infty]$, and that this number is invariant under affine transformations of U. Recall from [17] that $AE_{+\infty}(U)$ is a well-defined number in $[-\infty, 1]$, which is invariant under affine transformations of U, provided $U(\infty)$ remains strictly positive.

Here are some examples: For the exponential utility $U(x) = 1 - e^{-x}$ we obtain $AE_{-\infty}(U) = \infty$ and $AE_{+\infty}(U) = 0$; for utility functions of the form $U(x) = -|x|^\alpha$, as $x \to -\infty$, where $\alpha > 1$ is a fixed constant, we obtain $AE_{-\infty}(U) = \alpha$; finally, for utility functions of the form $U(x) = x\ln(-x)$, as $x \to -\infty$, we obtain $AE_{-\infty}(U) = 1$. Recall from [17] that for utility functions of the form $U(x) = \frac{x}{\ln(x)}$, as $x \to \infty$, we have $AE_{+\infty}(U) = 1$.

The economic interpretation of the asymptotic elasticities $AE_{+\infty}(U)$ and $AE_{-\infty}(U)$ is very similar: it is the limit of the ratio between the marginal utility $U'(x)$ and the average utility $\frac{U(x)}{x}$, as $x \to \infty$ or $x \to -\infty$ respectively. The extreme cases $AE_{+\infty}(U) = 1$ and $AE_{-\infty}(U) = 1$ correspond to the case when the marginal utility in the limit equals the average utility, as $x \to \infty$ and $x \to -\infty$ respectively. From an economic point of view this property of a utility function seems unreasonable (in both cases $x \to \infty$ and $x \to -\infty$). Economic intuition suggests that the marginal utility $U'(x)$ should be substantially smaller than the average utility $\frac{U(x)}{x}$, as $x \to \infty$, and substantially bigger as $x \to -\infty$. This leads us to the following definition.

DEFINITION 1.5. A utility function $U : \mathbb{R} \to \mathbb{R}$ satisfying (1) has *reasonable asymptotic elasticity* if $AE_{+\infty}(U) < 1$ and $AE_{-\infty}(U) > 1$.

Although this is not the issue of the present paper, in order to keep the definitions in [17] consistent with the present paper we propose to say that a utility function U: $\mathbb{R}_+ \to \mathbb{R}$ satisfying the Inada conditions $U'(0) = \infty$ and $U'(\infty) = 0$ has *reasonable asymptotic elasticity* if $AE_{+\infty}(U) < 1$.

With this notation we shall see that the condition of *reasonable asymptotic elasticity* is the crucial condition for the existence of the optimal solution to the maximization problem (6): for the case of utility functions $U : \mathbb{R}_+ \to \mathbb{R}$ satisfying the Inada conditions this was shown in [17] and for the case of utility functions $U : \mathbb{R} \to \mathbb{R}$ satisfying (1) this is the main result of the present paper (Theorem 2.2 below).

To formulate the dual problem to (6) we define the conjugate function $V(y)$ of the function $U(x)$ by

$$V(y) = \sup_{x\in\mathbb{R}}(U(x) - xy), \qquad y > 0, \tag{11}$$

which, under condition (1), is a smooth, convex function satisfying

$$V(0) = U(\infty), \quad V(\infty) = \infty, \quad V'(0) = -\infty, \quad V'(\infty) = \infty. \tag{12}$$

We have the relation $U' = (-V')^{-1}$ and we denote by I the inverse function $(U')^{-1}$ (which is equal to $-V'$) (compare [20], [16], [17]).

We also note the formula

$$V(y) = U(I(y)) - yI(y), \tag{13}$$

which will be used several times below.

To give a concrete example: for $U(x) = -e^{-x}$ we obtain $V(y) = y(\ln(y)-1)$, $U'(x) = e^{-x}$ and $V'(y) = \ln(y)$.

A by now classical route to solve the (primal) optimization problem (6) is to pass to the dual problem (see, e.g., [3], [18], [16]):

$$v(y) = \inf_{Q\in\mathscr{M}^a(S)} E\left[V\left(y\frac{dQ}{dP}\right)\right]. \tag{14}$$

Again the question arises about the appropriate domain over which the dual optimization problem is minimized. "Morally speaking" the proper set consists of the equivalent martingale probability measures for the process S: by this we mean that under sufficiently strong hypotheses [e.g., requiring that Ω is finite (compare [19])] the duality between (6) and (14) works out perfectly if one minimizes in (14) over the set of equivalent martingale measures only (i.e., the set of probability measures Q, equivalent to P, such that S is a Q-martingale).

But in order to obtain general results one has to enlarge this set: already Definition 1.1 of $\mathscr{M}^e(S)$ and $\mathscr{M}^a(S)$ refer to the concept of local martingales rather than martingales. In [17] a further enlargement of the set $\mathscr{M}^e(S)$ was necessary in order to obtain good duality results: we had to introduce a certain class $\mathscr{Y}(1)$ of supermartingales extending the class of density processes of equivalent local martingale measures.

It turns out, however, that this latter enlargement is not necessary in the present setting: we shall see that the optimal solution to the dual problem is automatically attained by a probability measure; in fact, in many cases we can assert that the optimal solution necessarily is in $\mathscr{M}^e(S)$, that is, equivalent to P. But there are also cases where we cannot assert this and have to consider the larger set $\mathscr{M}^a(S)$.

Clearly we could adopt the same philosophy as in the formulation of the primal problem (6): we could first formulate the dual problem by optimizing over some "big" set (such as in [17]) containing $\mathscr{M}^a(S)$ and subsequently show that the optimal solution lies already in $\mathscr{M}^e(S)$ or $\mathscr{M}^a(S)$ respectively. But for the dual problem we refrain from doing so as we consider it only as a technical gimmick for solving the primal problem, which is the question of our original concern.

For all these reasons we decided to formulate (14) as an optimization problem over $\mathscr{M}^a(S)$.

The fact that, under appropriate conditions, the optimal solution to (14) is attained for an element $\widehat{Q}$ of $\mathscr{M}^e(S)$ or $\mathscr{M}^a(S)$ respectively was already obtained in the paper [2] of F. Bellini and M. Fritelli (see Remark 2.4 below).

Let us end the introduction by an overview of the paper: the basic theme is to find conditions under which the formal duality between (6) and (14) can be turned into precise theorems, and to identify the optimal solutions to these two optimization problems and their mutual relations.

In Section 2 we shall prove two theorems on these lines: the (easier) case of a complete financial market is dealt with in Theorem 2.1 while the case of an incomplete financial market is treated in Theorem 2.2 which is the main result of the paper. For the latter theorem to hold true we crucially need the assumption of reasonable asymptotic elasticity of the utility function U. The question to which extent a theorem analogous to Theorem 2.1 of [17] holds true, dealing with the case of incomplete markets and possibly unreasonable asymptotic elasticity, is left to future research.

In Section 3 we give some examples showing what may go wrong if one drops the assumption of reasonable asymptotic elasticity, and in section 4 we give some characterizations of the property $AE_{-\infty}(U) > 1$. These results turn out to be rather straightforward variations of the theme treated in [17], and are therefore presented as briefly as possible.

2. The main results. We shall approximate the utility function $U : \mathbb{R} \to \mathbb{R}$ satisfying (1) by an increasing sequence $U^{(n)}$ of utility functions such that $U^{(n)}(x) = -\infty$, for $x \le -(n+1)$, in order to relate the present setting to the results from [17]: Then clearly the optimization problem with respect to $U^{(n)}$ is essentially the same as the one treated in [17], modulo the shift of the singularity of $U^{(n)}$ from zero to $-(n+1)$. For the sake of clarity we resume the situation in some detail:

Fix a utility function U satisfying (1) and an increasing sequence $(U^{(n)})_{n=1}^{\infty}$ of strictly concave, smooth utility functions, $U^{(n)} \le U$, such that $U^{(n)}$ coincides with U on $[-n, +n]$. On the negative end of the real line we require that $U^{(n)}(x) > -\infty$ for $x > -(n+1)$ and $\lim_{x \searrow -(n+1)} U^{(n)}(x) = -\infty$; on the positive end of the real line we impose the requirement $AE_{+\infty}(U^{(n)}) < 1$. If U already satisfies $AE_{+\infty}(U) < 1$, we don't have to modify U on $\mathbb{R}_+$ and therefore assume in this case that $U^{(n)}(x) = U(x)$, for $x \ge 0$.

In the case when U has reasonable asymptotic elasticity and $U(0) > 0$, we choose the $U^{(n)}$'s in addition in such a way that the estimates in Corollary 4.2 below hold true, uniformly in $n \in \mathbb{N}$. This technical issue will be used in Step 4 of the proof of Theorem 2.2 and it is easy to verify that this is always possible.

Denote by $V^{(n)}$ the conjugate function of $U^{(n)}$ and observe that $U^{(n)}$ and $V^{(n)}$ increase stationarily to U and V respectively.

The function $\widetilde{U}^{(n)}(x) = U^{(n)}(x - (n+1))$ is finitely valued for $x > 0$, and satisfies the requirements of Theorem 2.2 of [17]; therefore, fixing $x > 0$, and using the notation of [17], there exists a unique optimal solution $\widetilde{X}^{(n)}(x) = x + (H^n \cdot S) \in \mathcal{X}(x)$ to the optimization problem

$$\tilde{u}^{(n)}(x) = \sup_{X \in \mathcal{X}(x)} E\left[\widetilde{U}^{(n)}(X_T)\right], \qquad x > 0. \tag{15}$$

Note that it does not matter in the above optimization problem whether we optimize over the set $\mathcal{X}(x)$ of non-negative processes as introduced in [17] or the set $\mathcal{X}^b(x)$ as defined in 1.2 above: the assumption $\mathcal{M}^e(S) \neq \emptyset$ implies that for a process of the form $X = x + (H \cdot S)$ which is uniformly bounded from below and such that $X_T \geq 0$ we already have $X_t \geq 0$, for all $0 \leq t \leq T$ (compare [7] for this easy fact).

Hence by a simple shift on the real line, for $x > -(n+1)$, the process $\widehat{X}^{(n)}(x) := \widetilde{X}^{(n)}(x+n+1) - (n+1)$ is the optimal solution to the optimization problem

$$u^{(n)}(x) = \sup_{X \in \mathcal{X}^b(x)} E\left[U^{(n)}(X_T)\right], \qquad x > -(n+1). \tag{16}$$

Clearly we have

$$u^{(n)}(x) = \widetilde{u}^{(n)}(x+n+1), \qquad x > -(n+1). \tag{17}$$

Passing to the dual problem, fix $x > -(n+1)$ and let $y = (u^{(n)})'(x) = (\widetilde{u}^{(n)})'(x+n+1)$. Denoting by $\widetilde{V}^{(n)}$ (resp. $\widetilde{v}^{(n)}$) the conjugate function to $\widetilde{U}^{(n)}$ (resp. $\widetilde{u}^{(n)}$) and letting $\widetilde{Y}_T^{(n)}(y) = (\widetilde{U}^{(n)})'(\widetilde{X}_T^{(n)}(x+n+1)) = (U^{(n)})'(\widehat{X}_T^{(n)}(x))$ we infer from Theorem 2.2 of [17] that $\widetilde{Y}_T^{(n)}(y)$ is an element of $\mathcal{D}(y)$ (this definition is recalled after (20) below) and that it satisfies

$$\widetilde{v}^{(n)}(y) = \inf_{Y_T \in \mathcal{D}(y)} E\left[\widetilde{V}^{(n)}(Y_T)\right] = E\left[\widetilde{V}^{(n)}\left(\widetilde{Y}_T^{(n)}(y)\right)\right], \qquad y > 0. \tag{18}$$

To relate $\widetilde{V}^{(n)}$ and $\widetilde{v}^{(n)}$ to the conjugate functions $V^{(n)}$ and $v^{(n)}$ of $U^{(n)}$ and $u^{(n)}$ respectively, observe the simple equalities

$$\begin{aligned} V^{(n)}(y) &= \widetilde{V}^{(n)}(y) + (n+1)y, \\ v^{(n)}(y) &= \widetilde{v}^{(n)}(y) + (n+1)y, \end{aligned} \tag{19}$$

which directly follow from the conjugacy relations.

Hence we obtain

$$\begin{aligned} v^{(n)}(y) &= \inf_{Y_T \in \mathcal{D}(y)} E\left[\widetilde{V}^{(n)}(Y_T)\right] + (n+1)y \\ &= E\left[\widetilde{V}^{(n)}\left(\widetilde{Y}_T^{(n)}(y)\right)\right] + (n+1)y \\ &= E\left[V^{(n)}\left(\widetilde{Y}_T^{(n)}(y)\right)\right] + (n+1)\left(y - E\left[\widetilde{Y}_T^{(n)}(y)\right]\right). \end{aligned} \tag{20}$$

The last formula merits some comment: recall from [17] that $\mathcal{Y}(y)$ consists of the non-negative supermartingales $(Y_t)_{0 \leq t \leq T}$ starting at $Y_0 = y$ and such that $X_t Y_t$ is a supermartingale, for each $X \in \mathcal{X}(1)$, and that $\mathcal{D}(y)$ denotes the set of all non negative random variables dominated by some terminal value Y_T, where Y ranges through $\mathcal{Y}(y)$. In particular $E[Y_T] \leq y$ and we have equality iff Y_T/y is the Radon-Nikodym derivative of some $Q \in \mathcal{M}^a(S)$, in which case $Y_t = E[\frac{dQ}{dP}|F_t]$.

If $E[\widetilde{Y}_T^{(n)}(y)] = y$ then the above formula reduces to the pleasant equality

$$v^{(n)}(y) = E\left[V^{(n)}\left(\widetilde{Y}_T^{(n)}(y)\right)\right], \tag{21}$$

the formula one would expect naively. In general, however, the additional term $(n+1)\,(y - E[\widetilde{Y}_T^{(n)}(y)])$ has to be added; for an interpretation of this term as the action of the singular part of the optimal solution $\widetilde{Y}^{(n)}(y)$ on the function $V^{(n)}$ we refer to [5].

