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# The harmonic measure of balls in critical Galton–Watson trees with infinite variance offspring distribution

Shen Lin*

**Abstract**

We study properties of the harmonic measure of balls in large critical Galton–Watson trees whose offspring distribution is in the domain of attraction of a stable distribution with index $\alpha \in (1,2]$. Here the harmonic measure refers to the hitting distribution of height $n$ by simple random walk on the critical Galton–Watson tree conditioned on non-extinction at generation $n$. For a ball of radius $n$ centered at the root, we prove that, although the size of the boundary is roughly of order $n^{\frac{1}{\alpha-1}}$, most of the harmonic measure is supported on a boundary subset of size approximately equal to $n^{\beta_\alpha}$, where the constant $\beta_\alpha \in (0, \frac{1}{\alpha-1})$ depends only on the index $\alpha$. Using an explicit expression of $\beta_\alpha$, we are able to show the uniform boundedness of $(\beta_\alpha, 1 < \alpha \leq 2)$. These are generalizations of results in a recent paper of Curien and Le Gall [5].

**Keywords:** critical Galton–Watson tree; harmonic measure; Hausdorff dimension; invariant measure; simple random walk and Brownian motion on trees.
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## 1 Introduction

Recently, Curien and Le Gall have studied in [5] the properties of harmonic measure on generation $n$ of a critical Galton–Watson tree, whose offspring distribution has finite variance and which is conditioned to have height greater than $n$. They have shown the existence of a universal constant $\beta < 1$ such that, with high probability, most of the harmonic measure on generation $n$ of the tree is concentrated on a set of approximately $n^\beta$ vertices, although the number of vertices at generation $n$ is of order $n$. Their approach is based on the study of a similar continuous model, where it is established that the Hausdorff dimension of the (continuous) harmonic measure is almost surely equal to $\beta$.

In this paper, we continue the above work by extending their results to the critical Galton–Watson trees whose offspring distribution has infinite variance. To be more

*Université Paris-Sud XI, France. Currently at École Normale Supérieure, France.
E-mail: shen.lin.math@gmail.com

precise, let $\rho$ be a non-degenerate probability measure on $\mathbb{Z}_+$ with mean one, and we assume throughout this paper that $\rho$ is in the domain of attraction of a stable distribution of index $\alpha \in (1,2]$, which means that

$$\sum_{k\geq 0} \rho(k) r^k = r + (1-r)^\alpha L(1-r) \qquad \text{for any } r \in [0,1), \tag{1.1}$$

where the function $L(x)$ is slowly varying as $x \to 0^+$. We point out that the finite variance condition for $\rho$ is sufficient for the previous statement to hold with $\alpha = 2$. When $\alpha \in (1,2)$, by results of [8, Chapters XIII and XVII], the condition (1.1) is satisfied if and only if the tail probability

$$\sum_{k\geq x} \rho(k) = \rho([x,+\infty))$$

varies regularly with exponent $-\alpha$ as $x \to +\infty$. See e.g. [4] for the definition of regularly varying functions.

Under the probability measure $\mathbb{P}$, for every integer $n \geq 0$, we let $\mathsf{T}^{(n)}$ be a Galton–Watson tree with offspring distribution $\rho$, conditioned on non-extinction at generation $n$. Conditionally given the tree $\mathsf{T}^{(n)}$, we consider simple random walk on $\mathsf{T}^{(n)}$ starting from the root. The probability distribution of the first hitting point of generation $n$ by random walk will be called the harmonic measure $\mu_n$, which is supported on the set $\mathsf{T}^{(n)}_n$ of all vertices of $\mathsf{T}^{(n)}$ at generation $n$.

Let $q_n > 0$ be the probability that a critical Galton–Watson tree $\mathsf{T}^{(0)}$ survives up to generation $n$. It is shown in [16] that, as $n \to \infty$, the probability $q_n$ decreases as $n^{-\frac{1}{\alpha-1}}$ up to multiplication by a slowly varying function, and $q_n \# \mathsf{T}^{(n)}_n$ converges in distribution to a non-trivial limit distribution on $\mathbb{R}_+$, whose Laplace transform can be written explicitly in terms of the parameter $\alpha$. The following theorem generalizes the result [5, Theorem 1] in the finite variance case ($\alpha = 2$) to all $\alpha \in (1,2]$.

**Theorem 1.1.** *If the offspring distribution $\rho$ has mean one and belongs to the domain of attraction of a stable distribution of index $\alpha \in (1,2]$, there exists a constant $\beta_\alpha \in (0, \frac{1}{\alpha-1})$, which only depends on $\alpha$, such that for every $\delta > 0$, we have the convergence in $\mathbb{P}$-probability*

$$\mu_n\Big(\big\{v \in \mathsf{T}^{(n)}_n : n^{-\beta_\alpha-\delta} \leq \mu_n(v) \leq n^{-\beta_\alpha+\delta}\big\}\Big) \xrightarrow[n\to\infty]{(\mathbb{P})} 1\,. \tag{1.2}$$

*Consequently, for every $\varepsilon \in (0,1)$, there exists, with $\mathbb{P}$-probability tending to $1$ as $n \to \infty$, a subset $A_{n,\varepsilon}$ of $\mathsf{T}^{(n)}_n$ such that $\# A_{n,\varepsilon} \leq n^{\beta_\alpha+\delta}$ and $\mu_n(A_{n,\varepsilon}) \geq 1-\varepsilon$. Conversely, the maximal $\mu_n$-measure of a set of cardinality bounded by $n^{\beta_\alpha-\delta}$ tends to $0$ as $n \to \infty$, in $\mathbb{P}$-probability.*

The last two assertions of the preceding theorem are easy consequences of the convergence (1.2), as explained in [5].

