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Escape probability and transience for SLE*

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Abstract

We give estimates for the probability that a chordal, radial or two-sided radial SLE_κ curve retreats far from its terminal point after coming close to it, for $\kappa \leq 4$. The estimates are uniform over all initial segments of the curve, and are sharp up to a universal constant.

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1 Introduction

In this paper we will prove some estimates on the continuity of a Schramm–Loewner evolution (SLE_κ) curve at the terminal point. Before stating the estimates, we will describe why they are useful.

We will consider three types of SLE_κ. These are probability measures on curves in a simply connected domain with specified start and end points. Chordal SLE_κ is a measure on curves from one boundary point to another. Radial SLE_κ is a measure on curves from a boundary point to an interior point. Two-sided radial SLE_κ is essentially chordal SLE_κ conditioned to pass through an interior point (the target point); in this paper we will only consider the curve stopped when it hits that point. The question we will be asking is roughly "given that the curve is very close to its target point, what is the probability that it goes far away before reaching the target point?" There are various parametrizations of the curves, but the questions we discuss will be independent of the choice.

We will restrict our consideration to $0 < \kappa < 8$, for which the curves have Hausdorff dimension $d := 1 + \frac{\kappa}{8} < 2$. If $0 < \kappa \leq 4$, the measure is supported on simple curves while for $4 < \kappa < 8$, the curves have self-intersections.

For chordal SLE_κ, continuity at the endpoint is equivalent to transience of SLE_κ in the upper half-plane from 0 to ∞, which was proved in [8]. We improve this result for $\kappa \leq 4$ by giving an upper bound for the probability of returning to the ball of radius ϵ after the curve has reached the circle of radius 1. It is a uniform estimate over all realizations

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of the curve up to the first visit to the unit circle. For the radial case, a similar result was stated in [3]. However, as pointed out by Dapeng Zhan, there is an error in one of the proofs. We reprove this estimate here discussing the error and how it is corrected. We also give a uniform estimate for the two-sided radial case which is analogous to the chordal case. While the endpoint continuity of two-sided radial SLE was established in [3], and that of a generalization called radial $\mathrm{SLE}_\kappa^\mu(\rho)$ in [7], no uniform estimates were given.

We will now make a precise statement of our results. Throughout this paper we let

$$0<\kappa<8 \qquad \text{and let} \qquad a=\frac{2}{\kappa}.$$

Let

$$\mathbb{D}_r=\{z\in\mathbb{C}:|z|<e^{-r}\},\qquad C_r=\partial\mathbb{D}_r,\qquad C=C_0,$$
$$\mathbb{H}=\{z\in\mathbb{C}:\operatorname{Im} z>0\},\qquad \text{and}\qquad \mathbb{D}=\mathbb{D}_0.$$

The convention in this paper is that letters r and s denote the negative logarithm of a disk's radius rather than the radius itself. If $\gamma:[0,\infty)\to\mathbb{C}$ is a curve, we will write γ_t for the path $\gamma[0,t]$ and $\gamma(t)$ for the value of the curve at time t. We will write $\gamma=\gamma_\infty$ for the entire path.

Suppose that $\gamma:[0,\infty)\to\overline{\mathbb{H}}$ is a chordal SLE_κ curve from 0 to ∞ in the half-plane $\mathbb{H}$. The definition of chordal SLE_κ states that γ is a random curve characterized by the following property. For each t, let H_t be the unbounded component of $\mathbb{H}\setminus\gamma_t$, and let g_t denote the unique conformal transformation of H_t onto $\mathbb{H}$ satisfying $g_t(z)-z\to 0$ as $z\to\infty$. Then γ has been parametrized so that

$$g_t(z)=z+\frac{at}{z}+O(|z|^{-2}),\qquad z\to\infty,$$

and there exists a standard one-dimensional Brownian motion U such that

$$\partial_t g_t(z)=\frac{a}{g_t(z)-U_t},\qquad g_0(z)=z.$$

See [2] for more information. If $\kappa\le 4$, then γ is a simple curve, whereas if $\kappa\ge 8$, the curve is plane-filling.

The uniform estimate for transience is the following theorem.

Theorem 1.1. *If $0<\kappa\le 4$, there exists $c<\infty$ such that if γ is a chordal SLE_κ curve from 0 to ∞ in $\mathbb{H}$, $T=\inf\{t:\gamma(t)\in C\}$, and $r>0$, then*

$$\mathbf{P}\{\gamma[T,\infty)\cap C_r\neq\emptyset\mid\gamma_T\}\le c\,e^{-r(4a-1)}.$$

There is a similar result for radial SLE_κ. Let γ be a radial SLE_κ curve from 1 to 0 in the unit disk $\mathbb{D}$. This is defined in terms of the conformal transformation $h_t:\mathbb{D}\setminus\gamma_t\to\mathbb{D}$ with $h_t(0)=0$ and $h_t'(0)>0$. Then the curve is parametrized so that $h_t'(0)=e^{2at}$ and $L_t(z):=\frac{1}{2i}\log h_t(e^{2iz})$ satisfies

$$\partial_t L_t(z)=a\cot(L_t(z)-U_t),\qquad L_0(z)=z, \tag{1.1}$$

where U is a standard Brownian motion. If we write $g_t(e^{2i\theta})=e^{2iV_t}$ and let $\Theta_t=V_t-U_t$, then Θ_t satisfies

$$d\Theta_t=a\cot\Theta_t\,dt-dU_t.$$

Theorem 1.2. *If $0<\kappa\le 4$, there exists $c<\infty$ such that if γ is a radial SLE_κ curve from 1 to 0 in $\mathbb{D}$, $0<s<r$, and $\tau_r=\inf\{t:\gamma(t)\in C_r\}$, then*

$$\mathbf{P}\{\gamma[\tau_r,\infty)\cap C_s\neq\emptyset\mid\gamma_{\tau_r}\}\le c\,e^{-(r-s)(4a-1)/2}.$$

As mentioned before, this last result is stated in [3]; we prove it here by correcting the argument from that paper.

A two-sided radial SLE_κ curve from 1 to $e^{2i\theta}$ through 0 in $\mathbb{D}$ can be thought of as chordal SLE_κ from 1 to $e^{2i\theta}$ in $\mathbb{D}$, conditioned to pass through 0. For the purposes of this paper the curve will always be stopped when it reaches 0. The curve can be defined by weighting chordal SLE_κ in the sense of the Girsanov theorem by the SLE_κ Green's function in the slit domain at 0. Explicitly, the Green's function for SLE_κ from z to w in a domain D is

$$G_D(\zeta; z, w) = \lim_{\epsilon \downarrow 0} \epsilon^{d-2}\, \mathbf{P}\{\mathrm{dist}(\zeta, \gamma) < \epsilon\},$$

where γ is the entire curve. The limit is known to exist [5, 8] and is given by

$$G_{\mathbb{H}}(\zeta; 0, \infty) = \hat{c}\, [\mathrm{Im}\, \zeta]^{d-2}\, [\sin \arg \zeta]^{4a-1},$$

where $\hat{c}$ is a constant depending only on κ. For other domains, one uses the conformal covariance rule

$$G_D(\zeta; z, w) = |f'(\zeta)|^{2-d}\, G_{f(D)}(f(\zeta); f(z), f(w)).$$

In particular,

$$G_{\mathbb{D}}(0; 1, e^{2i\theta}) = \hat{c}\, \sin^{4a-1}\, \theta.$$

After weighting by the appropriate martingale, we see that

$$d\Theta_t = 2a \cot \Theta_t\, dt + dW_t, \qquad \Theta_0 = \theta, \qquad -dU_t = a \cot \Theta_t\, dt + dW_t,$$

where W is a standard Brownian motion in the new measure. See [6] for more details.