Summing up what we have obtained so far for the dual problem to the primal problem (16): using the notation from [17], the conjugate function $v^{(n)}$ to the value function $u^{(n)}$ defined in (16) is given by

$$v^{(n)}(y) = \inf_{Y_T \in \mathscr{D}(y)} E\left[V^{(n)}(Y_T)\right] + (n+1)\,(y - E[Y_T]), \qquad y > 0, \tag{22}$$

and, for $y = (u^{(n)})'(x)$, the unique optimal solution to (22) is given by $\widehat{Y}_T^{(n)} := \widetilde{Y}_T^{(n)} \in \mathscr{D}(y)$ via the formula

$$\widehat{Y}_T^{(n)}(y) = \left(U^{(n)}\right)'\left(\widehat{X}_T^{(n)}(x)\right). \tag{23}$$

We now can formulate the theorem pertaining to the case of complete financial markets:

THEOREM 2.1 (Complete case). *Assume that* (1) *and* (9) *hold true and that* $\mathscr{M}^e(S) = \{Q\}$ *where* Q *is a probability measure equivalent to* P. *Then*:

(i) *The value function* $u(x)$ *defined by* (8) *is a continuously differentiable concave function defined and finitely valued on the real line* $\mathbb{R}$; *the value function* $v(y)$ *defined by* (14) *is finitely valued, for at least one* $y > 0$; *it is continuously differentiable and strictly convex on the interior of the interval* $\{y : v(y) < \infty\}$. *The functions* u *and* v *are conjugate, that is,*

$$\begin{aligned} v(y) &= \sup_{x\in\mathbb{R}}[u(x) - xy], \qquad y > 0,\\ u(x) &= \inf_{y>0}[v(y) + xy], \qquad x \in \mathbb{R}. \end{aligned} \tag{24}$$

(ii) *Denote by* $]\alpha, \beta[\subseteq \mathbb{R}_+$ *the* (*possibly empty*) *interior of the interval* $\{y : v(y) < \infty\}$ *and denote by* $]a, b[\subseteq \mathbb{R}$ *the* (*possibly empty*) *image of the interval* $]\alpha, \beta[$ *under the map* $-v'$. *For a real number* x *in the closure of* $]a, b[$, *let* $y = u'(x)$; *the optimal solution* $F_T(x) \in \mathscr{C}_U(x)$ *to* (6) *exists, is unique and given by the formula*

$$F_T(x) = I\left(y\frac{dQ}{dP}\right). \tag{25}$$

The random variable $F_T(x)$ *equals the terminal value* $\widehat{X}_T(x)$ *of a uniformly integrable* Q-*martingale* $(\widehat{X}_t(x))_{0\le t\le T}$ *starting at* $\widehat{X}_0(x) = x$, *which is of the form* $\widehat{X}(x) = x + (H\cdot S)$, *for a predictable* S-*integrable process* H.

If $]\alpha, \beta[$ *is not empty, then, for* $x \in \mathbb{R}\setminus[a, b]$, *the optimal solution to* (6) *does not exist.*

(iii) *The value function* $u(x)$ *is strictly concave on* $]a, b[$ *and affine on the* (*possibly empty*) *intervals* $]-\infty, a]$ *and* $[b, \infty[$.

(iv) *For* y *in the closure of* $]\alpha, \beta[$ *and* x *in the closure of* $]a, b[$ *we have the relations*

$$v'(y) = E\left[\frac{dQ}{dP} V'\left(y\frac{dQ}{dP}\right)\right],$$
$$xu'(x) = E\left[\widehat{X}_T(x) U'\left(\widehat{X}_T(x)\right)\right], \tag{26}$$

where in the boundary case $y = \alpha$ (*resp.* $y = \beta$) *the term* $v'(y)$ *has to be interpreted as a right* (*resp. left*) *derivative.*

PROOF. Consider the sequence of value functions $v^{(n)} : \mathbb{R}_+ \to \mathbb{R}$ introduced at the beginning of this section. As $\mathscr{M}^e(S) = \{Q\}$ we clearly have

$$v^{(n)}(y) = E\left[V^{(n)}\left(y\frac{dQ}{dP}\right)\right] \tag{27}$$

and

$$v(y) = E\left[V\left(y\frac{dQ}{dP}\right)\right]. \tag{28}$$

It follows from the monotone convergence that, for $y > 0$, $v^{(n)}(y)$ increases to $v(y)$; as regards $(u^{(n)}(x))_{n=1}^{\infty}$, for $x \in \mathbb{R}$, it is obvious that this sequence increases to a limit - let us denote it by $u^{(\infty)}(x)$ - for which we have $u^{(\infty)}(x) \leq u(x)$ as $u^{(n)}(x) \leq u(x)$, for each $n \in \mathbb{N}$.

To verify that $u^{(\infty)}(x)$ indeed equals $u(x)$, fix $x \in \mathbb{R}$ and find, for $\varepsilon > 0$, $G_T \in \mathscr{C}_U^b(x)$ such that $u(x) < E[U(G_T)] + \varepsilon$. If X_T is the terminal value of a process $X \in \mathcal{X}^b(x)$ such that $G_T \leq X_T$ we have $u(x) < E[U(X_T)] + \varepsilon < \infty$ where the last inequality follows from (9). As X_T is uniformly bounded from below we have that $E[U^{(n)}(X_T)]$ is finite, for n sufficiently large, and by the monotone convergence theorem, this sequence converges to $E[U(X_T)]$, which readily implies that $u(x) < u^{(n)}(x) + 2\varepsilon$, for n large enough.

Summing up: we have shown that $u^{(n)}$ and $v^{(n)}$ increase monotonically to u and v respectively. As $u^{(n)}$ and $v^{(n)}$ are conjugate it follows that u and v are conjugate too.

Next we observe that $(u(x))_{x\in\mathbb{R}}$ is a finitely valued, concave non-decreasing function. Noting that the inequality $u(x) \geq U(x)$ implies that $u(\infty) = U(\infty)$, we deduce from (9) that $u(x) < u(\infty)$, for each $x \in \mathbb{R}$, which implies that u is strictly increasing. Hence there is at least one $y > 0$ such that, for c sufficiently large, the affine function $h(x) = c + yx$ dominates the function $u(x)$; hence we have that $\{y > 0 : v(y) < \infty\}$ is a non-empty interval and we denote by $\alpha \geq 0$ and $\beta \leq \infty$ the left and right endpoints of this interval.

We also deduce from the strict convexity of V that v is strictly convex on $]\alpha, \beta[$ and therefore, using elementary properties of the duality relation of conjugate functions, that u is continuously differentiable on $\mathbb{R}$. □

We now shall distinguish the cases that $]\alpha, \beta[$ is degenerate or not:

Case 1. $0 \leq \alpha < \beta \leq \infty$.

To verify formula (26) for $v'(y)$ and $y \in]\alpha, \beta[$, let $(y_n)_{n=1}^{\infty}$ be a sequence in $]\alpha, \beta[$ converging monotonically to y. We then have

$$\begin{aligned}\lim_{n\to\infty} \frac{v(y_n) - v(y)}{y_n - y} &= \lim_{n\to\infty} \frac{E\left[V\left(y_n \frac{dQ}{dP}\right)\right] - E\left[V\left(y \frac{dQ}{dP}\right)\right]}{y_n - y} \\ &= \lim_{n\to\infty} E\left[\frac{dQ}{dP} V'\left(\tilde{y}_n \frac{dQ}{dP}\right)\right] \\ &= E\left[\frac{dQ}{dP} V'\left(y \frac{dQ}{dP}\right)\right]\end{aligned} \tag{29}$$

where we have used the mean-value theorem of differential calculus and $\tilde{y}_n = \tilde{y}_n(\omega)$ is a random variable taking values in the interval between y and y_n; the last equality follows from the continuity of V' and the monotone convergence theorem, noting that, if $(y_n)_{n=1}^{\infty}$ monotonically converges to y, the sequence of random variables $(\tilde{y}_n)_{n=1}^{\infty}$ does so too.

Hence we have proved that

$$v'(y) = E\left[\frac{dQ}{dP} V'\left(y \frac{dQ}{dP}\right)\right] \quad \text{for } y \in]\alpha, \beta[, \tag{30}$$

which shows in particular that v is a continuously differentiable function on $]\alpha, \beta[$.

Letting $-b = \lim_{y \searrow \alpha} v'(y)$ and $-a = \lim_{y \nearrow \beta} v'(y)$, observe that $-v'$ induces a bijection between $]\alpha, \beta[$ and $]a, b[$.

What happens at the boundary points α and β? We only discuss the left limit point $\alpha \geq 0$, the case of the right limit point β being analogous: if the left limit

$$v(\alpha_{+0}) := \lim_{y \searrow \alpha} v(y) \tag{31}$$

is finite, then we have

$$v(\alpha_{+0}) = E\left[V\left(\alpha \frac{dQ}{dP}\right)\right] = v(\alpha). \tag{32}$$

Similarly, if the left limit

$$v'(\alpha_{+0}) := \lim_{y \searrow \alpha} v'(y) \tag{33}$$

is finite, then we have

$$v'(\alpha_{+0}) = E\left[\frac{dQ}{dP} V'\left(\alpha \frac{dQ}{dP}\right)\right]. \tag{34}$$

In the case $v(\alpha) < \infty$ we also have $v'(\alpha_{+0}) = v'_r(\alpha)$, where $v'_r(\alpha)$ denotes the right derivative of v at α, while, in the case $v(\alpha) = \infty$ we obtain $v'(\alpha_{+0}) = \infty$ which also can be interpreted as a right derivative of v at α.

To verify the above assertions observe that (34) follows immediately from (30) and the monotone convergence theorem using the monotonicity of V'. The assertion (32) similarly follows from the finiteness of $v(y) = E[V(y\frac{dQ}{dP})]$ on $]\alpha, \beta[$ and a slight adaptation of the monotone convergence theorem: first note that, in the case $\alpha = 0$, the formula $v(0) = E[V(0\frac{dQ}{dP})] = V(0)$ trivially holds true, hence we may assume that $\alpha > 0$. Writing $y_{\min} = \operatorname{argmin} V(y) = U'(0)$, split Ω into the sets $A_1 = \{\alpha\frac{dQ}{dP} > y_{\min}\}$, $A_2 = \{\alpha\frac{dQ}{dP} < y_{\min}/2\}$ and $A_3 = \{\alpha\frac{dQ}{dP} \in [y_{\min}/2, y_{\min}]\}$. We then verify

$$\lim_{n\to\infty} E\left[V\left(y_n \frac{dQ}{dP}\chi_{A_i}\right)\right] = E\left[V\left(y\frac{dQ}{dP}\chi_{A_i}\right)\right], \qquad i = 1, 2, 3. \tag{35}$$

In the cases $i = 1$ and $i = 2$ we apply the monotone convergence theorem, where we consider n large enough such that $y_n < 2\alpha$. In the case $i = 3$ we apply Lebesgue's theorem, for n large enough such that $y_n < 2\alpha$, noting that V is bounded on $[y_{\min}/2, 2y_{\min}]$.

We have thus proved assertions (32) and (34) and the subsequent remark on the right derivative $v'_r(\alpha)$ now follows too.

For $x \in \mathbb{R} \cap [a, b]$ and $y = u'(x) \in [\alpha, \beta]$ we define

$$\widehat{X}_T(x) := I\left(y\frac{dQ}{dP}\right) \tag{36}$$

so that

$$E_Q\left[\widehat{X}_T(x)\right] = E\left[\frac{dQ}{dP} I\left(y\frac{dQ}{dP}\right)\right] = -v'(y) = x. \tag{37}$$

By the preceding discussion, the above equality also holds true in the limiting cases $0 < \alpha = y$ and $y = \beta < \infty$, provided that $v'(y)$ is finite [where in these boundary cases $v'(y)$ has to be interpreted as the right or left derivative of v at α and β respectively].

Using the hypotheses $\mathscr{M}^e(S) = \{Q\}$ we may apply the martingale representation theorem (see [21]): equality (37) implies that $\widehat{X}_T(x)$ is the terminal value of a uniformly integrable Q-martingale $\left(\widehat{X}_t(x)\right)_{0\le t\le T}$ starting at $\widehat{X}_0(x) = x$.

We still have to verify that $\widehat{X}_T(x) =: F_T(x)$ is in $\mathscr{C}_U(x)$; we shall show that there is in a sequence $X^{(n)}(x) \in \mathfrak{X}^b(x)$ such that $U(X_T^{(n)}(x))$ converges to $U(\widehat{X}_T(x))$ in the norm of $L^1(P)$.

For $x \in \mathbb{R}$ and, for $n \in \mathbb{N}$, verifying $n > -x$, define $X_T^{(n)}(x)$ by

$$X_T^{(n)}(x) = (\widehat{X}_T(x) \vee (-n)) - \delta_n \chi_{\{\widehat{X}_T > x\}} \tag{38}$$

where $\delta_n \geq 0$ is chosen in such a way that $E_Q[X_T^{(n)}(x)] = x$. Clearly $X_T^{(n)}(x)$ is bounded from below by $-n$. By the martingale representation theorem $X_T^{(n)}(x)$ is the terminal value of a uniformly integrable Q-martingale $X^{(n)}(x) \in \mathfrak{X}^b(x)$.

As $E_Q[\widehat{X}_T(x)] = x$ it follows that δ_n decreases to zero; this easily implies that $E[|U(\widehat{X}_T(x)) - U(X_T^{(n)}(x))|]$ tends to zero.

To see that $\widehat{X}_T(x)$ really is the optimal solution to (6) we note that, for $x \in \mathbb{R} \cap [a, b]$and $y = u'(x)$, we have

$$\begin{aligned} E\left[U\left(\widehat{X}_T(x)\right)\right] &= E\left[U\left(I\left(y\frac{dQ}{dP}\right)\right)\right] \\ &= E\left[U\left(I\left(y\frac{dQ}{dP}\right)\right)\right] - yE\left[\frac{dQ}{dP}I\left(y\frac{dQ}{dP}\right)\right] \\ &\quad + yE\left[\frac{dQ}{dP}I\left(y\frac{dQ}{dP}\right)\right] \\ &= E\left[V\left(y\frac{dQ}{dP}\right)\right] + yx \\ &= v(y) + yx \\ &= u(x). \end{aligned} \tag{39}$$

The uniqueness of $\widehat{X}(x)$ is a consequence of the strict concavity of U and the fact that $\widehat{X}(x)$ may be written as $\widehat{X}(x) = x + (H \cdot S)$ follows from Yor's theorem (see [21] and [15] for the vector valued case). Finally, the strict concavity of u on the interval $]a, b[$ now follows from the existence of the optimal solutions $\widehat{X}(x)$, for $x \in [a, b] \cap \mathbb{R}$ by using Scholium 5.1 of [17]. The latter result also implies that, for $x \in \mathbb{R} \backslash [a, b]$, the optimal solution $F_T \in \mathscr{C}_U(x)$ does not exist. Indeed, the fact that u is affine on the intervals $]-\infty, a]$ and $[b, \infty[$ is a straightforward consequence of the conjugacy of u and v. Supposing that there exists $x < a$ (or $x > b$) such that $\widehat{X}(x)$ exists, then by considering $(\widehat{X}(x) + \widehat{X}(a))/2$ (or $(\widehat{X}(x) + \widehat{X}(b))/2$) and using the argument of Scholium 5.1 one arrives at a contradiction to $u\left(\frac{x+a}{2}\right) = \frac{u(x)+u(a)}{2}$ (or $u\left(\frac{x+b}{2}\right) = \frac{u(x)+u(b)}{2}$).

We still have to show the second formula in (26)

$$xu'(x) = E\left[U'\left(\widehat{X}_T(x)\right)\widehat{X}_T(x)\right], \qquad x \in [a, b] \cap \mathbb{R}. \tag{40}$$

But (40) simply is a reformulation of formula (30) observing that $x = -v'(y)$, $u'(x) = y$, $\widehat{X}_T(x) = -V'(y\frac{dQ}{dP})$ and $U'(\widehat{X}_T(x)) = y\frac{dQ}{dP}$.

