Multivariate juggling probabilities[*]

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Abstract

We consider refined versions of Markov chains related to juggling introduced by Warrington. We further generalize the construction to juggling with arbitrary heights as well as infinitely many balls, which are expressed more succinctly in terms of Markov chains on integer partitions. In all cases, we give explicit product formulas for the stationary probabilities. The normalization factor in one case can be explicitly written as a homogeneous symmetric polynomial. We also refine and generalize enriched Markov chains on set partitions. Lastly, we prove that in one case, the stationary distribution is attained in bounded time.

1 Introduction

Although juggling as a human endeavour has been around since time immemorial, it is fairly recently that mathematicians have taken an active interest in exploring the field. Combinatorialists became interested in juggling towards the end of the last century after an article in the *Amer. Math. Monthly* by Buhler, Eisenbud, Graham and Wright [4], where they enumerate what they call *juggling sequences* and relate it to other known combinatorial structures. Since then, their results have been q-ified [9] and further refined in various ways [20, 21, 6, 7, 5]. Other connections between juggling and

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mathematics appear for instance in algebraic geometry [8, 14]. A mathematical history of juggling is given in the fascinating book by Polster [18].

Probabilists, on the other hand, are newer to this game, possibly because no one is able to perform totally random juggling. Coincidentally, it was another popular article in the *Amer. Math. Monthly* by Warrington [22] on natural stochastic models inspired by juggling that got the attention of a few other members of the community. Exact combinatorial formulas for the stationary distribution of these models were found, and proved by a beautiful argument involving an auxiliary model defined on a larger state space whose elements may be viewed as set partitions, for which the stationary distribution is the uniform distribution.

The simplest model considered by Warrington, where the number of balls is conserved, has been generalized by Leskelä and Varpanen as the so-called Juggler's Exclusion Process (JEP), where the balls can be thrown arbitrarily high so that the state space is infinite [16]. These authors showed that the JEP converges to a (unique) stationary distribution under very mild assumptions, could obtain an exact expression for this stationary distribution in the particular case of a "memoryless" (geometric) height distribution, and noted an intriguing phenomenon of "ultrafast convergence" in this case. More recently, a q-deformation of Warrington's original finite model was considered by Engström, Leskelä and Varpanen [10], who also obtained exact expressions for the stationary distribution via an enriched chain formulated in terms of rook placements (in bijection with some set partitions).

In this paper, we provide multivariate generalizations of all the models introduced in [22], namely *juggling, add-drop juggling* and *annihilation juggling*. Furthermore, in the case of the juggling model with a conserved number of balls, we investigate the limiting case where balls can be thrown arbitrarily high, which corresponds to the so-called infinite juggling model suggested by Warrington. We also consider the limiting case where the number of balls tends to infinity. In all these cases, we obtain an exact formula for the stationary distribution (in particular, in the case where balls can be thrown arbitrarily high, we find an "exactly solvable" instance of the general JEP of [16] with countably many parameters) and the normalization factor. In one case the normalization factor can be explicitly written as a homogeneous symmetric polynomial. Our proofs were obtained using two approaches. The direct approach involves guessing the general formulas (for instance by a computation for small system sizes) and then proving the results in a straightforward manner by considering the juggling process itself. The other approach is more combinatorial, and consists in introducing an enriched chain whose stationary distribution is simpler, and which yields the original chain by a projection or "lumping" procedure on set partitions or words.

We remark that juggling models (especially the add-drop and annihilation versions) have a natural interpretation in statistical physics, where one could think of the balls as coming from a reservoir of particles. A model in this vein has been proposed recently, see [1].

The rest of the paper is organized as follows. In Section 2, we concentrate on the simplest version of our model, which we call the Multivariate Juggling Markov Chain (MJMC). The model is defined in Section 2.1, and we also discuss the uniqueness of the stationary distribution (a technicality deemed obvious in [22]). The expression for the stationary distribution of the Markov chain is stated in Section 2.2. For pedagogical purposes, we decide to provide two independent proofs by the direct and the combinatorial approaches. The direct proof is given in Section 2.3, via an interesting reformulation of the MJMC in terms of integer partitions restricted to lie within a rectangle. The combinatorial proof comes in Section 2.4 and involves set partitions with a prescribed number of elements and blocks. We then turn to extended models. Extensions to infinite state

spaces are considered in Section 3: the case of a finite number of balls but *unbounded* heights (UMJMC) is discussed in Section 3.1, while the case of an *infinite* number of balls (IMJMC) is considered in Section 3.2. In both cases, we find the stationary measure by the direct approach. Extensions to a fluctuating number of balls (but with a finite state space) are considered in Section 4: we provide the multivariate extension of the add-drop and the annihilation models introduced in [22], in the respective Sections 4.1 and 4.2. These models have the same transition graph, only the transitions probabilities differ. In both cases we find the stationary distribution by the combinatorial approach. In the case of the annihilation model, we further observe the interesting property that the stationary distribution is attained in bounded time. We end with some remarks and questions for future study in Section 5.

The claims of this paper can be verified by downloading the Maple$^{\text{TM}}$ program RandomJuggling either from the arXiv source or the first author's (A.A.) homepage.

2   The finite Multivariate Juggling Markov Chain

2.1   Definition

In this section, we introduce our juggling model in the simplest setting, i.e. a Markov chain on a finite state space. We start by explaining the model in colloquial terms, and refer to [22] for further motivation. Consider a person, called *Magnus* with no loss of generality, who is juggling with a fixed finite number ℓ of balls. Time is discretized in, say, steps of one second and we assume that, at each second, Magnus is able to catch at most one ball, and then throws it back immediately. Besides this limitation Magnus juggles *perfectly*, i.e. in such a way that the ball will always return to him after some time (an imperfect juggler could drop a ball or throw it in a wrong direction, for instance). Magnus controls the velocity at which he sends the ball, which determines how long it will take for the ball to return to him. We suppose for now that the launch velocity is bounded or, in other words, that the number of seconds before the ball returns to Magnus is bounded by an integer h.

Ignoring further spatial constraints, a simplified description of the state of the balls at a given time consists in associating to each ball the (integer) number of seconds remaining before it is caught by Magnus. This is now known as the *siteswap* notation. Of course, to be able to juggle for an indefinite amount of time, Magnus must choose the successive launch velocities in such a way that no two balls arrive to him at the same time. Thus the numbers associated to different balls shall be distinct and, treating the balls as indistinguishable, there are $\binom{h}{\ell}$ possible ball states. It is convenient to think of a state as a configuration of ℓ non overlapping particles on a one-dimensional lattice with h sites, where the i-th site (read from the left) is occupied if and only if a ball is scheduled to arrive i seconds in the future. However, beware that sites do no correspond to actual spatial positions, but to the "remaining flight times" of the balls. We denote by $k = h - \ell$ the number of empty (unoccupied) sites.

In this language, the time evolution of a state is easy to describe: at each time step, all particles are moved one site to the left. If there is no particle on the first site (i.e. Magnus catches no ball), then nothing else has to be done. Otherwise the particle on the first site, which would exit the lattice if moved to the left, is instead taken away and *reinserted* at one of the $k + 1$ available (empty) sites on the lattice (determined by the launch velocity chosen by Magnus). This defines the transition graph of our model, illustrated on Figure 1 for $h = 4$ and $\ell = k = 2$ (ignoring edge labels for now).

We now assume that Magnus juggles at random: each reinsertion is made randomly at one of the $k + 1$ available sites. In our model, we assume that the reinsertion is made at the i-th available site (read from the left) with probability x_{i-1}, independently of the

past, so that our model is a Markov chain. Here, $x_0, \ldots, x_k$ are fixed nonnegative real numbers such that $x_0 + \cdots + x_k = 1$. This defines the *Multivariate Juggling Markov Chain* (MJMC), which generalizes the model considered in [22], obtained by taking $x_0 = \cdots = x_k = 1/(k+1)$, but is a particular case of the general "Juggling Exclusion Process" defined in [16]: beyond the extension to infinitely many empty sites which we will discuss in the next section, the main difference is that, in the model of Leskelä and Varpanen, the x_i's are allowed to depend on the current state.
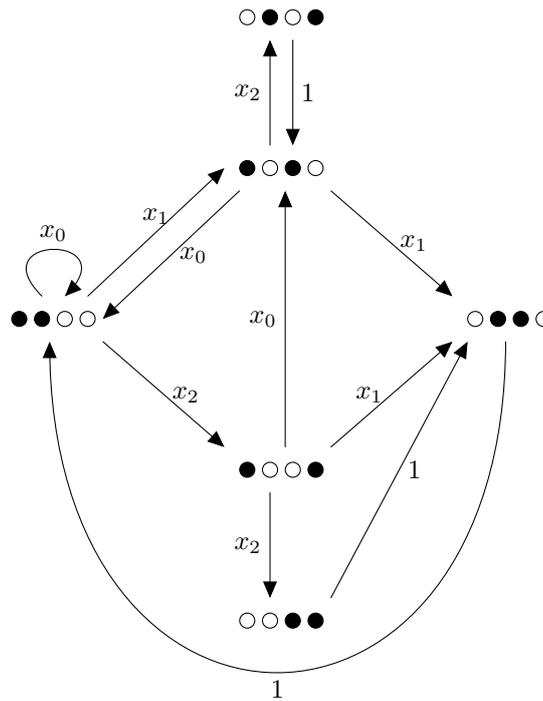

Figure 1: The Markov chain with $h = 4$ and $\ell = k = 2$.

We now provide a more formal mathematical definition of the MJMC. Following Warrington's notation, let St_h denote the set of words of length h on the alphabet $\{\bullet, \circ\}$, and let $\mathrm{St}_{h,k} \subset \mathrm{St}_h$ be the subset of words containing exactly k occurrences of $\circ$. For $A \in \mathrm{St}_{h,k+1}$ and $i \in \{0, \ldots, k\}$, we let $T_i(A) \in \mathrm{St}_{h,k}$ be the word obtained by replacing the $(i+1)$-th occurrence of $\circ$ in A by $\bullet$.

Definition 2.1. *Given h, k nonnegative integers such that $h \geq k$, and $x_0, \ldots, x_k$ nonnegative real numbers such that $x_0 + \cdots + x_k = 1$, the* Multivariate Juggling Markov Chain *is the Markov chain on the state space $\mathrm{St}_{h,k}$ for which the transition probability from $A = a_1 a_2 \cdots a_h$ to B reads*

$$P_{A,B} = \begin{cases} 1 & \text{if } a_1 = \circ \text{ and } B = a_2 \cdots a_h \circ, \\ x_i & \text{if } a_1 = \bullet \text{ and } B = T_i(a_2 \cdots a_h \circ), \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Example 2.2. Figure 1 illustrates the Markov chain in the case $h = 4$, $k = 2$, and the

transition matrix in the basis $(\bullet\,\bullet\,\circ\circ, \bullet\,\circ\,\bullet\circ, \bullet\,\circ\,\circ\bullet, \circ\,\bullet\,\bullet\circ, \circ\,\bullet\,\circ\bullet, \circ\,\circ\,\bullet\bullet)$ reads

$$
\begin{pmatrix}
x_0 & x_1 & x_2 & 0 & 0 & 0 \\
x_0 & 0 & 0 & x_1 & x_2 & 0 \\
0 & x_0 & 0 & x_1 & 0 & x_2 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}
\tag{2.2}
$$

Note that $(1, x_1 + x_2, x_2, (x_1 + x_2)^2, x_2(x_1 + x_2), x_2^2)$ is a left eigenvector with eigenvalue 1.

To compare our model with that of [16], note that $\mathrm{St}_{h,k}$ may be canonically identified with the set of subsets of $\{1, \ldots, h\}$ with $\ell = h - k$ elements (corresponding to the positions of the occupied sites). For two such subsets S and S', the transition probability from S to S' reads

$$
P_{S,S'} = \begin{cases}
1 & \text{if } S' = S - 1, \\
x_i & \text{if } S' = (S - 1) \setminus \{0\} \cup \{j\} \text{ with } i = |\{1, \ldots, j\} \setminus S|, \\
0 & \text{otherwise,}
\end{cases}
\tag{2.3}
$$

where $S - 1$ is the set whose entries are those of S with 1 subtracted.

Proposition 2.3. *If $x_0 > 0$, then the MJMC has a unique closed communicating class, whose all states are aperiodic. If furthermore $x_k > 0$, then the MJMC is irreducible.*

Proof. Assume that $x_0 > 0$ and let $\mathcal{E} = \bullet^\ell \circ^k$ be the "lowest" state. It is clear that, starting from any initial state, we may obtain $\mathcal{E}$ by a sequence of transitions of probabilities x_0 or 1, simply by doing any reinsertion at the first available site. Since $\mathcal{E}$ jumps to itself with probability x_0, there is a unique closed communicating class, and all states in this class are aperiodic.