The two-sided radial estimate is almost the same as the radial estimate.

Theorem 1.3. *If $0 < \kappa \leq 4$, there exists $c < \infty$ such that if $0 < \theta < \pi$, γ is a two-sided radial SLE_κ curve from 1 to $e^{2i\theta}$ through 0 in $\mathbb{D}$ stopped when it reaches 0, $0 < s < r$, and $\tau_r = \inf\{t : \gamma(t) \in C_r\}$, then*

$$\mathbf{P}\{\gamma[\tau_r, \infty) \cap C_s \neq \emptyset \mid \gamma_{\tau_r}\} \leq c\, e^{-(r-s)(4a-1)/2}.$$

More generally, chordal, radial and two-sided radial SLE_κ can be defined in simply connected domains by conformal invariance. If D is a simply connected domain and $z \in \partial D$, $w \in \overline{D} \setminus \{z\}$, we will say SLE_κ from z to w in D with the implication that it is chordal SLE_κ if $w \in \partial D$ and radial SLE_κ if $w \in D$. This convention is not just an arbitrary compression of notation, as the law of radial SLE targeted at an interior point converges to the law of chordal SLE as the point approaches the boundary; see [4, Section 8] for a precise formulation of this result.

2 Definitions and notation

Any time that we use the letter c, we will mean a finite positive constant, depending only on κ, that may differ from all previous uses of c. A *crosscut* of a domain D is a simple curve $\eta : (0, t_\eta) \to D$ with $\eta(0+), \eta(t_\gamma-) \in \partial D$. We will often write just η for the image $\eta(0, t_\eta)$.

We will take all domains to be in the Riemann sphere $\hat{\mathbb{C}}$, and let $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \subset \hat{\mathbb{C}}$. Let $\mathbb{R}_+ = (0, \infty)$ and $\mathbb{R}_- = (-\infty, 0)$. For a domain D, let $D^c = \hat{\mathbb{C}} \setminus D$ denote its complement. We always let B be a complex Brownian motion. For a domain D and an arbitrary set S, let

$$\tau_D = \inf\{t : B_t \notin D\}, \quad \sigma_S = \inf\{t : B_t \in S\}.$$

We write h_D for the harmonic measure, that is, if $V \subset \partial D$,

$$h_D(z, V) = \mathbf{P}^z\{\tau_D \in V\}.$$

If D is a domain and $V, W \subset \partial D$ are analytic boundary arcs, then the *excursion measure* in D between V and W is defined as

$$\mathcal{E}_D(V,W) = \int_V \mathcal{E}_D(v,W)\,|dv|,$$

where, if n_v is the inward pointing normal at v,

$$\mathcal{E}_D(v,W) = \lim_{\epsilon\downarrow 0} \epsilon^{-1}\, h_D(v+\epsilon n_v, W).$$

It turns out that $\mathcal{E}_D(V,W)$ is conformally invariant (see [2, Proposition 5.8]), and therefore it makes sense even when V, W are boundary arcs that are not analytic. Also, the measure is symmetric, $\mathcal{E}_D(V,W) = \mathcal{E}_D(W,V)$. As a conformal invariant, the excursion measure differs from the classical extremal length. One can see, however, that they determine one another in the case of a conformal rectangle: in the rectangle $D = [0,L]\times[0,\pi]$, the excursion measure between the sides of length π is e^{-L}, whereas the extremal length of all curves joining these two sides is L/π.

We extend this notation in two convenient ways. First, if D is not connected, we set $\mathcal{E}_D(V,W) = \mathcal{E}_{D_1}(V,W)$ where D_1 is the unique connected component of D from which V and W are both accessible, or zero if there is no such component. (Strictly speaking, this component is unique only once an orientation of the inward pointing normal is chosen on V and on W, but this rarely needs to be specified.) Second, if V, W are not boundary arcs but merely the images of simple curves (that may pass through D) then we set $\mathcal{E}_D(V,W) = \mathcal{E}_{D\setminus(V\cup W)}(V,W)$.

For fixed V, W, we can view $\mathcal{E}_D(V,W)$ as a measure $\mathcal{E}_{D\setminus W}(V,\cdot)$ on W of total mass $\mathcal{E}_D(V,W)$. The strong Markov property for Brownian motion gives the following rule. Suppose η is a curve in D that separates V and W. Then

$$\mathcal{E}_D(V,W) = \int_\eta h_D(z,W)\,\mathcal{E}_{D\setminus\eta}(V,dz),$$

and hence,

$$\mathcal{E}_D(V,W) \le \mathcal{E}_D(V,\eta)\,\sup_{z\in\eta} h_{D_1}(z,W), \tag{2.1}$$

$$\mathcal{E}_D(V,W) \ge \mathcal{E}_{D\setminus\eta}(V,\eta')\,\inf_{z\in\eta'} h_{D_1}(z,W) \qquad \text{if} \quad \eta'\subset\eta. \tag{2.2}$$

In particular, we see that for any domain D and any $r \ge 2$,

$$\mathcal{E}_D(C,C_r) \le c\,\sup_{z\in C_1} h_{D\setminus C_r}(z,C_r). \tag{2.3}$$

The upper bound (2.1) implies the following two estimates for excursion measure. Using (2.3) and the Poisson kernel in $\mathbb{H}\setminus\mathbb{D}_r$, we can see that

$$\mathcal{E}_{\mathbb{H}}(C,C_r) \le c\,e^{-r}. \tag{2.4}$$

By the Beurling estimate (see, e.g., [2, Theorem 3.76]), there exists $c<\infty$ such that if $\mathbb{C}\setminus D$ includes a curve connecting C with C_r, then

$$h_D(z,C_r) \le c\,e^{-r/2}, \qquad z\in C_1,$$

and hence (2.3) gives

$$\mathcal{E}_D(C,C_r) \le c\,e^{-r/2}. \tag{2.5}$$

3 Boundary intersection exponent for SLE

The following is the basic boundary intersection estimate for SLE. We will state it in a unified form that combines radial and chordal cases.

