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Stochastic evolution equations with multiplicative noise

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Abstract

We study parabolic stochastic partial differential equations (SPDEs), driven by two types of operators: one linear closed operator generating a C_0-semigroup and one linear bounded operator with Wick-type multiplication, all of them set in the infinite dimensional space framework of white noise analysis. We prove existence and uniqueness of solutions for this class of SPDEs. In particular, we also treat the stationary case when the time-derivative is equal to zero.

Keywords: generalized stochastic process; chaos expansion; stochastic evolution equation; Wick product; white noise; C_0-semigroup; infinitesimal generator; stochastic operator.
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1 Introduction and definitions

We consider a stochastic Cauchy problem of the form

$$\begin{aligned} \frac{d}{dt}U(t,x,\omega) &= \mathbf{A}U(t,x,\omega) + \mathbf{B}\Diamond U(t,x,\omega) + F(t,x,\omega) \\ U(0,x,\omega) &= U^0(x,\omega), \end{aligned} \tag{1.1}$$

where $t \in (0,T]$, $\omega \in \Omega$, and $U(t,\cdot,\omega)$ belongs to some Banach space X. The operator $\mathbf{A}$ is densely defined, generating a C_0-semigroup and $\mathbf{B}$ is a linear bounded operator which combined with the Wick product $\Diamond$ introduces convolution-type perturbations into the equation. All stochastic processes are considered in the setting of Wiener-Itô chaos expansions. A comprehensive explanation of the action of the operators $\mathbf{A}$ and $\mathbf{B}$ in this framework will be provided in Section 2.

Our investigations in this paper are inspired by [12] where the authors provide a comprehensive analysis of equations of the form

$$\frac{d}{dt}u(t,x,\omega) = \mathbf{A}u(t,x,\omega) + \delta(\mathbf{M}u(t,x,\omega)) = \mathbf{A}u(t,x,\omega) + \int \mathbf{M}u(t,x,\omega)\Diamond W(x,\omega)\, dx,$$

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where δ denotes the Skorokhod integral and W denotes the spatial white noise process. In Proposition 2.8 we prove that for every operator $\mathbf{M}$ there exists a corresponding operator $\mathbf{B}$ such that $\mathbf{B}\lozenge u = \delta(\mathbf{M}u)$. On the other hand, the class of operators $\mathbf{B}$ is much larger. This holds also for the class of operators $\mathbf{A}$ we consider (a comprehensive analysis of all operators is given in Section 2.1). Thus, we extend the results of [12] and [13] to a more general class of stochastic differential equations which are driven by two linear multiplicative operators: $\mathbf{A}$ acting with ordinary multiplication, while $\mathbf{B}\lozenge$ is acting with the convolution-type Wick product.

We have studied elliptic SPDEs, in particular the stochastic Dirichlet problem of the form $\mathbf{L}\lozenge u + f = 0$ in our previous papers [11], [18], [19]. As a conclusion to this series of papers we study parabolic SPDEs of the form (2.1). Such equations also include as a special case equations of the form $\frac{d}{dt}u = \mathbf{L}u + f$ and $\frac{d}{dt}u = \mathbf{L}\lozenge u + f$, where L is a strictly elliptic second order partial differential operator. These equations describe the heat conduction in random media (inhomogeneous and anisotropic materials), where the properties of the material are modeled by a positively definite stochastic matrix.

Other special cases of (2.1) include the heat equation with random potential $\frac{d}{dt}u = \Delta u + \mathbf{B}\lozenge u$, the Schrödinger equation $(i\hbar)\frac{d}{dt}u = \Delta u + \mathbf{B}\lozenge u + f$, the transport equation $\frac{d}{dt}u = \frac{d^2}{dx^2}u + W\lozenge \frac{d}{dx}u$ driven by white noise as in [20], the generalized Langevin equation $\frac{d}{dt}u = \mathbf{J}u + \mathbf{C}(Y')$, where Y is a Lévy process, $\mathbf{J}$ the infinitesimal generator of a C_0-semigroup and $\mathbf{C}$ a bounded operator, which was studied in [1], as well as the equation $\frac{d}{dt}u = \mathbf{L}u + W\lozenge u$, where $\mathbf{L}$ is a strictly elliptic partial differential operator as studied in [3] and [8].

Equations of the form $\frac{d}{dt}u = \mathbf{A}u + \mathbf{B}W$ were also studied in [14] and [15], where $\mathbf{A}$ is not necessarily generating a C_0-semigroup, but an r-integrated or a convolution semigroup.

In order to solve (2.1) we apply the method of Wiener-Itô chaos expansions, also known as the propagator method. With this method we reduce the SPDE to an infinite triangular system of PDEs, which can be solved by induction. Summing up all coefficients of the expansion and proving convergence in an appropriate weight space, one obtains the solution of the initial SPDE.

We also consider the case of stationary equations $\mathbf{A}U + \mathbf{B}\lozenge U + F = 0$. In particular, elliptic SPDEs have been studied in [11], [13], [18] and [19]. With the method of chaos expansions one can also treat hyperbolic SPDEs [9] and SPDEs with singularities [21]. One of its advantages is that it provides explicit solutions in terms of a series expansion, which can be easily implemented also to numerical approximations and computational simulations.

1.1 C_0-semigroups

We recall some well-known facts which will be used in the sequel (see [16]). Let X be a Banach space. If B is a bounded linear operator on X and A is the infinitesimal generator of a C_0-semigroup $\{T_t\}_{t\geq 0}$ satisfying $||T_t||_{L(X)} \leq Me^{wt},\ t\geq 0,$ for some $M, w > 0$, then $A + B$ is the infinitesimal generator of a C_0-semigroup $\{S_t\}_{t\geq 0}$, on X satisfying

$$\|S_t\|_{L(X)} \leq Me^{(w+M\|B\|_{L(X)})t},\ t\geq 0.$$

Let $u(0) = u^0 \in D = Dom(A)$ and $f \in C([0,\infty), X)$. Recall that $u : [0,T] \to X$ is a *(classical) solution* on $[0,T]$ to

$$\frac{d}{dt}u(t) = Au(t) + f(t),\ t\in(0,T],\quad u(0) = u^0, \tag{1.2}$$

if u is continuous on $[0,T]$, continuously differentiable on $(0,T]$, $u(t)\in D,\ t\in(0,T]$ and the equation is satisfied on $(0,T]$. If $f \in L^1((0,T),X)$, then $u(t) = T_t u^0 + \int_0^t T_{t-s}f(s)ds, t \in$

$[0,T]$ belongs to $C([0,T],X)$, and it is called a *mild solution*. Clearly, a mild solution that is continuously differentiable on $(0,T]$ is a classical solution.

Let $f \in L^1((0,T),X) \cap C((0,T],X)$ and $v(t) = \int_0^t T_{t-s} f(s)ds, t \in [0,T]$. The initial value problem has a solution u for every $u^0 \in D$ if one of the following conditions is satisfied (see [16]):

(i) v is continuously differentiable on $(0,T)$.

(ii) $v(t) \in D$ for $0 < t \leq T$ and $Av(t)$ is continuous on $(0,T]$.

If the initial value problem has a solution on $[0,T]$ for some $u^0 \in D$, then $v(t)$ satisfies both (i) and (ii). Note that if $f \in C^1([0,T],X)$ then conditions (i) and (ii) are fulfilled. Moreover, if $f \in C^1([0,T],X)$ and $u^0 \in D(A)$, then for the solution u of (1.2) we have that $u \in C^1([0,T],X)$ and $\frac{d}{dt}u(0) = Au^0 + f(0)$.

1.2 Generalized stochastic processes

Denote by $(\Omega, \mathcal{F}, P)$ the Gaussian white noise probability space $(S'(\mathbb{R}), \mathcal{B}, \mu)$, where $S'(\mathbb{R})$ denotes the space of tempered distributions, $\mathcal{B}$ the Borel sigma-algebra generated by the weak topology on $S'(\mathbb{R})$ and μ the Gaussian white noise measure corresponding to the characteristic function

$$\int_{S'(\mathbb{R})} e^{i\langle \omega, \phi \rangle} d\mu(\omega) = \exp\left[-\frac{1}{2}\|\phi\|^2_{L^2(\mathbb{R})}\right], \qquad \phi \in S(\mathbb{R}),$$

given by the Bochner-Minlos theorem.

We recall the notions related to $L^2(\Omega,\mu)$ (see [7]) where $\Omega = S'(\mathbb{R})$ and μ is Gaussian white noise measure. Define the set of multi-indices $\mathcal{I}$ to be $(\mathbb{N}_0^{\mathbb{N}})_c$, i.e. the set of sequences of non-negative integers which have only finitely many nonzero components. Especially, we denote by $\mathbf{0} = (0,0,0,\ldots)$ the multi-index with all entries equal to zero. The length of a multi-index is $|\alpha| = \sum_{i=1}^{\infty} \alpha_i$ for $\alpha = (\alpha_1, \alpha_2, \ldots) \in \mathcal{I}$, and it is always finite. Similarly, $\alpha! = \prod_{i=1}^{\infty} \alpha_i!$, and all other operations are also carried out componentwise. We will use the convention that $\alpha - \beta$ is defined if $\alpha_n - \beta_n \geq 0$ for all $n \in \mathbb{N}$, i.e., if $\alpha - \beta \geq \mathbf{0}$, and leave $\alpha - \beta$ undefined if $\alpha_n < \beta_n$ for some $n \in \mathbb{N}$.

The Wiener-Itô theorem (sometimes also referred to as the Cameron-Martin theorem) states that one can define an orthogonal basis $\{H_\alpha\}_{\alpha \in \mathcal{I}}$ of $L^2(\Omega,\mu)$, where H_α are constructed by means of Hermite orthogonal polynomials h_n and Hermite functions ξ_n,

$$H_\alpha(\omega) = \prod_{n=1}^{\infty} h_{\alpha_n}(\langle \omega, \xi_n \rangle), \quad \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n \ldots) \in \mathcal{I}, \quad \omega \in \Omega = S'(\mathbb{R}).$$

Then, every $F \in L^2(\Omega,\mu)$ can be represented via the so called *chaos expansion*

$$F(\omega) = \sum_{\alpha \in \mathcal{I}} f_\alpha H_\alpha(\omega), \quad \omega \in S'(\mathbb{R}), \quad \sum_{\alpha \in \mathcal{I}} |f_\alpha|^2 \alpha! < \infty, \quad f_\alpha \in \mathbb{R}, \quad \alpha \in \mathcal{I}.$$

Denote by $\varepsilon_k = (0,0,\ldots,1,0,0,\ldots)$, $k \in \mathbb{N}$ the multi-index with the entry 1 at the kth place. Denote by $\mathcal{H}_1$ the subspace of $L^2(\Omega,\mu)$, spanned by the polynomials $H_{\varepsilon_k}(\cdot)$, $k \in \mathbb{N}$. The subspace $\mathcal{H}_1$ contains Gaussian stochastic processes, e.g. Brownian motion is given by the chaos expansion $B(t,\omega) = \sum_{k=1}^{\infty} \int_0^t \xi_k(s)ds\, H_{\varepsilon_k}(\omega)$.

Denote by $\mathcal{H}_m$ the mth order chaos space, i.e. the closure of the linear subspace spanned by the orthogonal polynomials $H_\alpha(\cdot)$ with $|\alpha| = m$, $m \in \mathbb{N}_0$. Then the Wiener-Itô chaos expansion states that $L^2(\Omega,\mu) = \bigoplus_{m=0}^{\infty} \mathcal{H}_m$, where $\mathcal{H}_0$ is the set of constants in $L^2(\Omega,\mu)$.

It is well-known that the time-derivative of Brownian motion (white noise process) does not exist in the classical sense. However, changing the topology on $L^2(\Omega,\mu)$ to a weaker one, T. Hida [6] defined spaces of generalized random variables containing the white noise as a weak derivative of the Brownian motion. We refer to [6], [7] for white noise analysis (as an infinite dimensional analogue of the Schwartz theory of deterministic generalized functions).

Let $(2\mathbb{N})^{\alpha} = \prod_{n=1}^{\infty}(2n)^{\alpha_n}, \quad \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots) \in \mathcal{I}$. We will often use the fact that the series $\sum_{\alpha\in\mathcal{I}}(2\mathbb{N})^{-p\alpha}$ converges for $p > 1$. Define the Banach spaces

$$(S)_{1,p} = \{F = \sum_{\alpha\in\mathcal{I}} f_\alpha H_\alpha \in L^2(\Omega,\mu) : \ \|F\|^2_{(S)_{1,p}} = \sum_{\alpha\in\mathcal{I}} (\alpha!)^2 |f_\alpha|^2 (2\mathbb{N})^{p\alpha} < \infty\}, \quad p \in \mathbb{N}_0.$$

Their topological dual spaces are given by

$$(S)_{-1,-p} = \{F = \sum_{\alpha\in\mathcal{I}} f_\alpha H_\alpha : \ \|F\|^2_{(S)_{-1,-p}} = \sum_{\alpha\in\mathcal{I}} |f_\alpha|^2 (2\mathbb{N})^{-p\alpha} < \infty\}, \quad p \in \mathbb{N}_0.$$

The Kondratiev space of generalized random variables is $(S)_{-1} = \bigcup_{p\in\mathbb{N}_0}(S)_{-1,-p}$ endowed with the inductive topology. It is the strong dual of $(S)_1 = \bigcap_{p\in\mathbb{N}_0}(S)_{1,p}$, called the Kondratiev space of test random variables which is endowed with the projective topology. Thus,

$$(S)_1 \subseteq L^2(\Omega,\mu) \subseteq (S)_{-1}$$

forms a Gelfand triplet.