We observe that the hitting distribution $\mu_n$ of generation $n$ by simple random walk on $\mathsf{T}^{(n)}$ is unaffected if we remove the branches of $\mathsf{T}^{(n)}$ that do not reach height $n$. Thus in order to establish the preceding result, we may consider simple random walk on $\mathsf{T}^{*n}$, the reduced tree associated with $\mathsf{T}^{(n)}$, which consists of all vertices of $\mathsf{T}^{(n)}$ that have at least one descendant at generation $n$.

When the critical offspring distribution $\rho$ has infinite variance, scaling limits of the discrete reduced trees $\mathsf{T}^{*n}$ have been studied in [17] and [18]. If we scale the graph distances by the factor $n^{-1}$, the discrete reduced trees $n^{-1}\mathsf{T}^{*n}$ converge to a random compact rooted $\mathbb{R}$-tree $\Delta^{(\alpha)}$ that we now describe. For every $\alpha \in (1,2]$, we define the

$\alpha$-offspring distribution $\theta_\alpha$ as follows. For $\alpha = 2$, we let $\theta_2 = \delta_2$ be the Dirac measure at 2. If $\alpha < 2$, $\theta_\alpha$ is the probability measure on $\mathbb{Z}_+$ given by

$$\begin{aligned}\theta_\alpha(0) &= \theta_\alpha(1) = 0,\\ \theta_\alpha(k) &= \frac{\alpha\,\Gamma(k-\alpha)}{k!\,\Gamma(2-\alpha)} = \frac{\alpha(2-\alpha)(3-\alpha)\cdots(k-1-\alpha)}{k!}, \quad \forall k \geq 2,\end{aligned}$$

where $\Gamma(\cdot)$ is the Gamma function. We let $U_\varnothing$ be a random variable uniformly distributed over $[0,1]$, and let $K_\varnothing$ be a random variable distributed according to $\theta_\alpha$, independent of $U_\varnothing$. To construct $\Delta^{(\alpha)}$, one starts with an oriented line segment of length $U_\varnothing$, whose origin will be the root of the tree. We call $K_\varnothing$ the offspring number of the root $\varnothing$. Correspondingly, at the other end of the first line segment, we attach the origins of $K_\varnothing$ oriented line segments with respective lengths $U_1, U_2, \ldots, U_{K_\varnothing}$, such that, conditionally given $U_\varnothing$ and $K_\varnothing$, the variables $U_1, U_2, \ldots, U_{K_\varnothing}$ are independent and uniformly distributed over $[0, 1-U_\varnothing]$. This finishes the first step of the construction. In the second step, for the first of these $K_\varnothing$ line segments, we independently sample a new offspring number $K_1$ distributed as $\theta_\alpha$, and attach $K_1$ new line segments whose lengths are again independent and uniformly distributed over $[0, 1-U_\varnothing-U_1]$, conditionally on all the random variables appeared before. For the other $K_\varnothing - 1$ line segments, we repeat this procedure independently. We continue in this way and after an infinite number of steps we get a random non-compact rooted $\mathbb{R}$-tree, whose completion is the random compact rooted $\mathbb{R}$-tree $\Delta^{(\alpha)}$. See Fig. 1 in Section 2.1 for an illustration. We will call $\Delta^{(\alpha)}$ the reduced stable tree of parameter $\alpha$. Notice that all the offspring numbers involved in the construction of $\Delta^{(2)}$ are a.s. equal to 2, which correspond to the binary branching mechanism. In contrast, this is no longer the case when $1 < \alpha < 2$.

We denote by $\mathbf{d}$ the intrinsic metric on $\Delta^{(\alpha)}$. By definition, the boundary $\partial\Delta^{(\alpha)}$ consists of all points of $\Delta^{(\alpha)}$ at height 1. As the continuous analogue of simple random walk, we can define Brownian motion on $\Delta^{(\alpha)}$ starting from the root and up to the first hitting time of $\partial\Delta^{(\alpha)}$. It behaves like linear Brownian motion as long as it stays inside a line segment of $\Delta^{(\alpha)}$. It is reflected at the root of $\Delta^{(\alpha)}$ and when it arrives at a branching point, it chooses each of the adjacent line segments with equal probabilities. We define the (continuous) harmonic measure $\mu_\alpha$ as the (quenched) distribution of the first hitting point of $\partial\Delta^{(\alpha)}$ by Brownian motion.

