Electronic Journal of Probability

Electron. J. Probab. **20** (2015), no. 98, 1–20.
ISSN: 1083-6489 DOI: 10.1214/EJP.v20-3866

The fragmentation process of an infinite recursive tree and Ornstein-Uhlenbeck type processes*

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Abstract

We consider a natural destruction process of an infinite recursive tree by removing each edge after an independent exponential time. The destruction up to time t is encoded by a partition $\Pi(t)$ of $\mathbb{N}$ into blocks of connected vertices. Despite the lack of exchangeability, just like for an exchangeable fragmentation process, the process Π is Markovian with transitions determined by a splitting rates measure $\mathbf{r}$. However, somewhat surprisingly, $\mathbf{r}$ fails to fulfill the usual integrability condition for the dislocation measure of exchangeable fragmentations. We further observe that a time-dependent normalization enables us to define the weights of the blocks of $\Pi(t)$. We study the process of these weights and point at connections with Ornstein-Uhlenbeck type processes.

Keywords: Random recursive tree; destruction of graphs; fragmentation process; cluster sizes; Ornstein-Uhlenbeck type process.
AMS MSC 2010: 60J25; 60J80; 60K35.
Submitted to EJP on October 15, 2014, final version accepted on September 15, 2015.
Supersedes arXiv:1501.01400v1.
Supersedes HAL:hal-01071463v1.

1 Introduction

The purpose of this work is to investigate various aspects of a simple and natural fragmentation process on an infinite tree, which turns out to exhibit nonetheless some rather unexpected features.

Specifically, we first construct a tree $\mathbb{T}$ with set of vertices $\mathbb{N} = \{1, \ldots\}$ by incorporating vertices one after the other and uniformly at random: That is, 1 is the root, and for each vertex $i \geq 2$, we pick its parent u_i according to the uniform distribution in $\{1, \ldots, i-1\}$, independently of the other vertices. We call $\mathbb{T}$ an infinite (random) recursive tree. Recursive trees are especially useful in computer science where they arise as data structures; see, for example, the survey by Mahmoud and Smythe [16] for background.

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We next destroy $\mathbb{T}$ progressively by removing each edge e_i connecting i to its parent u_i at time ϵ_i, where the sequence $(\epsilon_i : i \geq 2)$ consists of i.i.d. standard exponential variables, which are further independent of $\mathbb{T}$. Panholzer [19] investigated costs related to this destruction process, whereas in a different direction, Goldschmidt and Martin [11] used it to provide a remarkable construction of the Bolthausen-Sznitman coalescent. We also refer to Kuba and Panholzer [14] for the study of a related algorithm for isolation of nodes, and to our survey [4] for further applications and many more references.

Roughly speaking, we are interested here in the fragmentation process that results from the destruction. We represent the destruction of $\mathbb{T}$ up to time t by a partition $\Pi(t)$ of $\mathbb{N}$ into blocks of connected vertices. In other words, if we view the fragmentation of $\mathbb{T}$ up to time t as a Bernoulli bond-percolation with parameter e^{-t}, then the blocks of $\Pi(t)$ are the percolation clusters. Clearly $\Pi(t)$ gets finer as t increases, and it is easily seen from the so-called fundamental splitting property of random recursive trees that the process $\Pi = (\Pi(t) : t \geq 0)$ is Markovian (see the explanations given in the proof of Proposition 2.3). In this context, we recall that Aldous and Pitman [1] have considered a similar logging of the Continuum Random Tree (CRT) that yields a notable fragmentation process, dual to the standard additive coalescent. More precisely, they split the skeleton of the CRT into subtrees according to a Poisson process of cuts with some intensity $t \geq 0$ per unit length. By considering the ranked masses of the tree components, Aldous and Pitman obtain a fragmentation process (as a process in t). Moreover, the time change $t \mapsto \mathrm{e}^{-t}$ turns the latter into the standard additive coalescent. In this regard, we also point at our remark below the proof of Theorem 2.2. We further mention the very recent work of Kalay and Ben-Naim [12], in which the effects of repeated random removal of nodes (instead of edges) in a finite random recursive tree are analyzed.

It turns out that Π shares many features with homogeneous fragmentation processes as defined in [5, 6]. In particular, the transition kernels of Π are very similar to those of a homogeneous fragmentation; they are entirely determined by the splitting rates $\mathbf{r}$, which define an infinite measure on the space of partitions of $\mathbb{N}$. However, there are also major differences: exchangeability, which is a key requirement for homogeneous fragmentation processes, fails for Π, and perhaps more notably, the splitting rates measure $\mathbf{r}$ does not fulfill the fundamental integral condition (2.4) which the splitting rates of homogeneous fragmentation processes have to satisfy.

It is known from the work of Kingman [13] that exchangeability plays a fundamental role in the study of random partitions, and more precisely, it lies at the heart of the connection between exchangeable random partitions (which are discrete random variables), and random mass-partitions (which are continuous random variables). In particular, the distribution of an exchangeable random partition is determined by the law of the asymptotic frequencies of its blocks B,

$$|B| = \lim_{n\to\infty} n^{-1} \#\{i \leq n : i \in B\}. \tag{1.1}$$

Even though $\Pi(t)$ is not exchangeable for $t > 0$, it is elementary to see that every block of $\Pi(t)$, say $B(t)$, has an asymptotic frequency. However, this asymptotic frequency is degenerate, $|B(t)| = 0$ (note that if Π were exchangeable, this would imply that all the blocks of $\Pi(t)$ would be singletons). We shall obtain a finer result and show that the limit

$$\lim_{n\to\infty} n^{-\mathrm{e}^{-t}} \#\{i \leq n : i \in B(t)\} \tag{1.2}$$

exists in $(0, \infty)$ almost surely. We will refer to the latter as the weight of the block $B(t)$ (we stress that this definition depends on the time t at which the block is taken), and another natural question about the destruction of $\mathbb{T}$ is thus to describe the process $\mathbf{X}$ of the weights of the blocks of the partition-valued process Π.

In this regard, let us point to related results of Möhle [18], which appeared almost simultaneously and independently of our work. With $\Pi_1(t)$ denoting the block of $\Pi(t)$ containing 1, he interprets the quantity $\#\{j \leq n : j \in \Pi_1(t)\}$ as the number of blocks $N_t^{(n)}$ at time t in a Bolthausen-Sznitman coalescent on $[n]$ (see the second proof of Theorem 3.1(ii) below). Möhle shows that $(n^{-\mathrm{e}^{-t}} N_t^{(n)} : t \geq 0)$ converges in distribution in the Skorohod topology to a limiting process which he terms Mittag-Leffler process. In particular, his results imply that with probability one, $\Pi_1(t)$ has weights simultaneously for all $t \geq 0$, and that these weights are càdlàg in t. For homogeneous fragmentations, a general result is known: By [6, Proposition 3.6], each block of a standard homogeneous fragmentation process possesses asymptotic frequencies simultaneously for all $t \geq 0$ a.s., and if B denotes such a block, then the process $t \mapsto |B(t)|$ is càdlàg.

Because Π resembles homogeneous fragmentations, but with splitting rates measure $\mathbf{r}$ which does not fulfill the integral condition of the former, and because the notion (1.2) of the weight of a block depends on the time t, one might expect that $\mathbf{X}$ should be an example of a so-called compensated fragmentation which was recently introduced in [7]. Although this is not exactly the case, we shall see that $\mathbf{X}$ fulfills closely related properties. Using well-known connections between random recursive trees, Yule processes, and Pólya urns, cf. [4], we shall derive a number of explicit results about its distribution. In particular, we shall show that upon a logarithmic transform, $\mathbf{X}$ can be viewed as a branching Ornstein-Uhlenbeck process. At this point, let us mention that for a homogeneous fragmentation, the process of the logarithms of the asymptotic frequencies forms a branching random walk when evaluated at integer times, say. More on this can be found in the book [6].

The rest of this paper is organized as follows. In Section 2, we study the structure of the partition-valued process Π which stems from the destruction of $\mathbb{T}$, stressing the resemblances and the differences with exchangeable fragmentations. In Section 3, we observe that after a suitable renormalization that depends on t, the blocks of the partition $\Pi(t)$ possess a weight, and we relate the process of these weights to Ornstein-Uhlenbeck type processes.

2 Destruction of $\mathbb{T}$ and fragmentation of partitions

The purpose of this section is to show that, despite the lack of exchangeability, the partition valued process Π induced by the fragmentation of $\mathbb{T}$ can be analyzed much in the same way as a homogeneous fragmentation. We shall present the main features and merely sketch proofs, referring to Section 3.1 in [6] for details.

We start by recalling that a partition π of $\mathbb{N}$ is a sequence $(\pi_i : i \in \mathbb{N})$ of pairwise disjoint blocks, indexed in the increasing order of their smallest elements, and such that $\cup_{i\in\mathbb{N}}\pi_i = \mathbb{N}$. We write $\mathcal{P}$ for the space of partitions of $\mathbb{N}$, which is a compact hypermetric space when endowed with the distance

$$\mathrm{d}(\pi, \pi') = 1/\max\{n \in \mathbb{N} : \pi_{|[n]} = \pi'_{|[n]}\},$$

where $\pi_{|B}$ denotes the restriction of π to a subset $B \subseteq \mathbb{N}$ and $[n] = \{1, \ldots, n\}$ is the set of the n first integers. The space $\mathcal{P}_B$ of partitions of B is defined similarly.