The time-derivative of the Brownian motion exists in the generalized sense and belongs to the Kondratiev space $(S)_{-1,-p}$ for $p \geq \frac{5}{12}$. We refer to it as to *white noise* and its formal expansion is given by $W(t,\omega) = \sum_{k=1}^{\infty}\xi_k(t)H_{\varepsilon_k}(\omega)$.

We extended in [17] the definition of stochastic processes also to processes of the chaos expansion form $U(t,\omega) = \sum_{\alpha\in\mathcal{I}} u_\alpha(t)H_\alpha(\omega)$, where the coefficients u_α are elements of some Banach space X. We say that U is an *X-valued generalized stochastic process*, i.e. $U(t,\omega) \in X \otimes (S)_{-1}$ if there exists $p > 0$ such that $\|U\|^2_{X\otimes(S)_{-1,-p}} = \sum_{\alpha\in\mathcal{I}} \|u_\alpha\|^2_X (2\mathbb{N})^{-p\alpha} < \infty$.

The *Wick product* of stochastic processes $F = \sum_{\alpha\in\mathcal{I}} f_\alpha H_\alpha, G = \sum_{\beta\in\mathcal{I}} g_\beta H_\beta \in X \otimes (S)_{-1}$ is

$$F\lozenge G = \sum_{\gamma\in\mathcal{I}} \sum_{\alpha+\beta=\gamma} f_\alpha g_\beta H_\gamma = \sum_{\alpha\in\mathcal{I}} \sum_{\beta\leq\alpha} f_\beta g_{\alpha-\beta} H_\alpha,$$

and the nth Wick power is defined by $F^{\lozenge n} = F^{\lozenge(n-1)}\lozenge F$, $F^{\lozenge 0} = 1$. Note that $H_{n\varepsilon_k} = H^{\lozenge n}_{\varepsilon_k}$ for $n \in \mathbb{N}_0$, $k \in \mathbb{N}$.

For example, let $X = C^k[0,T]$, $k \in \mathbb{N}$. In [18] we proved that differentiation of a stochastic process can be carried out componentwise in the chaos expansion, i.e. due to the fact that $(S)_{-1}$ is a nuclear space it holds that $C^k([0,T],(S)_{-1}) = C^k[0,T] \otimes (S)_{-1}$. This means that a stochastic process $U(t,\omega)$ is k times continuously differentiable if and only if all of its coefficients $u_\alpha(t)$, $\alpha \in \mathcal{I}$ are in $C^k[0,T]$.

The same holds for Banach space valued stochastic processes i.e. elements of $C^k([0,T],X) \otimes (S)_{-1}$, where X is an arbitrary Banach space. By the nuclearity of $(S)_{-1}$, these processes can be regarded as elements of the tensor product space

$$C^k([0,T], X\otimes(S)_{-1}) = C^k([0,T],X) \otimes (S)_{-1} = \bigcup_{p=0}^{\infty} C^k([0,T],X) \otimes (S)_{-1,-p}.$$

2 Stochastic operators

Definition 2.1. *Let X be a Banach space and $\mathbf{O} : X \otimes (S)_{-1} \to X \otimes (S)_{-1}$ an operator acting on the space of stochastic processes. We will call $\mathbf{O}$ to be a* coordinatewise operator *if there exists a family of operators $o_\alpha : X \to X$, $\alpha \in \mathcal{I}$, such that $\mathbf{O}(\sum_{\alpha\in\mathcal{I}} f_\alpha H_\alpha) = \sum_{\alpha\in\mathcal{I}} o_\alpha(f_\alpha)H_\alpha$ for all $F = \sum_{\alpha\in\mathcal{I}} f_\alpha H_\alpha \in X \otimes (S)_{-1}$.*

Clearly, not all operators are coordinatewise, for example $\mathbf{O}(F) = F^{\lozenge 2}$ can not be written in this form.

Definition 2.2. *The subclass of* simple coordinatewise operators *consists of operators for which* $o_\alpha = o_\beta = o$, $\alpha, \beta \in \mathcal{I}$, *that is, they can be written in form of* $\mathbf{O}(\sum_{\alpha\in\mathcal{I}} f_\alpha H_\alpha) = \sum_{\alpha\in\mathcal{I}} o(f_\alpha) H_\alpha$ *for some operator* $o : X \to X$.

For example, the operator of differentiation [18] and the Fourier transform [21] are simple coordinatewise operators. The Ornstein-Uhlenbeck operator is a coordinatewise operator but it is not a simple coordinatewise operator.

Note that even if all o_α, $\alpha \in \mathcal{I}$, are bounded linear operators, the coordinatewise operator $\mathbf{O}$ itself does not need to be bounded. If o_α, $\alpha \in \mathcal{I}$, are uniformly bounded by some $C > 0$, then $\mathbf{O}$ is also a bounded operator. This follows from

$$\begin{aligned}\|\mathbf{O}(F)\|^2_{X\otimes(S)_{-1,-p}} &\leq \sum_{\alpha\in\mathcal{I}} \|o_\alpha\|^2_{L(X)} \|f_\alpha\|^2_X (2\mathbb{N})^{-p\alpha} \\ &\leq C^2 \sum_{\alpha\in\mathcal{I}} \|f_\alpha\|^2_X (2\mathbb{N})^{-p\alpha} = C^2 \|F\|^2_{X\otimes(S)_{-1,-p}} < \infty,\end{aligned}$$

for $F \in X \otimes (S)_{-1,-p}$.

This condition is sufficient, but not necessary, and can be loosened by the embedding $(S)_{-1,-p} \subseteq (S)_{-1,-q}$, $q \geq p$.

Lemma 2.3. *Let* $\mathbf{O}$ *be a coordinatewise operator for which all* o_α, $\alpha \in \mathcal{I}$, *are polynomially bounded i.e.* $\|o_\alpha\|_{L(X)} \leq R(2\mathbb{N})^{r\alpha}$ *for some* $r, R > 0$. *Then, there exists* $q \geq p$ *such that* $\mathbf{O} : X \otimes (S)_{-1,-p} \to X \otimes (S)_{-1,-q}$ *is bounded.*

Proof. Let $q \geq p + 2r$. Then,

$$\begin{aligned}\|\mathbf{O}(F)\|^2_{X\otimes(S)_{-1,-q}} &\leq R^2 \sum_{\alpha\in\mathcal{I}} (2\mathbb{N})^{2r\alpha} \|f_\alpha\|^2_X (2\mathbb{N})^{-q\alpha} = R^2 \sum_{\alpha\in\mathcal{I}} \|f_\alpha\|^2_X (2\mathbb{N})^{-(q-2r)\alpha} \\ &\leq R^2 \sum_{\alpha\in\mathcal{I}} \|f_\alpha\|^2_X (2\mathbb{N})^{-p\alpha} = R^2 \|F\|^2_{X\otimes(S)_{-1,-p}} < \infty.\end{aligned}$$

Thus, $\|\mathbf{O}\|_{L(X)\otimes(S)_{-1}} \leq R$. □

Note that the condition $\|o_\alpha\|_{L(X)} \leq R(2\mathbb{N})^{r\alpha}$ for some $r, R > 0$ is actually equivalent to stating that there exists $r > 0$ such that $\sum_{\alpha\in\mathcal{I}} \|o_\alpha\|^2_{L(X)} (2\mathbb{N})^{-r\alpha} < \infty$.

Throughout the paper we will consider the equation

$$\begin{aligned}&\frac{d}{dt} U(t,\omega) = \mathbf{A} U(t,\omega) + \mathbf{B} \lozenge U(t,\omega) + F(t,\omega), \quad t \in (0,T], \omega \in \Omega, \\ &U(0,\omega) = U^0(\omega),\end{aligned} \tag{2.1}$$

where both operators $\mathbf{A}$ and $\mathbf{B}$ are assumed to be coordinatewise operators, i.e. composed out of a family of operators $\{A_\alpha\}_{\alpha\in\mathcal{I}}$, $\{B_\alpha\}_{\alpha\in\mathcal{I}}$, respectively. The operators A_α, $\alpha \in \mathcal{I}$, are assumed to be infinitesimal generators of C_0-semigroups with a common domain D dense in X and the action of $\mathbf{A}$ is given by $\mathbf{A}(U) = \sum_{\alpha\in\mathcal{I}} A_\alpha(u_\alpha) H_\alpha$, for $U = \sum_{\alpha\in\mathcal{I}} u_\alpha H_\alpha \in Dom(\mathbf{A}) \subseteq D \otimes (S)_{-1}$, where

$$Dom(\mathbf{A}) = \{U = \sum_{\alpha\in\mathcal{I}} u_\alpha H_\alpha \in D \otimes (S)_{-1} : \exists p_U > 0, \sum_{\alpha\in\mathcal{I}} \|A_\alpha(u_\alpha)\|^2_X (2\mathbb{N})^{-p_U\alpha} < \infty\}.$$

The operators B_α, $\alpha \in \mathcal{I}$, are assumed to be bounded and linear on X, and the action of the operator $\mathbf{B}\lozenge : X \otimes (S)_{-1} \to X \otimes (S)_{-1}$ is defined by

$$\mathbf{B}\lozenge(U) = \sum_{\alpha\in\mathcal{I}} \sum_{\beta\leq\alpha} B_\beta(u_{\alpha-\beta}) H_\alpha = \sum_{\gamma\in\mathcal{I}} \sum_{\alpha+\beta=\gamma} B_\alpha(u_\beta) H_\gamma.$$

In the next two lemmas we provide two sufficient conditions that ensure the operator $\mathbf{B}\lozenge$ to be well-defined. Both conditions are actually equivalent to the fact that B_α, $\alpha \in \mathcal{I}$, are polynomially bounded, but they provide finer estimates on the stochastic order (Kondratiev weight) of the domain and codomain of $\mathbf{B}\lozenge$.

Lemma 2.4. *If the operators B_α, $\alpha \in \mathcal{I}$, satisfy $\sum_{\alpha\in\mathcal{I}} \|B_\alpha\|^2_{L(X)} (2\mathbb{N})^{-r\alpha} < \infty$, then $\mathbf{B}\lozenge$ is well-defined as a mapping $\mathbf{B}\lozenge : X \otimes (S)_{-1,-p} \to X \otimes (S)_{-1,-(p+r+m)}$, $m > 1$.*

Proof. For $U \in X \otimes (S)_{-1,-p}$ and $q = p + r + m$ we have

$$\begin{aligned}
\sum_{\gamma\in\mathcal{I}} \| \sum_{\alpha+\beta=\gamma} B_\alpha(u_\beta)\|^2_X (2\mathbb{N})^{-q\gamma} &\leq \sum_{\gamma\in\mathcal{I}} \Big[\sum_{\alpha+\beta=\gamma} \|B_\alpha\|_{L(X)} \|u_\beta\|_X \Big]^2 (2\mathbb{N})^{-(p+r+m)\gamma} \\
&= \sum_{\gamma\in\mathcal{I}} (2\mathbb{N})^{-m\gamma} \left(\sum_{\alpha+\beta=\gamma} \|B_\alpha\|^2_{L(X)} (2\mathbb{N})^{-r\gamma} \right) \left(\sum_{\alpha+\beta=\gamma} \|u_\beta\|^2_X (2\mathbb{N})^{-p\gamma} \right) \\
&\leq M \left(\sum_{\alpha\in\mathcal{I}} \|B_\alpha\|^2_{L(X)} (2\mathbb{N})^{-r\alpha} \right) \left(\sum_{\beta\in\mathcal{I}} \|u_\beta\|^2_X (2\mathbb{N})^{-p\beta} \right) < \infty,
\end{aligned}$$

where $M = \sum_{\gamma\in\mathcal{I}} (2\mathbb{N})^{-m\gamma} < \infty$, for $m > 1$. □

Lemma 2.5. *If the operators B_α, $\alpha \in \mathcal{I}$, satisfy $\sum_{\alpha\in\mathcal{I}} \|B_\alpha\|_{L(X)} (2\mathbb{N})^{-\frac{r}{2}\alpha} < \infty$, for some $r > 0$, then $\mathbf{B}\lozenge$ is well-defined as a mapping $\mathbf{B}\lozenge : X \otimes (S)_{-1,-r} \to X \otimes (S)_{-1,-r}$.*

Proof. For $U \in X \otimes (S)_{-1,-r}$, we have by the generalized Minkowski inequality that

$$\begin{aligned}
\sum_{\gamma\in\mathcal{I}} \| \sum_{\alpha+\beta=\gamma} B_\alpha(u_\beta)\|^2_X (2\mathbb{N})^{-r\gamma} &\leq \sum_{\gamma\in\mathcal{I}} \Big[\sum_{\alpha+\beta=\gamma} \|B_\alpha\|_{L(X)} \|u_\beta\|_X \Big]^2 (2\mathbb{N})^{-r\gamma} \\
&\leq \sum_{\gamma\in\mathcal{I}} \Big[\sum_{\alpha+\beta=\gamma} \|B_\alpha\|_{L(X)} (2\mathbb{N})^{-\frac{r}{2}\alpha} \|u_\beta\|_X (2\mathbb{N})^{-\frac{r}{2}\beta} \Big]^2 \\
&\leq \left(\sum_{\alpha\in\mathcal{I}} \|B_\alpha\|_{L(X)} (2\mathbb{N})^{-\frac{r}{2}\alpha} \right)^2 \sum_{\beta\in\mathcal{I}} \|u_\beta\|^2_X (2\mathbb{N})^{-r\beta} < \infty.
\end{aligned}$$

□

2.1 Special cases and relationship to other works

Some of the most important operators of stochastic calculus are the operators of the Malliavin calculus. We recall their definitions in the generalized $S'(\mathbb{R})$ setting [10].