Case 2. $0 < \alpha = \beta < \infty$.

First we note that this case may indeed occur due to the lack of reasonable asymptotic elasticity (see proposition 3.5 below).

In this case we have $v(\alpha) < \infty$, while $v(y) = \infty$, for all $y \neq \alpha$. Assertions (ii) and (iv) then are vacuous and assertions (i) and (iii) are trivially satisfied as–by the conjugacy of u and v –the function $u(x)$ is affine in $x \in \mathbb{R}$.

The proof of Theorem 2.1 now is complete. □

We now turn to the case of incomplete markets, that is, when $\mathscr{M}^e(S)$ is not reduced to a singleton $\{Q\}$.

THEOREM 2.2 (Incomplete case, reasonable asymptotic elasticity). *Assume that* (1), (2) *and* (9) *hold true and that U has reasonable asymptotic elasticity*. *Then*:

(i) *The value functions u and v are finitely valued, strictly concave (resp. convex), differentiable functions defined on $\mathbb{R}$ (resp. $\mathbb{R}_+$); they are conjugate and satisfy*

$$u'(\infty) = 0, \qquad v'(0) = -\infty,$$
$$u'(-\infty) = \infty, \qquad v'(\infty) = \infty. \tag{41}$$

The value function u has reasonable asymptotic elasticity.

(ii) *For $y > 0$, the optimal solution $\widehat{Q}(y) \in \mathscr{M}^a(S)$ to the dual problem* (14) *exists, is unique and the map $y \mapsto \widehat{Q}(y)$ is continuous in the variation norm*.

(iii) *For $x \in \mathbb{R}$ the optimal solution $\widehat{F}_T(x) \in \mathscr{C}_U(x)$ to the primal problem* (7) *exists, is unique and is given by*

$$\widehat{F}_T(x) = I\left(y\frac{d\widehat{Q}(y)}{dP}\right), \tag{42}$$

where $u'(x) = y$.

(iv) *If $\widehat{Q}(y) \in \mathscr{M}^e(S)$ and $x = -v'(y)$, then $\widehat{F}_T(x)$ equals the terminal value $\widehat{X}_T(x)$ for a process of the form $\widehat{X}_t(x) = x + (H\cdot S)_t$, where H is predictable and S-integrable, such that $\widehat{X}$ is a uniformly integrable martingale under $\widehat{Q}(y)$.*

(v) *We have the formulae*

$$v'(y) = E\left[\frac{d\widehat{Q}(y)}{dP}V'\left(y\frac{d\widehat{Q}(y)}{dP}\right)\right],$$
$$xu'(x) = E\left[\widehat{X}_T(x)\,U'\left(\widehat{X}_T(x)\right)\right], \tag{43}$$

where the usual rule $0\cdot\infty = 0$ is applied, if the integrands are of this form.

PROOF. Without loss of generality we assume that $U(0) > 0$ so that $V(y) > 0$, for all $y > 0$. Let $U^{(n)}$, $V^{(n)}$, $u^{(n)}$ and $v^{(n)}$ be defined as in the beginning of this section. Recall that we have chosen $U^{(n)}$ such that the estimates of Corollary 4.2 hold true, uniformly in $n \in \mathbb{N}$.

We break the proof into several steps:

Step 1. $\lim_{n\to\infty} v^{(n)}(y) < \infty$, for each $y > 0$, and the function $v^{(\infty)}(y) = \lim_{n\to\infty} v^{(n)}(y)$ is bounded on compact subsets of $]0,\infty[$. If $(y_n)_{n=1}^\infty$ tends to $y > 0$, then $v^{(n)}(y_n)$ tends to $v^{(\infty)}(y)$.

First note that $(v^{(n)}(y))_{n=1}^\infty$ is bounded for at least one $\overline{y} > 0$: indeed, similarly as in the proof of Theorem 2.1 we deduce from the fact that $u(x) < U(\infty)$, for each $x \in \mathbb{R}$, that the conjugate function to u is finite for at least one $\overline{y} > 0$. As $u^{(n)} \leq u$, for all $n \in \mathbb{N}$, and $v^{(n)}$ is conjugate to $u^{(n)}$ we have $\lim_{n\to\infty} v^{(n)}(\overline{y}) < \infty$.

Using Theorem 2.2 of [17] we may find $Q^{(n)} \in \mathscr{M}^e(S)$ such that

$$\sup_n E\left[V^{(n)}\left(\overline{y}\frac{dQ^{(n)}}{dP}\right)\right] < \infty.$$

Now we use the assumption that U has reasonable asymptotic elasticity. It follows from Corollary 4.2 below and the discussion at the beginning of this section that, for $y > 0$, we can find a constant $C > 0$, s.t. the estimate

$$\begin{aligned}\lim_{n\to\infty} v^{(n)}(y) &\leq \lim_{n\to\infty} E\left[V^{(n)}\left(y\frac{dQ^{(n)}}{dP}\right)\right]\\ &\leq \sup_n CE\left[V^{(n)}\left(\overline{y}\frac{dQ^{(n)}}{dP}\right)\right] < \infty,\end{aligned} \tag{44}$$

holds true uniformly in $n \in \mathbb{N}$ and y ranging in a compact subset of $]0,\infty[$ (the constant C depending on this compact subset of $]0,\infty[$). This also implies the last assertion of Step 1 in view of the convexity and monotone convergence of the functions $v^{(n)}$.

Step 2. Denote by $(\widehat{Y}_T^{(n)}(y_n))_{n=1}^\infty \in \mathscr{D}(y_n)$ the optimal solution to the optimization problem (22) and let $(y_n)_{n=1}^\infty$ tend to $y > 0$.

Then $(\widehat{Y}_T^{(n)}(y_n))_{n=1}^\infty$ converges in the norm of $L^1(\Omega, \mathscr{F}_T, P)$ to a random variable $\widehat{Y}_T(y)$ which satisfies

$$y = E\left[\widehat{Y}_T(y)\right]. \tag{45}$$

From (22) we have the formula

$$v^{(n)}(y_n) = E\left[V^{(n)}\left(\widehat{Y}_T^{(n)}(y_n)\right)\right] + (n+1)\left(y_n - E\left[\widehat{Y}_T^{(n)}(y_n)\right]\right). \tag{46}$$

As $(v^{(n)}(y_n))_{n=1}^\infty$ tends to $v^{(\infty)}(y) < \infty$, as $V^{(n)} \geq 0$ and $E[\widehat{Y}_T^{(n)}(y_n)] \leq y_n$, we immediately obtain that

$$\lim_{n\to\infty}\left(y_n - E\left[\widehat{Y}_T^{(n)}(y_n)\right]\right) = 0. \tag{47}$$

In fact, we have a stronger result, namely

$$v^{(\infty)}(y) = \lim_{n\to\infty} E\left[V^{(n)}\left(\widehat{Y}_T^{(n)}(y_n)\right)\right], \tag{48}$$

or, what by (46) amounts to the same,

$$\lim_{n\to\infty}(n+1)\left(y_n - E\left[\widehat{Y}_T^{(n)}(y_n)\right]\right) = 0. \tag{49}$$

Indeed, if (49) were wrong, we could find $\alpha > 0$ such that, for infinitely many $n \in \mathbb{N}$,

$$(n+1)\left(y_n - E\left[\widehat{Y}_T^{(n)}(y_n)\right]\right) > \alpha. \tag{50}$$

Find $\varepsilon > 0$ such that $\sup_k v^{(k)}(y_k) < \frac{\alpha(1-\varepsilon)}{4\varepsilon}$, and find $n > m$ such that (50) holds true and $\lambda = \frac{y_m}{y_n}$ is close enough to 1 so that (151) below (see corollary 4.2) is satisfied uniformly for all $V^{(n)}$, and such that $(n+1-(m+1)\frac{y_m}{y_n}) > (n+1)/2$ and $v^{(n)}(y_n) < v^{(m)}(y_m) + \alpha/4$ hold true to estimate

$$\begin{aligned} v^{(m)}(y_m) &\le E\left[V^{(m)}\left(\frac{y_m}{y_n}\widehat{Y}_T^{(n)}(y_n)\right)\right] + (m+1)\left(y_m - E\left[\frac{y_m}{y_n}\widehat{Y}_T^{(n)}(y_n)\right]\right) \\ &\le E\left[V^{(n)}\left(\frac{y_m}{y_n}\widehat{Y}_T^{(n)}(y_n)\right)\right] + (m+1)\frac{y_m}{y_n}\left(y_n - E\left[\widehat{Y}_T^{(n)}(y_n)\right]\right) \\ &\le (1+\varepsilon)E\left[V^{(n)}\left(\widehat{Y}_T^{(n)}(y_n)\right)\right] + (n+1)\left(y_n - E\left[\widehat{Y}_T^{(n)}(y_n)\right]\right) \\ &\quad - \left(n+1-(m+1)\frac{y_m}{y_n}\right)\left(y_n - E\left[\widehat{Y}_T^{(n)}(y_n)\right]\right) \\ &\le (1+\varepsilon)v^{(n)}(y_n) - \alpha/2 \\ &< v^{(m)}(y_m), \end{aligned} \tag{51}$$

a contradiction showing (48).

To show that the sequence $(\widehat{Y}_T^{(n)}(y_n))_{n=1}^\infty$ converges in the norm of $L^1(P)$ we adopt a strategy which will turn out to be useful several times in the subsequent proof: we shall show that this sequence is uniformly integrable and Cauchy in the topology of convergence in measure; this will readily imply the convergence of $(\widehat{Y}_T^{(n)}(y_n))_{n=1}^\infty$ in the norm of $L^1(P)$ to a random variable $\widehat{Y}_T(y) \in L^1(P)$.

We start with the uniform integrability:

So suppose that $(\widehat{Y}_T^{(n)}(y_n))_{n=1}^\infty$ fails to be uniformly integrable, that is, there is $\alpha > 0$ such that, for each $C > 0$,

$$\limsup_{n\to\infty} E\left[\widehat{Y}_T^{(n)}\chi_{\{\widehat{Y}_T^{(n)}\ge C\}}\right] > \alpha. \tag{52}$$

It follows from the inequality

$$V^{(n)}(z) \ge U^{(n)}(-n) + nz \tag{53}$$

and the assumption $U^{(n)}(-n) > -\infty$ that

$$\lim_{z\to\infty}\frac{V^{(n)}(z)}{z} \ge n. \tag{54}$$

Fix $n \in \mathbb{N}$, find $C^{(n)} > 0$ such that $V^{(n)}(z) \geq (n-1)z$, for $z \geq C^{(n)}$, and find $m > n$ such that

$$E[\widehat{Y}_T^{(m)}(y_m)\chi_{\{\widehat{Y}_T^{(m)}(y_m)\geq C^{(n)}\}}] > \alpha. \tag{55}$$

Using (22),

$$\begin{aligned} v^{(m)}(y_m) &\geq E\left[V^{(m)}\left(\widehat{Y}_T^{(m)}(y_m)\right)\right] \\ &\geq E\left[V^{(n)}\left(\widehat{Y}_T^{(m)}(y_m)\right)\right] \\ &\geq E\left[V^{(n)}\left(\widehat{Y}_T^{(m)}(y_m)\right)\chi_{\{\widehat{Y}_T^{(m)}(y_m)\geq C^{(n)}\}}\right] \\ &\geq E\left[(n-1)\widehat{Y}_T^{(m)}(y_m)\chi_{\{\widehat{Y}_T^{(m)}(y_m)\geq C^{(n)}\}}\right] \\ &\geq (n-1)\alpha, \end{aligned} \tag{56}$$

which contradicts the boundedness of $(v^{(m)}(y_m))_{m=1}^{\infty}$ showing the uniform integrability of $(\widehat{Y}_T^{(n)}(y_n))_{n=1}^{\infty}$.

To show that $(\widehat{Y}_T^{(n)}(y_n))_{n=1}^{\infty}$ is Cauchy with respect to the topology of convergence in measure, suppose to the contrary that there is $\alpha > 0$ such that there are arbitrarily large n and m verifying

$$P\left[|\widehat{Y}_T^{(n)}(y_n) - \widehat{Y}_T^{(m)}(y_m)| > \alpha\right] > \alpha. \tag{57}$$

As $V^{(n)}$ increases to V stationarily on compact subsets of $]0,\infty[$ and, in the case $V(0) < \infty$, stationarily on compact subsets of $[0,\infty[$, we may use the boundedness of $(v^{(n)}(y_n))_{n=1}^{\infty}$ to find $N \in \mathbb{N}$ and a compact set K contained in $\{y \geq 0 : V^N(y) = V(y) < \infty\}$ such that, for $n \geq N$,

$$P\left[\widehat{Y}_T^{(n)}(y_n) \notin K\right] < \alpha/3. \tag{58}$$

By the strict convexity of V and the compactness of K we may find $\eta > 0$ such that, for $y_1, y_2 \in K$, $|y_1 - y_2| > \alpha$, we have

$$V\left(\frac{y_1+y_2}{2}\right) \leq \frac{V(y_1)+V(y_2)}{2} - \eta. \tag{59}$$

Find $\varepsilon > 0$ small enough such that $v^{(k)}(y_k) < \frac{\alpha\eta}{6\varepsilon}$, for all $k \in \mathbb{N}$, and find $n > m \geq N$, such that (57) holds true, $v^{(n)}(y_n) < v^{(m)}(y_m) + \frac{\alpha\eta}{3}$, and such that $\lambda = \frac{y_m}{y_n}$ is close enough to 1 so that $\lambda < 1 + \varepsilon$ and (151) holds true uniformly for all $V^{(n)}$ to estimate

$$\begin{aligned}
&v^{(m)}(y_m)\\
&\le E\left[V^{(m)}\left(\frac{\frac{y_m}{y_n}\widehat{Y}_T^{(n)}(y_n)+\widehat{Y}_T^{(m)}(y_m)}{2}\right)\right]\\
&\quad+(m+1)\left(y_m-E\left[\frac{\frac{y_m}{y_n}\widehat{Y}_T^{(n)}(y_n)+\widehat{Y}_T^{(m)}(y_m)}{2}\right]\right)\\
&\le\left(E\left[V^{(m)}\left(\frac{y_m}{y_n}\widehat{Y}_T^{(n)}(y_n)\right)\right]+(m+1)\left(y_m-\frac{y_m}{y_n}E\left[\widehat{Y}_T^{(n)}(y_n)\right]\right)\right.\\
&\quad\left.+E\left[V^{(m)}\left(\widehat{Y}_T^{(m)}(y_m)\right)\right]+(m+1)\left(y_m-E\left[\widehat{Y}_T^{(m)}(y_m)\right]\right)\right)/2-\frac{\alpha\eta}{3}\\
&\le\left(E\left[V^{(n)}\left(\frac{y_m}{y_n}\widehat{Y}_T^{(n)}(y_n)\right)\right]+(n+1)\frac{y_m}{y_n}\left(y_n-E\left[\widehat{Y}_T^{(n)}(y_n)\right]\right)\right.\\
&\quad\left.+E\left[V^{(m)}\left(\widehat{Y}_T^{(m)}(y_m)\right)\right]+(m+1)\left(y_m-E\left[\widehat{Y}_T^{(m)}(y_m)\right]\right)\right)/2-\frac{\alpha\eta}{3}\\
&\le\left((1+\varepsilon)v^{(n)}(y_n)+v^{(m)}(y_m)\right)/2-\frac{\alpha\eta}{3}\\
&\le v^{(m)}(y_m)-\frac{\alpha\eta}{6}.
\end{aligned}\tag{60}$$

This contradiction shows that $(\widehat{Y}_T^{(n)}(y_n))_{n=1}^{\infty}$ is Cauchy in measure and therefore converges in the norm of $L^1(P)$ to a random variable which we denote by $\widehat{Y}_T(y)$.