We next introduce some spaces of functions on $\mathcal{P}$. First, for every $n \geq 1$, we write D_n for the space of functions $f : \mathcal{P} \to \mathbb{R}$ which remain constant on balls with radius $1/n$, that is such that $f(\pi) = f(\eta)$ whenever the restrictions $\pi_{|[n]}$ and $\eta_{|[n]}$ of the partitions π and η to $[n]$ coincide. Plainly, $D_n \subset D_{n+1}$, and we set

$$D_\infty = \bigcup_{n\geq 1} D_n.$$

Observe that D_∞ is a dense subset of the space $\mathcal{C}(\mathcal{P})$ of continuous functions on $\mathcal{P}$.

In order to describe a family of transition kernels which appear naturally in this study, we first need some notation. For every block $B \subseteq \mathbb{N}$, write $B(j)$ for the j-th smallest element of B (whenever it makes sense), and then, for every partition $\pi \in \mathcal{P}$, $B \circ \pi$ for the partition of B generated by the blocks $B(\pi_i) = \{B(j) : j \in \pi_i\}$ for $i \in \mathbb{N}$. In other words, $B \circ \pi$ is simply the partition of B induced by π when one enumerates the elements of B in their natural order. Of course, if the cardinality of B is finite, say equal to $k \in \mathbb{N}$, then $B \circ \pi$ does only depend on π through $\pi_{|[k]}$, so that we may consider $B \circ \pi$ also for $\pi \in \mathcal{P}_{[k]}$ (or $\pi \in \mathcal{P}_{[\ell]}$ for any $\ell \geq k$).

In the same vein, for partitions $\eta \in \mathcal{P}_B$ and every integer $i \geq 1$, we write $\eta \underset{i}{\circ} \pi$ for the partition of B that results from fragmenting the i-th block of η by π, that is replacing the block η_i in η by $\eta_i \circ \pi$. Again, if $k = \#B < \infty$, we may also take $\pi \in \mathcal{P}_{[\ell]}$ for $\ell \geq k$.

Finally, for every $k \geq 2$, we consider a random partition of $\mathbb{N}$ that arises from the following Pólya urn. At the initial time, the urn contains $k-1$ black balls labeled $1, \ldots, k-1$ and a single red ball labeled k. Balls with labels $k+1, k+2, \ldots$ are colored black or red and then incorporated to the urn one after the other according to the following rule: For $n \geq k$, the color given to the $n+1$-th ball is that of a ball picked uniformly at random when the urn contains n balls. This yields a random binary partition of $\mathbb{N}$ into black and red balls; we write $\mathbf{p}_k$ for its law. We set

$$\mathbf{r} = \sum_{k=2}^{\infty} \mathbf{p}_k, \tag{2.1}$$

which is thus an infinite measure on the set of binary partitions of $\mathbb{N}$.

Remark 2.1. Comparing the construction of an infinite random recursive tree $\mathbb{T}$ with the dynamics of the above Pólya urn starting from $k-1$ black balls and one red ball, it should be clear that the set of (labels of) red balls can be identified with the set of vertices of the subtree of $\mathbb{T}$ that stems from the vertex k. This connection will allow us to express the jump rates of the restrictions $\Pi_{|[n]}$ in terms of $\mathbf{r}$, see Proposition 2.4 below.

Recall that each edge of $\mathbb{T}$ is deleted at an exponentially distributed random time, independently of the other edges. This induces, for every $t \geq 0$, a random partition $\Pi(t)$ of $\mathbb{N}$ into blocks corresponding to the subsets of vertices which are still connected at time t. Observe that, by construction and the very definition of the distance on $\mathcal{P}$, the process Π has càdlàg paths.

We are now able to state the main result of this section.

Theorem 2.2. (i) *The process* $\Pi = (\Pi(t) : t \geq 0)$ *is Markovian and has the Feller property. We write* G *for its infinitesimal generator.*
(ii) *For every* $n \geq 1$*,* D_n *is invariant and therefore* D_∞ *is a core for* G*.*
(iii) *For every* $f \in D_\infty$ *and* $\eta \in \mathcal{P}$*, we have*

$$\mathrm{G}f(\eta) = \int_{\pi \in \mathcal{P}} \mathbf{r}(\mathrm{d}\pi) \sum_i \left(f(\eta \underset{i}{\circ} \pi) - f(\eta) \right).$$

We stress that this characterization of the law of the process Π is very close to that of a homogeneous fragmentation. Indeed, one can rephrase well-known results (cf. Section 3.1.2 in [6]) on the latter as follows. Every homogeneous fragmentation process $\Gamma = (\Gamma_t : t \geq 0)$ is a Feller process on $\mathcal{P}$, such that the sub-spaces D_n are invariant (and hence D_∞ is a core). Further, its infinitesimal generator A is given in the form

$$\mathrm{A}f(\eta) = \int_{\pi \in \mathcal{P}} \mathbf{s}(\mathrm{d}\pi) \sum_i \left(f(\eta \underset{i}{\circ} \pi) - f(\eta) \right)$$

for every $f \in D_\infty$ and $\eta \in \mathcal{P}$, where $\mathbf{s}$ is some *exchangeable* measure on $\mathcal{P}$, that is $\mathbf{s}$ is invariant under permuting finitely many integers. Moreover, $\mathbf{s}(\{\mathbf{1}_{\mathbb{N}}\}) = 0$, where for every block $B \subseteq \mathbb{N}$, $\mathbf{1}_B \in \mathcal{P}_B$ denotes the neutral partition which has a single non-empty block B, and

$$\mathbf{s}\left(\left\{\pi \in \mathcal{P} : \pi_{|[n]} \neq \mathbf{1}_{[n]}\right\}\right) < \infty \qquad \text{for all } n \geq 2. \tag{2.2}$$

Observe that the measure $\mathbf{r}$ fails to be exchangeable, but it fulfills (2.2); indeed, one has

$$\mathbf{r}\left(\left\{\pi \in \mathcal{P} : \pi_{|[n]} \neq \mathbf{1}_{[n]}\right\}\right) = \sum_{k=2}^{n} \mathbf{p}_k(\mathcal{P}) = n - 1.$$

We shall now prepare the proof of Theorem 2.2. For this reason, it is convenient to introduce some further notation. Consider an arbitrary block $B \subseteq \mathbb{N}$, a partition $\eta \in \mathcal{P}_B$ and a sequence $\pi^{(\cdot)} = (\pi^{(i)} : i \in \mathbb{N})$ in $\mathcal{P}$. We write $\eta \circ \pi^{(\cdot)}$ for the partition of B whose family of blocks is given by those of $\eta \underset{i}{\circ} \pi^{(i)}$ for $i \in \mathbb{N}$. In words, for each $i \in \mathbb{N}$, the i-th block of η is split according to the partition $\pi^{(i)}$. Next, consider a probability measure $\mathbf{q}$ on $\mathcal{P}$ and a sequence $(\pi^{(i)} : i \in \mathbb{N})$ of i.i.d. random partitions with common law $\mathbf{q}$. We associate to $\mathbf{q}$ a probability kernel $\mathrm{Fr}(\cdot, \mathbf{q})$ on $\mathcal{P}_B$, by denoting the distribution of $\eta \circ \pi^{(\cdot)}$ by $\mathrm{Fr}(\eta, \mathbf{q})$ for every $\eta \in \mathcal{P}_B$. We point out that if $\mathbf{q}$ is exchangeable, then $\eta \circ \pi^{(\cdot)}$ has the same distribution as the random partition whose blocks are given by the restrictions $\pi^{(i)}_{|\eta_i}$ of $\pi^{(i)}$ to η_i for $i \in \mathbb{N}$, and $\mathrm{Fr}(\cdot, \mathbf{q})$ thus coincides with the fragmentation kernel that occurs for homogeneous fragmentations, see Definition 3.2 on page 119 in [6]. Of course, the assumption of exchangeability is crucial for this identification to hold.

Note that the restriction of partitions of $\mathbb{N}$ to $[n]$ is compatible with the fragmentation operator $\mathrm{Fr}(\cdot, \cdot)$, in the sense that

$$\left(\eta \circ \pi^{(\cdot)}\right)_{|[n]} = \eta_{|[n]} \circ \pi^{(\cdot)}{}_{|[n]}. \tag{2.3}$$

Proposition 2.3. *For every $n \in \mathbb{N}$, the process $\Pi_{|[n]} = (\Pi_{|[n]}(t) : t \geq 0)$ obtained by restricting Π to $[n]$, is a continuous time Markov chain on $\mathcal{P}_{[n]}$.*

Its semigroup can be described as follows: for every $s, t \geq 0$, the conditional distribution of $\Pi_{|[n]}(s + t)$ given $\Pi_{|[n]}(s) = \eta$ is $\mathrm{Fr}(\eta, \mathbf{q}_t)$, where $\mathbf{q}_t$ denotes the distribution of $\Pi(t)$.

Proof. The case $n = 1$ is clear since $\Pi_{|[1]}(t) = (\{1\}, \emptyset, \dots)$ for all times $t \geq 0$. Assume now $n \geq 2$. The proof relies crucially on the splitting property of random recursive trees that we now recall (see, e.g., Section 2.2 of [4]). Given a subset $B \subseteq \mathbb{N}$, the image of $\mathbb{T}$ by the map $j \mapsto B(j)$ which enumerates the elements of B in the increasing order, is called a random recursive tree on B and denoted by $\mathbb{T}_B$. In particular, for $B = [n]$, the restriction of $\mathbb{T}$ to the first n vertices is a random recursive tree on $[n]$. Imagine now that we remove k fixed edges (i.e. edges with given indices, say $i_1, \dots, i_k$, where $2 \leq i_1 < \dots < i_k \leq n$) from $\mathbb{T}_{[n]}$. Then, conditionally on the induced partition of $[n]$, say $\eta = (\eta_1, \dots, \eta_{k+1})$, the resulting $k + 1$ subtrees are independent random recursive trees on their respective sets of vertices η_j, $j = 1, \dots, k + 1$.