- The *Malliavin derivative*, $\mathbb{D}$, as a stochastic gradient in the direction of white noise, is a linear and continuous mapping $\mathbb{D} : X \otimes (S)_{-1} \to X \otimes S'(\mathbb{R}) \otimes (S)_{-1}$ given by

$$\mathbb{D}u = \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} \alpha_k \, u_\alpha \otimes \xi_k \otimes H_{\alpha-\varepsilon_k}, \quad \text{for } u = \sum_{\alpha\in\mathcal{I}} u_\alpha \otimes H_\alpha.$$

  In terms of quantum theory it corresponds to the annihilation operator reducing the order of the chaos space ($\mathbb{D} : \mathcal{H}_m \to \mathcal{H}_{m-1}$).

- The *Skorokhod integral*, δ, as an extension of the Itô integral to non-anticipating processes, is a linear and continuous mapping $\delta : X \otimes S'(\mathbb{R}) \otimes (S)_{-1} \to X \otimes (S)_{-1}$ given by

$$\delta(F) = \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} f_\alpha \otimes v_{\alpha,k} \otimes H_{\alpha+\varepsilon_k}, \quad \text{for } F = \sum_{\alpha\in\mathcal{I}} f_\alpha \otimes \left(\sum_{k\in\mathbb{N}} v_{\alpha,k}\, \xi_k \right) \otimes H_\alpha.$$

It is the adjoint operator of the Malliavin derivative and in terms of quantum theory it corresponds to the creation operator increasing the order of the chaos space ($\delta : \mathcal{H}_m \to \mathcal{H}_{m+1}$).

- The *Ornstein-Uhlenbeck operator*, $\mathcal{R}$, as the composition of the previous ones $\delta \circ \mathbb{D}$, is the stochastic analogue of the Laplacian. It is a linear and continuous mapping $\mathcal{R} : X \otimes (S)_{-1} \to X \otimes (S)_{-1}$ given by

$$\mathcal{R}(u) = \sum_{\alpha\in\mathcal{I}} |\alpha| u_\alpha \otimes H_\alpha, \quad \text{for } u = \sum_{\alpha\in\mathcal{I}} u_\alpha \otimes H_\alpha.$$

In terms of quantum theory it corresponds to the number operator. It is a selfadjoint operator $\mathcal{R} : \mathcal{H}_m \to \mathcal{H}_m$ with eigenvectors equal to the basis elements H_α, $\alpha \in \mathcal{I}$, i.e. $\mathcal{R}(H_\alpha) = |\alpha| H_\alpha$, $\alpha \in \mathcal{I}$. Thus, Gaussian processes with zero expectation are the only fixed points for the Ornstein-Uhlenbeck operator.

Clearly, the Ornstein-Uhlenbeck operator is a coordinatewise operator, while the Malliavin derivative and the Skorokhod integral are not coordinatewise operators.

The Ornstein-Uhlenbeck operator is the infinitesimal generator of the semigroup $T_t = e^{t\mathcal{R}}$, $t \geq 0$, given by $T_t(u) = \sum_{\alpha\in\mathcal{I}} e^{-|\alpha|t} u_\alpha \otimes H_\alpha$, for $u = \sum_{\alpha\in\mathcal{I}} u_\alpha \otimes H_\alpha \in X \otimes (S)_{-1}$.

It is also closely connected to the Ornstein-Uhlenbeck process. The Ornstein-Uhlenbeck process is the solution of the SDE $du(t,\omega) = -u(t,\omega)dt + dB(t,\omega)$, $u(0,\omega) = u_0(x,\omega)$, and it is given by $u(t,\omega) = e^{-t}u_0(\omega) + \int_0^t e^{t-s} dB(s,\omega)$. It is a Markov process with transition semigroup $\{T_t\}_{t\geq 0}$ [2]. The solution of the generalized heat equation $\frac{d}{dt}u + \mathcal{R}(u) = 0$, $u(0) = u_0$, is given by $u = T_t(u_0)$, i.e. $u(t,x) = (T_t u_0)(x)$ and $(T_t\varphi)(x) = E(\varphi(u(t,x))$ for any $\varphi \in C_b(\mathbb{R})$ and u is the Ornstein-Uhlenbeck process.

Now we turn to our equation

$$\frac{d}{dt} U(t,\omega) = \mathbf{A} U(t,\omega) + \mathbf{B} \Diamond U(t,\omega) + F(t,\omega), \tag{2.2}$$

where $\mathbf{A}$ and $\mathbf{B}$ are coordinatewise operators as described in Section 2, composed out of a family of operators $\{A_\alpha\}_{\alpha\in\mathcal{I}}$, $\{B_\alpha\}_{\alpha\in\mathcal{I}}$, respectively, where A_α are infinitesimal generators on X and B_α are bounded linear operators on X, both families being polynomially bounded, and their actions given by

$$\mathbf{A} U = \sum_{\alpha\in\mathcal{I}} A_\alpha(u_\alpha) H_\alpha, \qquad \text{for } U = \sum_{\alpha\in\mathcal{I}} u_\alpha H_\alpha, \tag{2.3}$$

$$\mathbf{B} \Diamond U = \sum_{\alpha\in\mathcal{I}} \sum_{\beta\leq\alpha} B_\beta(u_{\alpha-\beta}) H_\alpha, \qquad \text{for } U = \sum_{\alpha\in\mathcal{I}} u_\alpha H_\alpha. \tag{2.4}$$

Some important special cases include the following:

I) Special cases for $\mathbf{A}$:

1) $\mathbf{A}$ is a simple coordinatewise operator, i.e. $A_\alpha = A, \alpha \in \mathcal{I}$, where A is the infinitesimal generator of a C_0-semigroup on X. Such operators are, for

example the Laplacian Δ on $X = W^{2,2}(\mathbb{R}^n)$ or any strictly elliptic linear partial differential operator of even order $P(x,D) = \sum_{|\iota|\leq 2m} a_\iota(x)D^\iota$. For example, second order elliptic operators can be written in divergence form $L = \nabla\cdot(Q\nabla\cdot+b) + c\nabla\cdot$, where Q is a positively definite function matrix.

2) $A_\alpha = A + R_\alpha, \alpha\in\mathcal{I}$, where A is as in 1), while $R_\alpha, \alpha\in\mathcal{I}$, are bounded linear operators on X so that $\mathbf{R}$ is a coordinatewise operator

$$\mathbf{R}U(t,\omega) = \sum_{\alpha\in\mathcal{I}} R_\alpha u_\alpha(t)H_\alpha(\omega).$$

Especially, if we take $A = 0$ and R_α to be multiplication operators $R_\alpha(x) = r_\alpha\cdot x$, $x\in X$, then the resulting operator $\mathbf{R}$ is a self-adjoint operator with eigenvalues r_α corresponding to the eigenvectors H_α and thus represents a natural generalization of the Ornstein-Uhlenbeck operator. For $r_\alpha = |\alpha|$, $\alpha\in\mathcal{I}$, we retrieve the Ornstein-Uhlenbeck operator $\mathcal{R}$.

Finally, we note that every bounded linear coordinatewise operator $\mathbf{R}$ can be written in the form $\mathbf{R}u = \delta(\mathbf{M}u)$ where $\mathbf{M}$ is a generalization of the Malliavin derivative. This will be done in Proposition 2.6.

II) Special cases for $\mathbf{B}$:

1) $\mathbf{B}$ is an operator acting as a multiplication operator with a deterministic function, i.e. $B_\alpha = b$ for $\alpha = (0,0,0,0,\dots)$ and $B_\alpha = 0$ for all other $\alpha\in\mathcal{I}$. Its action is thus

$$\mathbf{B}\lozenge U(t,\omega) = \sum_{\alpha\in\mathcal{I}} b\cdot u_\alpha(t)H_\alpha(\omega).$$

For example, we may take $X = L^2(\mathbb{R}^n)$ and $b = b(x)$, $x\in\mathbb{R}^n$, for an essentially bounded function b.

2) $\mathbf{B}$ is multiplication with spatial white noise on $X = L^2(\mathbb{R}^n)$. Let $B_k := B_{\varepsilon_k} = \xi_k$, $k\in\mathbb{N}$, and $B_\alpha = 0$ for $\alpha\neq\varepsilon_k$, i.e. $B_k(v(x)) = \xi_k(x)\cdot v(x)$, $k\in\mathbb{N}$. Then,

$$\mathbf{B}\lozenge U(t,\omega) = W(x,\omega)\lozenge U(t,\omega).$$

Clearly,

$$\begin{aligned}\mathbf{B}\lozenge U &= \sum_{\gamma\in\mathcal{I}}\sum_{k\in\mathbb{N}} B_k(u_{\alpha-\varepsilon_k})H_\gamma = \sum_{\gamma\in\mathcal{I}}\sum_{k\in\mathbb{N}} u_{\alpha-\varepsilon_k}\xi_k H_\gamma\\ &= \sum_{\gamma\in\mathcal{I}}\sum_{\alpha+\varepsilon_k=\gamma} u_\alpha\xi_k H_\gamma = W\lozenge U.\end{aligned}$$

Multiplication with spatial white noise is important for applications since it describes stationary perturbations.

3) $\mathbf{B}$ is of the form $B_{\varepsilon_k} = B_k$, $k\in\mathbb{N}$, and $B_\alpha = 0$ for $\alpha\neq\varepsilon_k$, where $B_k : X\to X$, $k\in\mathbb{N}$, are bounded linear operators.

Note that in this case there is a one-to-one correspondence between operators of the form $\mathbf{B}\lozenge$ and operators of the form $\delta(\mathbf{M}u)$ where $\mathbf{M}$ is a simple coordinatewise operator. This will be done in Proposition 2.8.

4) $\mathbf{B}$ is a simple coordinatewise operator, i.e. $B_\alpha = B, \alpha\in\mathcal{I}$, where B is a bounded linear operator on X. Alternatively, we may also regard operators as $B : X\to X'$ in order to make them bounded; such operators are for example the divergence $\nabla\cdot$ as a mapping from $X = W^{1,2}(\mathbb{R}^n)$ to $X' = W^{-1,2}(\mathbb{R}^n)$.

5) $\mathbf{B}\lozenge = \nabla\cdot\lozenge(Q\lozenge\nabla\cdot+b\lozenge) + c\lozenge\nabla\cdot$ as a strictly elliptic second order operator with random coefficients. This operator is obtained for $B_\alpha = \nabla\cdot(Q_\alpha\nabla\cdot+b_\alpha) + c_\alpha\nabla\cdot$, $\alpha\in\mathcal{I}$, and was studied in [18] and [19].

Proposition 2.6. *Let $\mathbf{R} : X \otimes (S)_{-1} \to X \otimes (S)_{-1}$ be a bounded linear coordinatewise operator defined by $\mathbf{R}u(t,\omega) = \sum_{\alpha\in\mathcal{I}} R_\alpha u_\alpha(t) H_\alpha(\omega)$.*

1. *There exists an operator $\mathbf{M} : X \otimes (S)_{-1} \to X \otimes S'(\mathbb{R}) \otimes (S)_{-1}$ of the form*
$$\mathbf{M}u = \sum_{k=1}^{\infty} \mathbf{M}_k u \otimes \xi_k, \quad u \in X \otimes (S)_{-1},$$
for some coordinatewise operators $\mathbf{M}_k : X \otimes (S)_{-1} \to X \otimes (S)_{-1}$, $k \in \mathbb{N}$, such that
$$\mathbf{R}u = \delta(\mathbf{M}u)$$
holds.
2. *Especially, if $\mathbf{R}$ is a selfadjoint operator, then $\mathbf{M}$ is a generalization of the Malliavin derivative.*

Proof. a) In [10] we proved that the Skorokhod integral is invertible, i.e. there exists a unique solution to equations of the form $\delta(v) = f$. Considering the equation $\delta(\mathbf{M}u) = \sum_{\alpha\in\mathcal{I}} R_\alpha u_\alpha \, H_\alpha$ and applying the result from [10], we obtain $\mathbf{M}u$ in the form
$$\mathbf{M}u = \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} (\alpha_k + 1) \frac{R_{\alpha+\varepsilon_k}(u_{\alpha+\varepsilon_k})}{|\alpha+\varepsilon_k|} \otimes \xi_k \otimes H_\alpha.$$
By defining
$$\mathbf{M}_k u = \sum_{\alpha\in\mathcal{I}} (\alpha_k + 1) \frac{R_{\alpha+\varepsilon_k}(u_{\alpha+\varepsilon_k})}{|\alpha+\varepsilon_k|} \otimes H_\alpha, \quad k \in \mathbb{N},$$
we obtain the assertion.

b) Let $\mathbf{R}$ be a self-adjoint operator with eigenvalues r_α and eigenvectors H_α, $\alpha \in \mathcal{I}$, i.e., an operator of the form $\mathbf{R}u = \sum_{\alpha\in\mathcal{I}} r_\alpha u_\alpha H_\alpha$. Assume that $r_\alpha = \sum_{k\in\mathbb{N}} r_{k,\alpha}$ for some $r_{k,\alpha} \in \mathbb{R}$, $k \in \mathbb{N}$, $\alpha \in \mathcal{I}$, is an arbitrary decomposition of the value r_α.