Proposition 3.1. *If $0 < \kappa < 8$, there exists $c < \infty$ such that the following holds. Suppose D is a simply connected domain, $z \in \partial D$, and $w \in \overline{D} \setminus \{z\}$. Suppose η is a crosscut of D, and $\tilde{\gamma} : (0,1) \to D$ is a simple curve with $\tilde{\gamma}(0+) = z$, $\tilde{\gamma}(1-) = w$. If γ is an SLE$_\kappa$ curve from z to w in D, then*

$$\mathbf{P}\{\gamma \cap \eta \neq \emptyset\} \leq c\, \mathcal{E}_D(\eta, \tilde{\gamma})^{4a-1}.$$

Proof. By conformal invariance it suffices to consider the chordal case with $D = \mathbb{H}$, $z = 0$, $w = \infty$ and the radial case with $D = \mathbb{D}$, $z = 1$, $w = 0$.

For the chordal case, without loss of generality assume that the endpoints of η are on the positive real axis. Then, standard estimates for excursion measure (e.g., Corollary 5.2) show that

$$\mathcal{E}_{\mathbb{H}}(\eta, \tilde{\gamma}) \geq \mathcal{E}_{\mathbb{H}}(\eta, \mathbb{R}_-) \geq c \left(\frac{\operatorname{diam}(\eta)}{\operatorname{dist}(0,\eta)} \wedge 1\right).$$

A proof of

$$\mathbf{P}\{\gamma \cap \eta \neq \emptyset\} \leq c \left(\frac{\operatorname{diam}(\eta)}{\operatorname{dist}(0,\eta)} \wedge 1\right)^{4a-1}$$

can be found in [1].

For the radial case, if $\operatorname{diam}(\eta) \geq 1/10$, then $\mathcal{E}_{\mathbb{D}}(\eta, \tilde{\gamma}) \geq c$ and the result is immediate. Hence we assume that $\operatorname{diam}(\eta) \leq 1/10$. A proof of

$$\mathbf{P}\{\gamma \cap \eta \neq \emptyset\} \leq c \left(\frac{\operatorname{diam}(\eta)}{\operatorname{dist}(1,\eta)}\right)^{4a-1}$$

can be found in [3]. Roughly speaking, the only difficult case is when η is near 1, and in this case the path looks locally like chordal SLE from 1 to -1, for which the chordal estimate holds. By mapping to the half-plane case, we can see that

$$\mathcal{E}_{\mathbb{H}}(\eta, \tilde{\gamma}) \geq c\, \frac{\operatorname{diam}(\eta)}{\operatorname{dist}(1,\eta)}. \qquad \square$$

It follows immediately that if $\eta_1, \eta_2, \ldots$ is a countable collection of crosscuts and $A = \bigcup_j \eta_j$, then

$$\mathbf{P}\{\gamma \cap A \neq \emptyset\} \leq c \sum_{j=1}^{\infty} \mathcal{E}_D(\eta_j, \tilde{\gamma})^{4a-1}.$$

If $0 < \kappa \leq 4$, then $4a - 1 \geq 1$. This is exactly the condition that we need in order to make the estimate

$$\sum_{j=1}^{\infty} \mathcal{E}_D(\eta_j, \tilde{\gamma})^{4a-1} \leq \left[\sum_{j=1}^{\infty} \mathcal{E}_D(\eta_j, \tilde{\gamma})\right]^{4a-1},$$

and hence we can conclude the following.

Proposition 3.2. *If $0 < \kappa \leq 4$, there exists $c < \infty$ such that the following holds. Suppose D is a simply connected domain, $z \in \partial D$, $w \in \overline{D} \setminus \{z\}$. Suppose $\eta_1, \eta_2, \ldots$ are crosscuts of D. Suppose $\tilde{\gamma} : (0,1) \to D$ is a simple curve with $\tilde{\gamma}(0+) = z$, $\tilde{\gamma}(1-) = w$. If γ is an SLE$_\kappa$ curve from z to w in D, then*

$$\mathbf{P}\left\{\gamma \cap \bigcup_{j=1}^{\infty} \eta_j \neq \emptyset\right\} \leq c \left[\sum_{j=1}^{\infty} \mathcal{E}_D(\eta_j, \tilde{\gamma})\right]^{4a-1}.$$

This was the strategy in [3] and there was no problem in the argument at this point. However, as pointed out by Dapeng Zhan, when applying the argument, an upper bound was established for

$$\mathcal{E}_D\left(\bigcup_{j=1}^{\infty}\eta_j,\tilde{\gamma}\right), \tag{3.1}$$

rather than for

$$\sum_{j=1}^{\infty}\mathcal{E}_D(\eta_j,\tilde{\gamma}). \tag{3.2}$$

In general, it may not be easy to bound (3.2) in terms of (3.1). Fortunately, we will need such a bound only when $\{\eta_j\}$ is the set of crosscuts of D contained in a particular circle about the origin. The next lemma shows that in this case the quantity (3.2) is at most twice the quantity (3.1).

Lemma 3.3. *Let D be a simply connected domain and S the set of crosscuts of D that are subsets of the circle C_s. Then*

$$\sum_{\eta\in S}\mathcal{E}_D(C_r,\eta)\leq 2\,\mathcal{E}_D(C_r,C_s).$$

Proof. See Section 4. □

Theorems 1.1 and 1.2 follow from the following propositions. We also include another case which arises in current research. Recall that $\mathbb{D}_r = e^{-r}\,\mathbb{D}$ and $C_r = \partial\mathbb{D}_r$.

Proposition 3.4. *If $0<\kappa\leq 4$, there exists $c<\infty$ such that the following holds. Suppose that $D\subset\mathbb{H}$ is a simply connected domain such that $(\mathbb{H}\setminus D)\cap(\mathbb{H}\setminus\mathbb{D})=\{z\}$. Let γ be an SLE$_\kappa$ curve in D from z to ∞. Then,*

$$\mathbf{P}\{\gamma\cap C_r\neq\emptyset\}\leq c\,e^{-r(4a-1)}.$$

In particular, if $\kappa\leq 4$, γ is an SLE_κ curve from 0 to ∞ in $\mathbb{H}$, and $\rho=\inf\{t:|\gamma(t)|=1\}$, then

$$\mathbf{P}\{\gamma(\rho,\infty)\cap C_r\neq\emptyset\mid\gamma_\rho\}\leq c\,e^{-r(4a-1)}.$$

It is not known whether or not this estimate holds for $4<\kappa<8$.