It follows easily from the lack of memory of the exponential distribution and the compatibility property (2.3) that the restricted process $\Pi_{|[n]} = \left(\Pi_{|[n]}(t) : t \geq 0\right)$ is a continuous time Markov chain. More precisely, we deduce from the argument given in the preceding paragraph that the conditional distribution of $\Pi_{|[n]}(s+t)$ given $\Pi_{|[n]}(s) = \pi$ is $\mathrm{Fr}(\pi, \mathbf{q}_t)$, where $\mathrm{Fr}(\cdot, \mathbf{q}_t)$ is here viewed as a probability kernel on $\mathcal{P}_{[n]}$. $\square$

In order to describe the infinitesimal generator of the restricted processes $\Pi_{|[n]}$ for $n \in \mathbb{N}$, we consider its rates of jumps, which are defined by

$$r_\pi = \lim_{t \to 0+} t^{-1} \mathbb{P}(\Pi_{|[n]}(t) = \pi),$$

where now π denotes a generic partition of $[n]$ which has at least two (non-empty) blocks. The rates of jumps r_π determine the infinitesimal generator G_n of the restricted chain $\Pi_{|[n]}$, specifically we have for $f : \mathcal{P}_{[n]} \to \mathbb{R}$ and $\eta \in \mathcal{P}_{[n]}$

$$\mathrm{G}_n f(\eta) = \sum_{\pi \in \mathcal{P}_{[n]}} \sum_i \Big(f(\eta \underset{i}{\circ} \pi) - f(\eta) \Big) r_\pi$$

(recall that $\eta \underset{i}{\circ} \pi$ denotes the partition that results from fragmenting the i-th block of η according to π). This determines the distribution of the restricted chain $\Pi_{|[n]}$, and hence, letting n vary in $\mathbb{N}$, also characterizes the law of Π. Recall also that the measure $\mathbf{r}$ on $\mathcal{P}$ has been defined by (2.1).

Proposition 2.4. *For every $n \geq 2$ and every partition π of $[n]$ with at least two (non-empty) blocks, there is the identity*

$$r_\pi = \mathbf{r}(\mathcal{P}_\pi),$$

where $\mathcal{P}_\pi = \{\eta \in \mathcal{P} : \eta_{|[n]} = \pi\}$.

Proof. This should be intuitively straightforward from the connection between the construction of random recursive trees and the dynamics of Pólya urns. Specifically, fix $n \geq 2$ and consider a partition $\pi \in \mathcal{P}_{[n]}$. If π consists in three or more non-empty blocks, then we clearly have

$$\lim_{t \to 0+} t^{-1} \mathbb{P}(\Pi_{|[n]}(t) = \pi) = 0,$$

since at least two edges have to be removed from $\mathbb{T}_{|[n]}$ in order to yield a partition with three or more blocks. Assume now that π is binary with non-empty blocks π_1 and π_2, and let $k = \min \pi_2$. Then only the removal of the edge e_k may possibly induce the partition π, and more precisely, if we write η for the random partition of $[n]$ resulting from the removal of e_k, then the probability that $\eta = \pi$ is precisely the probability that in a Pólya urn containing initially $k-1$ black balls labeled $1, \ldots, k-1$ and a single red ball labeled k, after $n-k$ steps, the red balls are exactly those with labels in π_2. Since the edge e_k is removed at unit rate, this gives

$$\lim_{t \to 0+} t^{-1} \mathbb{P}(\Pi_{|[n]}(t) = \pi) = r_\pi = \mathbf{p}_k(\mathcal{P}_\pi)$$

in the notation of the statement. Note that the right-hand side can be also written as $\mathbf{r}(\mathcal{P}_\pi)$, since $\mathbf{p}_\ell(\mathcal{P}_\pi) = 0$ for all $\ell \neq k$. □

Proposition 2.4 should be compared with Proposition 3.2 in [6]; we refer henceforth to $\mathbf{r}$ as the splitting rate of Π. We have now all the ingredients necessary to establish Theorem 2.2.

Proof of Theorem 2.2. From Proposition 2.3, we see that the transition semigroup of Π is given by $(\mathrm{Fr}(\cdot, \mathbf{q}_t) : t \geq 0)$, and it is easily checked that the latter fulfills the Feller property; cf. Proposition 3.1(i) in [6]. Point (ii) is immediate from the compatibility of restriction with the fragmentation operator, see (2.3). Concerning (iii), let $f \in D_n$. Since f is constant on $\{\eta \in \mathcal{P} : \eta_{|[n]} = \pi\}$, it can naturally be restricted to a function $f : \mathcal{P}_{[n]} \to \mathbb{R}$. By the compatibility property (2.3), with $r_{\pi'} = \mathbf{r}(\mathcal{P}_{\pi'})$ for a partition π' of $[n]$ defined as in Proposition 2.4, we obtain

$$\int_{\pi \in \mathcal{P}} \mathbf{r}(\mathrm{d}\pi) \sum_i \Big(f(\eta \underset{i}{\circ} \pi) - f(\eta) \Big) = \sum_{\pi' \in \mathcal{P}_{[n]}} \sum_i \Big(f(\eta_{|[n]} \underset{i}{\circ} \pi') - f(\eta_{|[n]}) \Big) r_{\pi'} = \mathrm{G}_n f(\eta_{|[n]}),$$

where G_n is the infinitesimal generator of the restricted chain $\Pi_{|[n]}$ found above. This readily yields (iii). □

Remark 2.5. It may be interesting to recall that the standard exponential law is invariant under the map $t \mapsto -\ln(1-\mathrm{e}^{-t})$, and thus, if we set $\hat{\epsilon}_i = -\ln(1-\exp(-\epsilon_i))$ (recall that ϵ_i is the instant at which the edge connecting the vertex i to its parent is removed), then $(\hat{\epsilon}_i)_{i\geq 2}$ is a sequence of i.i.d. exponential variables. The time-reversal $t \mapsto -\ln(1-\mathrm{e}^{-t})$ transforms the destruction process of $\mathbb{T}$ into a construction process of $\mathbb{T}$ defined as follows. At each time $\hat{\epsilon}_i$, we create an edge between $i \geq 2$ and its parent which is chosen uniformly at random in $\{1, \ldots, i-1\}$. It follows that the time-reversed process $\hat{\Pi}(t) = \Pi(-\ln(1-\mathrm{e}^{-t})-)$, $t \geq 0$, is a binary coalescent process such that the rate at which two blocks, say B and B' with $\min B < \min B'$, merge, is given by $\#\{j \in B : j < \min B'\}$. This can be viewed as a duality relation between fragmentation and coalescent processes; see Dong, Goldschmidt and Martin [10] and references therein.

The next proposition underlines the fact that $\mathbf{r}$ is *not* the splitting rates measure of a homogeneous fragmentation.

Proposition 2.6. *For $\mathbf{r}$-almost all binary partitions $(B_1, B_2, \emptyset, \ldots) \in \mathcal{P}$, the blocks B_1 and B_2 have asymptotic frequencies, and more precisely, we have*

$$\int_{\mathcal{P}} f(|B_1|, |B_2|)\mathrm{d}\mathbf{r} = \int_0^1 f(1-x, x)x^{-2}\mathrm{d}x,$$

where $f : [0,1]^2 \to \mathbb{R}_+$ denotes a generic measurable function. In particular,

$$\int_{\mathcal{P}} (1 - \max\{|B_1|, |B_2|\})\mathrm{d}\mathbf{r} = \infty.$$

Proof. Indeed, it is a well-known fact of Pólya urns that for each $k \geq 2$, $\mathbf{p}_k$-almost every partition (B_1, B_2) has asymptotic frequencies with $|B_1| + |B_2| = 1$, and $|B_2|$ has the beta distribution with parameters $(1, k-1)$, i.e. with density $(k-1)(1-x)^{k-2}$ on $(0,1)$ (see e.g. [15, Theorem 3.2]). Our claims follow immediately since

$$\sum_{k=2}^{\infty} (k-1)(1-x)^{k-2} = x^{-2}, \qquad x \in (0,1).$$

□

Recall that the splitting rates $\mathbf{s}$ of a homogeneous fragmentation must fulfill the integrability condition

$$\int_{\mathcal{P}} (1 - \max\{|B_1|, |B_2|\})\mathrm{d}\mathbf{s} < \infty, \tag{2.4}$$

which thus fails for $\mathbf{r}$!

We next turn our attention to the Poissonian structure of the process Π, which can be rephrased in terms similar to those in Section 3.1 of [6]. For this reason, we introduce a random point measure

$$M = \sum_{i=2}^{\infty} \delta_{(\epsilon_i, \Delta_i, k_i)}$$

on $\mathbb{R}_+ \times \mathcal{P} \times \mathbb{N}$ as follows. Recall that ϵ_i is the time at which the edge e_i connecting the vertex $i \in \mathbb{N}$ to its parent in $\mathbb{T}$ is removed. Immediately before time ϵ_i, the vertex i belongs to some block of the partition $\Pi(\epsilon_i-)$, we denote the label of this block by k_i (recall that Π is càdlàg and that blocks of a partition are labeled in the increasing order of their smallest element). Removing the edge e_i yields a partition of that block $B = \Pi_{k_i}(\epsilon_i-)$ into two sub-blocks, which can be expressed (uniquely) in the form $B \circ \Delta_i$. Note that the size of the block B is almost surely infinite, so that the binary partition Δ_i and hence the point measure M is indeed defined unambiguously. The process Π

can be recovered from M, in a way similar to that explained on pages 117-118 in [6]. Roughly speaking, for every atom of M, say (t, Δ, k), $\Pi(t)$ results from partitioning the k-th block of $\Pi(t-)$ using Δ, that is by replacing $\Pi_k(t-)$ by $\Pi_k(t-) \circ \Delta$. For M, we have the following characterization.