Define
$$\mathbf{M}_k u = \sum_{\alpha\in\mathcal{I}} r_{k,\alpha} u_\alpha \otimes H_{\alpha-\varepsilon_k}.$$
Then $\mathbf{M}u = \sum_{k\in\mathbb{N}} \mathbf{M}_k u \otimes \xi_k = \sum_{k\in\mathbb{N}} \sum_{\alpha\in\mathcal{I}} r_{k,\alpha} u_\alpha \otimes H_{\alpha-\varepsilon_k} \otimes \xi_k$ and
$$\delta(\mathbf{M}u) = \sum_{k\in\mathbb{N}} \sum_{\alpha\in\mathcal{I}} r_{k,\alpha} u_\alpha \otimes H_\alpha = \sum_{\alpha\in\mathcal{I}} r_\alpha u_\alpha \otimes H_\alpha.$$
□

Remark 2.7. The converse is not true. Even if each $\mathbf{M}_k$, $k \in \mathbb{N}$, is a simple coordinatewise operator (and so is $\mathbf{M}$), $\mathbf{R} := \delta \circ \mathbf{M}$ does not need to be a coordinatewise operator. This would require that the system $R_\alpha(u_\alpha) = \sum_{k\in\mathbb{N}} m_k(u_{\alpha-\varepsilon_k})$, $\alpha \in \mathcal{I}$, is solvable for $R_\alpha(\cdot)$ given the functions $m_k(\cdot)$, $k \in \mathbb{N}$, which is not true in general.

Proposition 2.8. *Let $\mathbf{M} : X \otimes (S)_{-1} \to X \otimes S'(\mathbb{R}) \otimes (S)_{-1}$ be of the form*
$$\mathbf{M}u = \sum_{k=1}^{\infty} \mathbf{M}_k u \otimes \xi_k, \quad u \in X \otimes (S)_{-1}, \tag{2.5}$$
for some simple coordinatewise operators $\mathbf{M}_k : X \otimes (S)_{-1} \to X \otimes (S)_{-1}$, $k \in \mathbb{N}$. Then, there exists a coordinatewise operator $\mathbf{B}$ such that $B_\alpha = 0$ for $\alpha \neq \varepsilon_k$, $k \in \mathbb{N}$, and
$$\delta(\mathbf{M}u) = \mathbf{B}\lozenge u$$
holds.

Conversely, for any coordinatewise operator $\mathbf{B}$ such that $B_\alpha = 0$ for $\alpha \neq \varepsilon_k$, $k \in \mathbb{N}$, there exists an operator $\mathbf{M}$ of the form $\mathbf{M}u = \sum_{k=1}^{\infty} \mathbf{M}_k u \otimes \xi_k$ for some simple coordinatewise operators $\mathbf{M}_k$, $k \in \mathbb{N}$, such that $\delta(\mathbf{M}u) = \mathbf{B}\lozenge u$ holds.

Proof. Let $\mathbf{M}$ be an operator as stated above and since $\mathbf{M}_k$ are simple coordinatewise operators, we can write them as

$$\mathbf{M}_k(u) = \sum_{\alpha \in \mathcal{I}} m_k(u_\alpha) H_\alpha, \qquad u = \sum_{\alpha \in \mathcal{I}} u_\alpha H_\alpha,$$

for some operators $m_k : X \to X$, $k \in \mathbb{N}$. Thus,

$$\mathbf{M}u = \sum_{k=1}^{\infty} \sum_{\alpha \in \mathcal{I}} m_k(u_\alpha) H_\alpha \otimes \xi_k$$

which further implies

$$\delta(\mathbf{M}u) = \sum_{k=1}^{\infty} \sum_{\alpha \in \mathcal{I}} m_k(u_\alpha) H_{\alpha+\varepsilon_k} = \sum_{k=1}^{\infty} \sum_{\alpha \in \mathcal{I}} m_k(u_{\alpha-\varepsilon_k}) H_\alpha. \tag{2.6}$$

On the other hand, if $\mathbf{B}$ is such that $B_\alpha = 0$ for $\alpha \neq \varepsilon_k$, $k \in \mathbb{N}$, and we denote by $B_k := B_{\varepsilon_k}$, $k \in \mathbb{N}$, the operators acting on X, then

$$\mathbf{B}\lozenge u = \sum_{\alpha \in \mathcal{I}} \sum_{k=1}^{\infty} B_k(u_{\alpha-\varepsilon_k}) H_\alpha. \tag{2.7}$$

From (2.6) and (2.7) it follows that $\delta(\mathbf{M}u) = \mathbf{B}\lozenge u$ if and only if $m_k = B_k$ for all $k \in \mathbb{N}$. Thus, there is a one-to-one correspondence between the operators $\mathbf{B}\lozenge$ and $\delta \circ \mathbf{M}$. $\square$

Remark 2.9. In [12] and [13] Rozovskii and Lototsky considered the equation $\frac{d}{dt} = \mathbf{A}u + \delta(\mathbf{M}u) + f$, where $\mathbf{M}$ is of the form (2.5). They implicitly assumed that all their operators $\mathbf{A}$ and $\mathbf{M}_k$, $k \in \mathbb{N}$, belong to our class of simple coordinatewise operators. This corresponds to our special cases I-1) and II-3).

Some special cases of stochastic differential equations covered by (2.2) include the following:

- The heat equation with random potential

$$\frac{d}{dt} u = \Delta u + \mathbf{B}\lozenge u.$$

  In particular, if the random potential is modeled by stationary perturbations, we may take spatial white noise as a model and obtain $\frac{d}{dt} u = \Delta u + W \lozenge u$. This corresponds to the special choice of operators I-1) and II-2).
- The heat equation in random (inhomogeneous and anisotropic) media, where the physical properties of the medium are modeled by a stochastic matrix Q. This corresponds to the case I-1) with $\mathbf{A} = 0$ and II-5) leading to an equation of the form

$$\frac{d}{dt} u = \nabla \cdot \lozenge (Q \lozenge \nabla \cdot u + b \lozenge u) + c \lozenge \nabla \cdot u + f.$$

- Taking $\mathbf{A} = 0$ and $B_k := B_{\varepsilon_k} = \xi_k \nabla \cdot$, $k \in \mathbb{N}$, (see special cases II-2) and II-4)) we obtain the transport equation driven by white noise

$$\frac{d}{dt} u = \Delta u + W \lozenge \nabla \cdot u.$$

- The Langevin equation

$$\frac{d}{dt} u = -\lambda u + W(t),$$

$\lambda > 0$, corresponding to the case I-1) with $A = -\lambda$, $f = W$ and $\mathbf{B} = 0$. Its solution is the Ornstein-Uhlenbeck process describing the spatial position of a Brownian particle in a fluid with viscosity λ.

In [1] the authors considered the generalized Langevin equation leading to generalized Ornstein-Uhlenbeck operators driven by Lévy processes

$$\frac{d}{dt}u = Ju + C(\frac{d}{dt}Y),$$

where Y is a Lévy process, J the infinitesimal generator of a C_0-semigroup and C a bounded operator. All processes are Hilbert space valued. This corresponds to our case with X being this Hilbert space, $\mathbf{A} = J$, $\mathbf{B} = 0$ and $f = C(Y')$.

- The equation $\frac{d}{dt} = \mathbf{A}u + \delta(\mathbf{M}u) + f$, that was extensively studied in [12] and [13]. This corresponds to our special cases I-1) and II-3).
- The equation
$$\frac{d}{dt}u = Lu + W\lozenge u,$$
where L is a strictly elliptic partial differential operator as studied in [3] and [8]. This corresponds to the special case I-1) and II-2).

3 Stochastic evolution equations

Now we turn to the general case of stochastic Cauchy problems of the form $\frac{d}{dt}U(t,\omega) = \mathbf{A}U(t,\omega) + \mathbf{B}\lozenge U(t,\omega) + F(t,\omega)$, $t \in (0,T]$, $\omega \in \Omega$, with initial value $U(0,\omega) = U^0(\omega)$, $\omega \in \Omega$, and all processes are X-valued for a Banach space X.

Definition 3.1. *It is said that U is a solution to (2.1) if $U \in C([0,T],X) \otimes (S)_{-1} \cap C^1((0,T],X) \otimes (S)_{-1}$ and U satisfies (2.1).*

Theorem 3.2. *Let $\mathbf{A}$ be a coordinatewise operator of the form (2.3), where the operators $A_\alpha,\ \alpha \in \mathcal{I}$, defined on the same domain D dense in X, are infinitesimal generators of C_0-semigroups $(T_t)_\alpha,\ t \geq 0,\ \alpha \in \mathcal{I}$, uniformly bounded by*

$$\|(T_t)_\alpha\|_{L(X)} \leq Me^{wt},\ t \geq 0, \quad \text{for some}\ \ M, w > 0. \tag{3.1}$$

Let $\mathbf{B}\lozenge$ be of the form (2.4), where $B_\alpha,\ \alpha \in \mathcal{I}$, are bounded linear operators on X so that there exists $p > 0$ such that

$$K := \sum_{\alpha\in\mathcal{I}} \|B_\alpha\|(2\mathbb{N})^{-p\frac{\alpha}{2}} < \infty. \tag{3.2}$$

Let the initial value $U^0 \in X \otimes (S)_{-1}$ be such that $U^0 \in Dom(\mathbf{A})$ i.e.

$$U^0(\omega) = \sum_{\alpha\in\mathcal{I}} u^0_\alpha H_\alpha(\omega) \in X \otimes (S)_{-1,-p},\ \text{satisfies} \quad \sum_{\alpha\in\mathcal{I}} \|u^0_\alpha\|^2_X (2\mathbb{N})^{-p\alpha} < \infty; \tag{3.3}$$

and

$$\mathbf{A}U^0(\omega) = \sum_{\alpha\in\mathcal{I}} A_\alpha u^0_\alpha H_\alpha(\omega) \in X \otimes (S)_{-1,-p},\ \text{satisfies} \quad \sum_{\alpha\in\mathcal{I}} \|A_\alpha u^0_\alpha\|^2_X (2\mathbb{N})^{-p\alpha} < \infty. \tag{3.4}$$

Moreover, let

$$F(t,\omega) = \sum_{\alpha\in\mathcal{I}} f_\alpha(t) H_\alpha(\omega) \in C^1([0,T],X) \otimes (S)_{-1}, \quad t \mapsto f_\alpha(t) \in C^1([0,T],X),\ \alpha \in \mathcal{I},$$

so that $$\sum_{\alpha\in\mathcal{I}} \|f_\alpha\|^2_{C^1([0,T],X)} (2\mathbb{N})^{-p\alpha} = \sum_{\alpha\in\mathcal{I}} \Big(\sup_{t\in[0,T]} \|f_\alpha(t)\|_X + \sup_{t\in[0,T]} \|f'_\alpha(t)\|_X \Big)^2 (2\mathbb{N})^{-p\alpha} < \infty. \tag{3.5}$$

Then, the stochastic Cauchy problem (2.1) has a unique solution U in $C^1([0,T],X) \otimes (S)_{-1,-p}$.