Proof. We may assume $r\geq 1$. We write

$$D\cap C_r=\bigcup_{j=1}^{\infty}\eta_j,$$

where η_j are crosscuts of D. Let $\tilde{\gamma}$ be any simple curve from z to ∞ in $\mathbb{H}$ that does not intersect $C\setminus\{z\}$. Then for each j,

$$\mathcal{E}_D(\eta_j,\tilde{\gamma})\leq\mathcal{E}_D(\eta_j,C),$$

and hence, using (2.4),

$$\sum_{j=1}^{\infty}\mathcal{E}_D(\eta_j,\tilde{\gamma})\leq\sum_{j=1}^{\infty}\mathcal{E}_D(\eta_j,C)\leq 2\,\mathcal{E}_D(C_r,C)\leq 2\,\mathcal{E}_{\mathbb{H}}(C_r,C)\leq c\,e^{-r}.$$

The second inequality above uses Lemma 3.3. □

Proposition 3.5. *If $0<\kappa\leq 4$, there exists $c<\infty$ such that if $0<s\leq r$, $D\subset\mathbb{D}$ is a simply connected domain, and one of the following hold:*

- $\partial D \cap \overline{\mathbb{D}}_r = \{z, w\}$, $z \neq w$, *and* γ *is an* SLE$_\kappa$ *curve in* D *from* z *to* w,
- $\partial D \cap \overline{\mathbb{D}}_r = \{z\}$, *and* γ *is an* SLE$_\kappa$ *curve in* D *from* z *to* 0,

then

$$\mathbf{P}\{\gamma \cap C_s \neq \emptyset\} \leq c\, e^{-(4a-1)(r-s)/2}.$$

Proof. We may assume that $r \geq s+1$. Let $\tilde{\gamma}$ be a straight line from z to w (in the first case) or 0 (in the second case). Let S be the set of crosscuts of D that are subsets of the circle C_s. Then for each $\eta \in S$, $\mathcal{E}_D(\eta, \tilde{\gamma}) \leq \mathcal{E}_D(\eta, C_r)$. Therefore, using (2.5),

$$\sum_{\eta \in S} \mathcal{E}_D(\eta, \tilde{\gamma}) \leq \sum_{\eta \in S} \mathcal{E}_D(\eta, C_r) \leq 2\, \mathcal{E}_D(C_s, C_r) \leq c\, e^{-(r-s)/2}.$$

The middle inequality uses Lemma 3.3 and the last inequality uses the Beurling estimate. The result then follows from Proposition 3.2. □

4 Proof of Lemma 3.3

Here we prove Lemma 3.3. The idea is simple, and is illustrated in Figure 1. Suppose D is a simply connected domain and η is a crosscut of D contained in a circle C_s. Suppose

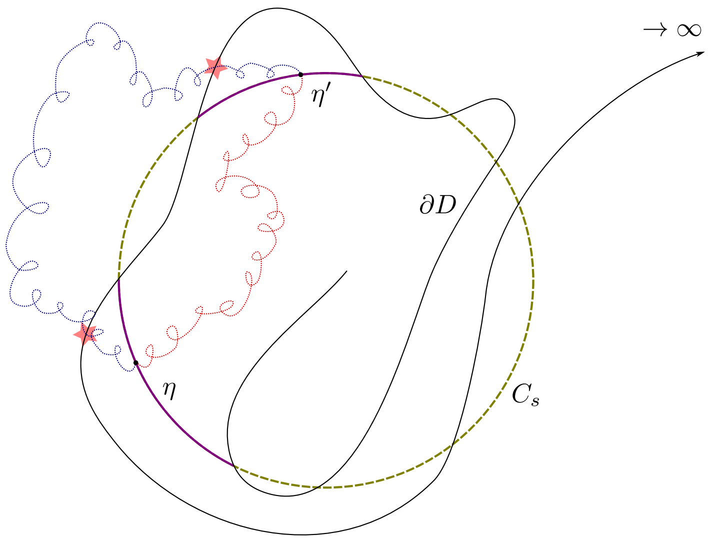

Figure 1: The idea behind Lemma 3.3.

we start a Brownian motion path on η and stop it when it reaches another crosscut η' on C_s. Consider the union of the path and its reflection about C_s. If the union is in D, then we have disconnected the boundary, contradicting the simple connectedness of D. Hence, for every such path, either the path or its reflection hits ∂D before reaching the new crosscut. In this section we make this argument precise.

Proposition 4.1. *Let* D *be a domain such that there is only one connected component of* D^c *that intersects* $\hat{\mathbb{R}}$. *Let* S *denote the set of crosscuts of* D *that are subsets of* $\hat{\mathbb{R}}$, *so that* $D \cap \hat{\mathbb{R}} = \bigcup_{\eta \in S} \eta$. *Let* $x \in D \cap \hat{\mathbb{R}}$ *be contained in crosscut* η_x, *let* $S_x = S \setminus \{\eta_x\}$ *and let*

$$E_x = \bigcup_{\eta \in S_x} \{\sigma_\eta < \tau_D\}$$

be the event that the Brownian motion reaches a crosscut other than η_x *before leaving the domain* D. *Then* $\mathbf{P}^x(E_x) \leq 1/2$.

We note the following.

- The conditions of the proposition hold if D is simply connected, or if D is the intersection of a simply connected domain with a domain containing $\hat{\mathbb{R}}$.
- The constant $1/2$ is optimal as can be seen by considering $D = \mathbb{C} \setminus e^{i\theta}[0,\infty)$ for $0 < \theta < \pi$. If $x > 0$, then $\mathbf{P}^x\{\sigma_{(-\infty,0)} < \tau_D\} = \theta/(\theta+\pi) \to 1/2$ as $\theta \to \pi$.

Proof. Let $\overline{D} = \{\overline{z} : z \in D\}$ be the reflection of D about the real line. Let $\overline{E}_x$ be the event that $\sigma_\eta < \tau_{\overline{D}}$ for some $\eta \in S_x$. In other words, this is the event E_x applied to the complex conjugate $\overline{B_t}$ rather than B_t. Since Brownian motion starting from x is invariant under complex conjugation, $\mathbf{P}^x(E_x) = \mathbf{P}^x(\overline{E}_x)$.

On the event $E_x \cap \overline{E}_x$, the path $B[0,\sigma_\eta] \cup \overline{B}[0,\sigma_\eta]$ lies entirely within D, but separates one endpoint of η from the other. But this is impossible, as the two endpoints of η lie in the same connected complement of D^c. Hence $\mathbf{P}^x(E_x \cap \overline{E}_x) = 0$, and so $\mathbf{P}^x(E_x) \leq 1/2$. ☐

Corollary 4.2. *Let D be a domain such that there is only one component of D^c that intersects C. Let S be the set of crosscuts of D that are subsets of the unit circle C. Let $\tau = \tau_D$. Let*

$$V = \sum_{\eta \in S} 1\{\sigma_\eta < \tau\}$$

be the number of distinct crosscuts visited by the Brownian motion by time τ. Then for any $z \in D$,

$$\mathbf{E}^z[V] \leq 2\,\mathbf{P}^z\{\sigma_C < \tau\}.$$

Proof. First observe that

$$\mathbf{E}^z[V] = \mathbf{E}^z\{V \mid \sigma_C < \tau\}\,\mathbf{P}^z\{\sigma_C < \tau\} \leq \mathbf{P}^z\{\sigma_C < \tau\} \sup_{w \in C} \mathbf{E}^w[V].$$

Consider a Möbius transformation of the Riemann sphere that sends C to $\mathbb{R}$. Since Brownian motion is conformally invariant, it follows from Proposition 4.1 that, for all $w \in C$,

$$\mathbf{P}^w\{\tau < \sigma_\eta \text{ for all } \eta \in S \text{ with } w \notin \eta\} \geq 1/2,$$

Therefore, the number of crosscuts hit before leaving D by a Brownian motion started from any $w \in C$ is stochastically dominated by a geometric random variable of parameter $1/2$, and hence $\mathbf{E}^w[V] \leq 2$. ☐