Proposition 2.7. *The random measure M is Poisson with intensity $\lambda \otimes \mathbf{r} \otimes \#$, where λ denotes Lebesgue measure on $\mathbb{R}_+$ and $\#$ the counting measure on $\mathbb{N}$.*

Proof. Recall that we write $\mathbf{1}_{[n]} = ([n], \emptyset, \dots)$ for the partition of $[n]$ which consists of a single non-empty block. Consider a Poisson random measure M' with intensity $\lambda \otimes \mathbf{r} \otimes \#$ as in the statement. Since the intensity measure is σ-finite and λ is diffuse, the superposition property of Poisson random measures implies that M has almost surely at most one atom in each fiber $\{t\} \times \mathcal{P} \times \mathbb{N}$. Furthermore, the discussion below (2.2) shows that for each $t' \geq 0$ and every $n \in \mathbb{N}$, the number of atoms (t, π, k) of M' with $t \leq t'$, $\pi_{|[n]} \neq \mathbf{1}_{[n]}$ and $k \leq n$ is finite. We may therefore define for fixed $n \in \mathbb{N}$ a $\mathcal{P}_{[n]}$-valued continuous time Markov chain $(\Pi'^{[n]}(t) : t \geq 0)$ starting from $\Pi'^{[n]}(0) = \mathbf{1}_{[n]}$ as follows: If t is a time at which the fiber $\{t\} \times \mathcal{P} \times \mathbb{N}$ carries an atom (t, π, k) of M' such that $\pi_{|[n]} \neq \mathbf{1}_{[n]}$ and $k \leq n$, then $\Pi'^{[n]}(t)$ results from $\Pi'^{[n]}(t-)$ by replacing its k-th block $\Pi_k'^{[n]}(t-)$ by $\Pi_k'^{[n]}(t-) \circ \pi_{|[n]}$.

The sequence $(\Pi'^{[n]}(t) : n \in \mathbb{N})$ is clearly compatible for every $t \geq 0$, in the sense that $\Pi'^{[n]}{}_{|[m]}(t) = \Pi'^{[m]}(t)$ for all integers $n \geq m$. We deduce as in the proof of Lemma 3.3 in [6] that there exists a unique $\mathcal{P}$-valued càdlàg function $(\Pi'(t) : t \geq 0)$ such that $\Pi'_{|[n]}(t) = \Pi'^{[n]}(t)$. Moreover, the i-th block $\Pi'_i(t)$ of $\Pi'(t)$ is given by the increasing union $\Pi'_i(t) = \cup_{n \in \mathbb{N}} \Pi_i'^{[n]}(t)$, and it follows from the very construction of $\Pi'^{[n]}(t)$ that the process Π' can be recovered from M' similarly to the description above the statement of the proposition. It remains to check that Π' and Π have the same law, which follows if we show that the restricted processes $\Pi'_{|[n]} = \Pi'^{[n]}$ and $\Pi_{|[n]}$ have the same law for each $n \in \mathbb{N}$. Fix $n \geq 2$, and denote by π a partition of $[n]$ with at least two non-empty blocks. From the Poissonian construction of $\Pi'^{[n]}$, with $\mathcal{P}_\pi$ as in the statement of Proposition 2.4, we first see that

$$\lim_{t \to 0+} t^{-1} \mathbb{P}\left(\Pi'^{[n]}(t) = \pi\right) = \mathbf{r}(\mathcal{P}_\pi).$$

Next, if $\pi' \neq \pi'' \in \mathcal{P}_{[n]}$, the jump rate of $\Pi'^{[n]}$ from π' to π'' is non-zero only if π'' can be obtained from π' by replacing one single block of π', say the k-th block π'_k, by $\pi'_k \circ \pi$, where π is some binary partition of $[n]$. This observation and the last display readily show that $\Pi'^{[n]}$ and $\Pi_{|[n]}$ have the same generator, and hence their laws agree. □

3 The process of the weights

Even though the splitting rates measure $\mathbf{r}$ of the fragmentation process Π fails to fulfill the integral condition (2.4), we shall see that we can nonetheless define the weights of its blocks. The purpose of this section is to investigate the process of the weights as time passes.

3.1 The weight of the first block as an O.U. type process

In this section, we focus on the first block $\Pi_1(t)$, that is the cluster at time t which contains the root 1 of $\mathbb{T}$. The next statement gathers its key properties, and in particular stresses the connection with an Ornstein-Uhlenbeck type process.

Theorem 3.1. (i) *For every $t \geq 0$, the following limit*

$$\lim_{n \to \infty} n^{-e^{-t}} \#\{j \leq n : j \in \Pi_1(t)\} = X_1(t)$$

exists in $(0,\infty)$ *a.s. The variable* $X_1(t)$ *has the Mittag-Leffler distribution with parameter* e^{-t}*,*

$$\mathbb{P}(X_1(t)\in \mathrm{d}x)/\mathrm{d}x = \frac{\mathrm{e}^t}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{k!}\Gamma(k\mathrm{e}^{-t}+1)x^{k-1}\sin(\pi k\mathrm{e}^{-t});$$

equivalently, its Mellin transform is given by

$$\mathbb{E}(X_1^q(t)) = \frac{\Gamma(q+1)}{\Gamma(\mathrm{e}^{-t}q+1)}, \qquad q\geq 0.$$

(ii) *The process* $(X_1(t):t\geq 0)$ *is Markovian and has the Feller property. Its semigroup* $P_t(x,\cdot)$ *is given by*

$$P_t(x,\cdot) = \mathbb{P}(x^{\mathrm{e}^{-t}}X_1(t)\in\cdot).$$

(iii) *The process* $Y(t)=\ln X_1(t)$*,* $t\geq 0$*, is of Ornstein-Uhlenbeck type. More precisely,*

$$L(t) = Y(t)+\int_0^t Y(s)\mathrm{d}s, \qquad t\geq 0,$$

is a spectrally negative Lévy process with cumulant-generating function

$$\kappa(q) = \ln\mathbb{E}(\exp(qL(1))), \qquad q\geq 0,$$

given by

$$\kappa(q) = q\psi(q+1),$$

where ψ *denotes the digamma function, that is the logarithmic derivative of the gamma function.*

Remark 3.2. We recall that Möhle obtained in [18] independently of our work and almost at the same time distributional convergence of the full process $(n^{-\mathrm{e}^{-t}}\#\{j\leq n: j\in\Pi_1(t)\}:t\geq 0)$ in the space of càdlàg paths equipped with the Skorohod topology.

In the following, we shall refer to $X_1(t)$ as the weight of the first block (or the root cluster) at time t. Before tackling the proof of Theorem 3.1, we make a couple of comments.

Firstly, observe from (i) that $\lim_{t\to\infty}\mathbb{E}(X_1(t)^q)=\Gamma(q+1)$, so that as $t\to\infty$, $Y(t)$ converges in distribution to the logarithm of a standard exponential variable. On the other hand, it is well known that the weak limit at ∞ of an Ornstein-Uhlenbeck type process is self-decomposable; cf. Section 17 in Sato [21]. So (iii) enables us to recover the fact that the log-exponential distribution is self-decomposable; see Shanbhag and Sreehari [22].

Secondly, note that the Lévy-Khintchin formula for κ reads

$$\kappa(q) = -\gamma q + \int_{-\infty}^{0}(\mathrm{e}^{qx}-1-qx)\frac{\mathrm{e}^x}{(1-\mathrm{e}^x)^2}\mathrm{d}x,$$

where $\gamma = 0.57721\ldots$ is the Euler-Mascheroni constant. Indeed, using a classical identity for the digamma function for the second equality and Tonelli's theorem in the third line, we have

$$\begin{aligned}
\kappa(q)+\gamma q = q\psi(q+1)+\gamma q = q\int_0^1 \frac{1-x^q}{1-x}\mathrm{d}x &= q\int_{-\infty}^0 \frac{1-\mathrm{e}^{qy}}{1-\mathrm{e}^y}\mathrm{e}^y\mathrm{d}y \\
&= q\int_{-\infty}^0 (1-\mathrm{e}^{qy})\left(\int_{-\infty}^y \frac{\mathrm{e}^x}{(1-\mathrm{e}^x)^2}\mathrm{d}x\right)\mathrm{d}y \\
&= q\int_{-\infty}^0 \left(\int_{-\infty}^0 (1-\mathrm{e}^{qy})\mathbb{1}_{\{y>x\}}\mathrm{d}y\right)\frac{\mathrm{e}^x}{(1-\mathrm{e}^x)^2}\mathrm{d}x \\
&= \int_{-\infty}^0 (\mathrm{e}^{qx}-1-qx)\frac{\mathrm{e}^x}{(1-\mathrm{e}^x)^2}\mathrm{d}x.
\end{aligned}$$

In turn, this enables us to identify the Lévy measure of L as

$$\Lambda(\mathrm{d}x) = \mathrm{e}^x(1-\mathrm{e}^x)^{-2}\mathrm{d}x, \qquad x\in(-\infty,0).$$

Since the jumps of L and of Y coincide, the Lévy-Itô decomposition entails that the jump process of $Y=\ln X_1$ is a Poisson point process with characteristic measure Λ. In this context, recall from Proposition 2.6 that the distribution of the asymptotic frequency of the first block under the measure $\mathbf{r}$ of the splitting rates of Π is $(1-y)^{-2}\mathrm{d}y$, $y\in(0,1)$, and observe that the image of the latter by the map $y\mapsto \ln y$ is precisely Λ. This should of course not come as a surprise.