Assume now that $x_k > 0$: to prove irreducibility we simply need to show that any state $A = a_1 \cdots a_h \in \mathrm{St}_{h,k}$ may be obtained from $\mathcal{E}$. The idea is to build the prefix of A of length i at the i-th step so that we get A in exactly h steps. For $i \in \{0, \ldots, h\}$, we let $A_i = \bullet^{\ell-i+n_i} \circ^{k-n_i} a_1 \cdots a_i$ where n_i is the number of occurrences of $\circ$ in $a_1 \cdots a_i$, so that $A_i \in \mathrm{St}_{h,k}$, with $A_0 = \mathcal{E}$ and $A_h = A$. For $i < h$, we have

$$
P_{A_i, A_{i+1}} = \begin{cases}
1 & \text{if } \ell = i - n_i, \\
x_0 & \text{if } \ell > i - n_i \text{ and } a_{i+1} = \circ, \\
x_k & \text{if } \ell > i - n_i \text{ and } a_{i+1} = \bullet.
\end{cases}
\tag{2.4}
$$

Note that when $a_1 = \circ$, $A_1 = \mathcal{E}$, but this is allowed since there is always a self-loop from $\mathcal{E}$ to itself. Thus A may be obtained from $\mathcal{E}$ by a sequence of possible transitions. $\quad\square$

Remark 2.4. When $x_0 = 0$, the chain may have several closed communicating classes. For example, see Figure 1 and let $x_0 = x_1 = 0$.

2.2 Stationary distribution

From now on we assume $x_0 > 0$. By Proposition 2.3, the MJMC admits a unique stationary probability distribution. Our main result is an explicit form for it, given as follows.

Theorem 2.5. *The stationary distribution π of the MJMC is given by*

$$
\pi(B) = \frac{1}{Z_{h,k}} \prod_{\substack{i \in \{1, \ldots, h\} \\ b_i = \bullet}} \left(x_{E_i(B)} + \cdots + x_k \right),
\tag{2.5}
$$

where $B = b_1 \cdots b_h \in \mathrm{St}_{h,k}$ and $E_i(B) = \#\{j < i \mid b_j = \circ\}$, and where $Z_{h,k} \equiv Z_{h,k}(x_0, \ldots, x_k)$ is determined by the condition that π be a probability distribution.

We will provide two proofs of this theorem, as they both lead to interesting generalizations to be studied later in Sections 3 and 4. The first proof, presented in Section 2.3, is a rather direct computational check that (2.5) indeed defines a stationary measure of the MJMC (but we find it clearer to first reformulate the chain in terms of integer partitions). The second proof, presented in Section 2.4, is more combinatorial and generalizes the approach of [22]: it consists in defining an "enriched" Markov chain on a larger state space, where the stationary probability of a given state will be given by a monomial in $x_0, \ldots, x_k$. The MJMC is then recovered by lumping together states of the enriched chain, which explains the form of (2.5).

Before proceeding to the proofs, let us discuss a few interesting cases.

Example 2.6. If the reinsertion is made uniformly among all $k + 1$ available sites, i.e. if we take $x_0 = \cdots = x_k = 1/(k+1)$, then we recover [22, Theorem 1] from Theorem 2.5 (see Corollary 2.11 below for a short rederivation of the normalization factor).

Example 2.7. If the reinsertion site is chosen according to a geometric distribution of parameter q conditioned to be smaller than or equal to $k + 1$, i.e. if we take $x_i = (1-q)q^i/(1-q^{k+1})$, $i = 0, \ldots, k$, then we recover the so-called bounded geometric JEP considered in [10]. It is a simple exercise to check that we recover [10, Theorem 2.1] from Theorem 2.5 (see also Corollary 2.11 below for the normalization factor).

Example 2.8. A slight variant of the previous example consists in picking the reinsertion site according to a "truncated" geometric distribution of parameter q, i.e. taking $x_i = (1-q)q^i$ for $i = 0, \ldots, k-1$ and $x_k = q^k$. In that case, the stationary probability of $B \in \mathrm{St}_{h,k}$ reads

$$\pi(B) = \frac{1}{\binom{h}{k}_q} \prod_{\substack{i \in \{1, \ldots, h\} \\ b_i = \bullet}} q^{E_i(B)} \tag{2.6}$$

(see again Corollary 2.11 below for the normalization factor).

We now move on to discuss properties of the normalization factor $Z_{h,k}$. Observe first that the expression

$$Z_{h,k}(x_0, \ldots, x_k) = \sum_{B \in \mathrm{St}_{h,k}} \prod_{\substack{i \in \{1, \ldots, h\} \\ b_i = \bullet}} \left(x_{E_i(B)} + \cdots + x_k \right) \tag{2.7}$$

allows us to define it for any $x_0, \ldots, x_k$ (not necessarily positive or subject to the condition $x_0 + \cdots + x_k = 1$) and $h, k \geq 0$. It vanishes for $h < k$ and is otherwise a homogeneous polynomial of degree $\ell = h - k$.

Proposition 2.9. *We have*

$$Z_{h,k} = \sum_{0 \leq i_1 \leq \cdots \leq i_\ell \leq k} y_{i_1} \cdots y_{i_\ell} = h_\ell(y_0, \ldots, y_k) \tag{2.8}$$

with h_ℓ the complete homogeneous symmetric polynomial of degree $\ell = h - k$, and

$$y_m = \sum_{j=m}^{k} x_j, \qquad m = 0, \ldots, k. \tag{2.9}$$

Again, there are several ways to establish this result. Perhaps the most combinatorial explanation comes from the integer partition approach of Section 2.3, see Remark 2.16 below. But, starting from the definition (2.7), we may easily check (2.8) by induction, using the following:

Lemma 2.10. *For $h, k \geq 0$, we have*

$$Z_{h,k}(x_0, \ldots, x_k) = Z_{h-1,k-1}(x_1, \ldots, x_k) + (x_0 + \cdots + x_k)Z_{h-1,k}(x_0, \ldots, x_k) \quad (2.10)$$

where, by convention, $Z_{h,k} = \delta_{h,k}$ if $h = -1$ or $k = -1$.

Proof. Obtained immediately by distinguishing whether $B \in \mathrm{St}_{h,k}$ starts with a $\circ$ or a $\bullet$ (for $h < k$ all terms vanish as wanted). $\qquad \square$

We readily verify by induction and homogeneity some explicit specializations:

Corollary 2.11. *We have*

$$Z_{h,k}(1, \ldots, 1) = \begin{Bmatrix} h+1 \\ k+1 \end{Bmatrix}, \quad (2.11)$$

$$Z_{h,k}(q^k, q^{k-1}, \ldots, 1) = \begin{Bmatrix} h+1 \\ k+1 \end{Bmatrix}_q, \quad (2.12)$$

$$Z_{h,k}(1, q, \ldots, q^k) = q^{k(h-k)} \begin{Bmatrix} h+1 \\ k+1 \end{Bmatrix}_{1/q}, \quad (2.13)$$

$$Z_{h,k}((1-q), (1-q)q, \ldots, (1-q)q^{k-1}, q^k) = \binom{h}{k}_q, \quad (2.14)$$

where $\begin{Bmatrix} \cdot \\ \cdot \end{Bmatrix}$ denotes Stirling numbers of the second kind, $\begin{Bmatrix} \cdot \\ \cdot \end{Bmatrix}_q$ their q-analogues as defined in [13] and $\binom{\cdot}{\cdot}_q$ q-binomial coefficients. Using homogeneity, we deduce the normalization factors for Examples 2.6 and 2.7.

For completeness, we mention another recursion relation for $Z_{h,k}$ of a different nature, as we remove x_k instead of x_0.

Proposition 2.12. *For $h, k \geq 0$, we have*

$$Z_{h,k}(x_0, \ldots, x_k) = \sum_{n=0}^{h-k} \binom{h}{n} x_k^n Z_{h-n-1,k-1}(x_0, \ldots, x_{k-1}) \quad (2.15)$$

where, by convention, $Z_{h,k} = \delta_{h,k}$ if $h = -1$ or $k = -1$.

Proof. Write

$$Z_{h,k} = h_\ell(y_0' + x_k, y_1' + x_k, \ldots, y_{k-1}' + x_k, x_k) \quad (2.16)$$

with $y_m' = \sum_{j=m}^{k-1} x_j$, and use the identity (easily verified e.g. by induction on $\ell + k$)

$$h_\ell(a_0 + a, a_1 + a, \ldots, a_k + a) = \sum_{n=0}^{\ell} \binom{\ell+k}{n} a^n h_{\ell-n}(a_0, \ldots, a_k). \quad (2.17)$$

$\qquad \square$

2.3 Reformulation of the MJMC in terms of integer partitions

There is a natural bijection between $\mathrm{St}_{h,k}$ and $\mathrm{Par}_{k,h-k}$, where $\mathrm{Par}_{k,\ell}$ is the set of integer partitions whose Young diagram fits within a $k \times \ell$ rectangle. In other words, an element of $\mathrm{Par}_{k,\ell}$ can be viewed as a nonincreasing sequence of ℓ nonnegative integers smaller than or equal to k. The bijection is given explicitly as follows: given a state in $\mathrm{St}_{h,k}$, we denote by $s_1 < \cdots < s_\ell$ the positions of $\bullet$'s, then the corresponding integer partition is $(s_\ell - \ell, s_{\ell-1} - (\ell-1), \ldots, s_1 - 1)$. Alternatively, an equivalent graphical construction is displayed on Figure 2. Upon identifying $\mathrm{St}_{h,k}$ and $\mathrm{Par}_{k,h-k}$ via this bijection, we may reformulate the MJMC as follows.
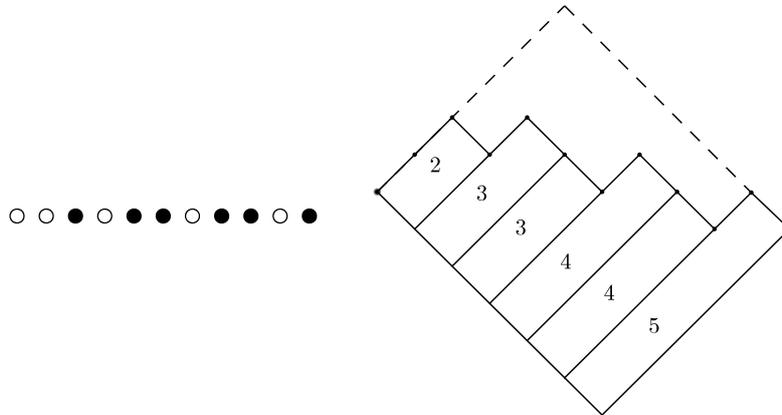
Figure 2: Bijection between juggling states and integer partitions. Each ○ is replaced by a north-east step, and each ● by a south-east step. The partition thus obtained is $(5, 4, 4, 3, 3, 2)$.

Proposition 2.13. *For $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ and μ two partitions in $\mathrm{Par}_{k,\ell}$, the transition probability from λ to μ reads*

$$P_{\lambda,\mu} = \begin{cases} 1 & \text{if } \lambda_\ell \neq 0 \text{ and } \mu = (\lambda_1 - 1, \ldots, \lambda_\ell - 1), \\ x_i & \text{if } \lambda_\ell = 0 \text{ and there exists } j \in \{1, \ldots, \ell\} \text{ such that} \\ & \mu = (\lambda_1 - 1, \ldots, \lambda_{j-1} - 1, i, \lambda_j, \ldots, \lambda_{\ell-1}), \\ 0 & \text{otherwise.} \end{cases} \tag{2.18}$$

The proof of this proposition is left to the reader. The interest of the reformulation in terms of integer partitions is that the stationary distribution of the MJMC takes a particularly simple form: indeed, Theorem 2.5 is equivalent to

Theorem 2.14. *The stationary distribution π of the MJMC is given by*

$$\pi(\lambda) = \frac{1}{Z_{h,k}} \prod_{i=1}^{\ell} y_{\lambda_i} \tag{2.19}$$

where $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ is an element of $\mathrm{Par}_{k,\ell}$ and where the y_m are as in (2.9).

Proof. Let $w(\lambda) = \prod_{i=1}^{\ell} y_{\lambda_i}$ be the unnormalized weight of $\lambda \in \mathrm{Par}_{k,\ell}$. We simply check that

$$w(\lambda) = \sum_{\mu \in \mathrm{Par}_{k,\ell}} w(\mu) P_{\mu,\lambda} \tag{2.20}$$

by considering the possible predecessors of λ:

- If $\lambda_1 = k$ then the only possible predecessor for λ is $\mu = (\lambda_2, \ldots, \lambda_\ell, 0)$, so that $w(\mu) = \prod_{i=2}^{\ell} y_{\lambda_i}$ (because $y_0 = 1$) and $P_{\mu,\lambda} = x_k = y_k$. Thus we have $w(\lambda) = P_{\mu,\lambda} w(\mu)$ as wanted.
- We now assume that $\lambda_1 < k$. Let μ be a predecessor of λ, μ can

– either be of the form $\mu = (\lambda_1 + 1, \ldots, \lambda_\ell + 1)$, i.e. is associated with a word that starts with a $\circ$, in which case we have $w(\mu) = \prod_{i=1}^{\ell} y_{\lambda_i+1} = \prod_{i=1}^{\ell} (y_{\lambda_i} - x_{\lambda_i})$ and $P_{\mu,\lambda} = 1$;

– or be of the form $\mu = (\lambda_1 + 1, \ldots, \lambda_{j-1} + 1, \lambda_{j+1}, \ldots, \lambda_\ell, 0)$ for any $1 \leq j \leq \ell$, i.e. is associated with a word that starts with a $\bullet$, in which case we have $w(\mu) = \prod_{i=1}^{j-1}(y_{\lambda_i} - x_{\lambda_i}) \prod_{i=j+1}^{\ell} y_{\lambda_i}$ and $P_{\mu,\lambda} = x_{\lambda_j}$.