Proof of Lemma 3.3. We may assume without loss of generality that $r > s$. Let $D_1 = D \setminus \overline{\mathbb{D}}_r$. Using the definition of excursion measure, it suffices to show that, for any $z \in C_r$ and $\epsilon > 0$ small, if $w = z(1+\epsilon)$,

$$\sum_{\eta \in S} h_{D_1 \setminus \eta}(w,\eta) \leq 2\, h_{D_1 \setminus C_s}(w, C_s) \tag{4.1}$$

Let D_2 be the connected component of D_1 that contains w, so that we may replace D_1 by D_2 in (4.1). Now either $\overline{\mathbb{D}}_r \subset D$, in which case D_2^c consists of $\overline{\mathbb{D}}_r$ (which does not intersect C_s) and one other component; or $\overline{\mathbb{D}}_r \not\subset D$, in which case D_1^c has only one connected component and hence D_2 is simply connected. In either case we see that Corollary 4.2 is applicable and says that

$$\sum_{\eta \in S} \mathbf{P}^w\{\sigma_\eta < \tau_{D_2}\} \leq 2\,\mathbf{P}^w\{\sigma_{C_s} < \tau_{D_2}\}.$$

This yields (4.1). ☐

5 Two-sided radial SLE

With Lemma 3.3 the proofs of Theorems 1.1 and 1.2 are complete. In this section we will prove Theorem 1.3.

Lemma 5.1. *Let $\gamma : [0,1] \to \overline{\mathbb{H}}$ be a simple curve with $\gamma(0) = 0$ and $\max_t |\gamma(t)| = r \leq 1/4$. Let $\eta : [0,1] \to \overline{\mathbb{H}}$ be a simple curve with $\eta(0) \in \mathbb{R}$ and $\operatorname{dist}(0, \eta) = 1$. Then*

$$\mathcal{E}_{\mathbb{H}}(\gamma, \eta) \asymp r \left(\operatorname{diam}(\eta) \wedge 1\right).$$

Proof. Let $D = \{z \in \mathbb{H} : |z| < 1/2\}$. If $|z| = 1/2$, one can deduce from basic properties of Brownian motion that

$$\mathcal{E}_{D\setminus\gamma}(z, \gamma) \asymp \operatorname{Im}(z) \operatorname{diam}(\gamma) \asymp r \operatorname{Im}(z),$$
$$h_{\mathbb{H}\setminus(\gamma\cup\eta)}(z, \eta) \asymp \operatorname{Im}(z) \left(\operatorname{diam}(\eta) \wedge 1\right).$$

Hence the result follows from (2.1) and (2.2). □

From this lemma we deduce an estimate for the excursion measure in the half-plane between two curves that touch the boundary.

Corollary 5.2. *If $\gamma, \eta : [0,1] \to \overline{\mathbb{H}}$ are disjoint simple curves with $\gamma(0), \eta(0) \in \mathbb{R}$, then*

$$\mathcal{E}_{\mathbb{H}}(\gamma, \eta) \wedge 1 \asymp \Big(\frac{\operatorname{diam}(\gamma)}{\operatorname{dist}(\gamma, \eta)} \wedge 1\Big)\Big(\frac{\operatorname{diam}(\eta)}{\operatorname{dist}(\gamma, \eta)} \wedge 1\Big).$$

Proof. We may assume without loss of generality that $\operatorname{diam}(\gamma) \leq \operatorname{diam}(\eta)$, $\gamma(0) = 0$ and $\operatorname{dist}(0, \eta) = 1$. Let $T = \inf\{t : \gamma(t) = 1/4\}$.

If $T = \infty$ then $\operatorname{dist}(\gamma, \eta) \in (3/4, 1]$ and hence Lemma 5.1 gives the claim.

If $T < \infty$, Lemma 5.1 shows that $\mathcal{E}_{\mathbb{H}}(\gamma[0,T], \eta) \asymp 1$, which implies the claim since $\operatorname{dist}(\gamma, \eta) \leq 1$. □

Lemma 5.3. *There exist $\delta > 0$ and $c < \infty$ such that if $\hat{\eta}$ is a crosscut of $\mathbb{H}$ and $\hat{I}$ is a simple curve in $\overline{\mathbb{H}}$ from 0 to $e^{i\theta}$ with $\mathcal{E}_{\mathbb{H}}(\hat{I}, \hat{\eta}) \leq \delta$, then*

$$\frac{\operatorname{diam}(\hat{\eta})}{\operatorname{dist}(\hat{\eta}, 0)} \wedge 1 < c \operatorname{dist}(e^{i\theta}, \hat{\eta})\, \mathcal{E}_{\mathbb{H}}(\hat{I}, \hat{\eta}).$$

Proof. Write $d = \operatorname{dist}(e^{i\theta}, \hat{\eta})$, $d_0 = \operatorname{dist}(\hat{\eta}, 0)$ and $\mathcal{E} = \mathcal{E}_{\mathbb{H}}(\hat{I}, \hat{\eta})$.

By Corollary 5.2, there is a universal constant $c_0 \in (0,1)$ such that

$$\mathcal{E} \wedge 1 > c_0 \Big(\frac{1}{d} \wedge 1\Big)\Big(\frac{\operatorname{diam}(\hat{\eta})}{d \wedge d_0} \wedge 1\Big). \tag{5.1}$$

If $d \geq 1$, the estimate (5.1) shows that

$$\mathcal{E} > c_0 \, \frac{1}{d}\Big(\frac{\operatorname{diam}(\hat{\eta})}{d_0} \wedge 1\Big),$$

and the conclusion follows.

Suppose that $d < 1$, a case which is illustrated in Figure 2. We will assume that $\mathcal{E} \leq \delta := c_0/4$. Then (5.1) implies that

$$\mathcal{E} > c_0 \, \frac{\operatorname{diam}(\hat{\eta})}{d} \tag{5.2}$$

and hence $\operatorname{diam}(\hat{\eta}) < d\, \mathcal{E}/c_0 < 1/4$. If $d_0 > 1/4$, the conclusion follows directly from (5.2). Otherwise, we see that $1/2 < d < 1$, in which case the conclusion follows from (5.1). □

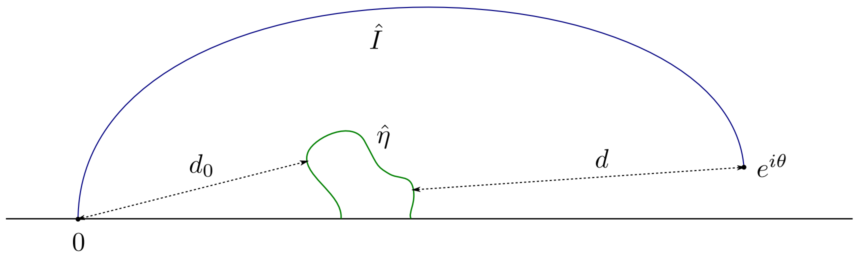

Figure 2: The case $d < 1$ in Lemma 5.3.