Proof. We seek for the solution in form of $U(t,\omega) = \sum_{\alpha\in\mathcal{I}} u_\alpha(t) H_\alpha(\omega)$. Then, the Cauchy problem (2.1) is equivalent to the infinite system:

$$
\begin{aligned}
&\frac{d}{dt}\, u_\alpha(t) = A_\alpha u_\alpha(t) + \sum_{\beta\leq\alpha} B_\beta u_{\alpha-\beta}(t) + f_\alpha(t), \quad t\in(0,T], \\
&\quad u_\alpha(0) = u_\alpha^0 \in D, \quad \alpha\in\mathcal{I}.
\end{aligned} \tag{3.6}
$$

Let $\mathbf{0}$ be the multi-index $\mathbf{0} = (0,0,...)$. We rewrite (3.6) as

$$
\begin{aligned}
&\frac{d}{dt}\, u_\alpha(t) = (A_\alpha + B_{\mathbf{0}}) u_\alpha(t) + \sum_{\mathbf{0}<\beta\leq\alpha} B_\beta u_{\alpha-\beta}(t) + f_\alpha(t), \quad t\in(0,T], \\
&\quad u_\alpha(0) = u_\alpha^0 \in D, \quad \alpha\in\mathcal{I}.
\end{aligned} \tag{3.7}
$$

Next, $A_\alpha + B_{\mathbf{0}}$ are infinitesimal generators of C_0-semigroups $(S_t)_\alpha$ in X such that

$$
\|(S_t)_\alpha\| \leq M e^{(w+M\|B_{\mathbf{0}}\|)t}, \; t\geq 0, \; \alpha\in\mathcal{I}. \tag{3.8}
$$

According to Subsection 1.1, if $f_\alpha,\ \alpha\in\mathcal{I}$, fulfills condition (i) or (ii), the inhomogeneous initial value problem (3.7) has a solution $u_\alpha(t)\in C([0,T],X)\cap C^1((0,T],X)$, $\alpha\in\mathcal{I}$, given by

$$
\begin{aligned}
u_{\mathbf{0}}(t) &= (S_t)_{\mathbf{0}} u_{\mathbf{0}}^0 + \int_0^t (S_{t-s})_{\mathbf{0}} f_{\mathbf{0}}(s)\,ds, \quad t\in[0,T] \\
u_\alpha(t) &= (S_t)_\alpha u_\alpha^0 + \int_0^t (S_{t-s})_\alpha \Big(\sum_{\mathbf{0}<\beta\leq\alpha} B_\beta u_{\alpha-\beta}(s) + f_\alpha(s) \Big) ds, \quad t\in[0,T].
\end{aligned} \tag{3.9}
$$

Since $f_\alpha\in C^1([0,T],X)$ it follows by induction on α that

$$
\sum_{\mathbf{0}<\beta\leq\alpha} B_\beta u_{\alpha-\beta}(s) + f_\alpha(s) \in C^1([0,T],X), \quad \text{for all} \quad \alpha\in\mathcal{I}.
$$

Thus, $u_\alpha\in C^1([0,T],X)$ and $\frac{d}{dt}u_\alpha(0) = (A_\alpha + B_{\mathbf{0}})u_\alpha^0 + \sum_{\mathbf{0}<\beta\leq\alpha} B_\beta u_{\alpha-\beta}^0 + f_\alpha(0),\ \alpha\in\mathcal{I}$.

Note that for each fixed $\alpha\in\mathcal{I}$, $u_\alpha(t)$ exists for all $t\in[0,T]$ and it is the unique (classical) solution on the whole interval $[0,T]$. It remains to prove that $\sum_{\alpha\in\mathcal{I}} u_\alpha(t)H_\alpha(\omega)$ converges in $C^1([0,T],X)\otimes(S)_{-1,-p}$.

First, we show that $U(t,\omega) = \sum_{\alpha\in\mathcal{I}} u_\alpha(t)H_\alpha(\omega) \in C^1([0,T_0],X)\otimes S_{-1,-p}$ for appropriate $T_0\in(0,T]$, i.e. we show that

$$
\sum_{\alpha\in\mathcal{I}} \|u_\alpha\|^2_{C^1([0,T_0],X)} (2\mathbb{N})^{-p\alpha} = \sum_{\alpha\in\mathcal{I}} \Big(\sup_{t\in[0,T_0]} \|u_\alpha(t)\|_X + \sup_{t\in[0,T_0]} \|\frac{d}{dt}u_\alpha(t)\|_X \Big)^2 (2\mathbb{N})^{-p\alpha} < \infty. \tag{3.10}
$$

Later on we will prove that the same holds if we take in (3.10) supremums over the intervals $[T_0, 2T_0]$, $[2T_0, 3T_0]$, ... etc. Since $[0,T]$ can be covered by finitely many intervals of the form $[kT_0,(k+1)T_0]$, $k\in\mathbb{N}_0$, we conclude that

$$
\sum_{\alpha\in\mathcal{I}} \|u_\alpha\|^2_{C^1([0,T],X)} (2\mathbb{N})^{-p\alpha} = \sum_{\alpha\in\mathcal{I}} \Big(\sup_{t\in[0,T]} \|u_\alpha(t)\|_X + \sup_{t\in[0,T]} \|\frac{d}{dt}u_\alpha(t)\|_X \Big)^2 (2\mathbb{N})^{-p\alpha} < \infty. \tag{3.11}
$$

In order to do this, we introduce a notation for subsets of multi-indices

$$\mathcal{I}_{n,m} = \{\alpha \in \mathcal{I} :\ |\alpha| \le n \wedge \mathrm{Index}(\alpha) \le m\},\ n,m \in \mathbb{N},$$

where, for $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m, 0, 0, \dots) \in \mathcal{I}$, we have $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_m$ and $\mathrm{Index}(\alpha)$ is last coordinate where α has a nonzero entry. For later reference, we introduce the function

$$C(t) = \frac{M^2}{(w + M\|B_{\mathbf{0}}\|)^2}(e^{(w+M\|B_{\mathbf{0}}\|)t} - 1)^2 \tag{3.12}$$

and fix $T_0 \in (0, T]$ such that $C(T_0) < \frac{1}{5K^2}$.

First, we show that

$$\sum_{\alpha\in\mathcal{I}} \|u_\alpha(t)\|^2_{C([0,T_0],X)}(2\mathbb{N})^{-p\alpha} = \sum_{\alpha\in\mathcal{I}} \sup_{t\in[0,T_0]} \|u_\alpha(t)\|^2_X (2\mathbb{N})^{-p\alpha} < \infty,$$

by proving that partial sums $\sum_{\alpha\in\mathcal{I}_{n,m}} \sup_{t\in[0,T_0]} \|u_\alpha(t)\|^2_X (2\mathbb{N})^{-p\alpha},\ n,m \in \mathbb{N}$, are bounded from above.

Using (3.9) we obtain

$$\begin{aligned}
\frac{1}{3}\sum_{\alpha\in\mathcal{I}_{n,m}} \|u_\alpha(t)\|^2_X (2\mathbb{N})^{-p\alpha} \le& \sum_{\alpha\in\mathcal{I}_{n,m}} \|(S_t)_\alpha\|^2 \|u^0_\alpha\|^2_X (2\mathbb{N})^{-p\alpha} \\
&+ \sum_{\alpha\in\mathcal{I}_{n,m}} \Big[\int_0^t \|(S_{t-s})_\alpha\| \sum_{\mathbf{0}<\beta\le\alpha} \|B_\beta u_{\alpha-\beta}(s)\|_X ds\Big]^2 (2\mathbb{N})^{-p\alpha} \\
&+ \sum_{\alpha\in\mathcal{I}_{n,m}} \Big[\int_0^t \|(S_{t-s})_\alpha\| \|f_\alpha(s)\|_X ds\Big]^2 (2\mathbb{N})^{-p\alpha}.
\end{aligned}$$

The first term on the right-hand side, for all $t \in [0, T_0]$, having in mind (3.3) and (3.8), satisfies

$$\begin{aligned}
\sum_{\alpha\in\mathcal{I}_{n,m}} \|(S_t)_\alpha\|^2 \|u^0_\alpha\|^2_X (2\mathbb{N})^{-p\alpha} &\le \sum_{\alpha\in\mathcal{I}} \|(S_t)_\alpha\|^2 \|u^0_\alpha\|^2_X (2\mathbb{N})^{-p\alpha} \\
&\le M^2 e^{2(w+M\|B_{\mathbf{0}}\|)T_0} \sum_{\alpha\in\mathcal{I}} \|u^0_\alpha\|^2_X (2\mathbb{N})^{-p\alpha} := Q_1 < \infty.
\end{aligned} \tag{3.13}$$

Similarly, for all $t \in [0, T_0]$, using (3.5) and (3.8), the third term satisfies

$$\begin{aligned}
\sum_{\alpha\in\mathcal{I}_{n,m}} \Big[\int_0^t \|(S_{t-s})_\alpha\| \|f_\alpha(s)\|_X ds\Big]^2 (2\mathbb{N})^{-p\alpha} &\le \sum_{\alpha\in\mathcal{I}} \Big[\int_0^t \|(S_{t-s})_\alpha\| \|f_\alpha(s)\|_X ds\Big]^2 (2\mathbb{N})^{-p\alpha} \\
\le \Big[\int_0^t M e^{(w+M\|B_{\mathbf{0}}\|)(t-s)} ds\Big]^2 & \sum_{\alpha\in\mathcal{I}} \sup_{s\in[0,t]} \|f_\alpha(s)\|^2_X (2\mathbb{N})^{-p\alpha} \\
\le \frac{M^2}{(w+M\|B_{\mathbf{0}}\|)^2}\Big(e^{(w+M\|B_{\mathbf{0}}\|)T_0} - 1\Big)^2 & \sum_{\alpha\in\mathcal{I}} \sup_{t\in[0,T]} \|f_\alpha(t)\|^2_X (2\mathbb{N})^{-p\alpha} := G < \infty.
\end{aligned} \tag{3.14}$$

Note that in (3.14) we took the supremum over the whole interval $[0, T]$.

For the second term, using (3.2), (3.8), (3.12) and the generalized Minkowski inequality, we obtain

$$\begin{aligned}
&\sum_{\alpha\in\mathcal{I}_{n,m}}\Big[\int_0^t \|(S_{t-s})_\alpha\|\sum_{\beta+\gamma=\alpha}\|B_\beta\|\|u_\gamma(s)\|_X ds\Big]^2(2\mathbb{N})^{-p\alpha}\\
&\quad\le \frac{M^2}{(w+M\|B_{\mathbf{0}}\|)^2}\Big(e^{(w+M\|B_{\mathbf{0}}\|)t}-1\Big)^2\sum_{\alpha\in\mathcal{I}_{n,m}}\Big[\sum_{\beta+\gamma=\alpha}\sup_{s\in[0,t]}\|B_\beta\|\|u_\gamma(s)\|_X\Big]^2(2\mathbb{N})^{-p\alpha}\\
&\quad\le C(T_0)\Big(\sum_{\beta\in\mathcal{I}_{n,m}}\|B_\beta\|(2\mathbb{N})^{-p\frac{\beta}{2}}\Big)^2\Big(\sum_{\gamma\in\mathcal{I}_{n,m}}\sup_{t\in[0,T_0]}\|u_\gamma(t)\|_X^2(2\mathbb{N})^{-p\gamma}\Big)\\
&\quad\le C(T_0)K^2\sum_{\alpha\in\mathcal{I}_{n,m}}\sup_{t\in[0,T_0]}\|u_\alpha(t)\|_X^2(2\mathbb{N})^{-p\alpha}. \qquad (3.15)
\end{aligned}$$

Finally, for all $n, m\in\mathbb{N}$, we obtain

$$\frac{1}{3}\sum_{\alpha\in\mathcal{I}_{n,m}}\sup_{t\in[0,T_0]}\|u_\alpha(t)\|_X^2(2\mathbb{N})^{-p\alpha}\le Q_1+G+C(T_0)K^2\sum_{\alpha\in\mathcal{I}_{n,m}}\sup_{t\in[0,T_0]}\|u_\alpha(t)\|_X^2(2\mathbb{N})^{-p\alpha}.$$

Since $\frac{1}{3}-C(T_0)K^2>\frac{1}{5}-C(T_0)K^2>0$, we have

$$\sum_{\alpha\in\mathcal{I}_{n,m}}\sup_{t\in[0,T_0]}\|u_\alpha(t)\|_X^2(2\mathbb{N})^{-p\alpha}\le\frac{Q_1+G}{\frac{1}{3}-C(T_0)K^2}. \qquad (3.16)$$

Let $(m_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of positive integers tending to infinity. Then,

$$\sum_{\alpha\in\mathcal{I}}\sup_{t\in[0,T_0]}\|u_\alpha(t)\|_X^2(2\mathbb{N})^{-p\alpha}=\lim_{n\to\infty}\sum_{\alpha\in\mathcal{I}_{n,m_n}}\sup_{t\in[0,T_0]}\|u_\alpha(t)\|_X^2(2\mathbb{N})^{-p\alpha}\le\frac{Q_1+G}{\frac{1}{3}-C(T_0)K^2},$$

since it is a series of positive numbers and thus does not depend on the order of summation.