We shall present two proofs of Theorem 3.1(i); the first relies on the well-known connection between random recursive trees and Yule processes and is based on arguments due to Pitman. Indeed, $\#\{j\le n : j\in\Pi_1(t)\}$ can be interpreted in terms of the two-type population system considered in Section 3.4 of [20], as the number of novel individuals at time t when the birth rate of novel offspring per novel individual is given by $\alpha=\mathrm{e}^{-t}$, and conditioned that there are n individuals in total in the population system at time t. Part (i) of the theorem then readily follows from Proposition 3.14 in connection with Corollary 3.15 and Theorem 3.8 in [20]. For the reader's convenience, let us nonetheless give a self-contained proof which is specialized to our situation. We further stress that variations of this argument will be used in the proofs of Proposition 3.7 and Corollary 3.9.

First proof of Theorem 3.1(i). Consider a population model started from a single ancestor, in which each individual gives birth to a new child at rate one (in continuous time). If the ancestor receives the label 1 and the next individuals are labeled $2,3,\dots$ according to the order of their birth times, then the genealogical tree of the entire population is a version of $\mathbb{T}$. Further, if we write $Z(s)$ for the number of individuals in the population at time s, then the process $(Z(s):s\ge 0)$ is a Yule process, that is a pure birth process with birth rate n when the population has size n. Moreover, it is readily seen that the Yule process Z and the genealogical tree $\mathbb{T}$ are independent.

It is well known (see Remark 5 on page 130 of Athreya and Ney [2]) that

$$\lim_{s\to\infty}\mathrm{e}^{-s}Z(s)=W \qquad \text{almost surely,}$$

where W has the standard exponential distribution. As a consequence, if we write $\tau_n=\inf\{s\ge 0: Z(s)=n\}$ for the birth-time of the individual with label n, then

$$\lim_{n\to\infty} n\mathrm{e}^{-\tau_n}=W \qquad \text{almost surely.} \tag{3.1}$$

Now we incorporate destruction of edges to this population model by killing each new-born child with probability $1-p\in(0,1)$, independently of the other children. The resulting population model is again a Yule process, say $Z^{(p)} = (Z^{(p)}(s) : s \geq 0)$, but now the rate of birth per individual is p. Therefore, we have also

$$\lim_{s\to\infty} \mathrm{e}^{-ps} Z^{(p)}(s) = W^{(p)} \qquad \text{almost surely,}$$

where $W^{(p)}$ is another standard exponential variable. We stress that $W^{(p)}$ is of course correlated to W and not independent of $\mathbb{T}$, in contrast to W.

In this framework, we identify for $p = \mathrm{e}^{-t}$

$$\#\{j \leq n : j \in \Pi_1(t)\} = Z^{(p)}(\tau_n)$$

and therefore

$$\lim_{n\to\infty} \mathrm{e}^{-p\tau_n} \#\{j \leq n : j \in \Pi_1(t)\} = W^{(p)} \qquad \text{almost surely.}$$

Combining with (3.1), we arrive at

$$\lim_{n\to\infty} n^{-p} \#\{j \leq n : j \in \Pi_1(t)\} = \frac{W^{(p)}}{W^p} \qquad \text{almost surely,}$$

which proves the first part of (i).

Since the left-hand side in the last display only depends on the genealogical tree $\mathbb{T}$ and on the exponential random variables ϵ_i attached to its edges, it is independent of the Yule process Z and *a fortiori* of W. Since both W and $W^{(p)}$ are standard exponentials, the second part of (i) now follows from the moments of exponential random variables. □

The second proof of Theorem 3.1(i) relies on more advanced features on the destruction of random recursive trees and Poisson-Dirichlet partitions.

Second proof of Theorem 3.1(i). It is known from the work of Goldschmidt and Martin [11] that the destruction of $\mathbb{T}$ bears deep connections to the Bolthausen-Sznitman coalescent. In this setting, the quantity

$$\#\{j \leq n : j \in \Pi_1(t)\}$$

can be viewed as the number of blocks at time t in a Bolthausen-Sznitman coalescent on $[n] = \{1, \ldots, n\}$ started from the partition into singletons. On the other hand, it is known that the latter is a so-called $(\mathrm{e}^{-t}, 0)$ partition; see Section 3.2 and Theorem 5.19 in Pitman [20]. Our claims now follow from Theorem 3.8 in [20]. □

Proof of Theorem 3.1(ii). Let Π'_1 be an independent copy of the process Π_1. Fix $s, t \geq 0$ and put $B = \Pi_1(s)$, $C = \Pi'_1(t)$. Recall that $B(j)$ denotes the j-th smallest element of B, and $B(C)$ stands for the subset $\{B(j) : j \in C\}$. By Proposition 2.3, there is the equality in distribution $\Pi_1(s + t) = B(C)$. From (i) we deduce that (with $B(n)$ playing the role of n in (i))

$$B(n) \sim (n/X_1(s))^{\mathrm{e}^s} \qquad \text{almost surely as } n \to \infty,$$

and similarly $C(n) \sim (n/X'_1(t))^{\mathrm{e}^t}$ as $n \to \infty$, where $X'_1(t)$ has the same law as $X_1(t)$ and is further independent of $(X_1(r) : r \geq 0)$. It follows that there are the identities

$$\begin{aligned}
X_1(s+t) &= \lim_{m\to\infty} m^{-\mathrm{e}^{-(s+t)}} \#\{j \leq m : j \in \Pi_1(s+t)\} \\
&= \lim_{n\to\infty} \left((B(C)(n))^{-\mathrm{e}^{-(s+t)}} n \right) \\
&= \lim_{n\to\infty} \left((B(C(n)))^{-\mathrm{e}^{-(s+t)}} n \right) \\
&= X_1^{\mathrm{e}^{-t}}(s) X'_1(t).
\end{aligned}$$

Here, in the next to last equality we have used the fact that the n-th smallest element of $B(C)$ is given by the $C(n)$-th smallest element of B, and for the last equality we have plugged in the asymptotic expressions for $B(n)$ and $C(n)$ that we found above. The Markov property now follows easily. For the Feller property, remark that from part (i) and Chebycheff's inequality, we obtain

$$\lim_{t\to 0+} X_1(t) = 1 \qquad \text{in probability.} \tag{3.2}$$

The theorem of dominated convergence then shows that P_t maps the space $\mathcal{C}_0$ of continuous functions vanishing at infinity to itself. Similarly, using again (3.2), one deduces that $\lim_{t\to 0} P_t f(x) = f(x)$ uniformly in $x \geq 0$, for $f \in \mathcal{C}_0$. □

We point out that, alternatively, the Markov property of X_1 can also be derived from the interpretation of $\#\{j \leq n : j \in \Pi_1(t)\}$ as the number of blocks at time t in a Bolthausen-Sznitman coalescent on $[n]$; see the second proof of Theorem 3.1(i) above.

Proof of Theorem 3.1(iii). We first observe from (ii) that the process Y is Markovian with semigroup $Q_t(y,\cdot)$ given by

$$Q_t(y,\cdot) = \mathbb{P}(\mathrm{e}^{-t}y + Y(t) \in \cdot). \tag{3.3}$$

Next, recall from the last remark made after Theorem 3.1 that the function $q \mapsto \kappa(q) = q\psi(q+1)$ is the cumulant-generating function of a spectrally negative Lévy process, say $L = (L(t) : t \geq 0)$. Consider then the Ornstein-Uhlenbeck type process $U = (U(t) : t \geq 0)$ that solves the stochastic differential equation

$$U(t) = L(t) - \int_0^t U(s)\mathrm{d}s,$$

that is, equivalently, $U(t) = \mathrm{e}^{-t}\int_0^t \mathrm{e}^s \mathrm{d}L(s)$. Then U is also Markovian with semigroup $R_t(u,\cdot)$ given by

$$R_t(u,\cdot) = \mathbb{P}(\mathrm{e}^{-t}u + U(t) \in \cdot).$$

So to check that the processes Y and U have the same finite-dimensional laws, it suffices to verify that they have the same one-dimensional distribution.