Equation (45) now follows from (47).

Step 3. For $(y_n)_{n=1}^{\infty}$ tending to $y>0$, the sequence $(V^{(n)}(\widehat{Y}_T^{(n)}(y_n)))_{n=1}^{\infty}$ tends to $V(\widehat{Y}_T(y))$ in the norm of $L^1(\Omega,\mathscr{F}_T,P)$.

For $y>0$, the probability measure $\widehat{Q}(y)$ defined by $\frac{d\widehat{Q}(y)}{dP}=\widehat{Y}_T(y)/y$ is an element of $\mathscr{M}^a(S)$ and the unique minimizer to the dual optimization problem (14).

The value function v defined in (14) satisfies $v(y)=v^{(\infty)}(y)=E[V(\widehat{Y}_T(y))]$, and this function is strictly convex.

The maps $y\mapsto\widehat{Q}(y)$ and $y\mapsto V(\widehat{Y}_T(y))$ are continuous in the variation norm.

Clearly the measure $\widehat{Q}(y)$ defined above is a probability measure absolutely continuous with respect to P.

To verify that S is a local martingale under $\widehat{Q}(y)$, first note that, using the proposition 3.1 of [17] and the notation introduced there, $\widehat{Y}_T(y)\in\mathscr{D}(y)$ as $\widehat{Y}_T(y)$ is the limit in probability of the elements $\frac{y}{y_n}\widehat{Y}_T^{(n)}(y_n)\in\mathscr{D}(y)$. If Z denotes the density process $Z_t=E[\frac{d\widehat{Q}(y)}{dP}|\mathscr{F}_t]$ and τ is a stopping time such that the stopped process $S^{\tau}=(\mathbb{1}_{]\!]0,\tau]\!]}\cdot S)$ is bounded, it follows from the definition of $\mathscr{D}(y)$ that $Z_t S_t^\tau$ as well as $-Z_t S_t^\tau$ are supermartingales and therefore martingales; by the local boundedness assumption on S this implies that S is a local martingale under $\widehat{Q}(y)$.

To check that $\widehat{Q}(y)$ is indeed the minimizer of (14), it suffices to show that $v^{(\infty)}(y) \geq E[V(\widehat{Y}_T(y))]$ which follows from (48) and Fatou's lemma:

$$\begin{aligned} v^{(\infty)}(y) &= \lim_{n\to\infty} E[V^{(n)}(\widehat{Y}_T^{(n)}(y_n))] \\ &\geq E[\lim_{n\to\infty} V^{(n)}(\widehat{Y}_T^{(n)}(y_n))] \\ &= E[V(\widehat{Y}_T(y))] \geq v(y). \end{aligned} \tag{61}$$

As $\lim_{n\to\infty} v^{(n)}(y) = v^{(\infty)}(y) \leq v(y)$ we have equalities above and obtain in particular that

$$v^{(\infty)}(y) = v(y). \tag{62}$$

The strict convexity of v now follows from the strict convexity of the function V and the convexity of the set $\mathscr{M}^a(S)$.

Noting that $(V^{(n)}(\widehat{Y}_T^{(n)}(y_n)))_{n=1}^\infty$ is a sequence of positive random variables in $L^1(P)$ converging to $V(\widehat{Y}_T(y))$ in measure and such that the expectations $E[V^{(n)}(\widehat{Y}_T^{(n)}(y_n))]$ converge to the expectation $E[V(\widehat{Y}_T(y))]$, we deduce that $(V^{(n)}(\widehat{Y}_T^{(n)}(y_n)))_{n=1}^\infty$ converges to $V(\widehat{Y}_T(y))$ in the norm of $L^1(P)$.

The continuity of the map $y \mapsto \widehat{Q}(y)$ is a straight-forward consequence of the results of step 2: indeed, it suffices to show that for $(y_k)_{k=1}^\infty$ tending to $y > 0$ we have

$$\lim_{k\to\infty} \|\widehat{Y}_T(y_k) - \widehat{Y}_T(y)\|_{L^1(P)} = 0. \tag{63}$$

Choosing an increasing sequence $(n_k)_{k=1}^\infty$ such that

$$\lim_{k\to\infty} \|\widehat{Y}_T(y_k) - \widehat{Y}_T^{(n_k)}(y_k)\|_{L^1(P)} = 0 \tag{64}$$

the result follows from step 2.

The continuity of $y \mapsto V(\widehat{Y}_T(y))$ follows in the same way from the convergence of $(V^{(n)}(\widehat{Y}_T^{(n)}(Y_n)))_{n=1}^\infty$ in the variation norm.

Step 4. The map $y \longmapsto \frac{d\widehat{Q}(y)}{dP} V'(y \frac{d\widehat{Q}(y)}{dP})$ is continuous in the variation norm. In fact, for $(y_n)_{n=1}^\infty$ tending to $y > 0$, the sequence $(\widehat{Y}_T^{(n)}(y_n)(V^{(n)})'(\widehat{Y}_T^{(n)}(y_n)))_{n=1}^\infty$ tends to $\widehat{Y}_T(y)V'(\widehat{Y}_T(y))$ in the variation norm.

The sequence $(\widehat{Y}_T(y)(V^{(n)})'(\widehat{Y}_T^{(n)}(y_n)))_{n=1}^\infty$ converges to $\widehat{Y}_T(y)V'(\widehat{Y}_T(y))$ in measure and the positive parts tend to the positive part in the variation norm; (this is just a preliminary result as we shall only be able in step 6 below to show that the negative parts converge in the variation norm too).

By Corollary 4.2(ii) below, there is a constant C such that

$$y\left|\left(V^{(n)}\right)'(y)\right| \leq C V^{(n)}(y) \quad \text{for } y \geq 0, \tag{65}$$

uniformly in $n \in \mathbb{N}$, where, in the case $y = 0$, we adopt the rule $0 \cdot \infty = 0$.

Hence the sequence of random variables $(\widehat{Y}_T^{(n)}(y_n)(V^{(n)})'(\widehat{Y}_T^{(n)}(y_n)))_{n=1}^\infty$ is dominated in absolute value by the L^1-convergent sequence $(CV^{(n)}(\widehat{Y}_T^{(n)}(y_n)))_{n=1}^\infty$ and therefore uniformly integrable. By the continuity of the map $y \mapsto yV'(y)$, which holds true, for $y > 0$, and, in the case $V(0) = U(\infty) < \infty$, for $y \geq 0$ too, we also have that $(\widehat{Y}_T^{(n)}(y_n)(V^{(n)})'(\widehat{Y}_T^{(n)}(y_n)))_{n=1}^\infty$ converges in measure to $\widehat{Y}_T(y)V'(\widehat{Y}_T(y))$, and therefore in the norm of $L^1(P)$.

The first assertion of step 4 now follows from the second one by the same easy argument as at the end of step 3 above.

We now turn to the last assertion of step 4. The convergence in measure of the sequence $(\widehat{Y}_T(y)(V^{(n)})'(\widehat{Y}_T^{(n)}(y_n)))_{n=1}^\infty$ towards $\widehat{Y}_T(y)V'(\widehat{Y}_T(x))$ being again obvious, we have to prove the uniform integrability of the positive parts

$$\begin{aligned}(66)\qquad &\widehat{Y}_T(y)\left(V^{(n)}\right)'\left(\widehat{Y}_T^{(n)}(y_n)\right)_+ \\ &= \widehat{Y}_T(y)\left(V^{(n)}\right)'\left(\widehat{Y}_T^{(n)}(y_n)\right)\chi_{\{(V^{(n)})'(\widehat{Y}_T^{(n)}(y_n))\geq 0\}} \quad \text{for } n \in \mathbb{N}.\end{aligned}$$

To do so, it clearly will suffice to show the uniform integrability of the double sequence

$$(67)\qquad \widehat{Y}_T^{(m)}(y_m)\left(V^{(n)}\right)'\left(\widehat{Y}_T^{(n)}(y_n)\right)\chi_{\{(V^{(n)})'(\widehat{Y}_T^{(n)}(y_n))\geq 0,\}} \qquad m, n \in \mathbb{N},\ m \geq n.$$

For this we use the inequality

$$\begin{aligned}(68)\qquad &\widehat{Y}_T^{(m)}(y_m)\left(V^{(n)}\right)'\left(\widehat{Y}_T^{(n)}(y_n)\right)\chi_{\{(V^{(n)})'(\widehat{Y}_T^{(n)}(y_n))\geq 0\}} \\ &\leq \max\Big\{\widehat{Y}_T^{(m)}(y_m)\left(V^{(m)}\right)'\left(\widehat{Y}_T^{(m)}(y_m)\right)\chi_{\{(V^{(n)})'(\widehat{Y}_T^{(n)}(y_n))\geq 0\}}, \\ &\qquad \widehat{Y}_T^{(n)}(y_n)\left(V^{(n)}\right)'\left(\widehat{Y}_T^{(n)}(y_n)\right)\chi_{\{(V^{(n)})'(\widehat{Y}_T^{(n)}(y_n))\geq 0\}},\Big\} \\ &\qquad\qquad m, n \in \mathbb{N},\ m \geq n,\end{aligned}$$

which is easily verified by distinguishing pointwise the cases $\widehat{Y}_T^{(m)}(y_m) \geq \widehat{Y}_T^{(n)}(y_n)$ and $\widehat{Y}_T^{(m)}(y_m) < \widehat{Y}_T^{(n)}(y_n)$.

As the family of functions on the right hand side of (68) is uniformly integrable we have completed the proof of step 4.

Step 5. We have the formula

$$(69)\qquad v'(y) = E\left[\frac{d\widehat{Q}(y)}{dP}V'\left(y\frac{d\widehat{Q}(y)}{dP}\right)\right] \quad \text{for } y > 0.$$

To prove this formula first observe that the term on the right hand side is a continuous function of $y > 0$ by step 4. As regards the term on the left hand side we deduce from the convexity of v that the derivative $v'(y)$ exists, for all but countably many y's. Hence it will suffice to show (69) under the additional assumption that $v'(y)$ exists.

To do so we proceed similarly as in the proof of Theorem 2.1 above (compare also [17], Lemma 3.10). Let $(y_n)_{n=1}^{\infty}$ be a sequence tending to $y > 0$ such that $v'(y)$ exists.

We shall show the two inequalities:

$$\limsup_{n\to\infty} \frac{v(y_n) - v(y)}{y_n - y} \le E\left[\frac{d\widehat{Q}(y)}{dP} V'\left(y \frac{d\widehat{Q}(y)}{dP}\right)\right] \tag{70}$$

and

$$\liminf_{n\to\infty} \frac{v(y_n) - v(y)}{y_n - y} \ge E\left[\frac{d\widehat{Q}(y)}{dP} V'\left(y \frac{d\widehat{Q}(y)}{dP}\right)\right]. \tag{71}$$

First we observe that it will suffice to show the above inequalities where the terms $\frac{v(y_n)-v(y)}{y_n-y}$ are replaced by $\frac{v^{(n)}(y_n)-v^{(n)}(y)}{y_n-y}$. Indeed, observe that, for $y > 0$ such that $v'(y)$ exists, we easily deduce form the convexity and the monotone convergence of the $v^{(n)}$'s that

$$\lim_{n\to\infty} \frac{v(y_n) - v(y)}{y_n - y} = \lim_{n\to\infty} \frac{v^{(n)}(y_n) - v^{(n)}(y)}{y_n - y}. \tag{72}$$

To prove (70) we therefore fix $y > 0$ such that $v'(y)$ exists and estimate

$$\begin{aligned} &\limsup_{n\to\infty} \frac{v(y_n) - v(y)}{y_n - y} \\ &\quad = \lim_{n\to\infty} \frac{v^{(n)}(y_n) - v^{(n)}(y)}{y_n - y} \\ &\quad \le \limsup_{n\to\infty} \frac{E\left[V^{(n)}\left(\frac{y_n}{y}\widehat{Y}_T^{(n)}(y)\right)\right] - E\left[V^{(n)}\left(\widehat{Y}_T^{(n)}(y)\right)\right]}{y_n - y} \\ &\quad \le \limsup_{n\to\infty} E\left[\frac{\widehat{Y}_T^{(n)}(y)(V^{(n)})'\left(\frac{\tilde{y}_n}{y}\widehat{Y}_T^{(n)}(y)\right)}{y}\right] \\ &\quad = E\left[\frac{d\widehat{Q}(y)}{dP} V'\left(y \frac{d\widehat{Q}(y)}{dP}\right)\right], \end{aligned} \tag{73}$$

where, as in the proof of Theorem 2.1, we have applied the mean value theorem of differential calculus and the random variables $\tilde{y}_n$ take values between y and y_n. The last equality follows from the fact that the sequence $\widehat{Y}_T^{(n)}(y)(V^{(n)})'(\frac{\tilde{y}_n}{y}\widehat{Y}_T^{(n)}(y))$ tends to the random variable $y\frac{d\widehat{Q}(y)}{dP}V'(y\frac{d\widehat{Q}(y)}{dP})$ almost surely and, using corollary 4.2(i) and (ii) uniformly in $n \in \mathbb{N}$ below, is dominated in absolute value by the uniformly integrable sequence $(CV^{(n)}(\widehat{Y}_T^{(n)}(y)))_{n=1}^{\infty}$, for some constant $C > 0$.

Regarding (71), fix again $y > 0$ such that $v'(y)$ exists and estimate

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{v(y_n) - v(y)}{y_n - y} \\
&\quad = \lim_{n\to\infty} \frac{v^{(n)}(y_n) - v^{(n)}(y)}{y_n - y} \\
&\quad \geq \liminf_{n\to\infty} \frac{E\left[V^{(n)}\left(\widehat{Y}_T^{(n)}(y_n)\right)\right] - E\left[V^{(n)}\left(\frac{y}{y_n}\widehat{Y}_T^{(n)}(y_n)\right)\right]}{y_n - y} \qquad (74)\\
&\quad = \liminf_{n\to\infty} E\left[\frac{\widehat{Y}_T^{(n)}(y_n)(V^{(n)})'\left(\frac{\tilde{y}_n}{y_n}\widehat{Y}_T^{(n)}(y_n)\right)}{y_n}\right] \\
&\quad = E\left[\frac{d\widehat{Q}(y)}{dP} V'\left(y \frac{d\widehat{Q}(y)}{dP}\right)\right],
\end{aligned}$$

where in the last equality follows as in (73) above, this time using the uniform integrability of the sequence $(CV^{(n)}(\widehat{Y}_T^{(n)}(y_n)))_{n=1}^{\infty}$.