Now we show that

$$\sum_{\alpha\in\mathcal{I}}\|\frac{d}{dt}u_\alpha(t)\|_{C([0,T_0],X)}^2(2\mathbb{N})^{-p\alpha}=\sum_{\alpha\in\mathcal{I}}\sup_{t\in[0,T_0]}\|\frac{d}{dt}u_\alpha(t)\|_X^2(2\mathbb{N})^{-p\alpha}<\infty.$$

In order to acomplish that, we differentiate (3.9) with respect to t, and obtain

$$\begin{aligned}
\frac{d}{dt}u_{\mathbf{0}}(t)&=(S_t)_{\mathbf{0}}(A_{\mathbf{0}}+B_{\mathbf{0}})u_{\mathbf{0}}^0+\int_0^t(S_{t-s})_{\mathbf{0}}\frac{d}{ds}f_{\mathbf{0}}(s)ds+(S_t)_{\mathbf{0}}f(0),\quad t\in[0,T],\\
\frac{d}{dt}u_\alpha(t)&=(S_t)_\alpha(A_\alpha+B_{\mathbf{0}})u_\alpha^0+\int_0^t(S_{t-s})_\alpha\Big(\sum_{\mathbf{0}<\beta\le\alpha}B_\beta\frac{d}{ds}u_{\alpha-\beta}(s)+\frac{d}{ds}f_\alpha(s)\Big)ds \qquad (3.17)\\
&\quad+(S_t)_\alpha\Big(\sum_{\mathbf{0}<\beta\le\alpha}B_\beta u_{\alpha-\beta}(0)+f_\alpha(0)\Big),\quad t\in[0,T],\quad\alpha\in\mathcal{I}.
\end{aligned}$$

In the sequel we estimate partial sums of $\sum_{\alpha\in\mathcal{I}}\sup_{t\in[0,T_0]}\|\frac{d}{dt}u_\alpha(t)\|_X^2(2\mathbb{N})^{-p\alpha}$. So,

$$
\begin{aligned}
\frac{1}{5}\sum_{\alpha\in\mathcal{I}_{n,m}}\|\frac{d}{dt}u_\alpha(t)\|_X^2(2\mathbb{N})^{-p\alpha} \le &\sum_{\alpha\in\mathcal{I}_{n,m}}\|(S_t)_\alpha\|^2\|(A_\alpha+B_{\mathbf{0}})u_\alpha^0\|_X^2(2\mathbb{N})^{-p\alpha}\\
&+\sum_{\alpha\in\mathcal{I}_{n,m}}\Big[\int_0^t\|(S_{t-s})_\alpha\|\sum_{\mathbf{0}<\beta\le\alpha}\|B_\beta\frac{d}{ds}u_{\alpha-\beta}(s)\|_X ds\Big]^2(2\mathbb{N})^{-p\alpha}\\
&+\sum_{\alpha\in\mathcal{I}_{n,m}}\Big[\int_0^t\|(S_{t-s})_\alpha\|\|\frac{d}{ds}f_\alpha(s)\|_X ds\Big]^2(2\mathbb{N})^{-p\alpha}\\
&+\sum_{\alpha\in\mathcal{I}_{n,m}}\|(S_t)_\alpha\|^2\Big[\sum_{\mathbf{0}<\beta\le\alpha}\|B_\beta u_{\alpha-\beta}(0)\|_X\Big]^2(2\mathbb{N})^{-p\alpha}\\
&+\sum_{\alpha\in\mathcal{I}_{n,m}}\|(S_t)_\alpha\|^2\|f_\alpha(0)\|_X^2(2\mathbb{N})^{-p\alpha}.
\end{aligned}
$$

According to (3.3) and (3.4), we obtain $\sum_{\alpha\in\mathcal{I}}(A_\alpha+B_{\mathbf{0}})u_\alpha^0 H_\alpha(\omega)\in X\otimes(S)_{-1,-p}$. So the first term on the right-hand side can be evaluated by

$$
\begin{aligned}
\sum_{\alpha\in\mathcal{I}_{n,m}}\|(S_t)_\alpha\|^2\|(A_\alpha+B_{\mathbf{0}})u_\alpha^0\|_X^2(2\mathbb{N})^{-p\alpha} &\le \sum_{\alpha\in\mathcal{I}}\|(S_t)_\alpha\|^2\|(A_\alpha+B_{\mathbf{0}})u_\alpha^0\|_X^2(2\mathbb{N})^{-p\alpha}\\
&\le M^2e^{2(w+M\|B_{\mathbf{0}}\|)T_0}\sum_{\alpha\in\mathcal{I}}\|(A_\alpha+B_{\mathbf{0}})u_\alpha^0\|_X^2(2\mathbb{N})^{-p\alpha}:=Q_1'<\infty.
\end{aligned} \tag{3.18}
$$

The third term, for all $t\in[0,T_0]$, satisfies

$$
\begin{aligned}
\sum_{\alpha\in\mathcal{I}_{n,m}}\Big[\int_0^t\|(S_{t-s})_\alpha\|\|\frac{d}{ds}f_\alpha(s)\|ds\Big]^2(2\mathbb{N})^{-p\alpha} &\le \sum_{\alpha\in\mathcal{I}}\Big[\int_0^t\|(S_{t-s})_\alpha\|\|\frac{d}{ds}f_\alpha(s)\|_X ds\Big]^2(2\mathbb{N})^{-p\alpha}\\
&\le \frac{M^2}{(w+M\|B_{\mathbf{0}}\|)^2}\Big(e^{(w+M\|B_{\mathbf{0}}\|)T_0}-1\Big)^2\sum_{\alpha\in\mathcal{I}}\sup_{t\in[0,T]}\|\frac{d}{ds}f_\alpha(t)\|_X^2(2\mathbb{N})^{-p\alpha}:=G'<\infty.
\end{aligned} \tag{3.19}
$$

The fourth term, using (3.2), (3.3), (3.8) and the generalized Minkowski inequality, can be estimated by

$$
\begin{aligned}
\sum_{\alpha\in\mathcal{I}_{n,m}}\|(S_t)_\alpha\|^2\Big[\sum_{\mathbf{0}<\beta\le\alpha}\|B_\beta u_{\alpha-\beta}(0)\|_X\Big]^2(2\mathbb{N})^{-p\alpha} &\le \sum_{\alpha\in\mathcal{I}}\|(S_t)_\alpha\|^2\Big[\sum_{\beta+\gamma=\alpha}\|B_\beta u_\gamma^0\|_X\Big]^2(2\mathbb{N})^{-p\alpha}\\
&\le M^2e^{2(w+M\|B_{\mathbf{0}}\|)t}\sum_{\alpha\in\mathcal{I}}\Big[\sum_{\beta+\gamma=\alpha}\|B_\beta\|\|u_\gamma^0\|_X\Big]^2(2\mathbb{N})^{-p\alpha}\\
&\le M^2e^{2(w+M\|B_{\mathbf{0}}\|)T_0}\Big(\sum_{\beta\in\mathcal{I}}\|B_\beta\|(2\mathbb{N})^{-p\frac{\beta}{2}}\Big)^2\Big(\sum_{\gamma\in\mathcal{I}}\|u_\gamma^0\|_X^2(2\mathbb{N})^{-p\gamma}\Big):=H_1'<\infty.
\end{aligned} \tag{3.20}
$$

For the fifth term, using (3.5) and (3.8), we have

$$
\begin{aligned}
\sum_{\alpha\in\mathcal{I}_{n,m}}\|(S_t)_\alpha\|^2\|f_\alpha(0)\|_X^2(2\mathbb{N})^{-p\alpha} &\le \sum_{\alpha\in\mathcal{I}}\|(S_t)_\alpha\|^2\|f_\alpha(0)\|_X^2(2\mathbb{N})^{-p\alpha}\\
&\le M^2e^{2(w+M\|B_{\mathbf{0}}\|)T_0}\sum_{\alpha\in\mathcal{I}}\sup_{t\in[0,T]}\|f_\alpha(t)\|_X^2(2\mathbb{N})^{-p\alpha}:=N'<\infty.
\end{aligned} \tag{3.21}
$$

Finally, for the second term, using (3.2), (3.8), (3.12) and the generalized Minkowski inequality, we obtain

$$\begin{aligned}
&\sum_{\alpha\in\mathcal{I}_{n,m}}\Big[\int_0^t\|(S_{t-s})_\alpha\|\sum_{\beta+\gamma=\alpha}\|B_\beta\|\|\frac{d}{ds}u_\gamma(s)\|_X ds\Big]^2(2\mathbb{N})^{-p\alpha}\\
&\quad\le\frac{M^2}{(w+M\|B_{\mathbf{0}}\|)^2}\Big(e^{(w+M\|B_{\mathbf{0}}\|)t}-1\Big)^2\sum_{\alpha\in\mathcal{I}_{n,m}}\Big[\sum_{\beta+\gamma=\alpha}\sup_{s\in[0,t]}\|B_\beta\|\|\frac{d}{ds}u_\gamma(s)\|_X\Big]^2(2\mathbb{N})^{-p\alpha}\\
&\quad\le C(t)\Big(\sum_{\beta\in\mathcal{I}_{n,m}}\|B_\beta\|(2\mathbb{N})^{-p\frac{\beta}{2}}\Big)^2\Big(\sum_{\gamma\in\mathcal{I}_{n,m}}\sup_{s\in[0,t]}\|\frac{d}{dt}u_\gamma(s)\|_X^2(2\mathbb{N})^{-p\gamma}\Big)\\
&\quad\le C(T_0)K^2\sum_{\alpha\in\mathcal{I}_{n,m}}\sup_{t\in[0,T_0]}\|\frac{d}{dt}u_\alpha(t)\|_X^2(2\mathbb{N})^{-p\alpha}. \qquad (3.22)
\end{aligned}$$

Finally, for all $n,m\in\mathbb{N}$, we obtain

$$\begin{aligned}
\frac{1}{5}\sum_{\alpha\in\mathcal{I}_{n,m}}\sup_{t\in[0,T_0]}\|\frac{d}{dt}u_\alpha(t)\|_X^2(2\mathbb{N})^{-p\alpha}\le& Q_1'+G'+H_1'+N'\\
&+C(T_0)K^2\sum_{\alpha\in\mathcal{I}_{n,m}}\sup_{t\in[0,T_0]}\|\frac{d}{dt}u_\alpha(t)\|_X^2(2\mathbb{N})^{-p\alpha}.
\end{aligned}$$

Since $\frac{1}{5}-C(T_0)K^2>0$, we have

$$\sum_{\alpha\in\mathcal{I}_{n,m}}\sup_{t\in[0,T_0]}\|\frac{d}{dt}u_\alpha(t)\|_X^2(2\mathbb{N})^{-p\alpha}\le\frac{Q_1'+G'+H_1'+N'}{\frac{1}{5}-C(T_0)K^2}. \qquad (3.23)$$

Again, taking $(m_n)_{n\in\mathbb{N}}$ to be an arbitrary sequence of positive integers tending to infinity, we have

$$\sum_{\alpha\in\mathcal{I}}\sup_{t\in[0,T_0]}\|\frac{d}{dt}u_\alpha(t)\|_X^2(2\mathbb{N})^{-p\alpha}=\lim_{n\to\infty}\sum_{\alpha\in\mathcal{I}_{n,m_n}}\sup_{t\in[0,T_0]}\|\frac{d}{dt}u_\alpha(t)\|_X^2(2\mathbb{N})^{-p\alpha}\le\frac{Q_1'+G'+H_1'+N'}{\frac{1}{5}-C(T_0)K^2}.$$

Therefore, we obtain

$$\begin{aligned}
&U(t,\omega)\in C^1([0,T_0],X)\otimes(S)_{-1,-p},\ \text{i.e.}\\
&\sum_{\alpha\in\mathcal{I}}\Big(\sup_{t\in[0,T_0]}\|u_\alpha(t)\|_X+\sup_{t\in[0,T_0]}\|\frac{d}{dt}u_\alpha(t)\|_X\Big)^2(2\mathbb{N})^{-p\alpha}\le\\
&2\sum_{\alpha\in\mathcal{I}}\Big(\sup_{t\in[0,T_0]}\|u_\alpha(t)\|_X^2+\sup_{t\in[0,T_0]}\|\frac{d}{dt}u_\alpha(t)\|_X^2\Big)(2\mathbb{N})^{-p\alpha}<\infty.
\end{aligned} \qquad (3.24)$$

Next, we consider in (3.24) supremums over the interval $[T_0,2T_0]$. On $[T_0,2T_0]$ one can rewrite the initial value problem (3.6) in the following equivalent form:

$$\begin{aligned}
&\frac{d}{dt}\,v_\alpha(t)=A_\alpha v_\alpha(t)+\sum_{\beta\le\alpha}B_\beta v_{\alpha-\beta}(t)+f_\alpha(T_0+t),\quad t\in(0,T_0]\\
&\quad v_\alpha(0)=v_\alpha^0:=u_\alpha(T_0),\quad \alpha\in\mathcal{I}.
\end{aligned} \qquad (3.25)$$

The semigroup corresponding to the generator $A_\alpha+B_{\mathbf{0}}$ in (3.25) is again the semigroup $(S_t)_\alpha,\ t\ge0$. Using (3.6) and (3.24), we have that $U(t,\omega)\in Dom(\mathbf{A})$, for all $t\in[0,T_0]$, and $\mathbf{A}U(t,\omega)\in X\otimes(S)_{-1,-p},\ t\in[0,T_0]$. According to this we have that $V^0(\omega)=U(T_0,\omega)=\sum_{\alpha\in\mathcal{I}}v_\alpha^0H_\alpha(\omega)\in Dom(\mathbf{A})$ and $\mathbf{A}V^0(\omega)\in X\otimes(S)_{-1,-p}$. Thus,

$$v_\alpha(t)=(S_t)_\alpha v_\alpha^0+\int_0^t(S_{t-s})_\alpha\Big(\sum_{\mathbf{0}<\beta\le\alpha}B_\beta v_{\alpha-\beta}(s)+f_\alpha(T_0+s)\Big)ds,\quad t\in[0,T_0],$$

and clearly $v_\alpha(t) = u_\alpha(T_0+t),\ t\in[0,T_0],\ \alpha\in\mathcal{I}$.