In [3], the boundary estimate for radial SLE$_\kappa$ is first proved for a finite time, and then extended to infinite time. We follow the same strategy here to prove the boundary estimate for two-sided radial SLE$_\kappa$.

Given a domain D and $z, w \in \partial D$, let g be a conformal map sending D, z, w to $\mathbb{H}, 0, \infty$. Recall that the SLE$_\kappa$ Green's function at $\zeta \in D$ is

$$G_D(\zeta; z, w) = \hat{c} \left(\frac{\operatorname{Im} g(\zeta)}{|g'(\zeta)|} \right)^{d-2} S_D(\zeta; z, w)^{4a-1},$$

where $S_D(\zeta; z, w) = \sin \arg g(\zeta)$. As before, we write γ_t for the image $\gamma[0, t]$.

Proposition 5.4. *If $0 < \kappa < 8$, there exists $c < \infty$ such that the following holds for all θ. Let γ be a two-sided radial SLE$_\kappa$ curve in $\mathbb{D}$ from 1 to $e^{-2i\theta}$ through 0 run until the stopping time $\rho = \inf\{t : |\gamma(t)| = \frac{1}{16}\}$. Let η be a crosscut of $\mathbb{D}$. Then*

$$\mathbf{P}\{\gamma[0, \rho] \cap \eta \neq \emptyset\} \leq c \left(\frac{\operatorname{diam}(\eta)}{\operatorname{dist}(\eta, 1)} \right)^{4a-1}.$$

Proof. It suffices to prove the result for $\operatorname{diam}(\eta)/\operatorname{dist}(\eta, 1)$ sufficiently small. For notational convenience we will write $\mathcal{E} = \mathcal{E}_{\mathbb{D}}([0, 1], \eta)$ and $\alpha = 1/(4a-1)$. Recall that $\mathcal{E} \wedge 1 \asymp \big(\operatorname{diam}(\eta)/\operatorname{dist}(\eta, 1)\big) \wedge 1$. Hence it suffices to prove the result for $\mathcal{E} \leq \delta$ where δ is as in Lemma 5.3 and $\operatorname{diam}(\eta) \leq (1/100) \operatorname{dist}(\eta, 1) \leq 1/50$. We assume throughout that η satisfies this and hence that $\mathcal{E} \asymp \operatorname{diam}(\eta)/\operatorname{dist}(\eta, 1)$.

Write $\tau = \inf\{t : \gamma(t) \in \eta\}$. Let $\mathbf{P}_0$ be a law under which γ is a chordal SLE$_\kappa$ curve in $\mathbb{D}$ from 1 to $e^{-2i\theta}$, once again stopped on reaching $\frac{1}{16}C$. It follows from the definition of two-sided radial SLE$_\kappa$ that for any stopping time $\sigma \leq \rho$,

$$\frac{d\mathbf{P}}{d\mathbf{P}_0}(\gamma_\sigma) = 1\{\sigma < \infty\} \frac{G_\sigma}{G_0} \asymp 1\{\sigma < \infty\} \left(\frac{S_\sigma}{S_0} \right)^{4a-1}.$$

since the conformal radius never changes by more than a factor of 16. Here we have denoted $G_\sigma = G_{\mathbb{D} \setminus \gamma_\sigma}(0; \gamma(\sigma), e^{-2i\theta})$ and S_σ similarly.

There is a unique conformal transformation g sending $\mathbb{D}$ to $\mathbb{H}$, 1 to 0, $e^{-2i\theta}$ to ∞ and 0 to $e^{i\theta}$. Indeed, this is just the fractional linear transformation

$$g(z) = e^{-i\theta} \frac{z - 1}{z - e^{-2i\theta}}.$$

We write $\hat{I} = g([0, 1])$, $\hat{\eta} = g(\eta)$, $\hat{\gamma} = g \circ \gamma$, and let $K_\theta = g(\{|z| \leq 1/16\})$. By a reflection we may assume that $0 < \theta \leq \pi/2$. Note that $|g'(0)| = 2 \sin \theta$; this can be seen be direct calculation or by noting that $|g'(0)|$ gives the conformal radius of $\mathbb{H}$ with respect to $e^{i\theta}$. Using the Koebe $1/4$ theorem on g^{-1} we can see that K_θ is contained in the disk of radius $[\sin \theta]/2$ about $e^{i\theta}$, and hence in the disk of radius 2θ about 1. By the definition of ρ, we have that $\rho = \inf\{t : \hat{\gamma}(t) \in K_\theta\}$.

The argument proceeds in two cases depending on the position of $\hat\eta$ in $\mathbb{H}$. Let

$$d = \mathrm{dist}(e^{i\theta}, \hat\eta).$$

Case 1 (trapped case): $d > 100\,\theta$ and the endpoints of $\hat\eta$ lie on $(0,1)$.

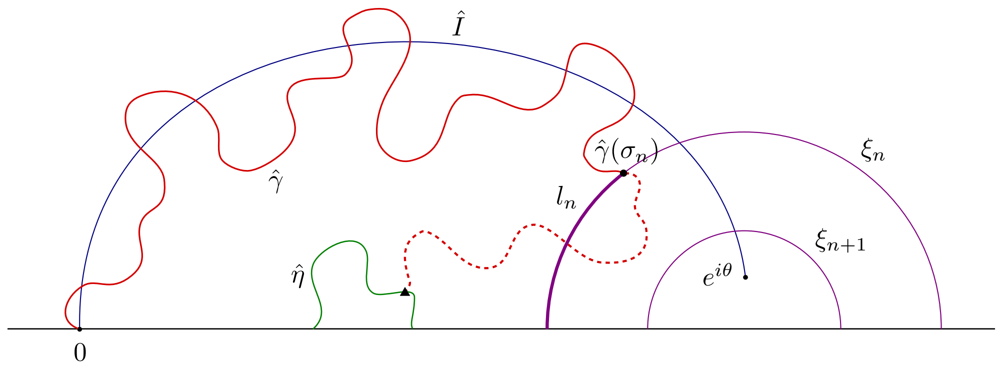

Figure 3: The trapped case in Proposition 5.4.