Thus, we have

$$
\sum_{\mu \in \mathrm{Par}_{k,\ell}} w(\mu) P_{\mu,\lambda} = \prod_{i=1}^{\ell}(y_{\lambda_i} - x_{\lambda_i}) + \sum_{j=1}^{\ell}\left(x_{\lambda_j}\prod_{i=1}^{j-1}(y_{\lambda_i} - x_{\lambda_i})\prod_{i=j+1}^{\ell} y_{\lambda_i}\right)
$$
$$
= \prod_{i=1}^{\ell} y_{\lambda_i} = w(\lambda)
$$
(2.21)

where, to go from the first line to the second, we write $x_{\lambda_j} = y_{\lambda_j} - (y_{\lambda_j} - x_{\lambda_j})$ so that the sum on the right hand side becomes telescopic.

$\square$

Example 2.15. We may now revisit Examples 2.6, 2.7 and 2.8 in the language of integer partitions. The corresponding stationary probabilities of $\lambda \in \mathrm{Par}_{k,\ell}$ read

$$
\pi(\lambda) = \begin{cases} \frac{1}{\left\{\begin{smallmatrix} h+1 \\ k+1 \end{smallmatrix}\right\}}\prod_{i=1}^{\ell}(k+1-\lambda_i) & \text{for } x_i = \frac{1}{k+1}, \\ \frac{1}{q^{k\ell}\left\{\begin{smallmatrix} h+1 \\ k+1 \end{smallmatrix}\right\}_{1/q}}\prod_{i=1}^{\ell} q^{\lambda_i}[k+1-\lambda_i]_q & \text{for } x_i = \frac{q^i}{[k+1]_q}, \\ \frac{1}{\binom{h}{k}_q}\prod_{i=1}^{\ell} q^{\lambda_i} & \text{for } x_i = (1-q)^{1-\delta_{i,k}}q^i. \end{cases}
$$
(2.22)

Remark 2.16. One can immediately recover Proposition 2.9 from Theorem 2.14, as

$$
Z_{h,k} = \sum_{\lambda \in \mathrm{Par}_{k,\ell}}\prod_{j=1}^{\ell} y_{\lambda_j} = \sum_{\substack{(m_0,\ldots,m_k)\in\mathbb{N}^{k+1} \\ \sum_{i=0}^{k} m_i=\ell}}\prod_{i=0}^{k} y_i^{m_i} = h_\ell(y_0,\ldots,y_k)
$$
(2.23)

where the m_i correspond to the part multiplicities. A slight extension of this reasoning consists in restricting the sum to partitions λ such that, say, $\lambda_j = n$ (with $1 \leq j \leq \ell$ and $0 \leq n \leq k$), so as to obtain the probability that the j-th part in the stationary distribution π is n:

$$
\pi(\{\lambda_j = n\}) = \frac{y_n h_{j-1}(y_n,\ldots,y_k) h_{\ell-j}(y_0,\ldots,y_n)}{h_\ell(y_0,\ldots,y_k)}.
$$
(2.24)

More generally, the joint distribution for a fixed number of parts reads

$$
\pi(\{\lambda_{j_1} = n_1, \ldots, \lambda_{j_m} = n_m\}) = \frac{\prod_{s=1}^{m+1} y_{n_s} h_{j_s-j_{s-1}-1}(y_{n_s},\ldots,y_{n_{s-1}})}{h_\ell(y_0, y_1, \ldots, y_k)}
$$
(2.25)

where $0 = j_0 < \cdots < j_{m+1} = \ell + 1$ and $k = n_0 \geq \cdots \geq n_{m+1} = 0$ (with $y_0 = 1$).

2.4 Enriched Markov chain on set partitions

We now provide another, more combinatorial, proof of Theorem 2.5 whose rough idea goes as follows: consider the stationary probability (2.5) and expand the product in the right hand side as a sum of monomials in the x_i's. We will interpret each of these monomials as the stationary probability of an "enriched" state belonging to the larger state space of another Markov chain. The MJMC will then be obtained as a projection of this enriched chain (see e.g. [17, Section 2.3.1] for a definition of this notion).

2.4.1 Definitions and basic properties

Our construction is a generalization of that of Warrington's. Rather than working with "landing/throwing-states", we prefer to work directly in the language of set partitions, see [22, Lemma 2]. Recalling that h stands for the total number of sites and k for the number of empty sites in the particle picture of the MJMC, let us introduce the shorthand notations

$$H = h + 1 \quad \text{and} \quad K = k + 1. \tag{2.26}$$

An *enriched state* will then be a partition of the set $\{1, \ldots, H\}$ into K subsets or *blocks*. We denote by $\mathcal{S}(H, K)$ the set of enriched states, and recall that $\left\{ {H \atop K} \right\} = |\mathcal{S}(H, K)|$ is a Stirling number of the second kind. To each enriched state σ, we associate a word $\psi(\sigma) = a_1 \ldots a_h$ by setting, for all i between 1 and h, $a_i = \circ$ if i is a *block maximum* of σ (i.e. the largest element of its block in σ), and $a_i = \bullet$ otherwise. Observe that ψ is a surjection from $\mathcal{S}(H, K)$ onto $\mathrm{St}_{h,k}$.

We now define the enriched Markov chain on $\mathcal{S}(H, K)$, which requires some notations. For an enriched state σ, we denote by $\sigma^\downarrow$ the partition of the set $\{1, \ldots, h\}$ obtained by removing 1 from σ (i.e. removing 1 from its block in σ, and removing this block from σ if it becomes empty), and shifting all the remaining elements of all blocks down by 1. Note that $\sigma \mapsto \sigma^\downarrow$ is a mapping from $\mathcal{S}(H, K)$ to $\mathcal{S}(h, k) \cup \mathcal{S}(h, K)$ (which is surjective). For $\tau \in \mathcal{S}(h, K)$ and $i \in \{0, \ldots, k\}$, we denote by $I_i(\tau)$ the set partition of $\{1, \ldots, H\}$ obtained by inserting H into the $(i+1)$-th block of τ, where the blocks are numbered by ascending order of their maxima (i.e. the first block has the smallest maximum among all blocks, etc. – this differs from the so-called standard form which consists in ordering blocks by ascending order of their minima). Note that $I_i(\tau) \in \mathcal{S}(H, K)$ and that the mapping $(\tau, i) \mapsto I_i(\tau)$ is injective.

Definition 2.17. *The* enriched chain *is the Markov chain on $\mathcal{S}(H, K)$ for which the transition probability from σ to τ is given by*

$$\tilde{P}_{\sigma,\tau} = \begin{cases} 1 & \text{if } \{1\} \in \sigma \text{ and } \tau = \sigma^\downarrow \cup \{H\}, \\ x_i & \text{if } \{1\} \notin \sigma \text{ and } \tau = I_i(\sigma^\downarrow) \text{ for some } i \in \{0, \ldots, k\}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.27}$$

The condition $x_0 + \cdots + x_k = 1$ and the above remarks ensure that $\tilde{P}$ is indeed a right stochastic matrix (i.e. each of its rows sums to 1).

Example 2.18. For $H = 8$ and $K = 3$:

- the enriched state $1 \,|\, 3, 5, 6 \,|\, 2, 4, 7, 8$ jumps with probability 1 to $2, 4, 5 \,|\, 1, 3, 6, 7 \,|\, 8$,
- the enriched state $\sigma = 3, 5 \,|\, 2, 6, 7 \,|\, 1, 4, 8$ reaches the intermediate state $\sigma^\downarrow = 2, 4 \,|\, 1, 5, 6 \,|\, 3, 7$ and jumps with probability:

  - x_0 to $1, 5, 6 \,|\, 3, 7 \,|\, 2, 4, 8$,
  - x_1 to $2, 4 \,|\, 3, 7 \,|\, 1, 5, 6, 8$,
  - x_2 to $2, 4 \,|\, 1, 5, 6 \,|\, 3, 7, 8$.

Note that here we write blocks in ascending order of their maxima, which differs from the standard notation of writing set partitions in ascending order of their minima.

Example 2.19. For $H = 4$ and $K = 2$, the transition graph of the enriched chain is illustrated on Figure 3.

The existence and uniqueness of the stationary distribution of the enriched chain results from:
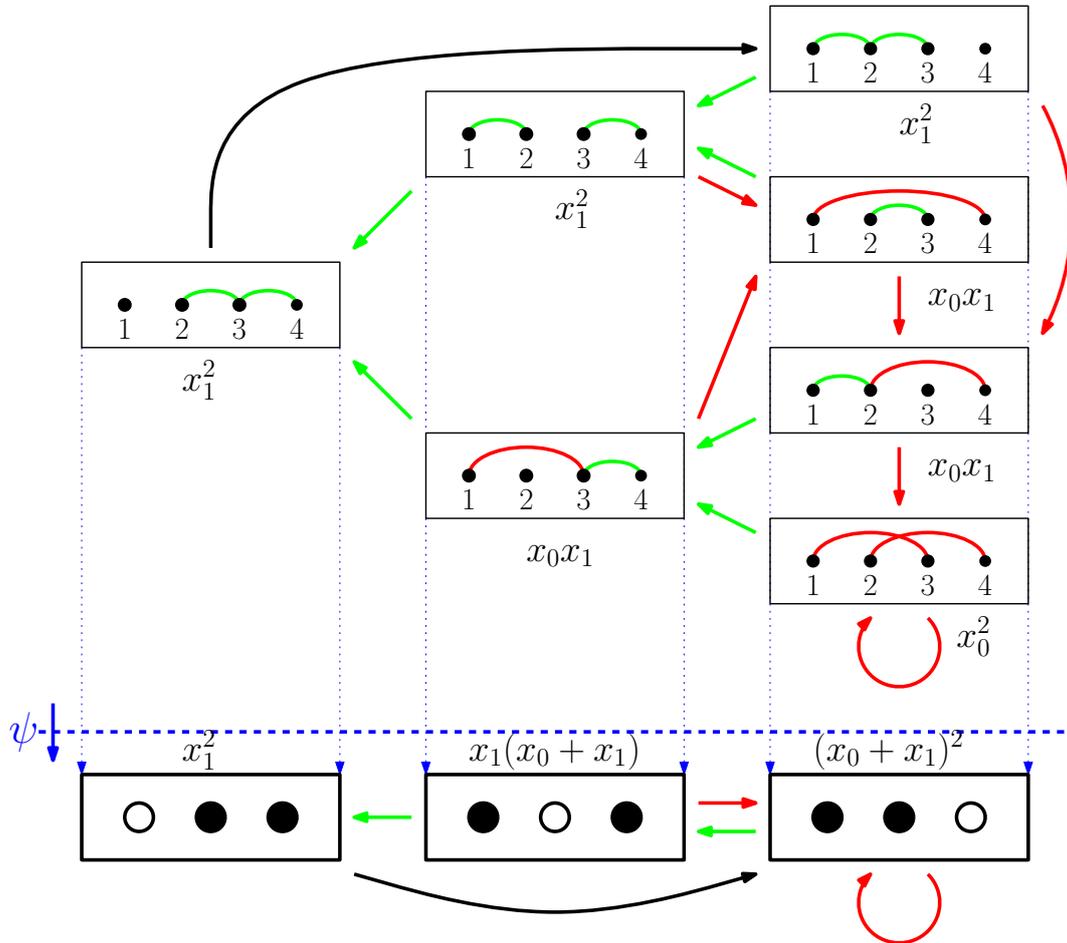
Figure 3: Transition graphs of the enriched chain (top part, above blue horizontal dashed line) and of the MJMC (bottom part) for $H = 4$ and $K = 2$ ($h = 3$ and $k = 1$). Enriched states (set partitions) are represented via "arches", see Definition 2.23. Red, green and black arrows represent transitions of respective probability x_0, x_1 and 1. Next to each state is displayed its unnormalized stationary weight; for the enriched chain it is obtained by attaching a weight x_1 (resp. x_0) to each arch covering 1 (resp. 2) blocks, displayed in green (resp. red). Enriched states in a same column map to the same MJMC state (displayed below) via ψ.

Proposition 2.20. *For $x_0 > 0$, the enriched chain has a unique closed communicating class, whose all states are aperiodic. The chain is irreducible if and only if all x_i's are nonzero.*

Proof. The first assertion is proved in the same way as that of Proposition 2.3, the "lowest" state being now the set partition

$$\tilde{\mathcal{E}} = \{\{1, \ldots, H\} \cap (j + K\mathbb{N})\}_{j=1,\ldots,K} \tag{2.28}$$

(precisely, starting from any initial state, we eventually obtain $\tilde{\mathcal{E}}$ by applying only transitions of probabilities x_0 and 1). Note that for $H = 4$ and $K = 2$, we have $\tilde{\mathcal{E}} = 1, 3 \,|\, 2, 4$ (which is the state with two red arches in Figure 3).