The calculations of Section 17 in Sato [21] (see Equation (17.4) and Lemma 17.1 there) show that for every $q \geq 0$,

$$\mathbb{E}\left(\exp(qU(t))\right) = \exp\left(\int_0^t \kappa(\mathrm{e}^{-s}q)\mathrm{d}s\right).$$

Now observe that

$$\int_0^t \kappa(\mathrm{e}^{-s}q)\mathrm{d}s = \int_0^t \mathrm{e}^{-s}q\psi(\mathrm{e}^{-s}q+1)\mathrm{d}s = \int_{\mathrm{e}^{-t}q+1}^{q+1} \psi(x)\mathrm{d}x = \ln\Gamma(q+1) - \ln\Gamma(\mathrm{e}^{-t}q+1),$$

where for the last equality we used that $\psi(x) = \frac{\mathrm{d}}{\mathrm{dx}}\ln\Gamma(x)$. We obtain

$$\mathbb{E}\left(\exp(qU(t))\right) = \frac{\Gamma(q+1)}{\Gamma(\mathrm{e}^{-t}q+1)} = \mathbb{E}\left(\exp(qY(t))\right),$$

where the second identity stems from Theorem 3.1(i). We conclude that the finite-dimensional laws of Y and U agree, and since we know from [18] that Y has càdlàg paths, the law of Y is determined. □

3.2 Fragmentation of weights as a branching O.U. process

We next turn our interest to the other blocks of the partition $\Pi(t)$; we shall see that they also have a weight, in the same sense as for the first block. It is convenient to write first $\mathbb{T}_i$ for the subtree of $\mathbb{T}$ rooted at $i \geq 1$; in particular $\mathbb{T}_1 = \mathbb{T}$. Then for $t \geq 0$, we write $T_i(t)$ the subtree of $\mathbb{T}_i$ consisting of vertices $j \in \mathbb{T}_i$ which are still connected to i after the edges e_k with $\epsilon_k \leq t$ have been removed. Note that for $i \geq 2$, the vertex set of $T_i(t)$ forms a block of the partition $\Pi(t)$ if and only if $\epsilon_i \leq t$, an event which has always probability $1 - \mathrm{e}^{-t}$ and is further independent of $T_i(t)$. All the blocks of $\Pi(t)$ arise in this form.

Lemma 3.3. *For every $t \geq 0$ and $i \in \mathbb{N}$, the following limit*

$$\lim_{n\to\infty} n^{-\mathrm{e}^{-t}} \#\{j \leq n : j \in T_i(t)\} = \rho_i(t)$$

exists in $(0, \infty)$ a.s. Moreover, the process

$$\rho_i = (\rho_i(t) : t \geq 0)$$

has the same law as

$$(\beta_i^{\mathrm{e}^{-t}} X_1(t) : t \geq 0),$$

where β_i denotes a beta variable with parameter $(1, i-1)$ and is further independent of $X_1(t)$. In particular, the positive moments of $\rho_i(t)$ are given by

$$\mathbb{E}(\rho_i^q(t)) = \frac{\Gamma(q+1)\Gamma(i)}{\Gamma(\mathrm{e}^{-t}q + i)}, \qquad q \geq 0.$$

Proof. The recursive construction of $\mathbb{T}$ and $\mathbb{T}_i$ has the same dynamics as a Pólya urn, and basic properties of the latter entail that the proportion β_i of vertices in $\mathbb{T}_i$ has the beta distribution with parameter $(1, i-1)$, see [15, Theorem 3.2]. Further, enumerating the vertices of $\mathbb{T}_i$ turns the latter into a random recursive tree. Our claim then follows readily from Theorem 3.1. □

Lemma 3.3 entails that for every $i \in \mathbb{N}$, the i-th block $\Pi_i(t)$ of $\Pi(t)$ has a weight in the sense of (1.2), a.s. We write $X_i(t)$ for the latter and set $\mathbf{X}(t) = (X_1(t), X_2(t), \ldots)$. We now investigate the process $\mathbf{X} = (\mathbf{X}(t) : t \geq 0)$.

We first draw our attention to the logarithms of the weights. Recall that for a homogeneous fragmentation, the process of the asymptotic frequencies of the blocks bears close connections with branching random walks. More precisely, the random point process with atoms at the logarithm of the asymptotic frequencies and observed, say at integer times, is a branching random walk; see [9] and references therein. This means that at each step, each atom, say y, is replaced by a random cloud of atoms located at $y + z$ for $z \in \mathcal{Z}$, independently of the other atoms, and where the random point process $\mathcal{Z}$ has a fixed distribution which does not depend on y nor on the step. In the same vein, we also point out that recently, a natural extension of homogeneous fragmentations, called compensated fragmentations, has been constructed in [8], and bears a similar connection with branching Lévy processes.

Our observations incite us to introduce

$$\mathcal{Y}_t = \sum_{i=1}^{\infty} \delta_{\ln X_i(t)}, \qquad t \geq 0.$$

Recall the characterization of $Y(t) = \ln X_1(t)$ given in Theorem 3.1(iii).

Theorem 3.4. *The process with values in the space of point measure* $\mathcal{Y} = (\mathcal{Y}_t : t \geq 0)$ *is a branching Ornstein-Uhlenbeck process started from* $\mathcal{Y}_0 = \delta_0$*, in the sense that it is Markovian and its transition probabilities can be described as follows: For every* $s, t \geq 0$*, the conditional law of* $\mathcal{Y}_{s+t}$ *given* $\mathcal{Y}_s = \sum_{i=1}^{\infty} \delta_{y_i}$ *is given by the distribution of*

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \delta_{\mathrm{e}^{-t} y_i + \zeta_j^{(i)}}$$

where the point measures

$$\mathcal{Z}^{(i)} = \sum_{j=1}^{\infty} \delta_{\zeta_j^{(i)}}$$

are independent and each has the same law as $\mathcal{Y}_t$*.*

Furthermore, the mean intensity of $\mathcal{Y}_t$ *is determined by*

$$\mathbb{E}\left(\int \mathrm{e}^{qy} \mathcal{Y}_t(\mathrm{d}y)\right) = \frac{(q-1)}{(\mathrm{e}^{-t}q - 1)} \frac{\Gamma(q)}{\Gamma(\mathrm{e}^{-t}q)}, \qquad q > \mathrm{e}^t.$$

Proof. Recall the proof of Theorem 3.1(ii) leading to the specific form of the semi-group (3.3). The first claim then follows from the splitting property and Proposition 2.3. Next, we deduce from Theorem 3.1(i) and Lemma 3.3 that

$$\begin{aligned}
\mathbb{E}\left(\sum_{i=1}^{\infty} X_i^q(t)\right) &= \mathbb{E}(X_1(t)^q) + \mathbb{E}\left(\sum_{i=2}^{\infty} \mathbb{1}_{\{\epsilon_i \leq t\}} \rho_i^q(t)\right) \\
&= \frac{\Gamma(q+1)}{\Gamma(\mathrm{e}^{-t}q+1)} + (1-\mathrm{e}^{-t})\Gamma(q+1) \sum_{i=2}^{\infty} \frac{\Gamma(i)}{\Gamma(\mathrm{e}^{-t}q+i)} \\
&= \mathrm{e}^{-t} \frac{\Gamma(q+1)}{\Gamma(\mathrm{e}^{-t}q+1)} + (1-\mathrm{e}^{-t}) \frac{\Gamma(q+1)}{\Gamma(\mathrm{e}^{-t}q)} \sum_{i=1}^{\infty} \frac{\Gamma(\mathrm{e}^{-t}q)\Gamma(i)}{\Gamma(\mathrm{e}^{-t}q+i)}.
\end{aligned}$$

Using the functional equation of the gamma function for the first term, and the integral representation of the beta function and Tonelli's theorem for the second, we rewrite the former quantity as

$$\begin{aligned}
& \frac{\Gamma(q)}{\Gamma(\mathrm{e}^{-t}q)} + (1-\mathrm{e}^{-t}) \frac{\Gamma(q+1)}{\Gamma(\mathrm{e}^{-t}q)} \int_0^1 x^{\mathrm{e}^{-t}q-1} \left(\sum_{i=1}^{\infty} (1-x)^{i-1}\right) \mathrm{d}x \\
= \; & \frac{\Gamma(q)}{\Gamma(\mathrm{e}^{-t}q)} + (1-\mathrm{e}^{-t}) \frac{q\Gamma(q)}{\Gamma(\mathrm{e}^{-t}q)} \int_0^1 x^{\mathrm{e}^{-t}q-2} \mathrm{d}x.
\end{aligned}$$

Provided that $q > \mathrm{e}^{-t}$, we finally obtain

$$\mathbb{E}\left(\int \mathrm{e}^{qy} \mathcal{Y}_t(\mathrm{d}y)\right) = \mathbb{E}\left(\sum_{i=1}^{\infty} X_i^q(t)\right) = \frac{\Gamma(q)}{\Gamma(\mathrm{e}^{-t}q)} + \frac{q(1-\mathrm{e}^{-t})\Gamma(q)}{(\mathrm{e}^{-t}q-1)\Gamma(\mathrm{e}^{-t}q)}, \tag{3.4}$$

which yields our last statement. □

The last display (3.4) entails that we can sort the weights $X_i(t)$ in the decreasing order, for every $t \geq 0$. We write $\mathbf{X}^{\downarrow}(t)$ for the sequence obtained from $\mathbf{X}(t)$ by ranking the weights $X_i(t)$ decreasingly, where as usual elements are repeated according to their multiplicity. For $q > 0$, let

$$\ell^{q\downarrow} = \left\{ \mathbf{x} = (x_1, \dots) : x_1 \geq x_2 \geq \dots \geq 0, \text{ and } \sum_{i=1}^{\infty} x_i^q < \infty \right\},$$

endowed with the ℓ^q-distance. Similarly, denote by $\ell^{\infty\downarrow}$ the space of ordered sequences of positive reals, endowed with the ℓ^∞-distance. For the process $\mathbf{X}^\downarrow$, we obtain the following characterization.