Let $\sigma_n = \inf\{t : |\hat\gamma(t) - 1| = 2^{-n}\,d\}$. Define integer k by $2\theta \le 2^{-k}\,d < 4\theta$ and note that $\sigma_k < \rho$ if $\sigma_k < \infty$. For $n > 0$, let V_n denote the event $\{\sigma_n < \infty, \sigma_n < \tau\}$. On the event V_n, let ξ_n be the circle of radius $2^{-n}\,d$ about 1, l_n the circular arc of ξ_n from $\hat\gamma(\sigma_n)$ to $1 - 2^{-n}\,d$, and $H_n = \mathbb{H} \setminus (\hat\gamma_{\sigma_n} \cup l_n)$. The boundary estimate (Proposition 3.1) implies that

$$\mathbf{P}_0(V_n)^\alpha \le \mathbf{P}_0\{\sigma_n < \infty\}^\alpha \asymp 2^{-n}\,d.$$

The curve l_n disconnects $\hat\eta$ from infinity, and hence on the event V_n,

$$\mathbf{P}_0\{\tau < \infty \mid \mathcal{F}_{\sigma_n}\}^\alpha < c\,\mathcal{E}_{H_n}(l_n, \hat\eta).$$

Using Corollary 5.2, we can see that if $n \ge 1$,

$$\mathcal{E}_{H_n}(l_n, \hat\eta) \le \mathcal{E}_{\mathbb{H}}(\xi_n, \hat\eta) \le c\,\frac{\mathrm{diam}(\hat\eta)}{\mathrm{dist}(\hat\eta, 1)}\,2^{-n} \le c\,2^{-n}\,\mathcal{E}.$$

Therefore,

$$\mathbf{P}_0\{\sigma_n < \tau < \rho\}^\alpha < c\,2^{-n}\,d \cdot 2^{-n}\,\mathcal{E}.$$

Note that $S_{\sigma_{n+1}} < c\,\theta/(2^{-n}\,d) \asymp S_0/(2^{-n}\,d)$. It follows that

$$\mathbf{P}\{\sigma_n < \tau < \sigma_{n+1}\}^\alpha < c\,2^{-n}\,\mathcal{E}, \qquad n < k.$$

Since $S_\rho \le 1 \asymp S_0/(2^{-k}\,d)$, we also get

$$\mathbf{P}\{\sigma_k < \tau < \rho\}^\alpha < c\,2^{-k}\,\mathcal{E}.$$

Similarly, $\mathbf{P}_0\{\tau < \infty\}^\alpha < c\,d\,\mathcal{E}$ and $S_{\sigma_1} < c\,\theta/d \asymp S_0/d$ if $\tau < \sigma_1$, and therefore $\mathbf{P}\{\tau < \sigma_1\}^\alpha < c\,\mathcal{E}$. Adding the estimates so obtained, we see that

$$\mathbf{P}\{\tau < \rho\}^\alpha < c\,\mathcal{E}.$$

Case 2 (untrapped case): any pair $(\theta, \hat\eta)$ not covered in Case 1.
In this case, it suffices to show that

$$\mathbf{P}_0\{\tau < \infty\}^\alpha < c\,d\,\mathcal{E}, \tag{5.3}$$

$$S_\tau < c\,\theta/d, \tag{5.4}$$

for we then have that $\mathbf{P}\{\tau < \rho\}^\alpha < c\,\mathcal{E}$. Estimate (5.3) follows, in light of Lemma 5.3, from the basic boundary estimate for chordal SLE$_\kappa$.

It remains only to prove estimate (5.4). This is trivial if $d \leq 100\,\theta$, so for the remainder we may assume that $d > 100\,\theta$. It then follows that $\hat{\eta}$ does not intersect the line segment J connecting 1 and $e^{i\theta}$, so $\hat{\eta} \subseteq \overline{\mathbb{H}} \setminus (\hat{I} \cup J)$. Because the sets $(0,1)$ and $\mathbb{R} \setminus [0,1]$ lie in different components of the disconnection $\overline{\mathbb{H}} \setminus (\hat{I} \cup J)$, it is impossible to have one endpoint of $\hat{\eta}$ in $(0,1)$ and the other outside $[0,1]$. Since $d > 100\,\theta$, this implies that both endpoints of $\hat{\eta}$ lie outside $[0,1]$ (else we would be in the trapped case). Since $\theta \leq \pi/2$, we can see that d is comparable to the radius of the largest disk centered at 1 that does not intersect $\hat{\eta}$.

Suppose $\hat{\gamma}_\tau$ is given and let B be a complex Brownian motion starting at $\zeta := e^{i\theta}$. Let U be the disk centered at ζ of radius $d/2$ and

$$T_1 = \inf\{t : B_t \notin \mathbb{H} \setminus \hat{\gamma}_\tau\},$$
$$T_2 = \inf\{t : |B_t - \zeta| = d/2\} = \inf\{t : B_t \notin U\}.$$

It is standard that $\mathbf{P}\{T_2 < T_1\} = O(\theta/d)$. Hence it suffices to prove that all paths with $T_1 \leq T_2$ leave on the same side of the domain $(\mathbb{H} \setminus \hat{\gamma}_\tau, \hat{\gamma}(\tau), \infty)$, in that the images of their endpoints under the conformal map that sends $\mathbb{H} \setminus \hat{\gamma}_\tau, \hat{\gamma}(\tau), \infty$ to $\mathbb{H}, 0, \infty$ are either all positive or all negative.

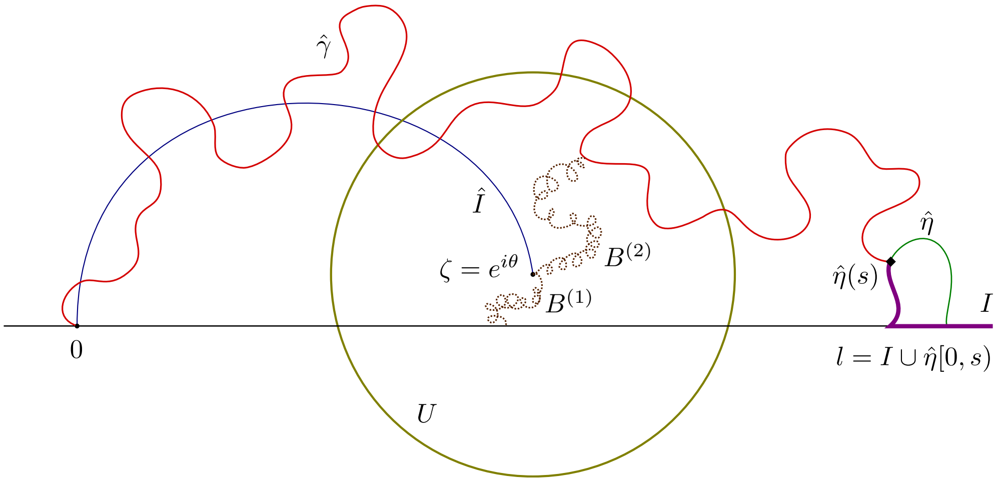

Figure 4: The untrapped case in Proposition 5.4.

To see this, suppose that $\hat{\gamma}(\tau) = \hat{\eta}(s)$. If $\hat{\eta}(0) > 1$, let $I = [\hat{\eta}(0), \infty)$; otherwise, $\hat{\eta}(0) < 0$ and we let $I = (-\infty, \hat{\eta}(0)]$. Then let $l = I \cup \hat{\eta}[0, s)$, which is the trace of a curve in $\overline{\mathbb{H} \setminus \hat{\gamma}_\tau}$ from $\hat{\gamma}(\tau)$ to ∞ that does not intersect U. Hence, any curve in $\mathbb{H} \setminus \hat{\gamma}_\tau$ that joins the two sides of $(\mathbb{H} \setminus \hat{\gamma}_\tau, \hat{\gamma}(\tau), \infty)$ must intersect l (possibly at an endpoint) and thus must leave U. Therefore, if two Brownian paths $B^{(1)}$ and $B^{(2)}$ starting at ζ both satisfy $T_1 \leq T_2$, then they leave on the same side of the domain. $\square$

We may deduce the basic boundary estimate for two-sided radial SLE$_\kappa$ run until it hits the target point. First, we prove a modification of Lemma 2.9 in [3].