For the second assertion, suppose that $x_i > 0$ for all i: we want to prove that the transition graph is strongly connected. We first replace the transitions of probability 1 by K transitions of probabilities $x_0, \ldots, x_k$, so that each state has K outgoing transitions, counting transitions to itself. It is then not difficult to check from the definition of the enriched chain that each state also has K incoming transitions. It follows that for any set of states, the number of incoming and outgoing transitions are the same. Since the state space is finite, all communicating classes are closed. As there is a unique closed communicating class, irreducibility follows. Conversely, if $x_i = 0$ for some i, then clearly some set partitions, e.g. $1 \,|\, 2 \,|\, \cdots \,|\, i \,|\, i + 2 \,|\, \cdots \,|\, K - 1 \,|\, K \, (K + 1) \cdots h \,|\, (i + 1) \, H$, are never produced. □

2.4.2 Projection, stationary distribution and second proof of Theorem 2.5

From the surjection ψ introduced above, we define a rectangular matrix Ψ with rows indexed by elements of $\mathcal{S}(H, K)$ and columns indexed by elements of $\mathrm{St}_{h,k}$ in the obvious manner (namely a coefficient of Ψ is 1 if its column index is the image by ψ of its row index, and 0 otherwise).

Lemma 2.21. *We have the intertwining relation*

$$\tilde{P}\Psi = \Psi P. \tag{2.29}$$

For clarity, the proofs of all lemmas in this section are postponed to Section 2.4.3. Note that, if we view the preimages by ψ of elements of $\mathrm{St}_{h,k}$ as equivalences classes for an equivalence relation on $\mathcal{S}(H, K)$, then (2.29) shows that the MJMC is the projection of the enriched chain [17, Lemma 2.5]. Noting that the sum of each row of Ψ is 1, we immediately deduce:

Corollary 2.22. *We have*

$$\pi = \tilde{\pi}\Psi \tag{2.30}$$

where $\tilde{\pi}$ is the stationary probability distribution of the enriched chain (viewed as a row vector), and π is that of the MJMC.

Our route to Theorem 2.5 is now clear. We want to find an explicit expression for $\tilde{\pi}$, then compute $\tilde{\pi}\Psi$. We first need to introduce some definitions and notations.

Definition 2.23. *Let σ be an enriched state and s, t two integers such that $1 \leq s < t \leq H$. We say that the pair (s, t) is an arch of σ if s and t belong to the same block β of σ, while no integer strictly between s and t belongs to β.*

Note that $\sigma \in \mathcal{S}(H, K)$ has exactly $\ell = H - K$ arches.

Notation 2.24. *For $1 \leq s < t \leq H$, we denote by $C_\sigma(s, t)$ the number of blocks containing at least one element in $\{s, s + 1, \ldots, t\}$ (when (s, t) is an arch, then we say that these blocks are those covered by (s, t)).*

We are now ready to express the stationary distribution of the enriched chain, see Figure 3 again for an illustration in the case $H = 4$, $K = 2$.

Lemma 2.25. *For $\sigma \in \mathcal{S}(H, K)$, the monomial*

$$\tilde{w}(\sigma) = \prod_{(s,t) \text{ arch of } \sigma} x_{K - C_\sigma(s,t)} \tag{2.31}$$

defines an unnormalized stationary measure of the enriched chain.

Remark 2.26. It is here natural to introduce the notation $z_i = x_{K-i}$, $i = 1, \ldots, K$, so that z_i is simply the weight per arch covering i blocks. In the juggling language, z_i is the probability of doing an insertion at the i-th available site starting from the right. This notation will be useful for the "add-drop" and "annihilation" generalizations of the MJMC, see Sections 4.1 and 4.2. However, the x_i notation is more convenient to study the $k \to \infty$ limit, as we do in Sections 3.1 and 3.2.

Noting that, for $\tilde{\mathcal{E}}$ as in (2.28), we have $\tilde{w}(\tilde{\mathcal{E}}) = x_0^\ell$, we find that

$$Z_{h,k} = \sum_{\sigma \in \mathcal{S}(H,K)} \tilde{w}(\sigma) \tag{2.32}$$

is positive whenever $x_0 > 0$, so that $\tilde{\pi} = \tilde{w}/Z_{h,k}$ is the stationary probability distribution of the enriched chain. For $B \in \mathrm{St}_{h,k}$, we set

$$w(B) = \sum_{\sigma \in \psi^{-1}(B)} \tilde{w}(\sigma), \tag{2.33}$$

that is $w = \tilde{w}\Psi$. This implies that $\pi = w/Z_{h,k}$ is the stationary probability distribution of the MJMC by Corollary 2.22. Theorem 2.5 then follows immediately from:

Lemma 2.27. *For $B = b_1 \cdots b_h \in \mathrm{St}_{h,k}$, we have*

$$w(B) = \prod_{\substack{i \in \{1, \ldots, h\} \\ b_i = \bullet}} \left(x_{E_i(B)} + \cdots + x_k \right). \tag{2.34}$$

where $E_i(B) = \#\{j < i | b_j = \circ\}$.

Example 2.28. Returning again to the case $H = 4$, $K = 2$ (i.e. $h = 3$, $k = 1$) illustrated on Figure 3, we find

$$w(\circ \bullet \bullet) = x_1^2, \qquad w(\bullet \circ \bullet) = (x_0 + x_1)x_1 = x_1, \qquad w(\bullet \bullet \circ) = (x_0 + x_1)^2 = 1. \tag{2.35}$$

Remark 2.29. The unnormalized weights $\tilde{w}(\sigma)$ are well-defined for arbitrary $x_0, \ldots, x_k$ (not necessarily positive or subject to the condition $x_0 + \cdots + x_k = 1$). For $x_0 = \cdots = x_k = 1$, we recover Warrington's combinatorial proof of the identity (2.11). For $x_i = q^{k-i}$, we have $\tilde{w}(\sigma) = q^{N(\sigma)}$ with

$$N(\sigma) = \sum_{(s,t) \text{ arch of } \sigma} (C_\sigma(s,t) - 1). \tag{2.36}$$

The identity (2.12) shows that $N(\sigma)$ is a so-called Mahonian statistic on set partitions, which seems different from the inversion number and the major index [19].

2.4.3 Technical proofs

We now prove Lemmas 2.21, 2.25 and 2.27.

Proof of Lemma 2.21. We check the relation (2.29) coefficientwise. Fix $\sigma \in \mathcal{S}(H, K)$ and $B \in \mathrm{St}_{h,k}$, and let $A = a_1 \cdots a_h = \psi(\sigma)$, where $\psi(\sigma)$ is defined in the beginning of Section 2.4.1. Observe that

$$\psi(\sigma^\downarrow \cup \{H\}) = a_2 \cdots a_h \circ \tag{2.37}$$

and that, for all i,

$$\psi(I_i(\sigma^\downarrow)) = T_i(a_2 \cdots a_h \circ) \tag{2.38}$$

(since inserting H into the $(i+1)$-th block of $\sigma^\downarrow$ transforms its maximum into a non-maximum, which means replacing the $(i+1)$-th occurrence of $\circ$ in $\psi(\sigma^\downarrow)\circ = a_2 \cdots a_h \circ$ by a $\bullet$). If $\{1\} \in \sigma$ (i.e. $a_1 = \circ$), then, by the definition of the transition matrices and (2.37),

$$(\tilde{P}\Psi)_{\sigma,B} = \delta_{\psi(\sigma^\downarrow \cup \{H\}),B} = \delta_{a_2 \cdots a_h \circ, B} = P_{A,B} = (\Psi P)_{\sigma,B}. \tag{2.39}$$

Otherwise, if $\{1\} \notin \sigma$ (i.e. $a_1 = \bullet$), we may write

$$(\tilde{P}\Psi)_{\sigma,B} = \sum_{\substack{\tau \in \mathcal{S}(H,K) \\ i \in \{0,\ldots,k\}}} x_i \delta_{I_i(\sigma^\downarrow),\tau} \delta_{\psi(\tau),B} = \sum_{i=0}^{k} x_i \delta_{\psi(I_i(\sigma^\downarrow)),B}$$

$$= \sum_{i=0}^{k} x_i \delta_{T_i(a_2 \cdots a_h \circ),B} = P_{A,B} = (\Psi P)_{\sigma,B} \tag{2.40}$$

where we use (2.38) to go from the first to the second line. In both cases, we have the wanted relation. $\square$

Proof of Lemma 2.25. We need to show that

$$\tilde{w}(\sigma) = \sum_{\tau \in \mathcal{S}(H,K)} \tilde{w}(\tau) \tilde{P}_{\tau,\sigma} \tag{2.41}$$

which is done by considering the possible predecessors of σ in the enriched chain. If $\{H\} \in \sigma$ then σ has a unique precedessor τ such that $\sigma = \tau^\downarrow \cup \{H\}$. The arches of σ and τ are clearly in one-to-one correspondence, and their weights are unchanged, thus $\tilde{w}(\sigma) = \tilde{w}(\tau)$ as wanted. Otherwise, there exists a unique pair $(\rho, i) \in \mathcal{S}(h, K) \times \{0, \ldots, k\}$ such that $\sigma = I_i(\rho)$ and it is easily seen that $\tilde{w}(\sigma) = x_i \tilde{w}(\rho)$ (since inserting H in the $(i+1)$-th block of ρ amounts to creating an arch covering $K - i$ blocks, the other arch weights being unaffected). The predecessors of σ are the τ such that $\tau^\downarrow = \rho$ and $\{1\} \notin \tau$, and we have $\tilde{P}_{\tau,\sigma} = x_i$ regardless of τ. There are exactly K predecessors, whose weights are $x_j \tilde{w}(\rho)$ with $j \in \{0, \ldots, k\}$ (indeed all these predecessors are obtained by shifting all elements of all blocks of ρ up by one, which preserves the arch weights, then inserting 1 into one of the K blocks, which creates a new arch covering an arbitrary number of blocks between 1 and K). From the condition $x_0 + \ldots + x_k = 1$, we conclude that (2.41) holds as wanted. $\square$

Proof of Lemma 2.27. We proceed by induction on $\ell = h - k = H - K$. For $\ell = 0$, the statement is true since $B = \circ^h$ has only one preimage, namely the set partition consisting of singletons $\{\{1\}, \{2\}, \ldots, \{H\}\}$ which has no arch, thus has a weight 1 consistent with (2.34). Let us now assume that $\ell > 0$ and consider the smallest i such that $b_i = \bullet$, i.e. $B = \circ^{i-1} \bullet b_{i+1} \ldots b_h$. Let $B' \in \mathrm{St}_{h,k+1}$ be the word obtained by replacing the i-th letter of B by $\circ$, i.e. $B' = \circ^i b_{i+1} \ldots b_h$.

Consider $\tau \in \psi^{-1}(B)$: it is easily seen from the definition of ψ that, for all $j < i$, the singleton $\{j\}$ is necessarily a block of τ, but the block β containing i contains at least another (larger) element, thus τ has an arch of the form (i, i') with weight $\tilde{w}_{i,i'} = x_n$ for some $n \geq E_i(B)$.

Splitting β in two blocks $\{i\}$ and $\beta \setminus \{i\}$, we obtain a set partition $\tau' \in \mathcal{S}(H, K+1)$, and a moment's thought shows that $\tau \mapsto (\tau', n)$ is a bijection between $\psi^{-1}(B)$ and $\psi^{-1}(B') \times \{E_i(B), \ldots, k\}$ such that $\tilde{w}(\tau) = x_n \tilde{w}(\tau')$. By the induction hypothesis we have

$$w(B') = \sum_{\tau' \in \psi^{-1}(B')} \tilde{w}(\tau') = \prod_{\substack{j \in \{i+1, \ldots, h\} \\ b_j = \bullet}} \left(x_{E_j(B)} + \cdots + x_k \right). \tag{2.42}$$

Multiplying this relation by x_n and summing over n, the desired relation (2.34) follows. $\square$

3 Extensions to infinite state spaces

In this section, we discuss extensions of the Multivariate Juggling Markov Chain to an infinite setting. More precisely, we first let the number of available sites k tend to infinity, keeping the number of balls ℓ fixed. This is the Unbounded MJMC discussed in Section 3.1. Further, we consider the case where the number of balls ℓ tends to infinity (with k infinite or not), which corresponds to the Infinite MJMC discussed in Section 3.2.