Corollary 3.5. *Let $T \in (0,\infty]$, and set $q = \mathrm{e}^T$ (with the convention $\mathrm{e}^\infty = \infty$). Then the process $\mathbf{X}^\downarrow = (\mathbf{X}^\downarrow(t) : t < T)$ takes its values in $\ell^{q\downarrow}$ and is Markovian. More specifically, its semigroup can be described as follows. For $s,t \geq 0$ with $s+t < T$, the law of $\mathbf{X}^\downarrow(s+t)$ conditioned on $\mathbf{X}^\downarrow(s) = (x_1,\dots)$ is given by the distribution of the decreasing rearrangement of the collection of real numbers $(x_i^{\mathrm{e}^{-t}} x_j^{(i)} : i,j \in \mathbb{N})$, where $((x_1^{(i)},\dots) : i \in \mathbb{N})$ is a sequence of independent random elements in $\ell^{q\downarrow}$, each of them distributed as $\mathbf{X}^\downarrow(t)$.*

Proof. The fact that $\mathbf{X}^\downarrow(t) \in \ell^{q\downarrow}$ for $t < T$ follows from (3.4). The specific form of the semigroup follows again from Proposition 2.3 and from the arguments given in the proof of Theorem 3.1(ii). See [6, Proposition 3.7] for a similar statement in the context of self-similar fragmentations. $\square$

Next we give a description of the finite dimensional laws of $\mathbf{X}(t) = (X_1(t), X_2(t),\dots)$ for $t > 0$ fixed. In this context, it is convenient to define two families of probability distributions.

The first family is indexed by $j \in \mathbb{N}$ and $t > 0$, and is defined as

$$\mu_{j,t}(k) = \binom{k-2}{k-j-1}(\mathrm{e}^{-t})^{k-j-1}(1-\mathrm{e}^{-t})^j, \qquad k \geq j+1.$$

Note that the shifted distribution $\tilde{\mu}_{j,t}(k) = \mu_{j,t}(k+1)$, $k \geq j$, is sometimes called the negative binomial distribution with parameters j and $1-\mathrm{e}^{-t}$, that is the law of the number of independent trials for j successes when the success probability is given by $1-\mathrm{e}^{-t}$.

The second family is indexed by $j \in \mathbb{N}$ and $k \geq j$, and can be described as follows. We denote by $\theta_{j,k}$ the probability measure on the discrete simplex $\Delta_{k,j} = \{(k_1,\dots,k_j) \in \mathbb{N}^j : k_1 + \dots + k_j = k\}$, such that $\theta_{j,k}\,(k_1,\dots,k_j)$ is the probability that on a random recursive tree of size k (that is on a random tree distributed as $\mathbb{T}_{|[k]}$), after $j-1$ of its edges chosen uniformly at random have been removed, the sequence of the sizes of the j subtrees, ordered according to the label of their root vertex, is given by $(k_1,\dots,k_j)$.

Remark 3.6. The distribution $\theta_{j,k}$ is equal to δ_k for $j=1$. For $j=2$, Meir and Moon [17] found the expression

$$\theta_{2,k}(k_1,k_2) = \frac{k}{k_2(k_2+1)(k-1)}, \qquad k_1,k_2 \in \mathbb{N} \text{ with } k_1+k_2 = k,$$

with $\theta_{2,k}(k_1,k_2) = 0$ for all other pairs (k_1,k_2). Generalizing the proof of this formula given in [17] to higher j, we find ($k \geq j \geq 3$ and $k_1 + \dots + k_j = k$)

$$\theta_{j,k}(k_1,k_2,\dots,k_j) = \frac{(k_1-1)!(k_2-1)!\cdots(k_j-1)!}{(k-1)!(k-1)\cdots(k-(j-1))}\sum_{\ell_j=j-1}^{k-k_j}\binom{k-\ell_j}{k_j}$$
$$\times \sum_{\ell_{j-1}=j-2}^{(k-k_j-k_{j-1})\wedge(\ell_j-1)}\binom{k-k_j-\ell_{j-1}}{k_{j-1}} \times \cdots \times \sum_{\ell_2=1}^{(k-\sum_{i=2}^j k_i)\wedge(\ell_3-1)}\binom{k-\sum_{i=3}^j k_i - \ell_2}{k_2}.$$

Proposition 3.7. *Let $j \in \mathbb{N}$, $q_1,\dots,q_{j+1} \geq 0$, and set $q = q_1 + \dots + q_{j+1}$, $k_{j+1} = 1$. The Mellin transform of the vector $(X_1(t),\dots,X_{j+1}(t))$ for fixed $t > 0$ is given by*

$$\mathbb{E}\left(X_1^{q_1}(t)\cdots X_{j+1}^{q_{j+1}}(t)\right) = \sum_{k=j+1}^{\infty}\mu_{j,t}(k)\sum_{\substack{k_1,\dots,k_j\geq 1,\\ k_1+\dots+k_j=k-1}}\theta_{j,k-1}\,(k_1,\dots,k_j)\,\frac{\Gamma(k)}{\Gamma(q\mathrm{e}^{-t}+k)}\prod_{i=1}^{j+1}\frac{\Gamma(q_i+k_i)}{\Gamma(k_i)}.$$

Remark 3.8. By plugging in the definition of $\mu_{j,t}(k)$, one checks that the right hand side is finite.

Proof. Fix $t > 0$, and set $p = \mathrm{e}^{-t}$. For ease of notation, we write Π_i and X_i instead of $\Pi_i(t)$ and $X_i(t)$. Furthermore, fix an integer $j \in \mathbb{N}$ and numbers $k_1, \dots, k_j \in \mathbb{N}$. For convenience, set $k = k_1 + \cdots + k_j + k_{j+1}$, with $k_{j+1} = 1$. We first work conditionally on the event

$$A_{k_1,\dots,k_j} = \left\{ \min \Pi_{j+1} = k, \, \#(\Pi_1 \cap [k]) = k_1, \dots, \#(\Pi_j \cap [k]) = k_j \right\}.$$

We shall adapt the first proof of Theorem 3.1(i). Here, we consider a multi-type Yule process starting from k individuals in total such that k_i of them are of type i, for each $i = 1, \dots, j+1$. The individuals reproduce independently of each other at unit rate, and each child individual adopts the type of its parent. Then, if $Z(s)$ stands for the total number of individuals at time s, we have that $\lim_{s\to\infty} \mathrm{e}^{-s} Z(s) = \gamma(k)$ almost surely, where $\gamma(k)$ is distributed as the sum of k standard exponentials, i.e. follows the gamma law with parameters $(k, 1)$. Now assume again that each new-born child is killed with probability $1 - p \in (0, 1)$, independently of each other. Writing $Z^{(i,p)}(s)$ for the size of the population of type i at time s (with killing), we obtain

$$\lim_{s\to\infty} \mathrm{e}^{-ps} Z^{(i,p)}(s) = \gamma^{(i,p)}(k_i), \qquad i = 1, \dots, j+1,$$

where the $\gamma^{(i,p)}(k_i)$ are independent gamma$(k_i, 1)$ random variables (they are however clearly correlated to the asymptotic total population size $\gamma(k)$). From the arguments given in the first proof of Theorem 3.1(i) it should be plain that conditionally on the event $A_{k_1,\dots,k_j}$, we have for the weights X_i the representation

$$X_i = \frac{\gamma^{(i,p)}(k_i)}{\gamma(k)^p}, \qquad i = 1, \dots, j+1,$$

and the X_i are independent of $\gamma(k)$. Now let $q_1, \dots, q_{j+1} \geq 0$ and put $q = q_1 + \cdots + q_{j+1}$. Using the expression for the X_i and independence, we calculate

$$\mathbb{E}\left(\gamma(k)^{qp}\right) \mathbb{E}\left(X_1^{q_1} \cdots X_{j+1}^{q_{j+1}} \mid A_{k_1,\dots,k_j}\right) = \prod_{i=1}^{j+1} \frac{\Gamma(q_i + k_i)}{\Gamma(k_i)}. \tag{3.5}$$

Therefore, again with $k_{j+1} = 1$,

$$\mathbb{E}\left(X_1^{q_1} \cdots X_{j+1}^{q_{j+1}}\right) = \sum_{k=j+1}^{\infty} \frac{\Gamma(k)}{\Gamma(qp + k)} \sum_{\substack{k_1,\dots,k_j \geq 1, \\ k_1 + \cdots + k_j = k-1}} \prod_{i=1}^{j+1} \frac{\Gamma(q_i + k_i)}{\Gamma(k_i)} \mathbb{P}\left(A_{k_1,\dots,k_j}\right).$$

With $k = k_1 + \cdots + k_{j+1}$ as above, we express the probability of $A_{k_1,\dots,k_j}$ as

$$\mathbb{P}\left(A_{k_1,\dots,k_j}\right) = \mathbb{P}\left(\#(\Pi_1 \cap [k]) = k_1, \dots, \#(\Pi_j \cap [k]) = k_j \mid \min \Pi_{j+1} = k\right) \mathbb{P}\left(\min \Pi_{j+1} = k\right).$$

By induction on j we easily deduce that $\min \Pi_{j+1} - 1$ is distributed as the sum of j independent geometric random variables with success probability $1 - p$, i.e. $\min \Pi_{j+1} - 1$ counts the number of trials for j successes, so that $\mathbb{P}\left(\min \Pi_{j+1} = k\right) = \mu_{j,t}(k)$. Moreover, it follows from the very definition of the blocks Π_i and the fact that the exponentials attached to the edges of $\mathbb{T}_{|[k]}$ are i.i.d. that

$$\mathbb{P}\left(\#(\Pi_1 \cap [k]) = k_1, \dots, \#(\Pi_j \cap [k]) = k_j \mid \min \Pi_{j+1} = k\right) = \theta_{j,k-1}(k_1, \dots, k_j).$$

This proves the proposition. $\square$

We finally look closer at the joint moments of $X_1(t)$ and $X_2(t)$ when t tends to zero. We observe $\theta_{1,k} = \delta_k$ and $\mu_{1,t}(k) = (\mathrm{e}^{-t})^{k-2}(1-\mathrm{e}^{-t})$, so that

$$\mathbb{E}\left(X_1^{q_1}(t)X_2^{q_2}(t)\right) = (1-\mathrm{e}^{-t})\Gamma(q_2+1)\sum_{k=2}^{\infty}(k-1)(\mathrm{e}^{-t})^{k-2}\frac{\Gamma(q_1+k-1)}{\Gamma((q_1+q_2)\mathrm{e}^{-t}+k)}.$$

Now assume $q_2 > 1$. From the last display we get

$$\begin{aligned}\lim_{t\to 0+}\frac{1}{t}\mathbb{E}\left(X_1^{q_1}(t)X_2^{q_2}(t)\right) &= \sum_{k=1}^{\infty}k\frac{\Gamma(q_1+k)\Gamma(q_2+1)}{\Gamma(q_1+q_2+k+1)}\\ &= \sum_{k=1}^{\infty}k\int_0^1(1-x)^{q_1+k-1}x^{q_2}\mathrm{d}x = \int_0^1(1-x)^{q_1}x^{q_2-2}\mathrm{d}x,\end{aligned}$$

which one could have already guessed from Proposition 2.6 (choose $f(1-x,x) = x^{q_1}x^{q_2}$ there).