Step 6. For a sequence $(y_n)_{n=1}^{\infty}$ converging to $y > 0$, the sequence

$$\left(\widehat{Y}_T(y)(V^{(n)})'(\widehat{Y}_T^{(n)}(y_n))\right)_{n=1}^{\infty}$$

converges toward $\widehat{Y}_T(y)V'(\widehat{Y}_T(y))$ in the variation norm.

Observe that

$$E\left[\widehat{Y}_T(y)(V^{(n)})'(\widehat{Y}_T^{(n)}(y_n))\right] = -E_{\widehat{Q}(y)}\left[\widehat{X}_T^{(n)}(x_n)\right] y \geq -x_n y \qquad (75)$$

where $x_n = -(v^{(n)})'(y_n)$ and where we have used that the processes $X^{(n)}$, which start at $X_0^{(n)} = x_n$ and are integrals on S and uniformly bounded from below, are $\widehat{Q}(y)$-supermartingales.

Similarly as in step 5 note that by the smoothness of v we have $\lim_{n\to\infty} x_n = \lim_{n\to\infty} -(v^{(n)})'(y_n) = -v'(y) = x$.

Hence the sequence $(\widehat{Y}_T(y)(V^{(n)})'(\widehat{Y}_T^{(n)}(y_n)))_{n=1}^{\infty}$ converges to $\widehat{Y}_T(y)V'(\widehat{Y}_T(y))$ in the norm of $L^1(P)$, as it converges in measure, the positive parts are uniformly integrable and for the expectations we have the inequality

$$\begin{aligned}
&\liminf_{n\to\infty} E\left[\widehat{Y}_T(y)\left(V^{(n)}\right)'\left(\widehat{Y}_T^{(n)}(y_n)\right)\right] \\
&\quad \geq \lim_{n\to\infty}(-x_n y) = -xy = v'(y)y \qquad (76)\\
&\quad = E\left[\widehat{Y}_T(y)V'\left(\widehat{Y}_T(y)\right)\right].
\end{aligned}$$

Hence (76) implies that the sequence $\widehat{Y}_T(y)\left(V^{(n)}\right)'\left(\widehat{Y}_T^{(n)}(y_n)\right)$ converges in the norm of $L^1(P)$.

Step 7. For $x \in \mathbb{R}$, denote by $\widehat{X}_T^{(n)}(x) \in \mathcal{X}^b(x)$ the optimal solution to the primal problem (16).

The sequence $(U(\widehat{X}_T^{(n)}(x)))_{n=1}^\infty$ converges in the variation norm to a random variable $U(\widehat{F}_T(x))$, where $\widehat{F}_T(x)$ is a $]-\infty,\infty]$-valued random variable, belonging to $\mathcal{C}_U(x)$.

We have

$$\lim_{n\to\infty} u^{(n)}(x) = u(x) = E\left[U\left(\widehat{F}_T(x)\right)\right], \tag{77}$$

hence $\widehat{F}_T$ is the unique maximizer to the primal problem (6). Its relation to the minimizer $\widehat{Q}(y)$ of the dual problem (14) is given, for $u'(x) = y$, by

$$\widehat{F}_T(x) = I\left(\widehat{Y}_T(y)\right) = I\left(y\frac{d\widehat{Q}(y)}{dP}\right). \tag{78}$$

We also have that u and v are conjugate, as $u^{(n)}$ and $v^{(n)}$ are so, and $u^{(n)}$ and $v^{(n)}$ converge monotonically to u and v, respectively.

Fix $x \in \mathbb{R}$ and deduce from (23) that $\widehat{X}_T^{(n)}(x)$ is given by

$$\widehat{X}_T^{(n)}(x) = I^{(n)}\left(\widehat{Y}_T^{(n)}(y_n)\right) \tag{79}$$

where y_n is determined via $(u^{(n)})'(x) = y_n$. To show that $(y_n)_{n=1}^\infty$ converges to y, observe that the concave functions $u^{(n)}$ increase to a function, which we denote by $u^{(\infty)}$, and which is conjugate to $v = v^{(\infty)}$. As we have seen that v is strictly convex on $\mathbb{R}_+$, the conjugate function $u^{(\infty)}$ is smooth and therefore $(u^{(n)})'$ converges to $(u^{(\infty)})'$ pointwise (in fact, uniformly on compact subsets of $\mathbb{R}$), which proves that $y_n \mapsto y := -v'(x)$.

Next we show that

$$\lim_{n\to\infty} \left\| U\left(\widehat{X}_T^{(n)}(x)\right) - U^{(n)}\left(\widehat{X}_T^{(n)}(x)\right)\right\|_{L^1(P)} = 0. \tag{80}$$

Indeed, otherwise we could find $\alpha > 0$ such that the above expression is bigger than α, for infinitely many n's, which gives rise to the following estimate for infinitely many n's

$$\begin{aligned} u^{(n+1)}(x) &\geq E\left[U^{(n+1)}\left(\widehat{X}_T^{(n)}(x)\right)\right] \\ &= E\left[U\left(\widehat{X}_T^{(n)}(x)\right)\right] \\ &\geq E\left[U^{(n)}\left(\widehat{X}_T^{(n)}(x)\right)\right] + \alpha \\ &= u^{(n)}(x) + \alpha, \end{aligned} \tag{81}$$

where in the equality above we have used that $U^{(n+1)}(x)$ coincides with $U(x)$, for $x \geq -(n+1)$ and that $\widehat{X}_T^{(n)}(x) \geq -(n+1)$. This contradiction to (9) shows (80).

Hence we obtain from Steps 3 and 4

$$
\begin{aligned}
&\lim_{n,m\to\infty} E\left[\left|U\left(\widehat{X}_T^{(n)}(x)\right) - U\left(\widehat{X}_T^{(m)}(x)\right)\right|\right] \\
&\quad = \lim_{n,m\to\infty} E\left[\left|U^{(n)}\left(\widehat{X}_T^{(n)}(x)\right) - U^{(m)}\left(\widehat{X}_T^{(m)}(x)\right)\right|\right] \\
&\quad = \lim_{n,m\to\infty} E\left[\left|U^{(n)}\left(I^{(n)}\left(\widehat{Y}_T^{(n)}(y_n)\right)\right) - U^{(m)}\left(I^{(m)}\left(\widehat{Y}_T^{(m)}(y_m)\right)\right)\right|\right] \\
&\quad \le \lim_{n,m\to\infty} \left(E\left[\left|U^{(n)}\left(I^{(n)}\left(\widehat{Y}_T^{(n)}(y_n)\right)\right) - \widehat{Y}_T^{(n)}(y_n) I^{(n)}\left(\widehat{Y}_T^{(n)}(y_n)\right)\right.\right.\right. \\
&\qquad\quad \left.\left. - \left(U^{(m)}\left(I^{(m)}\left(\widehat{Y}_T^{(m)}(y_m)\right)\right) - \widehat{Y}_T^{(m)}(y_m) I^{(m)}\left(\widehat{Y}_T^{(m)}(y_m)\right)\right)\right|\right] \qquad (82) \\
&\qquad\quad \left. + E\left[\left|\widehat{Y}_T^{(n)}(y_n) I^{(n)}\left(\widehat{Y}_T^{(n)}(y_n)\right) - \widehat{Y}_T^{(m)}(y_m) I^{(m)}\left(\widehat{Y}_T^{(m)}(y_m)\right)\right|\right]\right) \\
&\quad = \lim_{n,m\to\infty} \left(E\left[\left|V^{(n)}\left(\widehat{Y}_T^{(n)}(y_n)\right) - V^{(m)}\left(\widehat{Y}_T^{(m)}(y_m)\right)\right|\right]\right. \\
&\qquad\quad \left. + E\left[\left|\widehat{Y}_T^{(n)}(y_n) I^{(n)}\left(\widehat{Y}_T^{(n)}(y_n)\right) - \widehat{Y}_T^{(m)}(y_m) I^{(m)}\left(\widehat{Y}_T^{(m)}(y_m)\right)\right|\right]\right) \\
&\quad = 0.
\end{aligned}
$$

So $(U(\widehat{X}_T^{(n)}(x)))_{n=1}^{\infty}$ converges in the norm of $L^1(P)$ to a random variable which we may write as $U(\widehat{F}_T(x))$, where $\widehat{F}_T(x)$ is in $\mathscr{C}_U(x)$ by the very definition of this set.

As $E[U(\widehat{F}_T(x))] = u^{(\infty)}(x)$ we also have shown that $u = u^{(\infty)}$.

The formula

$$\widehat{F}_T(x) = I\left(\widehat{Y}_T(y)\right) \tag{83}$$

now follows from

$$\widehat{X}_T^{(n)}(x) = I^{(n)}\left(\widehat{Y}_T^{(n)}(y_n)\right), \qquad n \in \mathbb{N}, \tag{84}$$

where on both sides we pass to the limits in the topology of convergence in measure.

Step 8. We have the formula

$$xu'(x) = E\left[\widehat{F}_T(x)\, U'\left(\widehat{F}_T(x)\right)\right]. \tag{85}$$

Indeed, just as in the proof of Theorem 2.1 this formula now is just a reformulation of formula (69).

Step 9. $u(x)$ has reasonable asymptotic elasticity.

Indeed, by Corollary 4.2 we have that, for each $\lambda > 0$, there is a constant $C > 0$ such that

$$V(\lambda y) \le CV(y) \quad \text{for } y > 0, \tag{86}$$

and in view of the identity $v(y) = E[V(\widehat{Y}_T(y))]$ established in (61) this inequality passes over to the value function v:

$$v(\lambda y) \le Cv(y) \quad \text{for } y > 0. \tag{87}$$

It follows from Proposition 4.1 below and [17], Lemma 6.1 that the latter inequality implies the reasonable asymptotic elasticity of u and therefore in particular the assertions (41).

Step 10. Supposing that $\widehat{Q}(y)$ is equivalent to P and letting x verify $u'(x) = y$, the formula

$$\widehat{X}_t(x) = E_{\widehat{Q}(y)}\left[\widehat{F}_T(x)\,\middle|\mathscr{F}_t\right] \tag{88}$$

defines a uniformly integrable $\widehat{Q}(y)$-martingale, which is a stochastic integral on S starting at $X_0(x) = x$, for a predictable S-integrable integrand $(\widehat{H}_t(x))_{0\le t\le T}$,

$$\widehat{X}_t = x + (\widehat{H}(x)\cdot S)_t = x + \int_0^t \widehat{H}_u(x)dS_u, \qquad 0 \le t \le T. \tag{89}$$

First note that (88) well-defines a uniformly integrable $\widehat{Q}(y)$-martingale as we have

$$E_{\widehat{Q}(y)}\left[\left|\widehat{F}_T(x)\right|\right] = E\left[\left|\frac{d\widehat{Q}(y)}{dP} I\left(\widehat{Y}_T(y)\right)\right|\right] < \infty. \tag{90}$$

We also note that $\widehat{X}_0(x) = x$, as by (69) and (83) we have

$$E_{\widehat{Q}(y)}\left[\widehat{F}_T(x)\right] = E\left[\frac{d\widehat{Q}(y)}{dP} I\left(\widehat{Y}_T(y)\right)\right] = -v'(y) = x. \tag{91}$$

By Theorem 2.2 of [17] each $\widehat{X}^{(n)}(x)$ is a stochastic integral on S starting at $\widehat{X}_0^{(n)}(x) = x$ for some integrand $\widehat{H}^{(n)}(x)$:

$$\widehat{X}_t^{(n)} = x + \left(\widehat{H}^{(n)}(x)\cdot S\right)_t, \qquad 0 \le t \le T. \tag{92}$$

Our aim is to deduce the limiting formula (89). We know that the sequence of terminal values $(\widehat{X}_T^{(n)}(x))_{n=1}^\infty$ converges to $\widehat{X}_T(x)$ in the norm of $L^1(\widehat{Q}(y))$. Indeed letting $y_n = (u^{(n)})'(x)$ we have shown in step 6 that the sequence $-y\frac{d\widehat{Q}(y)}{dP}\widehat{X}_T^{(n)}(x) = (\widehat{Y}_T(y)(V^{(n)})'(\widehat{Y}_T^{(n)}(y_n)))_{n=1}^\infty$ converges to $\widehat{Y}_T(y)V'(\widehat{Y}_T(y))$ $= -y\frac{d\widehat{Q}(y)}{dP}\widehat{X}_T(x)$ in the norm of $L^1(P)$, which amounts to the same thing.

If we knew that the processes $\widehat{X}^{(n)}$ were uniformly integrable $\widehat{Q}(y)$-martingales we could apply Yor's theorem [21] to deduce (89) from (92). But unfortunately we only know that the processes $\widehat{X}^{(n)}$ are $\widehat{Q}(y)$-supermartingales [we only have shown that they are u.i. martingales under $\widehat{Q}(y_n)$] and, in fact, there is no reason why they should be $\widehat{Q}(y)$-martingales.

Hence we have to work a bit harder and apply the more general methodology as developed in [10].

First we want to control the negative parts of $\widehat{X}^{(n)}$. Using the $L^1(\widehat{Q}(y))$-convergence of $(\widehat{X}_T^{(n)}(x))_{n=1}^\infty$, we may pass to a subsequence, still denoted by

$(\widehat{X}^{(n)})_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} \|(\widehat{X}_T^{(n)}(x)) - (\widehat{X}_T^{(n+1)}(x))\|_{L^1(\widehat{Q}(y))} < \infty$. Hence the supremum over the negative parts

$$Z := \sup_n \left(\widehat{X}_T^{(n)}(x)\right)_- \le (\widehat{X}_T^{(1)}(x))_- + \sum_{n=1}^{\infty} \left| \left(\widehat{X}_T^{(n)}(x)\right) - \left(\widehat{X}_T^{(n+1)}(x)\right) \right| \tag{93}$$

has finite expectation under $\widehat{Q}(y)$. Denoting by $Z_t = E_{\widehat{Q}(y)}[Z|\mathscr{F}_t]$ the $\widehat{Q}(y)$-martingale with terminal value $Z = Z_T$, we define, for $N \in \mathbb{N}$, the stopping times

$$\tau_N := \inf\{t : 0 \le t \le T \text{ and } Z_t \ge N\}, \tag{94}$$

where we define the inf over the empty set to equal T, and the functions w_N by

$$w_N = Z_{\tau_N} \chi_{\{\tau_N < \infty\}} + N \chi_{\{\tau_N = T\}}. \tag{95}$$

It is easy to verify that τ_N increases almost surely to T, that $w_N \in L^1(\widehat{Q}(y))$, for each $N \in \mathbb{N}$, and that, for fixed N, the sequence $((\widehat{X}^{(n)}(x))^{\tau_N})_{n=1}^{\infty}$ of processes stopped at time τ_N is bounded from below by the function w_N.