3.1 Unbounded heights

As suggested in the conclusion of [22], a first natural extension is to allow Magnus to throw balls arbitrarily high, so that the ball flight times are unbounded. This corresponds to taking the limit $h \to \infty$ of the MJMC, keeping the number of balls ℓ fixed. In the particle picture, the sites are now labelled by the set of positive integers, and exactly ℓ sites are occupied by a particle. The time evolution is essentially unchanged: at each time step, all particles are moved one site to the left, and if there was a particle on the first site, it is reinserted at an available site anywhere on the lattice. We keep the MJMC prescription of choosing the i-th available site with probability x_{i-1}, but since there are now infinitely many available sites, we have an infinite sequence $(x_i)_{i \geq 0}$ of nonnegative real numbers such that $\sum_{i=0}^{\infty} x_i = 1$.

Formally, states can be viewed as infinite words on the alphabet $\{\bullet, \circ\}$ containing exactly ℓ occurrences of $\bullet$. We denote by $\mathrm{St}^{(\ell)}$ the set of such states (which can be viewed as the direct limit of the set sequence $(\mathrm{St}_{h,h-\ell})_{h \geq \ell}$). For $A \in \mathrm{St}^{(\ell-1)}$ and i a nonnegative integer, we let $T_i(A) \in \mathrm{St}^{(\ell)}$ be the word obtained by replacing the $(i+1)$-th occurrence of $\circ$ in A by $\bullet$.

Definition 3.1. *Given a nonnegative integer ℓ and a sequence $(x_i)_{i \geq 0}$ of nonnegative real numbers such that $\sum_{i=0}^{\infty} x_i = 1$, the* Unbounded Multivariate Juggling Markov Chain *(UMJMC) is the Markov chain on the state space $\mathrm{St}^{(\ell)}$ for which the transition probability from $A = a_1 a_2 a_3 \cdots$ to B reads*

$$P_{A,B} = \begin{cases} 1 & \text{if } a_1 = \circ \text{ and } B = a_2 a_3 \cdots, \\ x_i & \text{if } a_1 = \bullet \text{ and } B = T_i(a_2 a_3 \cdots), \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

Again, we might identify $\mathrm{St}^{(\ell)}$ with the set of ℓ-element subsets of $\mathbb{N}$ (corresponding to the positions of the occupied sites). For two such subsets S and S', the UMJMC transition probability from S to S' is still given by (2.3). Our model is thus a particular case of the general model considered in [16]. The following proposition is an immediate extension of Proposition 2.3.

Proposition 3.2. *If $x_0 > 0$, then the UMJMC has a unique closed communicating class all of whose states are aperiodic. Furthermore, if infinitely many x_i's are nonzero, then the UMJMC is irreducible.*

From now on, we assume that $x_0 > 0$. Note that, if $k = \sup\{i : x_i > 0\}$ is finite, then no insertion is ever made at a position larger than k. Thus, the states with particles at positions larger than k are transient, and upon removing them we recover the finite MJMC on $\mathrm{St}_{\ell+k,k}$. In this degenerate case, all the forthcoming statements remain true, but were already established in Section 2.

We are again interested in the stationary distribution, but the fact that the state space is now countably infinite requires a bit of care. Still, we might extend Theorem 2.5 in the following form.

Theorem 3.3. *The unique invariant measure (up to constant of proportionality) of the UMJMC is given by*

$$w(B) = \prod_{i \in \mathbb{N}, \, b_i = \bullet} y_{E_i(B)} \tag{3.2}$$

where $B = b_1 b_2 \cdots \in \mathrm{St}^{(\ell)}$, $E_i(B) = \#\{j < i | b_j = \circ\}$ *and* $y_m = \sum_{j=m}^{\infty} x_j$.

We prove this theorem by a straightforward extension of the discussion of Section 2.3. In the integer partition language, the extension simply consists in lifting the bound on part sizes. We denote by $\mathrm{Par}_{\infty,\ell}$ the set of integer partitions with at most ℓ parts and unbounded part sizes (which we can view as nonincreasing sequences of ℓ nonnegative integers). This set is naturally identified with $\mathrm{St}^{(\ell)}$: given a state in $\mathrm{St}^{(\ell)}$, we denote by $s_1 < s_2 < \ldots < s_{\ell-1} < s_\ell$ the positions of $\bullet$'s, so that the corresponding integer partition is $(s_\ell - \ell, s_{\ell-1} - (\ell - 1), \ldots, s_1 - 1)$. We let the reader verify that, in the integer partition language, the UMJMC transitions are still given by (2.18). Theorem 3.3 may then be reformulated as follows.

Theorem 3.4. *The unique invariant measure (up to constant of proportionality) of the UMJMC is given by*

$$w(\lambda) = \prod_{i=1}^{\ell} y_{\lambda_i} \tag{3.3}$$

where $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \mathrm{Par}_{\infty,\ell}$.

Proof. We simply check that

$$w(\lambda) = \sum_{\mu \in \mathrm{Par}_{\infty,\ell}} w(\mu) P_{\mu,\lambda}. \tag{3.4}$$

Any predecessor μ of λ is either:

- of the form $\mu = (\lambda_1 + 1, \ldots, \lambda_\ell + 1)$ in which case we have $w(\mu) = \prod_{i=1}^{\ell} y_{\lambda_i+1} = \prod_{i=1}^{\ell} (y_{\lambda_i} - x_{\lambda_i})$ and $P_{\mu,\lambda} = 1$;
- of the form $\mu = (\lambda_1 + 1, \ldots, \lambda_{j-1} + 1, \lambda_{j+1}, \ldots, \lambda_\ell, 0)$ for some $j \in \{1, \ldots, \ell\}$ in which case we have $w(\mu) = \prod_{i=1}^{j-1} (y_{\lambda_i} - x_{\lambda_i}) \prod_{i=j+1}^{\ell} y_{\lambda_i}$ and $P_{\mu,\lambda} = x_{\lambda_j}$.

The verification of (3.4) is then done exactly as in (2.21). $\qquad\square$

We have exhibited an invariant measure of the UMJMC which is clearly σ-finite. We might wonder whether it is actually finite, so that it may be normalized into a probability distribution.

Proposition 3.5. *The invariant measure w of the UMJMC is finite if and only if*

$$\sum_{i=0}^{\infty} i x_i < \infty \tag{3.5}$$

in which case its total mass reads

$$Z^{(\ell)} = h_\ell(y_0, y_1, \ldots). \tag{3.6}$$

Proof. Let $Z^{(\ell)} \in [0, \infty]$ be the total mass of w. Observe that

$$Z^{(1)} = \sum_{\lambda_1 = 0}^{\infty} y_{\lambda_1} = \sum_{i=0}^{\infty} (i+1)x_i \qquad (3.7)$$

thus the first assertion is obviously true for $\ell = 1$. It remains true for general ℓ as we have the inequalities

$$Z^{(1)} \leq Z^{(\ell)} \leq \left(Z^{(1)}\right)^{\ell} \qquad (3.8)$$

since $\mathrm{Par}_{\infty,\ell}$ contains the set of partitions with at most one nonzero part (which has mass $Z^{(1)}$ by (3.3) and the fact that $y_0 = 1$), but may be viewed as a subset of $\mathbb{N}^{\ell}$, for which we have $\sum_{\lambda \in \mathbb{N}^{\ell}} w(\lambda) = \left(Z^{(1)}\right)^{\ell}$ by (3.3).

The identity (3.6) is obtained by letting $k \to \infty$ in (2.23), since $(\mathrm{Par}_{k,\ell})_{k \geq 0}$ forms an increasing family of sets with union $\mathrm{Par}_{\infty,\ell}$ so that $Z_{k+\ell,k} \nearrow Z^{(\ell)}$. Note that the notation $h_\ell(y_0, y_1, \ldots)$ makes sense: h_ℓ, being a symmetric function, can be expressed as a polynomial in the power sum symmetric functions $(p_m)_{m \geq 1}$ that does not depend on its number of variables, and

$$p_m(y_0, y_1, \ldots) = y_0^m + y_1^m + \cdots \qquad (3.9)$$

is a finite real number for any $m \geq 1$ by (3.5). $\qquad \square$

By standard results from the theory of Markov chains, see e.g. [15, Chapter 13] or [17, Chapter 21], we deduce:

Corollary 3.6. *The UMJMC is positive recurrent if and only if* (3.5) *holds. In that case, there is a unique stationary probability distribution, and the chain started from any initial state converges to it in total variation as time tends to infinity.*

Example 3.7. Fix $q \in (0,1)$ and pick $x_i = (1-q)q^i$. We recover the "JEP with memoryless height distribution" with parameter q considered in [16]. Clearly (3.5) holds and we recover from Theorem 3.3 Leskelä-Varpanen's expression for the stationary probability distribution. Interestingly, in the integer partition language, we find that $w(\lambda) = q^{|\lambda|}$ for $\lambda \in \mathrm{Par}_{\infty,\ell}$, where $|\lambda|$ stands for the size (sum of all parts) of λ.

Remark 3.8. It is interesting to note that (3.5) is a necessary and sufficient condition for the UMJMC to be uniformly integrable in the sense of [16]. When (3.5) does not hold, the chain may either be null recurrent or transient. For $\ell = 1$, the state with the first site occupied is clearly recurrent, thus the chain is null recurrent. Figuring out the situation for $\ell > 1$ is an intriguing question. By somewhat heuristic arguments inspired by Polyá's theorem on recurrence of random walks, we expect that, for a sequence x_i decaying asymptotically as $i^{-1-\alpha}$ with $\alpha \in (0,1)$ (so that $\sum i x_i = \infty$), the UMJMC is null recurrent if $\ell(1-\alpha) \leq 1$ and transient if $\ell(1-\alpha) > 1$. A formal proof of this statement is beyond the scope of this paper.

3.2 Infinitely many balls

We now consider another limit in which Magnus juggles with infinitely many balls ($\ell \to \infty$). Let us first heuristically discuss this limit in the particle picture, a precise mathematical statement (Proposition 3.13) being given at the end of this section. Since the particles tend to accumulate on the left side of the lattice (i.e. balls tend to have low remaining flight times), we expect that, as ℓ becomes large, all sites at a finite distance from the first site will be occupied with high probability (in particular, Magnus receives a ball at every time step), and the first available site will then be typically at a position of order ℓ. It is convenient to relabel the sites by arbitrary (not necessarily

positive) integers, so that the leftmost site has label $-\ell$, and the lowest state corresponds to having particles on sites $-\ell, -\ell+1, \ldots, -1$. In the limit $\ell \to \infty$, the leftmost site is sent to $-\infty$, and the lowest state corresponds to having all negative sites filled and all nonnegative sites empty.

A general state is obtained by moving finitely many particles from occupied to empty sites, in other words it is a bi-infinite word $B = (b_i)_{i \in \mathbb{Z}} \in \{\bullet, \circ\}^{\mathbb{Z}}$ such that

$$\{i \geq 0 : b_i = \bullet\} = \{i < 0 : b_i = \circ\} < \infty \tag{3.10}$$

(such a word is sometimes called a "Maya diagram"). The time evolution is now easy to describe: at each time step, all particles are moved one site to the left and a new particle is inserted at an available site (so that the condition (3.10) is preserved). Again, we keep the MJMC prescription of picking the i-th available site from the left with probability x_{i-1} (this is well-defined since the set of available positions is bounded from below).

Rather than writing down the transitions formally, we prefer to work directly in the language of integer partitions. It is well-known that bi-infinite words subject to (3.10) are in one-to-one correspondence with arbitrary integer partitions (i.e. non-increasing sequences of integers that vanish eventually), see again Figure 2 and think about extending the displayed juggling state by adding infinitely many $\bullet$'s on the left and infinitely many $\circ$'s on the right, which does not change the corresponding integer partition. We denote by Par the set of all integer partitions and, for $\lambda = (\lambda_j)_{j \geq 1} \in \mathrm{Par}$ and $i \geq 0$, we set

$$\lambda^{(i)} = (\lambda_1 - 1, \ldots, \lambda_{j-1} - 1, i, \lambda_j, \lambda_{j+1}, \ldots) \tag{3.11}$$

with j the smallest index such that $\lambda_j \leq i$. This is the Par equivalent of the above time evolution with insertion at the $(i+1)$-th available site.