3.3 Asymptotic behaviors

We shall finally present some asymptotic properties of the process $\mathbf{X}$ of the weights. To start with, we consider the large time behavior.

Corollary 3.9. *As $t \to \infty$, there is the weak convergence*

$$(X_i(t) : i \in \mathbb{N}) \Longrightarrow (W_i : i \in \mathbb{N}),$$

where on the right-hand side, the W_i are i.i.d. standard exponential variables.

Remark 3.10. This result is a little bit surprising, as obviously $\Pi(\infty)$ is the partition into singletons. That is, $\Pi_i(\infty)$ is reduced to $\{i\}$ and hence has weight 1 if we apply (1.2) for $t = \infty$. In other words, the limits $n \to \infty$ and $t \to \infty$ may not be interchanged.

Proof. Fix $j \in \mathbb{N}$ arbitrarily and consider the event $A(t) = \{\min \Pi_{j+1}(t) = j+1\}$. Recall from the proof of Proposition 3.7 that $\mathbb{P}(A(t)) = \mu_{j,t}(j+1)$. Further, on the event $A(t)$, we have also $\Pi_i(t) \cap [j+1] = \{i\}$ for all $1 \le i \le j+1$, that is $A(t) = A_{1,\dots,1}$, again in the notation of the proof of Proposition 3.7. Now take $q_1, \dots, q_{j+1} \ge 0$. Applying (3.5), we get

$$\lim_{t\to\infty}\mathbb{E}\left(X_1^{q_1}(t)\cdots X_{j+1}^{q_{j+1}}(t) \mid A(t)\right) = \prod_{i=1}^{j+1}\Gamma(q_i+1),$$

noting that the additional factor in (3.5) converges to 1 as $t \to \infty$. Using additionally that $\mathbb{P}(A(t)) \to 1$ as $t \to \infty$, the last display entails that $(X_1(t), \dots, X_{j+1}(t))$ converge in distribution as $t \to \infty$ towards a sequence of $j+1$ independent exponentially distributed random variables. □

We next consider for $t > 0$ fixed the behavior of $X_n(t)$ as $n \to \infty$.

Corollary 3.11. *Let $t > 0$. As $n \to \infty$, there is the weak convergence*

$$n^{\mathrm{e}^{-t}}X_n(t) \Longrightarrow V^{\mathrm{e}^{-t}}X_1(t),$$

where V denotes an exponential random variable of parameter $(1-\mathrm{e}^{-t})^{-1}$ which is independent of $X_1(t)$.

Proof. In the notation of Lemma 3.3, we have $X_n(t) = \rho_{i(n,t)}(t)$, where (again with the convention $\epsilon_1 = 0$)

$$i(n,t) = \min\left\{j \ge 1 : \sum_{i=1}^{j}\mathbb{1}_{\{\epsilon_i \le t\}} = n\right\}.$$

From Lemma 3.3 we know that $X_n(t) = \beta_{i(n,t)}^{\mathrm{e}^{-t}} X_1(t)$ in distribution, where $i(n,t)$ and $\beta_{i(n,t)}$ are both independent of $X_1(t)$. By the law of large numbers, $i(n,t) \sim n(1-\mathrm{e}^{-t})^{-1}$ almost surely. Writing

$$\beta_{i(n,t)} = \left(i(n,t)\beta_{i(n,t)}\right) i(n,t)^{-1}$$

and using the fact that $k\beta_k$ converges in distribution to a standard exponential random variable as $k \to \infty$, the claim follows. □

Let us now look at the behavior of $\mathbf{X}(t)$ when $t \to 0+$. From (3.2) we already know that $X_1(t) \to 1$ in probability as $t \to 0+$. The weights $X_i(t)$ with $i \geq 2$ converge uniformly to zero:

Corollary 3.12. *As $t \to 0+$, there is the convergence in probability*

$$\sup_{i \geq 2} X_i(t) \longrightarrow 0.$$

Proof. Let $0 < \varepsilon < 1$. By Lemma 3.3, with β_i denoting a beta$(1, i-1)$ random variable,

$$\mathbb{P}\left(X_i(t) \geq \varepsilon \text{ for some } i \geq 2\right) \leq (1-\mathrm{e}^{-t}) \sum_{i=2}^{\infty} \mathbb{P}\left(\beta_i X_1^{\mathrm{e}^t}(t) \geq \varepsilon^{\mathrm{e}^t}\right),$$

where $1-\mathrm{e}^{-t}$ is the probability of the event $\{\epsilon_i \leq t\}$. Using independence and the expression for the moments of $X_1(t)$ from Theorem 2.2, we obtain for $t \geq 0$ such that $\mathrm{e}^t \leq 2$, $\mathbb{E}(\beta_i X_1^{\mathrm{e}^t}(t)) \leq 2/i$ and $\mathrm{Var}(\beta_i X_1^{\mathrm{e}^t}(t)) \leq 24/i^2$. Therefore, for such t and all $i \geq 2$, we obtain by Chebycheff's inequality

$$\mathbb{P}\left(\left|\beta_i X_1^{\mathrm{e}^t}(t) - \mathbb{E}(\beta_i X_1^{\mathrm{e}^t}(t))\right| \geq \varepsilon^{\mathrm{e}^t} - 2/i\right) \leq \frac{C}{i^2}$$

for some constant C depending only on ε. This shows

$$\mathbb{P}\left(X_i(t) \geq \varepsilon \text{ for some } i \geq 2\right) \leq C(1-\mathrm{e}^{-t}) \sum_{i=2}^{\infty} \frac{1}{i^2} \leq C't$$

for some $C' > 0$ independent of t. □

3.4 Application to cluster sizes of percolation

As it should be plain from the introduction, the sets of vertices

$$C_i^{(n)}(t) = \{j \leq n : j \in \Pi_i(t)\}$$

form the percolation clusters of a Bernoulli bond percolation with parameter $p = \mathrm{e}^{-t}$ on a random recursive tree on the vertexx set $\{1, \dots, n\}$. Our results of Section 3 can therefore be understood as results on the asymptotic sizes of these clusters when n tends to infinity. For example, Theorem 3.1(i) determines the asymptotic size of the root cluster.

Cluster sizes of random recursive trees were already studied in [7] when the percolation parameter p satisfies $p = p(n) = 1 - s/\ln n + o(1/\ln n)$ for $s > 0$ fixed. The analysis was extended in [3] to all regimes $p(n) \to 1$. It shows that in these regimes, the root cluster containing 1 has always the asymptotic size $\sim n^{p(n)}$, while the next largest cluster sizes, normalized by a factor $(1-p(n))^{-1} n^{-p(n)}$, are in the limit given by the (ranked) atoms of a Poisson random measure on $(0, \infty)$ with intensity $a^{-2}\mathrm{d}a$.

The regime of constant parameter $p = \mathrm{e}^{-t}$ considered here deserves the name "critical", since in this regime, the root cluster and the next largest clusters have the same order of magnitude, namely n^p. This is already apparent from Lemma 3.3. We can say more:

Corollary 3.13.

$$\lim_{t\uparrow\infty}\liminf_{n\to\infty}\mathbb{P}\left(\#C_i^{(n)}(t) > \#C_1^{(n)}(t) \text{ for some } i\geq 2\right) = 1.$$

Proof. Given $\varepsilon > 0$ and a sequence $(W_i : i \in \mathbb{N})$ of i.i.d. standard exponentials, we can find $k \geq 2$ such that

$$\mathbb{P}\left(W_i > W_1 \text{ for some } i = 2, \ldots, k\right) \geq 1 - \varepsilon/2.$$

Also, since $n^{-e^{-t}}\#C_i^{(n)}(t) \to X_i(t)$ almost surely as n tends to infinity by Lemma 3.3,

$$\liminf_{n\to\infty}\mathbb{P}\left(\#C_i^{(n)}(t) > \#C_1^{(n)}(t) \text{ for some } i\geq 2\right) \geq \mathbb{P}\left(X_i(t) > X_1(t) \text{ for some } i = 2, \ldots, k\right).$$

Together with the next to last display, Corollary 3.9 then shows that for t sufficiently large,

$$\liminf_{n\to\infty}\mathbb{P}\left(\#C_i^{(n)}(t) > \#C_1^{(n)}(t) \text{ for some } i\geq 2\right) \geq 1 - \varepsilon.$$

□

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Acknowledgments. We would like to thank two anonymous referees for their valuable comments. The first author is grateful to Grégory Miermont for stimulating discussions.