When approximating partial sums of $\sum_{\alpha\in\mathcal{I}}\sup_{t\in[0,T_0]}\|v_\alpha(t)\|_X^2(2\mathbb{N})^{-p\alpha}$, comparing to the previous calculations for $u_\alpha(t)$, only the constant Q_1 will be different, and here, we denote it by Q_2, so we again obtain

$$\sum_{\alpha\in\mathcal{I}}\sup_{t\in[0,T_0]}\|v_\alpha(t)\|_X^2(2\mathbb{N})^{-p\alpha} = \sum_{\alpha\in\mathcal{I}}\sup_{t\in[T_0,2T_0]}\|u_\alpha(t)\|_X^2(2\mathbb{N})^{-p\alpha} \le \frac{Q_2+G}{\frac{1}{3}-C(T_0)K^2}.$$

Similarly, for the derivative $\frac{d}{dt}V(t,\omega)$ we obtain

$$\sum_{\alpha\in\mathcal{I}}\sup_{t\in[0,T_0]}\|\frac{d}{dt}v_\alpha(t)\|_X^2(2\mathbb{N})^{-p\alpha} \le \frac{Q_2'+G'+H_2'+N'}{\frac{1}{5}-C(T_0)K^2},$$

where, comparing to the estimates of $\frac{d}{dt}U(t,\omega)$, only the constants Q_1' and H_1' have changed and we denoted them here by Q_2' and H_2'.

For arbitrary $T>0$, one can cover the interval $[0,T]$ by intervals of the form $[kT_0,(k+1)T_0],\ k\in\mathbb{N}_0$, in finitely many steps (say in l steps). So we have

$$\sum_{\alpha\in\mathcal{I}}\sup_{t\in[0,T]}\|u_\alpha(t)\|_X^2(2\mathbb{N})^{-p\alpha} \le \frac{Q+G}{\frac{1}{3}-C(T_0)K^2},$$

where $Q=\max_{1\le k\le l}\{Q_k\}$. Thus,

$$U(t,\omega) = \sum_{\alpha\in\mathcal{I}} u_\alpha(t)H_\alpha(\omega) \in C([0,T],X)\otimes(S)_{-1,-p}.$$

Also,

$$\sum_{\alpha\in\mathcal{I}}\sup_{t\in[0,T]}\|\frac{d}{dt}u_\alpha(t)\|_X^2(2\mathbb{N})^{-p\alpha} \le \frac{Q'+G'+H'+N'}{\frac{1}{5}-C(T_0)K^2},$$

where $Q'=\max_{1\le k\le l}\{Q_k'\}$, $H'=\max_{1\le k\le l}\{H_k'\}$. Since $\frac{d}{dt}u_\alpha(t)\in C([0,T],X)$, $\alpha\in\mathcal{I}$, we have

$$\frac{d}{dt}U(t,\omega) = \sum_{\alpha\in\mathcal{I}}\frac{d}{dt}u_\alpha(t)H_\alpha(\omega)\in C([0,T],X)\otimes(S)_{-1,-p}.$$

Therefore, $U(t,\omega)\in C^1([0,T],X)\otimes(S)_{-1,-p}$ and thus, U is a solution of (2.1) in the sense of Definition 3.1.

The solution U is unique due to the uniqueness of the coordinatewise (classical) solutions u_α in (3.9) and due to uniqueness in the Wiener-Itô chaos expansion.

□

Note that according to the previous theorem the solution U remains in the same stochastic order space $(S)_{-1,-p}$ where the input data $U^0,\ \mathbf{A}U^0$ and F belong to.

Example 3.3. We provide three examples of equation (2.1) where $\mathbf{A}$ is a uniformly bounded (not a simple) coordinatewise operator. Consider the Banach space $X = L^p(\mathbb{R}),\ 1\le p<\infty$, and the stochastic Cauchy problem

$$\begin{aligned}\frac{d}{dt}U(t,x,\omega) &= \mathbf{A}U(t,x,\omega)+W\Diamond U(t,x,\omega)+F(t,x,\omega),\\ U(0,x,\omega) &= U^0(x,\omega),\end{aligned}\qquad(3.26)$$

where the operator $\mathbf{A}: Dom(\mathbf{A})\to X\otimes(S)_{-1}$ is a coordinatewise operator composed out of a family of closed operators $\{A_\alpha\}_{\alpha\in\mathcal{I}}$ of the form $A_\alpha = a_\alpha D,\ \alpha\in\mathcal{I}$, where the functions $a_\alpha\in L^\infty(\mathbb{R}),\ \alpha\in\mathcal{I}$, are uniformly bounded, i.e. $\sup_{x\in\mathbb{R}}|a_\alpha(x)|\le M,\ \alpha\in\mathcal{I}$, for some $M > 0$, and D is one of the following differential operators: $\frac{\partial}{\partial x},\ \frac{\partial^2}{\partial x^2}$ or $\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x}$, and $W = \sum_{k\in\mathbb{N}} \xi_k H_{\varepsilon_k}$ represents spatial white noise. Then, (3.26) is equivalent to the infinite system

$$\begin{aligned}\frac{d}{dt}u_\alpha(t,x) &= A_\alpha u_\alpha(t,x) + \sum_{k\in\mathbb{N}} \xi_k(x) u_{\alpha-\varepsilon_k}(t,x) + f_\alpha(t,x)\\ u_\alpha(0,x) &= u_\alpha^0(x), \quad \alpha\in\mathcal{I}.\end{aligned}$$

The C_0-semigroup that corresponds to the closed operator D, denoted by $T_t,\ t\geq 0$, is, respectively,

$$\begin{aligned}&T_t g(x) = g(t+x),\ \ g\in L^p(\mathbb{R}), \qquad \text{for}\ \ D=\frac{\partial}{\partial x},\\ &T_t g(x) = \frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} g(x-y)e^{-\frac{y^2}{4t}}dy,\ \ g\in L^p(\mathbb{R}), \qquad \text{for}\ \ D=\frac{\partial^2}{\partial x^2},\\ &T_t g(x) = \frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} g(x-y)e^{-\frac{(y+t)^2}{4t}}dy,\ \ g\in L^p(\mathbb{R}), \qquad \text{for}\ \ D=\frac{\partial^2}{\partial x^2}+\frac{\partial}{\partial x}.\end{aligned}$$

In all cases, we have, using Young's inequality, that $\|T_t\|\leq 1,\ t\geq 0$. The C_0-semigroups corresponding to the operators $A_\alpha,\ \alpha\in\mathcal{I}$, are of the form $(S_t)_\alpha = a_\alpha T_t$. Thus, $\|(S_t)_\alpha\|\leq M,\ \alpha\in\mathcal{I}$. The operators $B_\alpha,\ \alpha\in\mathcal{I}$, are given by $B_{\varepsilon_k}=\xi_k,\ k\in\mathbb{N}$ and $B_\alpha = 0,\ \alpha\neq\varepsilon_k$. Thus, $\|B_\alpha\|\leq \sup_{k\in\mathbb{N}}\|\xi_k\|_{L^\infty(\mathbb{R})}\leq 1,\ \alpha\in\mathcal{I}$. Now, according to Theorem 3.2, equation (3.26) has a unique solution $U(t,x,\omega)=\sum_{\alpha\in\mathcal{I}} u_\alpha(t,x)H_\alpha(\omega)$, where

$$u_\alpha(t,x) = (S_t)_\alpha u_\alpha^0(x) + \int_0^t (S_{t-s})_\alpha (\sum_k \xi_k(x)u_{\alpha-\varepsilon_k}(s,x) + f_\alpha(s,x))ds,\ \alpha\in\mathcal{I}.$$

Example 3.4. Consider the Cauchy problem

$$\begin{aligned}\frac{d}{dt}U(t,\omega) &= \mathbf{A}U(t,\omega) + \mathbf{B}\lozenge U(t,\omega) + F(t,\omega)\\ U(0,\omega) &= U^0(\omega),\end{aligned}$$

where $\mathbf{A}$ is a simple coordinatewise operator $A_\alpha = A$, $\alpha\in\mathcal{I}$, generating a C_0-semigroup, $B_\alpha\neq 0$ only for $\alpha=\varepsilon_k,\ k\in\mathbb{N}$, are such that $\sum_{k\in\mathbb{N}}\|B_{\varepsilon_k}\|(2k)^{-\frac{p}{2}}<\infty$, and U^0 and F are deterministic functions, i.e. $u_\alpha^0 = 0$ and $f_\alpha = 0$ for all $\alpha\in\mathcal{I}\setminus\{\mathbf{0}\}$.

The solution of this system, according to Theorem 3.2, is

$$\begin{aligned}u_{\mathbf{0}}(t) &= T_t u_{\mathbf{0}}^0 + \int_0^t T_{t-s}f_{\mathbf{0}}(s)ds,\\ u_\alpha(t) &= \int_0^t T_{t-s}\Big(\sum_{k\in\mathbb{N}} B_{\varepsilon_k}u_{\alpha-\varepsilon_k}(s)\Big)ds, \quad \alpha\in\mathcal{I}\setminus\mathbf{0},\end{aligned}$$

the same form as it was obtained in [12].

We provide two generalisations of Theorem 3.2: one possibility is to allow the operators B_α to depend on the time variable t (except for $B_{\mathbf{0}}$ which must be free of t). This embraces for example SPDEs driven by space-time noises which have zero expectation (and are thus free of t). The other possibility is to allow $B_{\mathbf{0}}$ to be unbounded but satisfying certain properties so that $A_\alpha + B_{\mathbf{0}}$ are infinitesimal generators of C_0-semigroups. For example, if $A_\alpha = \frac{\partial^2}{\partial x^2}$ and $B_{\mathbf{0}} = \frac{\partial}{\partial x}$, then although B_0 is unbounded, $A_\alpha + B_{\mathbf{0}}$ is the generator of a contraction semigroup. Following [4] we will enlist some sufficient conditions which ensure that $A_\alpha + B_{\mathbf{0}}$ is the generators of a C_0-semigroup.

Remark 3.5. In Theorem 3.2 one can consider operators $B_\alpha(t),\ \alpha \in \mathcal{I}\setminus\{\mathbf{0}\}$, depending on t, so that $B_\alpha \in C^1([0,T],L(X)),\ \alpha\in\mathcal{I}\setminus\{\mathbf{0}\}$, $B_{\mathbf{0}}(t)=B_{\mathbf{0}}\in L(X)$, for all $t\in[0,T]$, and

$$\begin{aligned}K:&=\sum_{\substack{\alpha\in\mathcal{I},\\ \alpha>\mathbf{0}}}\|B_\alpha\|_{C^1([0,T],L(X))}(2\mathbb{N})^{-p\frac{\alpha}{2}}\\ &=\sum_{\substack{\alpha\in\mathcal{I},\\ \alpha>\mathbf{0}}}\Big(\sup_{t\in[0,T]}\|B_\alpha(t)\|_{L(X)}+\sup_{t\in[0,T]}\|\frac{d}{dt}B_\alpha(t)\|_{L(X)}\Big)(2\mathbb{N})^{-p\frac{\alpha}{2}}<\infty.\end{aligned}\tag{3.27}$$

Replacing (3.2) by (3.27) and retaining all other assumptions of Theorem 3.2, one can again obtain a unique solution U in $C^1([0,T],X)\otimes(S)_{-1,-p}$ of the corresponding Cauchy problem (2.1).

The solution is $U(t,\omega)=\sum_{\alpha\in\mathcal{I}}u_\alpha(t)H_\alpha(\omega)$, $u_\alpha(t)\in C^1([0,T],X)$, $\alpha\in\mathcal{I}$, where (see (3.9))

$$\begin{aligned}u_{\mathbf{0}}(t)&=(S_t)_{\mathbf{0}}u^0_{\mathbf{0}}+\int_0^t(S_{t-s})_{\mathbf{0}}f_{\mathbf{0}}(s)ds,\quad t\in[0,T],\\ u_\alpha(t)&=(S_t)_\alpha u^0_\alpha+\int_0^t(S_{t-s})_\alpha\Big(\sum_{\mathbf{0}<\beta\le\alpha}B_\beta(s)u_{\alpha-\beta}(s)+f_\alpha(s)\Big)ds,\quad t\in[0,T].\end{aligned}\tag{3.28}$$

Its derivative is $\frac{d}{dt}U(t,\omega)=\sum_{\alpha\in\mathcal{I}}\frac{d}{dt}u_\alpha(t)H_\alpha(\omega)$, where (see (3.17))

$$\begin{aligned}\frac{d}{dt}u_{\mathbf{0}}(t)&=(S_t)_{\mathbf{0}}(A_{\mathbf{0}}+B_{\mathbf{0}})u^0_{\mathbf{0}}+\int_0^t(S_{t-s})_{\mathbf{0}}\frac{d}{ds}f_{\mathbf{0}}(s)ds+(S_t)_{\mathbf{0}}f(0),\quad t\in[0,T],\\ \frac{d}{dt}u_\alpha(t)&=(S_t)_\alpha(A_\alpha+B_{\mathbf{0}})u^0_\alpha\\ &\quad+\int_0^t(S_{t-s})_\alpha\Big(\sum_{\mathbf{0}<\beta\le\alpha}\Big(B_\beta(s)\frac{d}{ds}u_{\alpha-\beta}(s)+\frac{d}{ds}B_\beta(s)u_{\alpha-\beta}(s)\Big)+\frac{d}{ds}f_\alpha(s)\Big)ds\\ &\quad+(S_t)_\alpha\Big(\sum_{\mathbf{0}<\beta\le\alpha}B_\beta(0)u_{\alpha-\beta}(0)+f_\alpha(0)\Big),\quad t\in[0,T],\quad\alpha\in\mathcal{I}.\end{aligned}\tag{3.29}$$

The proof can be performed in the same manner as in Theorem 3.2, now taking $T_0\in(0,T]$ to be small enough so that $C(T_0)<\frac{1}{6K^2}$, since now we have six summands in (3.29) instead of the previous five in (3.17).