Definition 3.9. *Given a sequence $(x_i)_{i \geq 0}$ of nonnegative real numbers such that $\sum_{i=0}^{\infty} x_i = 1$, the* Infinite Multivariate Juggling Markov Chain *(IMJMC) is the Markov chain on the state space* Par *for which the transition probability from λ to μ reads*

$$P_{\lambda,\mu} = \begin{cases} x_i & \text{if } \mu = \lambda^{(i)}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.12}$$

The existence and uniqueness (up to normalization) of an invariant measure of the IMJMC is ensured by the following:

Proposition 3.10. *If $x_0 > 0$, then the IMJMC has a unique closed communicating class, whose all states are aperiodic. They are precisely the integer partitions whose parts are smaller than or equal to $\sup\{i : x_i > 0\}$. In particular, if infinitely many x_i's are nonzero, then the IMJMC is irreducible.*

Proof. It is easily seen that, starting from any initial state and applying repeatedly the transition x_0, we end up with the empty partition (i.e. the partition corresponding to an empty Young diagram), which is a fixed point. This proves the first statement. For the second statement, let us prove that, for any k such that $x_k > 0$, we may obtain any partition μ with $\mu_1 \leq k$ from the empty partition using only the transitions x_0 and x_k. We proceed by double induction on the number of parts p of μ (i.e. the largest p such that $\mu_p > 0$) and on $k - \mu_1$. If $p = 0$, then μ is already the empty partition. Otherwise, if $k - \mu_1 = 0$, then μ is obtained from $(\mu_2, \mu_3, \ldots)$ (which has one less part) by the transition x_k. Finally, if $k - \mu_1 > 0$, then μ is obtained from $\mu' = (\mu_1 + 1, \ldots, \mu_p + 1, 0, 0, \ldots)$ (which still has p parts and $k - \mu_1' < k - \mu_1$) by the transition x_0. $\qquad\square$

Theorem 3.11. *An invariant measure of the IMJMC is given by*

$$w(\lambda) = \prod_{i=1}^{\infty} y_{\lambda_i} \tag{3.13}$$

where $\lambda \in \mathrm{Par}$ and $y_m = \sum_{j=m}^{\infty} x_i$. Its total mass reads

$$Z = \prod_{m=1}^{\infty} \frac{1}{1 - y_m} \tag{3.14}$$

and Z is finite (i.e. the IMJMC is positive recurrent) if and only if (3.5) *holds.*

Proof. Note that the right-hand side of (3.13) is well-defined since $y_0 = 1$ and λ_i vanishes eventually. By (3.11), any predecessor of $\lambda \in \mathrm{Par}$ is necessarily of the form $\mu = (\lambda_1 + 1, \ldots, \lambda_{j-1} + 1, \lambda_{j+1}, \lambda_{j+2}, \ldots)$ for some j, and $\lambda = \mu^{(\lambda_j)}$. We then have $w(\mu) = \prod_{i=1}^{j-1}(y_{\lambda_i} - x_{\lambda_i}) \prod_{i=j+1}^{\infty} y_{\lambda_i}$ and $P_{\mu,\lambda} = x_{\lambda_j}$. Telescoping as in (2.21), we deduce

$$\sum_{\mu \in \mathrm{Par}} w(\mu) P_{\mu,\lambda} = \sum_{j=1}^{\infty} \left(x_{\lambda_j} \prod_{i=1}^{j-1}(y_{\lambda_i} - x_{\lambda_i}) \prod_{i=j+1}^{\infty} y_{\lambda_i} \right) = \prod_{i=1}^{\infty} y_{\lambda_i} = w(\lambda) \tag{3.15}$$

(note that $\prod_{i=1}^{j-1}(y_{\lambda_i} - x_{\lambda_i}) \prod_{i=j+1}^{\infty} y_{\lambda_i} \leq y_1^{j-1}$ with $y_1 < 1$). This establishes the invariance of w.

The expression (3.14) follows from standard considerations on integer partitions, and is finite if and only if $\sum y_m < \infty$, i.e. if (3.5) holds. $\square$

Again, when (3.5) does not hold, the IMJMC is either null recurrent or transient, and it would be interesting to know which situation occurs.

Example 3.12. Consider again the geometric case $x_i = (1 - q)q^i$, $q \in (0, 1)$. Then, the stationary distribution is nothing but the "q^{size}" measure over arbitrary integer partitions. Note that, contrary to the case of finitely many balls (Example 3.7), the phenomenon of ultrafast convergence to stationarity observed in [16] cannot occur: the stationary distribution is supported on the set of all integer partitions with arbitrarily many parts, but since we may create at most one new part at each time step, the distribution at a finite time starting from a given initial partition is supported on a strictly smaller set.

So far we have not given a precise mathematical meaning to the fact that the IMJMC is the limit as $\ell \to \infty$ of the (U)MJMC. This is actually the case according to a certain notion of "local convergence", which again is easier to state in the language of integer partitions. Let us fix the sequence $(x_i)_{i \geq 0}$ and, for $\ell \in \mathbb{N} \cup \{\infty\}$, denote by $\Lambda(\ell; 0), \Lambda(\ell; 1), \ldots$ the Par-valued Markov chain corresponding to the UMJMC with ℓ balls for $\ell \in \mathbb{N}$, or the IMJMC for $\ell = \infty$, started at an arbitrary (deterministic or random) initial state $\Lambda(\ell, 0)$. (Note that the state space $\mathrm{Par}_{\infty,\ell}$ of $\Lambda(\ell, \cdot)$ with $\ell \in \mathbb{N}$ may naturally be viewed as a subset of Par, upon appending an infinite zero sequence to its elements.)

Proposition 3.13. *For any nonnegative integer t and any fixed partition ν, we have the convergence in distribution*

$$(\Lambda(\ell; 0), \ldots, \Lambda(\ell; t)) \xrightarrow[\ell \to \infty]{(d)} (\Lambda(\infty; 0), \ldots, \Lambda(\infty; t)) \tag{3.16}$$

when each chain is started at the deterministic state $\Lambda(\ell, 0) = \nu$ (assuming that ℓ is larger than the number of parts of ν).

Furthermore, if the condition (3.5) *for positive recurrence is satisfied, then the convergence* (3.16) *also holds when each chain is started at its stationary probability distribution.*

Proof. The transition probabilities for the chain $\Lambda(\ell, \cdot)$ are given by (2.18) when $\ell \in \mathbb{N}$ and by (3.12) when $\ell = \infty$: observe that they are equal whenever λ has strictly less than ℓ parts, and furthermore that any transition increases the number of parts by at most one. It follows that, when each chain is started at the deterministic state $\Lambda(\ell, 0) = \nu$, then the law of $(\Lambda(\ell; 0), \dots, \Lambda(\ell; t))$ does not depend on ℓ as soon as $\ell - t$ is larger than the number of parts of ν, which immediately implies the first statement.

To prove the second statement we note that the stationary distribution of $\Lambda(\ell, \cdot)$ converges in total variation to that of $\Lambda(\infty, \cdot)$ as $\ell \to \infty$ (so the statement holds for $t = 0$). This is a simple consequence of Theorems 3.4 and 3.11: the unnormalized invariant measure of $\Lambda(\ell, \cdot)$ for $\ell \in \mathbb{N}$ is the restriction of that of $\Lambda(\infty, \cdot)$ to the set of integer partitions with at most ℓ parts and, when (3.5) holds, their total masses are finite and tend to one another as $\ell \to \infty$. Since a partition has finitely many parts, the previous argument allows us to conclude that (3.16) holds for any t. $\qquad\square$

Remark 3.14. It is easier to state the convergence in terms of integer partitions since the state spaces of the chains $\Lambda(\ell, \cdot)$ are naturally included in one another. In terms of particles, this corresponds to embedding the state space $\mathrm{St}^{(\ell)}$ of the UMJMC with ℓ balls into that of the IMJMC by prepending infinitely many $\bullet$'s on the left, then shifting all letters in the resulting bi-infinite word by ℓ positions to the left so as to satisfy (3.10). This allows us to translate Proposition 3.13 in the particle language, and justifies the heuristic discussion at the beginning of this section.

4 Extensions with a fluctuating number of balls

We now consider extensions of the MJMC where the number of balls is not fixed but is allowed to fluctuate. These extensions are the natural multivariate generalizations of the so-called *add-drop* and *annihilation* models introduced in [22, Section 4] (to which we refer for motivations), and we thus keep the same denomination here. Both models are defined on the same state space and have the same transition graph, only the transitions probabilities differ. In both cases, we will provide an exact expression for the stationary distribution, whose validity will be proved by considering the enriched version of the chain.

Since many definitions and notations will be common to both models, we factorize their discussion here. The basic state space of the model will be $\mathrm{St}_h = \{\circ, \bullet\}^h$, with h a fixed nonnegative integer. As hinted in Remark 2.26, it will be convenient here to read a word from right to left.

Notation 4.1. *For $A \in \bigcup_{h \geq 0} \mathrm{St}_h$ and i a nonnegative integer, we let $S_i(A)$ be the word obtained by replacing the i-th occurrence of $\circ$ in A by $\bullet$, upon reading the word from the right (if i is equal to 0 or larger than the number of occurrences of $\circ$ in A then we set $S_i(A) = A$ by convention).*

Note that $S_i(A) = T_{k-i}(A)$ for $A \in \mathrm{St}_{h,k}$ and $1 \leq i \leq k$, with T as in Section 2.1. We now turn to the enriched model, whose state space is the set $\mathcal{S}(H)$ of all set partitions of $\{1, \dots, H\}$, with $H = h + 1$ (the number of enriched states is thus a Bell number). Recall the notations from Section 2.4.1: ψ is a surjection from $\mathcal{S}(H)$ onto St_h and $\sigma \mapsto \sigma^{\downarrow}$ is a mapping from $\mathcal{S}(H)$ to $\mathcal{S}(h)$.

Notation 4.2. *For $\tau \in \bigcup_{h \geq 0} \mathcal{S}(h)$ and i a nonnegative integer, we define $J_i(\tau)$ as follows. If $\tau \in \mathcal{S}(h)$ then we let $J_i(\tau) \in \mathcal{S}(h+1)$ be the set partition obtained by inserting $h+1$ into the i-th block of τ, now numbered by decreasing order of maxima (if i is equal to 0 or larger than the number of blocks of τ then we set $J_i(\tau) = \tau \cup \{h+1\}$ by convention).*

Note that $J_i(\tau) = I_{K-i}(\tau)$ for $\tau \in \mathcal{S}(h, K)$ and $1 \leq i \leq K$. Observe that, for any $i \geq 0$,

we have the fundamental "intertwining" relation

$$\psi(J_i(\tau)) = S_i(\psi(\tau)\circ) \tag{4.1}$$

which is nothing but a compact rewriting of (2.37) and (2.38). Note also that, when τ is non empty ($h \geq 1$), we have the commutation relation

$$J_i(\tau)^{\downarrow} = J_i(\tau^{\downarrow}). \tag{4.2}$$

The basic transition graph is defined as follows: for any $A = a_1 a_2 \ldots a_h \in \mathrm{St}_h$ and $i \geq 0$, we have an oriented edge from A to $S_i(a_2 \ldots a_h\circ)$ (we ignore edge multiplicities). See Figure 4 for $h = 2$. Similarly, the enriched transition graph is obtained by connecting each $\sigma \in \mathcal{S}(H)$ to $J_i(\sigma^{\downarrow})$ for all $i \geq 0$. It is not difficult to check that both transition graphs are strongly connected using Proposition 2.3 and Proposition 2.20.
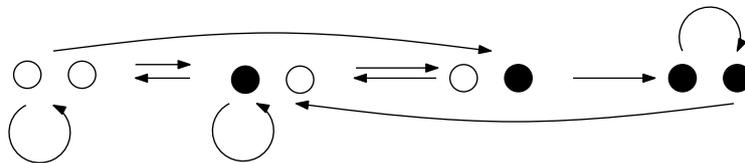

Figure 4: The basic transition graph for $h = 2$

4.1 Add-drop juggling

Definition 4.3. *Given h a nonnegative integer and $a = z_0, \ldots, z_h$ nonnegative real numbers, the (multivariate)* add-drop model *is the Markov chain on the state space St_h for which the transition probability from $A = a_1 \cdots a_h$ to B reads*

$$P_{A,B} = \begin{cases} \dfrac{z_i}{z_0 + \cdots + z_k} & \text{if } B = S_i(a_2 \ldots a_h\circ) \text{ for some } i \in \{0, \ldots, k\}, \\ 0 & \text{otherwise,} \end{cases} \tag{4.3}$$

with k the number of occurrences of $\circ$ in $a_2 \ldots a_h\circ$.

Warrington's add-drop model is recovered by taking $a = z_1 = \cdots = z_h = 1$. It is easily seen that, in general, the chain is aperiodic with a unique communicating class whenever $a > 0$, so that the stationary distribution is unique.

Theorem 4.4. *The stationary distribution of the add-drop model is given by*

$$\Pi(B) = \frac{a^k}{Z_h} \prod_{\substack{i=1 \\ b_i=\bullet}}^{h} \left(z_1 + \cdots + z_{\psi_i(B)+1}\right), \tag{4.4}$$

for $B = b_1 \ldots b_h \in \mathrm{St}_{h,k}$, with $\psi_i(B) = \#\{j : i < j \leq h, b_j = \circ\}$ and

$$Z_h = \sum_{k=0}^{h} a^k h_{h-k}(z_1, z_1 + z_2, \ldots, z_1 + \cdots + z_{k+1}), \tag{4.5}$$

where h_ℓ is the complete homogeneous symmetric polynomial of degree ℓ. (Note that z_{h+1} never appears in (4.4) since the product is empty for $k = h$.)