Hence we are in a position to apply, for fixed N, Theorem D of [10]. To avoid (mainly notational) difficulties pertaining to the repeated formation of convex combinations we isolate the easy diagonalization argument before we apply Theorem D of [10]: by Komlos' theorem (compare [10], Theorem 1.3) we may find a sequence, denoted by $(\xi^{1,n})_{n=1}^{\infty}$, of processes such that $\xi^{1,n} \in \operatorname{conv}(\widehat{X}^{(n)}(x), \widehat{X}^{(n+1)}(x), \ldots)$ such that the sequence of random variables $(\xi^{1,n}_{\tau_1})_{n=1}^{\infty}$ converges a.s.; next we form in a similar way a sequence of convex combinations $(\xi^{2,n})_{n=1}^{\infty}$ of the sequence of processes $(\xi^{1,n})_{n=1}^{\infty}$ such that $(\xi^{2,n}_{\tau_2})_{n=1}^{\infty}$ converges a.s.; note that the a.s. convergence of $(\xi^{2,n}_{\tau_1})_{n=1}^{\infty}$ still holds true. Continuing in an obvious way we obtain a sequence $((\xi^{k,n})_{n=1}^{\infty})_{k=1}^{\infty}$ of sequences of processes obtained by repeatedly taking convex combinations and such that $(\xi^{k,n}_{\tau_j})_{n=1}^{\infty}$ converges a.s., for each k and $j = 1, \ldots, k$. The diagonal sequence of processes $(\xi^{n,n})_{n=1}^{\infty}$ then is a sequence of convex combinations of the original sequence $(\widehat{X}^{(n)}(x))_{n=1}^{\infty}$ and has the property that $(\xi^{n,n}_{\tau_j})_{n=1}^{\infty}$ converges a.s., for each $j \in \mathbb{N}$.

Summing up, by passing to a sequence of convex combinations–which we now still denote by the sequence of processes $(\widehat{X}^{(n)}(x))_{n=1}^{\infty}$–we may and do assume that, for each $N \in \mathbb{N}$, the sequence of random variables $(\widehat{X}^{(n)}_{\tau_N}(x))_{n=1}^{\infty}$ converges a.s. to a random variable which we denote by X_{τ_N}. Also note that $(X_{\tau_N})_{N=1}^{\infty}$ converges a.s. to $\widehat{X}_T(x)$ as $\tau_N \to T$ almost surely.

After this preparation we may deduce from Theorem D of [10] that we can find a sequence $(H^{(N)})_{N=1}^{\infty}$ of S-integrable predictable processes supported by

$]\!]\tau_{N-1}, \tau_N]\!]$ such that the stochastic processes $H^{(N)} \cdot S$ are supermartingales and such that

$$x + (H^{(1)} \cdot S)_{\tau_1} \geq X_{\tau_1} \tag{96}$$

and

$$(H^{(N)} \cdot S)_{\tau_N} \geq X_{\tau_N} - X_{\tau_{N-1}} \quad \text{for } N > 1. \tag{97}$$

We now paste the $H^{(N)}$'s together and define the S-integrable predictable integrand

$$H := \sum_{N=1}^{\infty} H^{(N)}. \tag{98}$$

It follows from Theorem D of [10] that the resulting process

$$\widetilde{X}_t := x + (H \cdot S)_t \tag{99}$$

is a well-defined $\widehat{Q}(y)$-supermartingale, as $(\widetilde{X}_t)_{0 \leq t \leq T}$ is bounded from below by the uniformly integrable $\widehat{Q}(y)$-martingale $(-Z_t)_{0 \leq t \leq T}$. From (96) and (97) we obtain that

$$x + (H \cdot S)_T = \widetilde{X}_T \geq \widehat{X}_T(x). \tag{100}$$

On the other hand, using (88) and the $\widehat{Q}(y)$-super-martingale property of $(\widetilde{X}_t)_{0 \leq t \leq T}$ we obtain

$$x = E_{\widehat{Q}(y)}[\widehat{X}_T(x)] \leq E_{\widehat{Q}(y)}[\widetilde{X}_T] \leq x. \tag{101}$$

Hence we must have equality in (100) and the process $\widetilde{X}$ equals the process $\widehat{X}(x)$ and therefore is a uniformly integrable $\widehat{Q}(y)$-martingale, for which the representation (89) holds true if we define $\widehat{H}(x) := H$.

The proof of Theorem 2.2 now is complete. □

REMARK 2.3. We have assumed in item (iv) of Theorem 2.2 that $\widehat{Q}(y)$ is equivalent to P and left open the case, when $\widehat{Q}(y)$ only is absolutely continuous with respect to P.

Note that in the case, when $U(\infty) = \infty$, it is obvious from (14) and the equality $U(\infty) = V(0)$ that for the minimizer $\widehat{Q}(y) \in \mathscr{M}^a(S)$ we have that $\frac{\widehat{Q}(y)}{dP} > 0$ almost surely, that is, that $\widehat{Q}(y) \in \mathscr{M}^e(S)$, so that Theorem 2.2(iv) applies. But there are also other important cases where one may assert that $\widehat{Q}(y)$ is equivalent to P: for example, it follows from the work of I. Csiszar [4] that, for the exponential utility $U(x) = -e^{-x}$, the condition

$$\inf_{Q \in \mathscr{M}^e(S)} E\left[V\left(y \frac{dQ}{dP}\right)\right] < \infty \tag{102}$$

implies that $\widehat{Q}(y) \in \mathscr{M}^e(S)$. Note, however, that there is a slight difference between condition (102) and the finiteness of $v(y)$ as defined in (14).

In general, there is little reason why $\widehat{Q}(y)$ should be equivalent to P. A thorough discussion of this case is left to the future research.

We also note that the processes $X_t = x + (H \cdot S)_t$ which are uniformly integrable under some $Q \in \mathscr{M}^e(S)$, such as the optimal process $\widehat{X}(x)$ in Theorem 2.2(iv) above, have extensively been studied in [8], where they were shown to have a number of features which make them interesting in applications to Mathematical Finance.

REMARK 2.4. As mentioned in the introduction, F. Bellini and M. Fritelli have proved, using different methods, a result which implies that the assumptions (1), (2) and

$$\inf_{Q \in \mathscr{M}^a(S)} E\left[V\left(\frac{dQ}{dP}\right)\right] < \infty \tag{103}$$

imply that there is a unique minimizer $\widehat{Q} \in \mathscr{M}^a(S)$ to (103) which is in $\mathscr{M}^e(S)$ if $U(\infty) = \infty$.

We have reproved this result in step 2 above, but this proof is rather complicated as we had to prepare the ground for the proof of the other assertions of Theorem 2.2 too.

If one is only interested in the result of F. Bellini and M. Fritelli above one may proceed in a considerably simpler way and one does not need the assumption of reasonable asymptotic elasticity of U. For the sake of completeness we sketch the argument: applying Theorem 2.2 of [17] we may find, for each $n \in \mathbb{N}$, a measure $Q^{(n)} \in \mathscr{M}^e(S)$ satisfying

$$v^{(n)}(1) + \frac{1}{n} > E\left[V^{(n)}\left(\frac{dQ^{(n)}}{dP}\right)\right]. \tag{104}$$

The applicability of Theorem 2.2 of [17] is justified as we have chosen the sequence $(U^{(n)})_{n=1}^{\infty}$ such that each $U^{(n)}$ has reasonable asymptotic elasticity, even when U fails to do so.

Assumption (103) clearly implies that $\lim_{n\to\infty} v^{(n)}(1) < \infty$. We shall show that $(Q^{(n)})_{n=1}^{\infty}$ converges in the variation norm by showing that $(\frac{dQ^{(n)}}{dP})_{n=1}^{\infty}$ is uniformly integrable and converges in measure. As regards the uniform integrability we just copy the argument from (56) above. As regards the convergence in measure we have–at least–two possibilities: either we repeat the argument of (57) above showing directly that $(\frac{dQ^{(n)}}{dP})_{n=1}^{\infty}$ is Cauchy in measure; or we may also be lazy and avoid this argument by applying Komlos' theorem: passing to convex combinations of $(Q^{(n)})_{n=1}^{\infty}$ one can without further argument assume that this sequence of convex combinations, still denoted by $(Q^{(n)})_{n=1}^{\infty}$, is Cauchy in measure. Consequently $(\frac{dQ^{(n)}}{dP})_{n=1}^{\infty}$ is convergent in the norm of $L^1(P)$ to some $\frac{d\widehat{Q}}{dP}$.

As $\mathscr{M}^a(S)$ is closed in the variation norm we have that $\widehat{Q} \in \mathscr{M}^a(S)$ and Fatou's lemma implies that it is the minimizer to (103).

The uniqueness of $\widehat{Q}$ now follows from the strict convexity of V and the fact that $\widehat{Q} \in \mathscr{M}^e(S)$, if $U(\infty) = V(0) = \infty$, is obvious.

REMARK 2.5. The formula

$$u'(x) = \frac{E[\widehat{X}_T(x)U'(\widehat{X}_T(x))]}{x} \tag{105}$$

may be viewed as a special case of a general principle on the variation of the optimal investment $\widehat{X}_T(x)$. We interpret this *principle of valuing by marginal utility* in economic terms (compare [12] and [6]): suppose your initial endowment is changed from x to $x+h$, for some small $h \in \mathbb{R}$. What is the resulting change $u(x+h)-u(x)$ in expected utility, if we invest optimally in the financial market S?

One may use the additional endowment h to finance the contingent claim $\frac{h}{x}\widehat{X}_T(x)$ (we assume for simplicity $x \neq 0$), which can be replicated on the financial market at a cost of h. The resulting difference in expected utility then equals

$$E\left[U\left(\frac{x+h}{x}\widehat{X}_T(x)\right) - U\left(\widehat{X}_T(x)\right)\right] \approx \frac{h \cdot E[\widehat{X}_T(x)U'(\widehat{X}_T(x))]}{x}. \tag{106}$$

Economic intuition suggests that $\left(1+\frac{h}{x}\right)\widehat{X}_T(x)$ equals the optimal investment $\widehat{X}_T(x+h)$ up to terms of order $o(h)$, which leads us to conjecture

$$\frac{u(x+h)-u(x)}{h} \approx \frac{E[\widehat{X}_T(x)U'(\widehat{X}_T(x))]}{x}. \tag{107}$$

In the present paper as well as in [17] we have given precise and fairly general conditions making sure that this intuitive reasoning indeed leads to the precise formula (105).

Let us now consider an alternative use of the additional endowment h, namely investing it into the bond: this results in a terminal wealth $\widehat{X}_T(x)+h$. Again economic intuition suggests that this investment should be optimal, for given endowment $x+h$, up to terms of order $o(h)$. Hence we conjecture the relation

$$\frac{u(x+h)-u(x)}{h} \approx \frac{E[U(\widehat{X}_T(x)+h) - U(\widehat{X}_T(x))]}{h} \tag{108}$$

$$\approx E[U'(\widehat{X}_T(x))]. \tag{109}$$

Under which conditions does this–even simpler–relation result in a precise formula? The answer is: in the setting of the present paper it does hold true, while in the setting of [17] things may go wrong.

Indeed, using the relations $y = u'(x)$ and $U'(\widehat{X}_T(x)) = \widehat{Y}_T(y)$ the formula

$$u'(x) = E[U'(\widehat{X}_T(x))] \tag{110}$$

is equivalent to the formula

$$y = E[\widehat{Y}_T(y)]. \tag{111}$$

As we have seen, formula (111) always holds true under the assumptions of Theorem 2.2 of the present paper; on the other hand, it was shown in Example 5.1 of [17] that the assumptions of Theorem 2.2 of [17] are not sufficient to imply the validity of (111).

In a way, this situation is not too surprising, if one thinks in economic terms: Formula (105) reflects the logic of a *multiplicative* variation of the optimal endowment $\widehat{X}_T(x)$ while formula (110) reflects the logic of an *additive* variation. While, at least on an intuitive level, a multiplicative variation does fit well to utility functions $U : \mathbb{R} \to \mathbb{R}$ as well as to utility functions $U : \mathbb{R}_+ \to \mathbb{R}$, the additive variation is prone to lead to difficulties, when $U'(0) = \infty$, while it does fit well to utility functions U defined on all of $\mathbb{R}$.

We thank I. Klein for helpful comments on the theme of this remark.

REMARK 2.6. In the present paper we have assumed that the stock price process S is a locally bounded semi-martingale, a setting which is slightly less general then the one chosen in [17].

The reason for assuming local boundedness is that our present approach is based on approximating the optimal process $\widehat{X}(x)$ by a sequence $X^{(n)}(x)$ of processes which are bounded from below, thus modeling, for fixed n, the situation of an economic agent with a finite credit line. The assumption of local boundedness of S is crucial for this approach to work successfully, which again is not too surprising if one thinks in economic terms.

We give an easy example illustrating the new phenomena arising in the non locally bounded case: Let $S = (S_0, S_1)$ be a one period process with $S_0 = 0$ and S_1 normally distributed with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$. We consider S to be defined with respect to its natural filtration $(\mathscr{F}_0, \mathscr{F}_1)$. Given the endowment $x \in \mathbb{R}$ the set of random variables

$$\mathscr{K}(x) := \{x + (H \cdot S)_1 : H \text{ predictable and } S\text{-integrable}\} \tag{112}$$

trivially reduces to the set

$$\mathscr{K}(x) := \{x + \lambda S_1 : \lambda \in \mathbb{R}\}. \tag{113}$$

Let us now consider the exponential utility $U(x) = -e^{-x}$ and the optimization problem

$$\{E[U(X_1)],\ X_1 \in \mathscr{K}(x)\} \longmapsto \max! \tag{114}$$

An elementary calculation reveals the well-known result that the maximizer to this problem is given by $\widehat{X}_1 = x + \widehat{\lambda} S_1$, where $\widehat{\lambda} = \mu/\sigma$, which makes perfect sense economically: the higher the Sharpe-ratio μ/σ of the investment opportunity is, the more the utility-maximizing agent wants to invest in it.

But note that, for $\mu \neq 0$, this investment cannot be approximated by trading strategies on S which are uniformly bounded from below (i.e., which can be chosen by an agent with a finite credit line), as there are no such trading strategies except for the zero-investment.

Summing up: in the case of non locally bounded processes the modeling of the optimal investment as a limit of investments which are bounded from below does not work any more and a different methodology has to be developed. This theme is left to future research.