Remark 3.6. In Theorem 3.2 one can consider the operator $B_{\mathbf{0}}$ to be unbounded, densely defined on D (the same domain which is common for all A_α) so that either of the following holds:

(i) A_α, $\alpha\in\mathcal{I}$, are generating contraction semigroups (i.e. $M=1$, $w=0$), and $B_{\mathbf{0}}$ is dissipative, $A_\alpha-$bounded with $a^0_\alpha<1$ (i.e. there exist $a_\alpha,b_\alpha>0$ such that $\|B_{\mathbf{0}}x\|\le a_\alpha\|A_\alpha x\|+b_\alpha\|x\|$, $x\in D$, and $a^0_\alpha=\inf\{a_\alpha>0:\exists b_\alpha>0,\forall x\in D,\|B_{\mathbf{0}}x\|\le a_\alpha\|A_\alpha x\|+b_\alpha\|x\|\}$), for all $\alpha\in\mathcal{I}$,

(ii) $B_{\mathbf{0}}$ is closable, dissipative and $A_\alpha-$compact (i.e. $B:(D,\|\cdot\|_{A_\alpha})\to X$ is compact where $\|\cdot\|_{A_\alpha}$ denotes the graph norm), for all $\alpha\in\mathcal{I}$,

(iii) A_α are generating analytic semigroups (i.e. $w<0$), $\alpha\in\mathcal{I}$, and $B_{\mathbf{0}}$ is closable and $A_\alpha-$compact .

Then, $A_\alpha+B_{\mathbf{0}}$ is the infinitesimal generator of a C_0-semigroup (denote it $(S_t)_\alpha$) for all $\alpha\in\mathcal{I}$. If the semigroups $(T_t)_\alpha$ corresponding to A_α are uniformly bounded in α, then so will be $(S_t)_\alpha$. Retaining all other assumptions of Theorem 3.2, now we follow the same proof pattern with the semigroup $(S_t)_\alpha$, $\|(S_t)_\alpha\| \le \tilde{M} e^{\tilde{w}t}$, for some $\tilde{M} \ge 1$, $\tilde{w} \in \mathbb{R}$, independent of α.

Finally we note that in case (i) and (ii) $A_\alpha + B_{\mathbf{0}}$ will be generating contraction semigroups, while in case (iii) they will be generating analytic semigroups.

4 Stationary equations

In this section we consider stationary equations of the form

$$\mathbf{A}U + \mathbf{B}\lozenge U + F = 0, \tag{4.1}$$

where $\mathbf{A} : X \otimes (S)_{-1} \to X \otimes (S)_{-1}$ and $\mathbf{B}\lozenge : X \otimes (S)_{-1} \to X \otimes (S)_{-1}$ are coordinatewise operators as in (2.3) and (2.4). We assume that $\{A_\alpha\}_{\alpha\in\mathcal{I}}$ and $\{B_\alpha\}_{\alpha\in\mathcal{I}}$ are bounded operators and that A_α are of the form

$$A_\alpha = \widetilde{A}_\alpha + C_\alpha, \quad \alpha \in \mathcal{I},$$

where $B_{\mathbf{0}}$ and $\widetilde{A}_\alpha$, $\alpha \in \mathcal{I}$ are compact operators and C_α are self adjoint operators for all $\alpha \in \mathcal{I}$. Denote by r_α the eigenvalue corresponding to the orthogonal family of eigenvectors H_α, i.e. $C_\alpha(H_\alpha) = r_\alpha H_\alpha$, $\alpha \in \mathcal{I}$. Using classical results of elliptic PDEs and the Fredholm alternative (see [5]) we prove existence and uniqueness of the solution to (4.1).

Theorem 4.1. *Let X be a Banach space. Let $\mathbf{A} : X \otimes (S)_{-1} \to X \otimes (S)_{-1}$ and $\mathbf{B}\lozenge : X \otimes (S)_{-1} \to X \otimes (S)_{-1}$ be coordinatewise operators, for which the following assumptions hold:*

1. *$\mathbf{A}$ is of the form $\mathbf{A} = \widetilde{\mathbf{A}} + \mathbf{C}$, where $\widetilde{\mathbf{A}}(U) = \sum_{\alpha\in\mathcal{I}} \widetilde{A}_\alpha(u_\alpha)H_\alpha$ and $\widetilde{A}_\alpha : X \to X$ are compact operators for all $\alpha \in \mathcal{I}$, $\mathbf{C}(U) = \sum_{\alpha\in\mathcal{I}} r_\alpha u_\alpha H_\alpha$, $r_\alpha \in \mathbb{R}$, $\alpha \in \mathcal{I}$, and $\mathbf{B}$ is of the form (2.4), where $B_{\mathbf{0}} : X \to X$ is a compact operator. Assume there exists $K > 0$ such that:*

$$-\|\widetilde{A}_\alpha\| - \|B_{\mathbf{0}}\| - r_\alpha \ge 0, \quad \textit{for all} \quad \alpha \in \mathcal{I}, \tag{4.2}$$

   and

$$\sup_{\alpha\in\mathcal{I}} \left(\frac{1}{-r_\alpha - \|\widetilde{A}_\alpha\| - \|B_{\mathbf{0}}\|} \right) < K. \tag{4.3}$$

2. *$\mathbf{B}$ is of the form (2.4), where $B_\beta : X \to X$, $\beta \in \mathcal{I} \setminus \{\mathbf{0}\}$, are bounded operators and there exists $p > 0$ such that*

$$K \sum_{\substack{\beta\in\mathcal{I} \\ \beta>\mathbf{0}}} \|B_\beta\| \, (2\mathbb{N})^{\frac{-p\beta}{2}} < \frac{1}{\sqrt{2}}. \tag{4.4}$$

3. *For every $\alpha \in \mathcal{I}$*

$$\operatorname{Ker}\left(\widetilde{A}_\alpha + (1 + r_\alpha)\mathrm{Id} + B_{\mathbf{0}}\right) = \{0\}. \tag{4.5}$$

Then, for every $F \in X \otimes (S)_{-1,-p}$ there exists a unique solution $U \in X \otimes (S)_{-1,-p}$ to equation (4.1)*.*

Proof. Equation (4.1) is equivalent to $U - (\widetilde{\mathbf{A}}(U) + \mathbf{C}U + U + \mathbf{B}\lozenge U) = F$ and

$$\sum_{\gamma\in\mathcal{I}} \left(u_\gamma - \widetilde{A}_\gamma u_\gamma - (1 + r_\gamma)\, u_\gamma - \sum_{\alpha+\beta=\gamma} B_\alpha(u_\beta) \right) H_\gamma = \sum_{\gamma\in\mathcal{I}} f_\gamma H_\gamma.$$

Due to uniqueness of the Wiener-Itô chaos expansion this is equivalent to

$$u_\gamma - \left(\widetilde{A}_\gamma + (1+r_\gamma)Id + B_{\mathbf{0}}\right) u_\gamma = f_\gamma + \sum_{\mathbf{0}<\beta\leq\gamma} B_\beta(u_{\gamma-\beta}), \quad \gamma \in \mathcal{I}. \tag{4.6}$$

By (4.5) it follows that for each $\gamma \in \mathcal{I}$ the homogeneous equation

$$u_\gamma - \left(\widetilde{A}_\gamma + (1+r_\gamma)Id + B_{\mathbf{0}}\right) u_\gamma = 0$$

has only trivial solution $u_\gamma = 0$. Since the operator $\widetilde{A}_\gamma + (1+r_\gamma)Id + B_{\mathbf{0}}$ is compact, the classical Fredholm alternative implies that for each $\gamma \in \mathcal{I}$ there exists a unique u_γ that solves (4.6) and it is of the form

$$u_\gamma = (Id - ((r_\gamma+1)\, Id + \widetilde{A}_\gamma + B_{\mathbf{0}}))^{-1} \left(f_\gamma + \sum_{\beta>\mathbf{0}} B_\beta(u_{\gamma-\beta}) \right), \qquad \gamma \in \mathcal{I},$$

so that

$$\|u_\gamma\|_X \leq \frac{1}{-r_\gamma - \|\widetilde{A}_\gamma\| - \|B_{\mathbf{0}}\|} \cdot \left(\|f_\gamma\|_X + \sum_{\beta>\mathbf{0}} \|B_\beta\| \|u_{\gamma-\beta}\|_X \right), \quad \gamma \in \mathcal{I}.$$

We will prove that $\sum_{\gamma\in\mathcal{I}} u_\gamma \otimes H_\gamma$ converges in $X \otimes (S)_{-1}$. Indeed,

$$\begin{aligned}
\sum_{\gamma\in\mathcal{I}} \|u_\gamma\|_X^2 (2\mathbb{N})^{-p\gamma} &\leq K^2 \sum_{\gamma\in\mathcal{I}} \left(\|f_\gamma\|_X + \sum_{\gamma=\alpha+\beta,\alpha>\mathbf{0}} \|B_\alpha\| \|u_\beta\|_X \right)^2 (2\mathbb{N})^{-p\gamma} \\
&\leq 2K^2 \left(\sum_{\gamma\in\mathcal{I}} \|f_\gamma\|_X^2 (2\mathbb{N})^{-p\gamma} + \sum_{\gamma\in\mathcal{I}} (\sum_{\gamma=\alpha+\beta,\alpha>\mathbf{0}} \|B_\alpha\| \|u_\beta\|_X)^2 (2\mathbb{N})^{-p\gamma} \right) \\
&\leq 2K^2 \left(\sum_{\gamma\in\mathcal{I}} \|f_\gamma\|_X^2 (2\mathbb{N})^{-p\gamma} + (\sum_{\alpha>\mathbf{0}} \|B_\alpha\| (2\mathbb{N})^{-\frac{p\alpha}{2}})^2 \sum_{\beta\in\mathcal{I}} \|u_\beta\|_X^2 (2\mathbb{N})^{-p\beta} \right).
\end{aligned}$$

Therefore,

$$(1 - 2K^2(\sum_{\alpha>\mathbf{0}} \|B_\alpha\| (2\mathbb{N})^{-\frac{p\alpha}{2}})^2) \cdot \sum_{\gamma\in\mathcal{I}} \|u_\gamma\|_X^2 (2\mathbb{N})^{-p\gamma} \leq 2K^2 \sum_{\gamma\in\mathcal{I}} \|f_\gamma\|_X^2 (2\mathbb{N})^{-p\gamma}.$$

By assumption (4.4) we have that $M = 1 - 2K^2(\sum_{\alpha>\mathbf{0}} \|B_\alpha\| (2\mathbb{N})^{-\frac{p\alpha}{2}})^2 > 0$. This implies

$$\sum_{\gamma\in\mathcal{I}} \|u_\gamma\|_X^2 (2\mathbb{N})^{-p\gamma} \leq \frac{2K^2}{M} \sum_{\gamma\in\mathcal{I}} \|f_\gamma\|_X^2 (2\mathbb{N})^{-p\gamma} < \infty.$$

□

Example 4.2. We provide some special cases of equation (4.1).

1. If $A_\alpha = 0$ for all $\alpha \in \mathcal{I}$ and B_α, $\alpha \in \mathcal{I}$ are second order strictly elliptic partial differential operators in divergent form

$$B_\alpha = \sum_{i=1}^n D_i (\sum_{j=1}^n a_\alpha^{ij}(x) D_j + b_\alpha^i(x)) + \sum_{i=1}^n c_\alpha^i(x) D_i + d_\alpha(x) \tag{4.7}$$

with essentially bounded coefficients, then equation (4.1) reduces to the elliptic equation

$$\mathbf{B}\lozenge U = F,$$

which was solved in [18] and [19].

2. Let $\widetilde{A}_\alpha = 0$ for all $\alpha \in \mathcal{I}$ and let B_α, $\alpha \in \mathcal{I}$, be second order strictly elliptic partial differential operators in divergent form (4.7). Let $\mathbf{C} = c\,P(\mathcal{R})$, for some $c \in \mathbb{R}$, where $\mathcal{R}$ is the Ornstein-Uhlenbeck operator, P a polynomial of degree m with real coefficients and $P(\mathcal{R})$ the differential operator $P(\mathcal{R}) = p_m\mathcal{R}^m + p_{m-1}\mathcal{R}^{m-1} + ... + p_1\mathcal{R} + p_0 Id$. Then, the corresponding eigenvalues are $r_\alpha = cP(|\alpha|)$, $\alpha \in \mathcal{I}$. Hence, equation (4.1) transforms to the elliptic equation with a perturbation term driven by the polynomial of the Ornstein-Uhlenbeck operator

$$\mathbf{B}\lozenge U + cP(\mathcal{R})U = F,$$

that was solved in [11].

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