Example 4.5. The transition matrix for $h = 2$ in the lexicographically-ordered basis ($\bullet\bullet$, $\circ\bullet$, $\bullet\circ$, $\circ\circ$) is given by

$$
P = \begin{pmatrix}
\dfrac{z_1}{a + z_1} & 0 & \dfrac{a}{a + z_1} & 0 \\[2mm]
\dfrac{z_1}{a + z_1} & 0 & \dfrac{a}{a + z_1} & 0 \\[2mm]
0 & \dfrac{z_1}{a + z_1 + z_2} & \dfrac{z_2}{a + z_1 + z_2} & \dfrac{a}{a + z_1 + z_2} \\[2mm]
0 & \dfrac{z_1}{a + z_1 + z_2} & \dfrac{z_2}{a + z_1 + z_2} & \dfrac{a}{a + z_1 + z_2}
\end{pmatrix}, \tag{4.6}
$$

and the stationary probabilities are given by the normalized coordinates of the row eigenvector with eigenvalue 1, namely

$$
\frac{1}{Z_2} \left(z_1^2,\ a z_1,\ a(z_1 + z_2),\ a^2 \right), \tag{4.7}
$$

where $Z_2 = a^2 + a z_2 + 2\, a z_1 + z_1^2$.

Theorem 4.4 is the natural multivariate generalization of [22, Theorem 3, item 1]. The parameter a has the physical interpretation of the fugacity for a $\circ$. In other words the distribution Π is the grand-canonical version of the stationary distribution π of Theorem 2.5. We prove Theorem 4.4 by a straightforward extension of the construction of Section 2.4, which we now detail.

Definition 4.6. *The (multivariate)* enriched add-drop model *is the Markov chain on the state space $\mathcal{S}(H)$ for which the transition probability from σ to τ is given by*

$$
\tilde{P}_{\sigma,\tau} = \begin{cases}
\dfrac{z_i}{z_0 + \cdots + z_k} & \text{if } \tau = J_i(\sigma^\downarrow) \text{ for some } i \in \{0, \ldots, k\}, \\[3mm]
0 & \text{otherwise,}
\end{cases} \tag{4.8}
$$

with k the number of blocks of $\sigma^\downarrow$.

By (4.1) it is immediate that the add-drop model is indeed the projection of the enriched chain, and furthermore it is easily seen that the enriched chain is aperiodic with a unique communicating class for $z_0 = a > 0$.

Lemma 4.7. *For $\sigma \in \mathcal{S}(H)$ with, say, K blocks, the monomial*

$$
\tilde{W}(\sigma) = a^{K-1} \prod_{(s,t) \text{ arch of } \sigma} z_{C_\sigma(s,t)}, \tag{4.9}
$$

where $C_\sigma(s,t)$ is as in Notation 2.24, defines an unnormalized stationary distribution of the enriched chain.

Proof. We want to show that $\tilde{W}(\tau) = \sum_{\sigma \in \mathcal{S}(H)} \tilde{W}(\sigma) \tilde{P}_{\sigma,\tau}$ for any τ. Let $\tau^\uparrow$ be the set partition obtained by removing the element H from τ, so that the possible predecessors of τ are the σ such that $\sigma^\downarrow = \tau^\uparrow$. More precisely, any predecessor is obtained by shifting all elements of $\tau^\uparrow$ up by 1, then inserting the element 1 either as a singleton or into a preexisting block. Hence, there are $k + 1$ predecessors where k is the number of blocks of $\tau^\uparrow$, and we readily see that their weights are $z_i \tilde{W}(\tau^\uparrow)$ with $i = 0, \ldots, k$. On the other hand, let j be equal to 0 if $\{H\}$ is a singleton of τ, or equal to $C_\tau(r, H)$ where r is such that (r, H) is an arch of τ. We have $\tilde{W}(\tau) = z_j \tilde{W}(\tau^\uparrow)$ and $\tilde{P}_{\sigma,\tau} = z_j/(z_0 + \cdots + z_k)$ for any predecessor σ. We deduce that, as wanted,

$$
\sum_{\sigma \in \mathcal{S}(H)} \tilde{W}(\sigma) \tilde{P}_{\sigma,\tau} = \sum_{i=0}^{k} z_i \tilde{W}(\tau^\uparrow) \frac{z_j}{z_0 + \cdots + z_k} = \tilde{W}(\tau). \tag{4.10}
$$

$\square$

Example 4.8. The transition matrix for $H = 3$ in the ordered basis $(1, 2, 3;\ 1\,|\,2, 3;\ 2\,|\,1, 3;\ 1, 2\,|\,3;\ 1\,|\,2\,|\,3)$ is given by

$$\tilde{P} = \begin{pmatrix} \dfrac{z_1}{a+z_1} & 0 & 0 & \dfrac{a}{a+z_1} & 0 \\[2ex] \dfrac{z_1}{a+z_1} & 0 & 0 & \dfrac{a}{a+z_1} & 0 \\[2ex] 0 & \dfrac{z_1}{a+z_1+z_2} & \dfrac{z_2}{a+z_1+z_2} & 0 & \dfrac{a}{a+z_1+z_2} \\[2ex] 0 & \dfrac{z_1}{a+z_1+z_2} & \dfrac{z_2}{a+z_1+z_2} & 0 & \dfrac{a}{a+z_1+z_2} \\[2ex] 0 & \dfrac{z_1}{a+z_1+z_2} & \dfrac{z_2}{a+z_1+z_2} & 0 & \dfrac{a}{a+z_1+z_2} \end{pmatrix}, \qquad (4.11)$$

and admits the row vector $\left(z_1^2,\ az_1,\ az_2,\ az_1,\ a^2\right)$ as a left eigenvector of eigenvalue 1. Compare this with the stationary distribution in Example 4.5.

Proof of Theorem 4.4. For $B \in \mathrm{St}_{h,k}$, let

$$W(B) = \sum_{\sigma \in \psi^{-1}(B)} \tilde{W}(\sigma). \qquad (4.12)$$

By Lemma 2.27 and Remark 2.26 (note that $x_{E_i(B)} = z_{\psi_i(B)+1}$), we readily deduce

$$W(B) = a^k \prod_{\substack{i=1 \\ b_i = \bullet}}^{h} \left(z_1 + \cdots + z_{\psi_i(B)+1}\right). \qquad (4.13)$$

The total mass of W is clearly equal to $\sum_{k=0}^{h} a^k Z_{h,k}(z_{k+1}, z_k, \ldots, z_1)$ with $Z_{h,k}$ the normalization factor of the MJMC defined in Proposition 2.9. $\qquad\square$

4.2 Annihilation juggling

As the reader is by now familiar with our approach, we define the basic model and its enriched version at the same time.

Definition 4.9. *Given h a nonnegative integer and $z_1, \ldots, z_{h+1} = a$ nonnegative real numbers such that $z_1 + \cdots + z_h + a = 1$, the (multivariate) annihilation model is the Markov chain on the state space St_h for which the transition probability from $A = a_1 a_2 \ldots a_h$ to B reads*

$$P_{A,B} = \begin{cases} z_i & \text{if } B = S_i(a_2 \ldots a_h \circ) \text{ for some } i \in \{1, \ldots, k\}, \\ z_{k+1} + \cdots + z_h + a & \text{if } B = a_2 \ldots a_h \circ, \\ 0 & \text{otherwise,} \end{cases} \qquad (4.14)$$

with k the number of occurrences of $\circ$ in $a_2 \ldots a_h \circ$. Similarly, the (multivariate) enriched annihilation model is the Markov chain on the state space $\mathcal{S}(H)$ for which the transition probability from σ to τ is given by

$$\tilde{P}_{\sigma,\tau} = \begin{cases} z_i & \text{if } \tau = J_i(\sigma^{\downarrow}) \text{ for some } i \in \{1, \ldots, k\}, \\ z_{k+1} + \cdots + z_h + a & \text{if } \tau = \sigma^{\downarrow} \cup \{H\}, \\ 0 & \text{otherwise,} \end{cases} \qquad (4.15)$$

with k the number of blocks of $\sigma^{\downarrow}$.

Remark 4.10. By our convention for $S_i(A)$ (resp. $J_i(\tau)$) when i is larger than the number of occurrences of $\circ$ in A (resp. the number of blocks of τ), we have the more compact expression $P_{A,B} = \sum z_i$ (resp. $\tilde{P}_{\sigma,\tau} = \sum z_i$) where the sum runs over all $i \in \{1, \ldots, h+1\}$ such that $B = S_i(a_2 \ldots a_h \circ)$ (resp. $\tau = J_i(\sigma^{\downarrow})$).

Remark 4.11. Warrington's annihilation model is recovered by taking $z_1 = \cdots = z_h = a = 1/(h+1)$. We still call our multivariate generalization the annihilation model, but this requires some clarification. Indeed, in Warrington's uniform case, one can interpret the dynamics by saying that ball insertions are made at arbitrary (empty or occupied) sites, and that a ball inserted at an occupied site is annihilated. However, in our multivariate generalization the correct interpretation is to say that we pick an i between 1 and $h + 1$ and insert the ball at the i-th available site from the right, and that the ball is annihilated if there is no such site on the lattice.

Here is the multivariate generalization of [22, Theorem 3, item 2]:

Theorem 4.12. *The stationary distribution of the annihilation model is given by*

$$\Pi(B) = \prod_{\substack{i=1 \\ b_i=\bullet}}^{h} \left(z_1 + \cdots + z_{\psi_i(B)+1}\right) \prod_{j=1}^{k} (z_{j+1} + \cdots + z_h + a), \tag{4.16}$$

for $B = b_1 \ldots b_h \in St_{h,k}$, with $\psi_i(B) = \#\{j : i < j \leq h, b_j = \circ\}$ as before. Similarly, the stationary distribution of the enriched annihilation model is given by

$$\tilde{\Pi}(\sigma) = \prod_{(s,t) \text{ arch of } \sigma} z_{C_\sigma(s,t)} \prod_{i=1}^{K-1} (z_{i+1} + \cdots + z_{H-1} + a), \tag{4.17}$$

with $\sigma \in \mathcal{S}(H)$ and K its number of blocks. There is no normalization factor, as Π and $\tilde{\Pi}$ are already normalized for $z_1 + \cdots + z_h + a = 1$.

Example 4.13. The transition matrix of the basic annihilation model for $h = 2$ in the lexicographically-ordered basis ($\bullet\bullet$, $\circ\bullet$, $\bullet\circ$, $\circ\circ$) is given by

$$P = \begin{pmatrix} z_1 & 0 & z_2 + a & 0 \\ z_1 & 0 & z_2 + a & 0 \\ 0 & z_1 & z_2 & a \\ 0 & z_1 & z_2 & a \end{pmatrix}, \tag{4.18}$$

and admits the row vector $\left(z_1^2, z_1 (z_2 + a), (z_1 + z_2)(z_2 + a), a(z_2 + a)\right)$ as a left eigenvector of eigenvalue 1. The sum of the coordinates of this row vector is $(z_1 + z_2 + a)^2 = 1$. Similarly, the transition matrix of the enriched annihilation model for $H = 3$ in the same ordered basis as in Example 4.8 is given by

$$\tilde{P} = \begin{pmatrix} z_1 & 0 & 0 & z_2 + a & 0 \\ z_1 & 0 & 0 & z_2 + a & 0 \\ 0 & z_1 & z_2 & 0 & a \\ 0 & z_1 & z_2 & 0 & a \\ 0 & z_1 & z_2 & 0 & a \end{pmatrix}, \tag{4.19}$$

and admits the row vector $\left(z_1^2, z_1 (z_2 + a), z_2 (z_2 + a), z_1 (z_2 + a), a(z_2 + a)\right)$ as a left eigenvector of eigenvalue 1.

Proving Theorem 4.12 can be done by checking "by hand" the stationarity of $\tilde{\Pi}$, then deducing that of Π using the projection property (4.1). However, the fact that $\tilde{\Pi}(\sigma)$ is not a monomial suggests that there exists a "doubly enriched" chain that yields both P and $\tilde{P}$ by projection. It can be seen that, if we relax the condition $z_1 + \cdots + z_h + a = 1$, then

the total mass of Π (or of $\tilde{\Pi}$) is equal to $(z_1 + \cdots + z_h + a)^h$: this suggests that doubly enriched states should consist of h-tuples of elements in a set of cardinality $h + 1$. From now on we will write z_{h+1} in lieu of a.

Definition 4.14. *Given h a nonnegative integer and $z_1, \ldots, z_{h+1}$ nonnegative real numbers such that $z_1 + \cdots + z_{h+1} = 1$, the* doubly enriched annihilation model *is the Markov chain on the state space $\{1, \ldots, h + 1\}^h$ for which the transition probability from $W = w_1 w_2 \ldots w_h$ to W' reads*

$$\hat{P}_{W,W'} = \begin{cases} z_i & \text{if } W' = w_2 \cdots w_h i, \\ 0 & \text{otherwise.} \end{cases} \tag{4.20}$$

Remark 4.15. The doubly enriched annihilation model can be seen a specialization of the de Bruijn process considered in [3].

It is obvious that the stationary probability of $W = w_1 \cdots w_h$ in the doubly enriched annihilation model is

$$\hat{\Pi}(W) = z_{w_1} \cdots z_{w_h} \tag{4.21}$$

since we are basically moving a window of size h within a sequence of independent identically distributed random variables. The nontrivial fact is that this chain may be projected to the annihilation and the enriched annihilation models.