3. Examples. In this section we give some examples similar to example 5.2 in [17] where the assumption $AE_{-\infty}(U) = 1$ now replaces (or accompanies) the assumption $AE_{+\infty}(U) = 1$ of [17].

LEMMA 3.1. *Assume that* $U : \mathbb{R} \to \mathbb{R}$ *is a utility function satisfying* (1) *such that* $AE_{-\infty}(U) = 1$; *denote by* V *its conjugate function. Then there is a probability measure* Q *on* $\mathbb{R}_+$ *supported by a sequence* $(x_k)_{k\geq 0}$ *of positive numbers increasing to infinity such that*:

(i) $\int_0^\infty V(x)Q(dx) < \infty$.
(ii) $\int_0^\infty xV'(x)Q(dx) < \infty$.
(iii) $\int_0^\infty V(\gamma x)Q(dx) = \infty$ *for any* $\gamma > 1$.

PROOF (Compare [17], Lemma 5.1). Using Proposition 4.1(iii) below we may find a sequence $(y_n)_{n\geq 1}$ of positive numbers increasing to infinity such that, for any $\gamma > 1$,

$$\sum_{n=1}^{\infty} \frac{1}{2^{2n}} \frac{V(\gamma y_n)}{V(y_n)} = +\infty. \tag{115}$$

Denote

$$x_n = \frac{y_n}{1 + \frac{1}{2^n}} \quad \text{and} \quad p_n = \frac{K}{2^{2n} V(y_n)} \tag{116}$$

where the normalizing constant K is chosen such that $\sum_{n=1}^{\infty} p_n = 1$. We now are ready to define the measure Q, which is supported by the sequence $(x_n)_{n\geq 1}$:

$$Q(x_n) = p_n. \tag{117}$$

Let us check the assertions of our lemma. We have

$$\int_0^\infty V(x)Q(dx) = \sum_{n=1}^{\infty} p_n V(x_n) < \infty \tag{118}$$

where we have used that for all but finitely many n's we have $V(x_n) \leq V(y_n)$ and that $\sum_{n=1}^{\infty} p_n V(y_n) < \infty$. This proves 3.1(i). Regarding 3.1(ii), we use the inequality

$$xV'(x) \leq \frac{1}{\gamma - 1} (V(\gamma x) - V(x)) \leq \frac{1}{\gamma - 1} V(\gamma x), \tag{119}$$

which is valid for any $\gamma > 1$ and x sufficiently big, to get

$$x_n V'(x_n) \leq 2^n V(y_n), \tag{120}$$

for n sufficiently big, say $n \geq n_0$, and hence

$$\int_0^\infty x V'(x) Q(dx) = \sum_{n=1}^\infty p_n x_n V'(x_n) \leq \text{const } + \sum_{n=n_0}^\infty p_n 2^n V(y_n) < \infty. \tag{121}$$

Finally, (115) implies 3.1(iii): for any $\gamma > 1$

$$\int_0^\infty V(\gamma x) Q(dx) = \sum_{n=1}^\infty p_n V(\gamma x_n) = \infty. \tag{122}$$

The proof is complete. □

REMARK 3.2. Assertions (i)-(iii) of Lemma 3.1 are sensitive only to the behavior of Q near infinity. For example, we can always choose Q in such a way that $\int_0^\infty x Q(dx) = 1$. [Note that assertion 3.1(i) implies in particular that $\int_0^\infty x Q(dx) < \infty$.]

We now can formulate the analogue to example 5.2 of [17]:

PROPOSITION 3.3. *Let $U : \mathbb{R} \to \mathbb{R}$ be a utility function satisfying* (1) *and such that $AE_{-\infty}(U) = 1$. Then there is a complete continuous financial market $(S_t)_{0 \leq t \leq T}$ such that*:

(i) *there is $a \in \mathbb{R}$ such that, for $x \geq a$, the optimization problem* (6) *has a unique optimal solution $\widehat{X}(x)$, while, for $x < a$, no optimal solution to* (6) *exists.*

(ii) *u is continuously differentiable; it is strictly concave on $[a, \infty[$, while $u'(x) = 1$, for $x \leq a$.*

(iii) *v is continuously differentiable and strictly convex on $]0, 1]$ and the left derivative v'_l at $y = 1$ equals $v'_l(1) = -a$, while $v(y) = \infty$, for $y > 1$.*

PROOF. We proceed similarly as in [17], Example 5.2.

Let U be a utility function satisfying (1) and such that $AE_{-\infty}(U) = 1$. Let W be a standard Brownian motion with $W_0 = 0$ defined on a filtered probability space $(\Omega, \mathscr{F}_T, (\mathscr{F}_t)_{0 \leq t \leq T}, P)$, where $0 < T < \infty$ is fixed and the filtration $(\mathscr{F}_t)_{0 \leq t \leq T}$ is supposed to be generated by W. Let Q be a measure on $(0, \infty)$ for which the assertions (i)-(iii) of Lemma 3.1 hold true and such that (see Remark 3.2)

$$\int_0^\infty x Q(dx) = 1. \tag{123}$$

Let

$$-a = \int_0^\infty x V'(x) Q(dx). \tag{124}$$

and η an $\mathbb{R}_+$-valued random variable on $(\Omega, \mathscr{F}_T)$, whose distribution under P coincides with the measure Q. Clearly, (123) implies that $E[\eta] = 1$. The process

$$Z_t = E[\eta|\mathscr{F}_t], \qquad t \geq 0. \tag{125}$$

is a strictly positive martingale with initial value $Z_0 = 1$. From the integral representation theorem we deduce the existence of a predictable process $\mu = (\mu_t)_{t\geq 0}$ such that

$$Z_t = 1 + \int_0^t \mu_s Z_s dW_s \tag{126}$$

or, equivalently,

$$Z_t = \exp\left(\int_0^t \mu_s dW_s - \tfrac{1}{2}\int_0^t \mu_s^2 ds\right). \tag{127}$$

The stock price process S is now defined as

$$S_t = 1 + \int_0^t S_u\left(-\mu_u du + dW_u\right). \tag{128}$$

Standard arguments based on the integral representation theorem and the Girsanov theorem imply that the family of martingale measures for the process S consists of exactly one element (i.e., the market is complete) and that the density process of the unique martingale measure is equal to Z. The verification of the assertions (i)-(iii) of the proposition 3.3 now follows exactly the same lines as the proof of proposition 5.2 in [17] and therefore is omitted. □

We now turn to the case of utility functions $U : \mathbb{R} \to \mathbb{R}$ satisfying (1) and such that $AE_{+\infty}(U)$ as well as $AE_{-\infty}(U)$ equal 1; in this situation we may construct examples which – from an economic point of view – are even more puzzling than the above example described in Proposition 3.3.

First we need a combination of Lemma 3.1 above with Lemma 5.1 from [17]:

LEMMA 3.4. *Let* $U : \mathbb{R} \to \mathbb{R}$ *be a utility function satisfying* (1) *such that* $AE_{+\infty}(U) = AE_{-\infty}(U) = 1$. *There is a probability measure* Q *on* $\mathbb{R}_+$ *supported by an increasing sequence* $(x_k)_{k\in\mathbb{Z}}$ *such that* $\lim_{k\to-\infty} x_k = 0$, $\lim_{k\to\infty} x_k = \infty$ *such that*:

(i) $\int_0^\infty V(x)Q(dx) < \infty$ *and* $\int_0^\infty xQ(dx) = 1$.
(ii) $\int_0^\infty x|V'(x)|Q(dx) < \infty$.
(iii) $\int_0^1 V(\gamma x)Q(dx) = \infty$, *for any* $\gamma \neq 1$.

PROOF. The proof is a straightforward combination of the proofs of Lemma 3.1 above and Lemma 5.1 of [17]: for $(x_k)_{k\geq 0}$ we mimic the former and for $(x_k)_{k<0}$ the latter one. □

PROPOSITION 3.5. *Let $U : \mathbb{R} \to \mathbb{R}$ be a utility function satisfying* (1) *and such that $AE_{+\infty}(U) = AE_{-\infty}(U) = 1$. Then there is a complete continuous financial market $(S_t)_{0\le t\le T}$ such that*:

(i) *There is precisely one $x_0 \in \mathbb{R}$ for which the optimal solution* (6) *exists*; *moreover this optimal solution $\widehat{X}(x_0)$ is unique.*
(ii) *$u(x) = c + x$ for some constant $c \in \mathbb{R}$.*
(iii) *$v(1) < \infty$ while, for $y \neq 1$, we have $v(y) = \infty$.*

PROOF. We repeat the construction of the proof of Proposition 3.3 above where the measure Q on $\mathbb{R}_+$ now satisfies the properties listed in Lemma 3.4 rather than those of Lemma 3.1. Then it follows from Lemma 3.4(i) and (iii) that assertion 3.5(iii) is satisfied which in turn implies 3.5(ii) by the conjugacy of the functions u and v. As regards assertion 3.5(i) we deduce form Scholium 5.1 of [17] that there is at most one $x_0 \in \mathbb{R}$ such that the optimal solution $\widehat{X}(x_0)$ exists. To find this x_0 and the corresponding $\widehat{X}(x_0)$ we define the random variable

$$X := I(Z_T) = -V'(Z_T), \tag{129}$$

where Z_T is defined as in the proof of proposition 3.3 above. Then it follows from Lemma 3.4(ii) that

$$E[|X|Z_T] = \int_0^\infty |V'(x)|xQ(dx) < \infty \tag{130}$$

so that $x_0 = E[XZ_T]$ is well defined and, using the martingale representation theorem, we may write X as

$$X = x_0 + \int_0^T H_u dS_u. \tag{131}$$

Here the process $(x_0 + \int_0^T H_u dS_u)_{0\le t\le T}$ is a uniformly integrable martingale under the unique martingale measure for the financial market $(S_t)_{0\le t\le T}$ (the density of this martingale measure with respect to P is given by Z_T). In other words the random variable X equals the terminal value $\widehat{X}_T(x_0)$ of a process $\widehat{X}(x_0) \in \mathcal{X}(x_0)$.

To show that $\widehat{X}(x_0)$ is the unique solution to the optimization problem (6) (for $x = x_0$) it suffices to show that $E[U(\widehat{X}_T(x_0))] = u(x_0)$ holds true. Noting that $u(x_0) = v(1) + x_0$ this equality follows from

$$\begin{aligned} E\left[U(\widehat{X}_T(x_0))\right] &= E\left[U\left(I(Z_T)\right)\right] \\ &= E\left[U\left(I(Z_T)\right) - I(Z_t)Z_T\right] + E\left[I(Z_T)Z_T\right] \\ &= E\left[V(Z_T)\right] + E\left[XZ_T\right] \\ &= v(1) + x_0 \end{aligned} \tag{132}$$

which yields 3.5(i) and thus completes the proof. □

REMARK 3.6. One also may construct several variants of the example given in proposition 3.5 [always using the hypothesis $AE_{+\infty}(U) = AE_{-\infty}(U) = 1$]: for example, one may modify the construction of Lemma 3.4 such that assertion 3.4(ii) is replaced by:

3.4(ii′) $\int_0^\infty x|V'(x)|Q(dx) = \infty$,

while assertions 3.4(i) and 3.4(iii) remain uncharged.

Plugging a measure Q satisfying 3.4(i), (ii′) and (iii) into the construction of proposition 3.5, we again obtain the assertions 3.5(ii) and (iii) while (i) is replaced by:

3.5(i′) There is no $x \in \mathbb{R}$ for which the optimal solution $\widehat{X}(x_0)$ exists.

A second variation is to modify the construction of Lemma 3.4 so that we have:

3.4(i″)] $\int_0^\infty V(\gamma x)Q(dx) < \infty$ for $1 \leq \gamma \leq 2$ and $\int_0^\infty xQ(dx) = 1$.

3.4(ii″) $\int_0^\infty x|V'(\gamma x)|Q(dx) < \infty$ for $1 \leq \gamma \leq 2$.

3.4(iii″) $\int_0^\infty V(\gamma x)Q(dx) = \infty$ for $\gamma \notin [1, 2]$.

Using such a probability measure Q in the construction of Proposition 3.5 yields the following assertions:

3.5(i″) *There are numbers* $-\infty < a < b < \infty$ *such that, for* $x \in [a, b]$, *there exists a unique optimal solution* $\widehat{X}(x)$ *to* (6); *for* $x \notin [a, b]$ *the optimal solution to* (6) *does not exist.*

3.5(ii″) $u(x)$ *is a smooth function which is strictly concave on* $[a, b]$, *while* $u'(x) = 2$, *for* $x \leq a$, *and* $u'(x) = 1$, *for* $x \geq b$.

3.5(iii″) $v(y)$ *is a finitely valued, smooth and strictly convex function on the interval* $[1, 2]$ *while* $v(y) = \infty$, *for* $y \notin [1, 2]$. *The right derivative* $v'_r(1)$ *at* 1 *and the left derivative* $v'_l(2)$ *at* 2 *are finite and we have*

$$v'_r(1) = -b \quad \text{and} \quad v'_l(2) = -a.$$

The proofs of the above observations are rather straightforward variations of the preceding arguments and left to the energetic reader.

REMARK 3.7. The attentive reader certainly has noticed a slight difference between [17] and the present paper with respect to the condition insuring that the value function $u(x)$ as defined in (8) does not become degenerate: in (9) above we have required that

$$u(x) < U(\infty) \quad \text{for some } x \in \mathbb{R}, \tag{133}$$

while in [17] we have used the assumption

$$u(x) < \infty \quad \text{for some } x > 0. \tag{134}$$

Of course, it is natural that in the context of the present paper we allow x to vary in $\mathbb{R}$, while, in the context of [17], x varies in $\mathbb{R}_+$. But why do we have to impose in (133) the (stronger) requirement $u(x) < U(\infty)$ instead of $u(x) < \infty$? Clearly these two requirements coincide if $U(\infty) = \infty$. But what happens if $U(\infty) < \infty$? The point is that for a utility function $U : \mathbb{R}_+ \to \mathbb{R}$ satisfying $U'(0) = \infty$, as considered in [17] and such that $U(\infty) < \infty$ we may deduce already from our standing assumption 2, that is, $\mathscr{M}^e(S) \neq \emptyset$, that we automatically have $u(x) < U(\infty)$, for all $x > 0$. Indeed, the equality $u(x) = U(\infty)$, for some $x > 0$, implies that there is a sequence $(H^{(n)})_{n=1}^{\infty}$ of admissible integrands such that the processes $X^{(n)} = x + H^{(n)} \cdot S$ are nonnegative and such that $(X_T^{(n)})_{n=1}^{\infty}$ tends to $+\infty$ almost surely. This implies that the sequence of trading strategies $(H^{(n)})_{n=1}^{\infty}$ defines a "free lunch with bounded risk" as $H^{(n)} \cdot S \geq -x$, which is in contradiction to the assumption $\mathscr{M}^e(S) \neq \emptyset$ (see [7]).