Lemma 5.5. *There exists $c < \infty$ such that the following holds. Let η be a crosscut of $\mathbb{D}$ that lies outside $\frac{1}{2}\mathbb{D}$. Let $\gamma : (0,1] \to \mathbb{D}$ be a curve starting at 1 stopped when it first hits C_n, and suppose that $\gamma \cap \eta = \emptyset$. Let D be the connected component of $\mathbb{D} \setminus \gamma$*

containing 0 *and* $g : D \to \mathbb{D}$ *the conformal map with* $g(\gamma(1)) = 1$ *and* $g(0) = 0$*. Then*

$$\frac{\operatorname{diam}(g(\eta))}{\operatorname{dist}(g(\eta),1)} \leq c\, e^{-n/2}\, \operatorname{diam}(\eta).$$

Proof. Let H denote the connected component of $D \cap \mathbb{D}_1$ containing the origin. Note that $\partial H \cap C_1$ is a countable (perhaps infinite) collection of open subarcs. However, there is a unique one that separates η from 0 in H. Let l denote this subarc. Consider the conformal rectangle R that is the connected component of $D \setminus (l \cup C_n)$ that contains both C_n and l on its boundary. We can write $\partial R = C_n \cup \partial_+ \cup \partial_- \cup l$, which we consider as arcs of ∂R in the sense of prime ends. (For example, if γ is simple then each point of γ corresponds to two points of ∂D.) Note that $g(\partial_+)$, $g(\partial_-)$ are adjacent intervals in $\partial\mathbb{D}$ with common boundary point 1, and $g(\eta)$ is separated from 0 by $g(l)$. Using conformal invariance we can see that

$$\begin{aligned} \operatorname{dist}(g(\eta),1) &\geq c \min\{h_D(0,\partial_+), h_D(0,\partial_-)\}, \\ \operatorname{diam} g(\eta) &\leq c\, h_D(0,\eta). \end{aligned}$$

There is a conformal transformation

$$f : R \to \mathcal{R}_L := \{x + iy : 0 < x < L,\ 0 < y < \pi\}$$

that maps ∂_-, ∂_+ onto the two horizontal boundaries. The length L is a conformal invariant, but the Beurling estimate shows that $L \geq n/2 - c$. If $\zeta \in \mathcal{R}_L$ with $\operatorname{Re}\zeta \leq 1$, then, one can check directly that

$$h_{\mathcal{R}_L}(\zeta, L + i(0,\pi)) \leq c\, e^{-L}\, \min\bigl\{h_{\mathcal{R}_L}(\zeta,(0,L)), h_{\mathcal{R}_L}(\zeta, i\pi + (0,L))\bigr\}.$$

By starting a Brownian motion from 0 and considering the sequence of hitting times of C_n and $f^{-1}\{\operatorname{Re}\zeta = 1\}$ before it leaves H, it is not hard to see that

$$\begin{aligned} h_H(0,l) &\leq c\, e^{-n/2}\, \min\bigl\{h_H(0,\partial_+), h_H(0,\partial_-)\bigr\} \\ &\leq c\, e^{-n/2}\, \min\bigl\{h_D(0,\partial_+), h_D(0,\partial_-)\bigr\} \\ &\leq c\, e^{-n/2}\, \operatorname{dist}(g(\eta),1). \end{aligned}$$

Finally, a simple application of the Harnack inequality implies that if $|\zeta| \leq e^{-1}$, then

$$h_{D\setminus\eta}(\zeta,\eta) \leq h_{\mathbb{D}}(\zeta,\eta) \leq c\, h_{\mathbb{D}}(0,\eta) \leq c\, \operatorname{diam}(\eta),$$

and hence

$$\begin{aligned} \operatorname{diam}(g(\eta)) \asymp h_{D\setminus\eta}(0,\eta) &\leq h_{D\setminus\eta}(0,l) \sup_{\zeta\in l} h_{D\setminus\eta}(\zeta,\eta) \\ &\leq c\, e^{-n/2}\, \operatorname{diam}(\eta)\, \operatorname{dist}(g(\eta),1). \end{aligned}$$ □

Corollary 5.6. *If* $0 < \kappa < 8$*, there exists* $c < \infty$ *such that the following holds. Let* D *be a simply connected domain with* $z, w \in \partial D$ *and* $\zeta \in D$*. Let* η *be a crosscut of* D *and* $\tilde\gamma$ *be any curve in* D *from* z *to* ζ*. Let* γ *be a two-sided radial SLE*$_\kappa$ *curve in* D *from* z *to* w *through* ζ *stopped when it reaches* ζ*. Then*

$$\mathbf{P}\{\gamma \cap \eta \neq \emptyset\} \leq c\, \mathcal{E}_D(\tilde\gamma,\eta)^{4a-1}.$$

Proof. By conformal invariance it suffices to consider the case $D = \mathbb{D}$, $z = 1$, $\zeta = 0$, $w = e^{-2i\theta}$. Let D_t be the connected component of $\mathbb{D} \setminus \gamma_t$ containing 0 and let $g_t : D_t \to \mathbb{D}$

be the conformal map sending 0 to 0 and $\gamma(t)$ to 1. Let $\sigma_n = \inf\{t : |\gamma(t)| = 16^{-n}\}$ and $\eta_n = g_{\sigma_n} \circ \eta$.

On the event $\{\gamma_{\sigma_n} \cap \eta = \emptyset\}$, Proposition 5.4 implies that

$$\mathbf{P}\{\gamma[\sigma_n, \sigma_{n+1}] \cap \eta \neq \emptyset \mid \gamma_{\sigma_n}\} \leq c \left(\frac{\mathrm{diam}(\eta_n)}{\mathrm{dist}(1, \eta_n)}\right)^{4a-1},$$

and Lemma 5.5 gives

$$\frac{\mathrm{diam}(\eta_n)}{\mathrm{dist}(1, \eta_n)} \leq c \, \frac{\mathrm{diam}(\eta)}{\mathrm{dist}(1, \eta)} \, 4^{-n}.$$

Therefore,

$$\mathbf{P}\{\gamma[\sigma_n, \sigma_{n+1}] \mid \gamma_{\sigma_n} \cap \eta = \emptyset\} \leq c \left(\frac{\mathrm{diam}(\eta)}{\mathrm{dist}(1, \eta)} \, 4^{-n}\right)^{4a-1},$$

and summing these probabilities, we see that

$$\mathbf{P}\{\gamma \cap \eta \neq \emptyset\} \leq c \left(\frac{\mathrm{diam}(\eta)}{\mathrm{dist}(1, \eta)}\right)^{4a-1}. \qquad \square$$

The proof of Theorem 1.3 follows from Corollary 5.6 by the same argument as for radial SLE$_\kappa$, which is given in Proposition 3.5.

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