Theorem 4.16. *Let ϕ and $\tilde{\phi}$ be the mappings from $\bigcup_{j \geq 0} \{1, \ldots, h + 1\}^j$ to respectively $\bigcup_{j \geq 0} \mathrm{St}_j$ and $\bigcup_{j \geq 0} \mathcal{S}(j + 1)$ defined inductively by*

$$\phi(w_1 \cdots w_j) = \begin{cases} \emptyset & \text{for } j = 0, \\ S_{w_j}(\phi(w_1 \cdots w_{j-1}) \circ) & \text{for } j \geq 1, \end{cases} \tag{4.22}$$

where $\emptyset$ denotes the empty word, and

$$\tilde{\phi}(w_1 \cdots w_j) = \begin{cases} \{\{1\}\} & \text{for } j = 0, \\ J_{w_j}(\tilde{\phi}(w_1 \cdots w_{j-1})) & \text{for } j \geq 1. \end{cases} \tag{4.23}$$

Then, the restrictions of ϕ and $\tilde{\phi}$ to $\{1, \ldots, h+1\}^h$ yield projections of the doubly enriched chain onto respectively the annihilation model and the enriched annihilation model.

In other words, the projection ϕ (resp. $\tilde{\phi}$) consists in applying the composition of the mappings $B \mapsto S_{w_i}(B \circ)$ (resp. $\sigma \mapsto J_{w_i}(\sigma)$) with i running from 1 to h, to the "seed" $\emptyset$ (resp. $\{\{1\}\}$).

Example 4.17. The transition matrix of the doubly enriched annihilation model for $h = 2$ in the ordered basis $(11, 21, 31, 12, 22, 32, 13, 23, 33)$ is

$$\hat{P} = \begin{pmatrix} z_1 & 0 & 0 & z_2 & 0 & 0 & z_3 & 0 & 0 \\ z_1 & 0 & 0 & z_2 & 0 & 0 & z_3 & 0 & 0 \\ z_1 & 0 & 0 & z_2 & 0 & 0 & z_3 & 0 & 0 \\ 0 & z_1 & 0 & 0 & z_2 & 0 & 0 & z_3 & 0 \\ 0 & z_1 & 0 & 0 & z_2 & 0 & 0 & z_3 & 0 \\ 0 & z_1 & 0 & 0 & z_2 & 0 & 0 & z_3 & 0 \\ 0 & 0 & z_1 & 0 & 0 & z_2 & 0 & 0 & z_3 \\ 0 & 0 & z_1 & 0 & 0 & z_2 & 0 & 0 & z_3 \\ 0 & 0 & z_1 & 0 & 0 & z_2 & 0 & 0 & z_3 \end{pmatrix}. \tag{4.24}$$

The Markov chains with transition matrices $\hat{P}$, $\tilde{P}$ and P for $h = 2$ are displayed on Figure 5. The words 12 and 13 project via $\tilde{\phi}$ to the set partition $3 \,|\, 1, 2$ while 22 and 32 project to $1, 3 \,|\, 2$, and all four project via ϕ to the juggling state $\bullet\circ$.
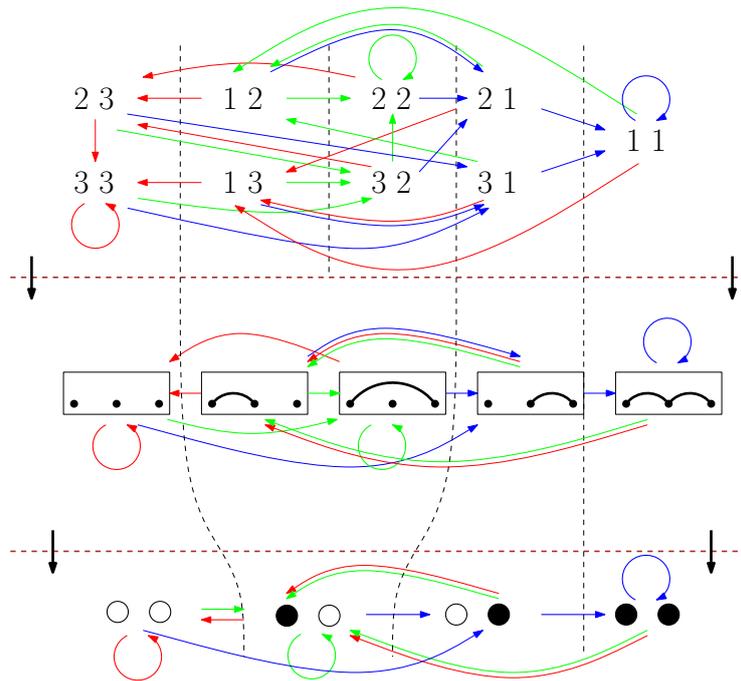
Figure 5: Transition graphs of the doubly enriched annihilation model (top), the enriched annihilation model (middle) and the annihilation model (bottom) for $h = 2$. Blue, green and red arrow represent transitions of respective probabilities z_1, z_2, z_3 (when several arrows have the same endpoints, the corresponding probabilities should be added).

Remark 4.18. The cardinality $h+1$ of the "alphabet" in the doubly enriched annihilation model is optimal in the sense that some states in St_h or $\mathcal{S}(h+1)$ are not attained with a smaller alphabet while, in a word of length h, all letters larger than $h+1$ can be replaced by $h+1$ without changing its image by ϕ or $\tilde{\phi}$. To be specific, if the alphabet has ℓ letters with $\ell \leq h$, the corresponding stationary probability of $B = b_1 \cdots b_h \in St_{h,k}$ in the annihilation model is

$$\Pi(B) = \prod_{\substack{i=1 \\ b_i=\bullet}}^{h} \left(z_1 + \cdots + z_{\psi_i(B)+1}\right) \prod_{j=1}^{k}(z_{j+1} + \cdots + z_\ell), \tag{4.25}$$

if $k < \ell$, and 0 otherwise. The corresponding stationary distribution of $\sigma \in \mathcal{S}(H, K)$ in the enriched annihilation model

$$\tilde{\Pi}(\sigma) = \prod_{(s,t) \text{ arch of } \sigma} z_{C_\sigma(s,t)} \prod_{i=1}^{K-1} (z_{i+1} + \cdots + z_\ell) \tag{4.26}$$

if $K \leq \ell$, and 0 otherwise. When $\ell \geq h + 1$, we might do the substitutions $z_{h+1} \leftarrow z_{h+1} + \cdots + z_\ell$ and $z_j \leftarrow 0$ for $j > h + 1$ without affecting Π and $\tilde{\Pi}$.

Proof of Theorem 4.16. It is not difficult to see that $\phi = \psi \circ \tilde{\phi}$ from (4.1); thus we need only check the statement for $\tilde{\phi}$. Consider the transition of probability z_i from $W = w_1 \cdots w_h$ to $W' = w_2 \ldots w_h i$ in the doubly enriched annihilation model: it is sufficient to have that $\tilde{\phi}(W)$ is sent to $\tilde{\phi}(W')$ by the corresponding transition of the enriched annihilation model, namely that

$$\tilde{\phi}(W') = J_i(\tilde{\phi}(W)^{\downarrow}). \tag{4.27}$$

But this is an easy consequence from the commutation relation (4.2) and the definition of $\tilde{\phi}$. □

Proof of Theorem 4.12. Let us first prove (4.17). We extend the definition (4.21) of $\tilde{\Pi}$ to words of arbitrary length by setting $\hat{\Pi}(w_1 \ldots w_j) = z_{w_1} \ldots z_{w_j}$. We will prove by induction on $j \leq h$ that, for any $\sigma \in \mathcal{S}(j+1)$, we have

$$\sum_{W \in \tilde{\phi}^{-1}(\sigma)} \hat{\Pi}(W) = \prod_{(s,t) \text{ arch of } \sigma} z_{C_\sigma(s,t)} \prod_{i=1}^{K-1} (z_{i+1} + \cdots + z_{h+1}) \tag{4.28}$$

with K the number of blocks of σ. The relation is true for $j = 0$ since both sides equal 1. For $j \geq 1$, let $\sigma' \in \mathcal{S}(j)$ be the set partition obtained from σ by removing the element $j + 1$. The preimages by $\tilde{\phi}$ of σ are then obtained from those of σ' by

- appending a unique letter $w \leq K$ if $j + 1$ is not a singleton in σ (note that σ has one arch more than σ', and w is the number of blocks that is covers),
- appending an arbitrary letter $w \geq K$ if $j + 1$ is a singleton in σ (note that σ has one block more than σ').

Then (4.28) follows from the induction hypothesis: it is deduced from the relation for σ' by adding an extra factor to the first product in the first case, and to the second product in the second case. We finally deduce (4.17) by taking $j = h$ (recall that $H = h + 1$ and $a = z_{h+1}$), noting that the left hand side of (4.28) is nothing but the wanted $\tilde{\Pi}(\sigma)$.

We then deduce (4.16) from the relation $\phi = \psi \circ \tilde{\phi}$, using again Lemma 2.27. □

Another nice property of the doubly enriched annihilation model which is easy to prove is that it is "memoryless": after h transitions we end up with a perfect sample of the stationary distribution, since all the initial letters have been flushed out. Therefore, h is a deterministic *strong stationary time* of the doubly enriched chain [17, Section 6.4], which is independent even of the initial distribution. In other words, for any initial probability distribution $\hat{\eta}$ over $\{1, \ldots, h+1\}^h$, we have

$$\hat{\eta}\hat{P}^h = \hat{\Pi}, \tag{4.29}$$

and this implies that the only eigenvalues of the transition matrix $\hat{P}$ are 1 (with multiplicity 1) and 0. These properties are clearly preserved by projection, which implies the following nontrivial and remarkable property of the annihilation model, that makes it distinct from the generic MJMC and add-drop models.

Theorem 4.19. *For any initial probability distributions η over St_h and $\tilde{\eta}$ over $\mathcal{S}(H)$, the distribution at time h is equal to the stationary distribution, namely*

$$\eta P^h = \Pi, \qquad \tilde{\eta}\tilde{P}^h = \tilde{\Pi}. \tag{4.30}$$

In particular, the only eigenvalues of P and $\tilde{P}$ are 1 (with multiplicity 1) and 0.

Remark 4.20. The fact that all other eigenvalues of the doubly enriched annihilation model are zero can also be proved directly from [3, Theorem 12] by setting $x_{a,m} = z_a$ so that $\beta_{a,m} = z_1 + \cdots + z_{h+1}$ for all $a \in [n], m \in [L]$ for words of length L on an alphabet of size n. However, the fact that the distribution at time L is the stationary distribution in general is not remarked in [3].

This result is stronger than a statement about mixing times since we reach the exact stationary state in bounded time! Another "ultrafast convergence to equilibrium" was observed in [16] for the model of Example 3.7, but its combinatorial origin seems rather different from that of the annihilation model (in our model the time needed to reach the stationary distribution is constant, while in Leskelä and Varpanen's model it depends on the initial state, and can be arbitrarily large).

5 Conclusion and discussion

We end with speculative ideas for further work along the direction of this paper. All three annihilation Markov chains (on juggling sequences, on set partitions and on words) have the property that all the eigenvalues of the transition matrices are nonnegative and trivially linear in the parameters z_i. Such Markov chains typically arise from an underlying structure which we now describe. If one considers the matrices P_i (resp. $\tilde{P}_i$) obtained by setting $z_i = 1$ and $z_j = 0$ for $j \neq i$ in the transition matrix P (resp. $\tilde{P}$) in (4.14) (resp. (4.15)) then the monoid generated by these matrices is $\mathscr{R}$-trivial. For a recent monograph on the connection between Markov chains and $\mathscr{R}$-trivial monoids, see [2]. One could reprove our statement about the eigenvalues in Theorem 4.19 using this connection. The general theory of free tree monoids described in [2], however, does not directly apply to the annihilation chains because the generators P_i (resp. $\tilde{P}_i$) do not always square to themselves (i.e. are not idempotents). It should be interesting to study when the results in [2] can be applied to situations when the generators are not idempotents.

This work suggests that it is worth studying a general framework for a class of problems in statistical physics, which we tentatively call "boundary-driven Markov chains". Just as the juggling and set-partition chains (2.1) and (2.27) studied in this paper, they have the property that the motion is deterministic for the most part (that is, in the bulk). Only when a ball (in the juggling context) reaches the boundary, i.e. Magnus' hand, something stochastic happens, namely the ball is thrown to a randomly chosen height. Similar things happen with the set-partition Markov chain.

A natural idea is to combine the juggling process with an exclusion process, for instance the Totally Asymmetric Exclusion Process (TASEP) where particles do not all move at the same time but, instead, at each time step one particle is selected randomly and moves if the next site is empty. When a particle on the first site is selected, it is then reinserted, say, uniformly at any available site on the lattice. This model was investigated via a physical hydrodynamic approach in [1], where a nontrivial phase diagram was found. It would be interesting to obtain more precise results on this model, by looking for instance for a possible exact solution.

Finally, the fact that the stationary distributions of the Markov chains considered in this paper all admit product forms is reminiscent of the Zero-Range Process admitting a "factorized stationary state" [11, 12], and one might wonder whether a connection exists.

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