Summing up: we have that for a utility function $U : \mathbb{R}_+ \to \mathbb{R}$ satisfying $U'(0) = \infty$ as in [17] under the assumption $\mathscr{M}^e(S) \neq \emptyset$ the conditions $u(x) < \infty$ and $u(x) < U(\infty)$, for some $x > 0$, are equivalent. Indeed, the case $U(\infty) = \infty$ this is true for trivial reasons, while in the case $U(\infty) < \infty$ both conditions are automatically satisfied.

In the context of the present paper the situation is not so pleasant any more. In the subsequent Lemma 3.8 we show that, for any utility function $U : \mathbb{R} \to \mathbb{R}$ satisfying (1) and such that $U(\infty) < \infty$ we may construct a financial market S satisfying $\mathscr{M}^e(S) \neq \emptyset$ but such that $u(x) = U(\infty)$, for all $x \in \mathbb{R}$. To exclude these cases we had to use assumption (133) in the present paper.

We thank C. Summer for helpful comments on the theme of this remark.

LEMMA 3.8. *Let $U : \mathbb{R} \to \mathbb{R}$ be a utility function satisfying* (1) *and such that $U(\infty) < \infty$. Then there is a complete continuous financial market $(S_t)_{0 \leq t \leq T}$ such that*:

$$u(x) = U(\infty) \quad \textit{for all } x \in \mathbb{R}. \tag{135}$$

PROOF. Fix a sequence $(p_n)_{n=1}^{\infty}$ of strictly positive numbers, $\sum_{n=1}^{\infty} p_n = 1$, such that

$$\lim_{n \to \infty} p_n U(-n2^n) = 0. \tag{136}$$

Letting $q_n = 2^{-n}$, define $x_n = \frac{q_n}{p_n}$. Now repeat the construction of proposition 3.3 to find a complete, continuous financial market $(S_t)_{0 \leq t \leq T}$ such that the Radon-Nykodym derivative Z_T of the unique equivalent martingale measure for the process S has the form

$$Z_T = \sum_{n=1}^{\infty} x_n \chi_{D_n} \tag{137}$$

where $(D_n)_{n=1}^{\infty}$ is a partition of Ω into $\mathscr{F}_T$-measurable sets, satisfying $P[D_n] = p_n$.

Now we consider the Arrow-Debreu-type security $A^{(n)}$, whose payoff at time T is defined as

$$A_T^{(n)} = -n2^n \chi_{D_n}, \tag{138}$$

that is, which pays to the holder $-n2^n$ at time $t = T$ (or, phrased the other way round, obliges the holder to pay $n2^n$) if the true state of the world ω lies in D_n and zero otherwise.

As our market is complete, the price of this security at time $t = 0$ equals

$$A_0^{(n)} = E\left[Z_T A_T^{(n)}\right] = p_n x_n \cdot (-n2^n) = -n. \tag{139}$$

On the other hand the expected utility of the security $A^{(n)}$ at time $t = T$, which equals $E[U(A_T^{(n)})] = p_n U(-n2^n)$, tends to zero as $n \to \infty$. Speaking informally: the Arrow–Debreu-type security $A^{(n)}$ is a very good deal for an agent whose utility is defined by U: she receives the amount n at time $t = 0$ while the possible loss of $n2^n$ at time $t = T$, if the true state of the world happens to lie in D_n, has little effect on the expected utility, as we have chosen p_n to be very small.

Now fix $C > 0$, and $m \in \mathbb{N}$ and define, for $n > m$, the security $X^{(n)}$ by

$$X_T^{(n)} = \sum_{j=1}^{m} C\chi_{D_j} - n2^n \chi_{D_n}. \tag{140}$$

Its price at time $t = 0$ is given by

$$X_0^{(n)} = C \sum_{j=1}^{m} q_j - n, \tag{141}$$

so that for fixed initial endowment $x \in \mathbb{R}$, we have $X_0^{(n)} < x$, for n sufficiently large, which means that $X_T^{(n)} \in \mathscr{C}_U^b(x)$. On the other hand, the expected utility of $X_T^{(n)}$ equals

$$E\left[U\left(X_T^{(n)}\right)\right] = U(C) \sum_{j=1}^{m} p_j - o(n) \tag{142}$$

which is arbitrarily close to $U(\infty)$, if we chose $C > 0$ and $m \in \mathbb{N}$ sufficiently large. Hence

$$u(x) = \sup_{X_T \in \mathscr{C}_U^b(x)} E[U(X_T)] = U(\infty). \tag{143}$$

□

4. The asymptotic elasticity at $-\infty$. In this section we give a characterization of the property that the asymptotic elasticity $AE_{-\infty}(U)$ at minus infinity of a utility function $U : \mathbb{R} \to \mathbb{R}$ is equal to 1. The result is entirely parallel to the characterization of the property that the asymptotic elasticity $AE_{+\infty}(U)$ at plus infinity is equal to 1 which were obtained in Section 6 of [17].

PROPOSITION 4.1. *Let $U : \mathbb{R} \to \mathbb{R}$ be a utility function satisfying* (1). *The following assertions are equivalent*:

(i) $AE_{-\infty}(U) > 1$.
(ii) *There is $x_0 < 0$, $\lambda > 1$ and $c > 1$ such that*

$$U(\lambda x) < c\lambda U(x) \quad \text{for } x < x_0. \tag{144}$$

(iii) *There is $y_0 > 0$, $\lambda > 1$ and $C < \infty$ such that*

$$V(\lambda y) < CV(y) \quad \text{for } y > y_0. \tag{145}$$

(iv) *There is $y_0 > 0$ and $C > 0$ such that*

$$V'(y) < C\frac{V(y)}{y} \quad \text{for } y > y_0. \tag{146}$$

PROOF. (i) $\Rightarrow$ (ii): Assuming $AE_{-\infty}(U) > 1$ we may find $\alpha > 0$ and $x_0 < 0$ such that $\frac{xU'(x)}{U(x)} > 1 + \alpha$, for $x < x_0$. Given $\lambda > 1$ we may estimate, for $x < x_0$:

$$\begin{aligned} U(\lambda x) &\leq U(x) + U'(x)(\lambda - 1)x \\ &\leq U(x) + U(x)(\lambda - 1)(1 + \alpha) \\ &\leq U(x)[1 + (\lambda - 1)(1 + \alpha)]. \end{aligned} \tag{147}$$

Noting that the term in the last bracket is strictly bigger than λ we have proved (ii).

(ii) $\Rightarrow$ (iii): Assuming that (ii) holds true for $x_0 < 0$, $\lambda > 1$ and $c > 1$ let $y_0 = U'(\lambda x_0)$ and estimate, for $y > y_0$ and $\mu = c > 1$:

$$\begin{aligned} V(y) &= \sup_{x<x_0}[U(x) - xy] \\ &> \sup_{x<x_0}\left[\frac{1}{c\lambda}U(\lambda x) - xy\right] \\ &= \sup_{x<x_0}\frac{1}{c\lambda}[U(\lambda x) - (\lambda x)(cy)] \\ &= \frac{1}{c\lambda}V(cy), \end{aligned} \tag{148}$$

which proves (iii).

(iii) $\Rightarrow$ (iv): Assuming that (iii) holds true for some $y_0 > 0$, $\lambda > 1$ and $C < \infty$ let $y > y_0$ and estimate:

$$\begin{aligned} V'(y) &\leq \frac{V(\lambda y) - V(y)}{\lambda y - y} \\ &\leq \frac{(C-1)V(y)}{(\lambda - 1)y} = C' \frac{V(y)}{y}, \end{aligned} \tag{149}$$

where $C' = \frac{(C-1)}{(\lambda-1)}$.

(iv) $\Rightarrow$ (i): Assuming that (iv) holds true for some $y_0 > 0$ and $C > 0$ let $x_0 = -V'(y_0)$ and estimate, for $x < x_0$:

$$\begin{aligned} U(x) &= \inf_{y > y_0} [V(y) + xy] \\ &= V\left(U'(x)\right) + xU'(x) \\ &> C^{-1} V'\left(U'(x)\right) U'(x) + xU'(x) \\ &> xU'(x)\left(1 - C^{-1}\right) \end{aligned} \tag{150}$$

showing that $AE_{-\infty}(U) \geq (1 - C^{-1})^{-1}$. □

The subsequent consequence of Proposition 4.1 was used several times in this paper:

COROLLARY 4.2. *If* $U : \mathbb{R} \to \mathbb{R}$ *is a utility function satisfying* (1), $U(0) > 0$, *having reasonable asymptotic elasticity*, *and* $[\lambda_0, \lambda_1]$ *is a compact interval contained in* $]0, \infty[$, *we may find constants* $C > 0$ *and* $K > 0$ *s.t.*:

(i) $V(\lambda y) \leq CV(y)$, *for* $y > 0$ *and* $\lambda_0 \leq \lambda \leq \lambda_1$.
(ii) $y|V'(y)| \leq CV(y)$, *for* $y > 0$.
(iii) *For* $\varepsilon > 0$ *we may find* $\delta > 0$ *s.t. for all* $(1 - \delta) < \lambda < (1 + \delta)$ *we have*

$$(1 - \varepsilon)V(y) < V(\lambda y) < (1 + \varepsilon)V(y) \quad \textit{for } y > 0. \tag{151}$$

PROOF. (i) It follows from Proposition 4.1(iii) above and Lemma 6.3(iii) in [17] that for a given interval $[\lambda_0, \lambda_1]$ we may find a constant C and $0 < y_0 < y_1 < \infty$ such that

$$V(\lambda y) \leq CV(y) \quad \text{for } 0 < y < y_0 \text{ and } y_1 < y < \infty. \tag{152}$$

For the y lying in the interval $[y_0, y_1]$ first note that the assumption $U(0) > 0$ implies that $V(y) > 0$, for all $y > 0$. Hence by a compactness argument we have

$$\lim_{C \to \infty} \inf_{y_0 \leq y \leq y_1, \lambda_0 \leq \lambda \leq \lambda_1} CV(y) - V(\lambda y) = \infty, \tag{153}$$

which implies that, for $C > 0$ sufficiently large, assertion (i) holds true.

(ii) The proof of inequality (ii) is analogous now applying Proposition 4.1(iv) above and Lemma 6.3(iv) of [17].

(iii) For the proof of (iii) observe that it suffices to prove one of the inequalities in (151), say

$$V(\lambda y) < (1+\varepsilon)V(y) \quad \text{for } y > 0. \tag{154}$$

Denoting by $\overline{y}$ the argmin of V, that is, where $V'(y_{\min}) = 0$, note that the above inequality is trivial for $\lambda < 1$ and $y > y_{\min}/(1-\delta)$ as well as for $\lambda > 1$ and $y < y_{\min}(1+\delta)$. The non-trivial cases are:

A. when $\lambda < 1$ and y is close to zero (say $0 < y < y_0$ for some $y_0 > 0$) and

B. when $\lambda > 1$ and y is close to infinity (say $y_1 < y$ for some $y_1 > 0$).

In Case A the validity of (154) follows from [17], Lemma 6.3(iii) and Case B follows from a refinement of proposition 4.1(iii) above, which is completely analogous to the situation of [17], Lemma 6.3(iii) and left to the reader.

Finally the extension to the case $\lambda < 1$ and $y_0 \le y \le y_{\min}/(1-\delta)$ as well as $\lambda > 1$ and $y_{\min}/(1+\delta) \le y \le y_1$ is obtained from the assumption $\inf_{y>0} V(y) = V(y_{\min}) > 0$ and a compactness argument similarly as in the proof of (i) above. □

REFERENCES

[1] ANSEL, J. P. and STRICKER, C.(1994). Couverture des actifs contingents et prix maximum. *Ann. Inst. H. Poincaré Statist. Probab.* **30** 303–315.

[2] BELLINI, F. and FRITELLI, M. (1998). On the existence of minimax martingale measures. Preprint, Univ. degli Studi di Milano-Bicocca.

[3] BISMUT, J. M. (1973). Conjugate convex functions in optimal stochastic control. *J. Math. Anal. Appl.* **44** 384–404.

[4] CSISZAR, I. (1975). 1-Divergence geometry of probability distributions and minimization problems. *Ann. Probab.* **3** 146–158.

[5] CVITANIC, J. and SCHACHERMAYER, W. and WANG, H. (1999). Utility maximization in incomplete markets with random endowment. *Finance and Stochastics.* To appear.

[6] DAVIS, M. (1997). Option pricing in incomplete markets. In *Mathematics of Derivative Securities* (M. A. H. Dempster and S. R. Pliska, eds.) 216–226. Cambridge Univ. Press.

[7] DELBAEN, F. and SCHACHERMAYER, W. (1994). A general version of the fundamental theorem of asset pricing. *Math. Ann.* **300** 463–520.

[8] DELBAEN, F. and SCHACHERMAYER, W. (1997). The Banach space of workable contingent claims in arbitrage theory. *Ann. Inst. H. Poincare Statist. Probab.* **33** 113–144.

[9] DELBAEN, F. and SCHACHERMAYER, W. (1998). The fundamental theorem of asset pricing for unbounded stochastic processes. *Math. Ann.* **312** 215–250.

[10] DELBAEN, F. and SCHACHERMAYER, W. (1999). A compactness principle for bounded sequences of martingales with applications. In *Proceedings of the Seminar on Stochastic Analysis. Random Fields and Applications* 137–173. Birkhäuser, Basel.

[11] EMERY, M. (1980). Compensation de processus á variation finie non localement intégrables. *Séminaire de Probabilité XIV. Lecture Notes in Math.* **784** 152–160. Springer, Berlin.

[12] FOLDES, L. P. (1990). Conditions for optimality in the infinite-horizon portfolio-cum-savings problem with semimartingale investments. *Stochastics Stochastics Rep.* **29** 133–171.

[13] HARRISON, J. M. and KREPS, D. M. (1979). Martingales and arbitrage in multiperiod securities markets. *J. Econom. Theory* **20** 381–408.

[14] HARRISON, J. M. and PLISKA, S. R. (1981). Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Processes Appl.* **11** 215–260.

[15] JACOD, J. (1979). *Calcul stochastique et problèmes de martingales. Lecture Notes in Math.* **714.** Springer, Berlin.
[16] KARATZAS, I., LEHOCZKY, J. P., SETHI, S. P. and XU, G. L. (1991). Martingale and duality methods for utility maximisation in an incomplete market. *J. Control Optim.* **29** 702–730.
[17] KRAMKOV, D. and SCHACHERMAYER, W. (1999). A condition on the asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Ann. Appl. Probab.* **9** 904–950.
[18] PLISKA, S. (1986). A stochastic calculus model of continuous trading: optimal portfolios. *Math. Oper. Res.* **11** 371–382.
[19] Pliska, S., *Introduction to Mathematical Finance.* Blackwell, Oxford.
[20] ROCKAFELLAR, R. T. (1970). *Convex Analysis.* Princeton Univ. Press.
[21] YOR, M. (1978). Sous-espaces denses dans L^1 ou H^1. *Séminaire de Probabilité XII. Lecture Notes in Math.* **649** 265–309. Springer, Berlin.

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