**Theorem 1.2.** *For every index $\alpha \in (1,2]$, with the same constant $\beta_\alpha$ as in Theorem 1.1, we have $\mathbb{P}$-a.s. $\mu_\alpha(\mathrm{d}x)$-a.e.,*

$$\lim_{r\downarrow 0} \frac{\log \mu_\alpha(\mathcal{B}_{\mathbf{d}}(x,r))}{\log r} = \beta_\alpha\,, \tag{1.3}$$

*where $\mathcal{B}_{\mathbf{d}}(x,r)$ stands for the closed ball of radius $r$ centered at $x$ in the metric space $(\Delta^{(\alpha)}, \mathbf{d})$. Consequently, the Hausdorff dimension of $\mu_\alpha$ is $\mathbb{P}$-a.s. equal to $\beta_\alpha$.*

According to Lemma 4.1 in [12], the last assertion of the preceding theorem follows directly from (1.3). As another direct consequence of (1.3), we have that $\mathbb{P}$-a.s. for $\mu_\alpha(\mathrm{d}x)$-a.e. $x \in \partial\Delta^{(\alpha)}$, $\mu_\alpha(\mathcal{B}_{\mathbf{d}}(x,r)) \to 0$ as $r \downarrow 0$, which is equivalent to non-atomicity of $\mu_\alpha$.

Since it has been proved in [7, Theorem 1.5] that the Hausdorff dimension of $\partial\Delta^{(\alpha)}$ with respect to $\mathbf{d}$ is a.s. equal to $\frac{1}{\alpha-1}$, the previous theorem implies that the harmonic measure has a.s. strictly smaller Hausdorff dimension than that of the whole boundary of the reduced stable tree. This phenomenon of dimension drop has been shown in [5, Theorem 2] for the special case of binary branching $\alpha = 2$.

We prove Theorem 1.2 in Section 2.5, where our approach is different and shorter than the one developed in [5] for the special case $\alpha = 2$.

Notice that the Hausdorff dimension of the boundary $\partial\Delta^{(\alpha)}$ increases to infinity when $\alpha \downarrow 1$. However, it is an interesting fact that the Hausdorff dimension of the harmonic measure remains bounded when $\alpha \downarrow 1$.

**Theorem 1.3.** *There exists a constant $C > 0$ such that for any $\alpha \in (1, 2]$, we have $\beta_\alpha < C$.*

Our proof of Theorem 1.3 relies on the fact that the constant $\beta_\alpha$ in Theorems 1.1 and 1.2 can be expressed in terms of the conductance of $\Delta^{(\alpha)}$. Informally, if we think of the random tree $\Delta^{(\alpha)}$ as a network of resistors with unit resistance per unit length, the effective conductance between the root and the boundary $\partial\Delta^{(\alpha)}$ is a random variable which we denote by $\mathcal{C}^{(\alpha)}$. From a probabilistic point of view, it is the mass under the Brownian excursion measure for the excursion paths away from the root that hit height 1. Following the definition of $\Delta^{(\alpha)}$ and the above electric network interpretation, the distribution of $\mathcal{C}^{(\alpha)}$ satisfies the recursive distributional equation

$$\mathcal{C}^{(\alpha)} \overset{\text{(d)}}{=\!=} \left( U + \frac{1-U}{\mathcal{C}_1^{(\alpha)} + \mathcal{C}_2^{(\alpha)} + \cdots + \mathcal{C}_{N_\alpha}^{(\alpha)}} \right)^{-1}, \tag{1.4}$$

where $(\mathcal{C}_i^{(\alpha)})_{i\geq 1}$ are i.i.d. copies of $\mathcal{C}^{(\alpha)}$, the integer-valued random variable $N_\alpha$ is distributed according to $\theta_\alpha$, and $U$ is uniformly distributed over $[0, 1]$. All these random variables are supposed to be independent.

**Proposition 1.4.** *For any $\alpha \in (1, 2]$, the distribution $\gamma_\alpha$ of the conductance $\mathcal{C}^{(\alpha)}$ is characterized in the class of all probability measures on $[1, \infty)$ by the distributional equation (1.4). The constant $\beta_\alpha$ appearing in Theorems 1.1 and 1.2 is given by*

$$\beta_\alpha = \frac{1}{2}\left( \frac{\left(\int \gamma_\alpha(\mathrm{d}s)s\right)^2}{\iint \gamma_\alpha(\mathrm{d}s)\gamma_\alpha(\mathrm{d}t)\frac{st}{s+t-1}} - 1 \right). \tag{1.5}$$

Interestingly, formula (1.5) expresses the exponent $\beta_\alpha$ as the same function of the distribution $\gamma_\alpha$, for all $\alpha \in (1, 2]$. In the course of the proof, we obtain two other formulas for $\beta_\alpha$ (see (2.18) and (2.19) below), but they both depend on $\alpha$ in a more complicated way, which also involves the distribution $\theta_\alpha$.

The paper is organized as follows. In Section 2 below, we study the continuous model of Brownian motion on $\Delta^{(\alpha)}$. A formal definition of the reduced stable tree $\Delta^{(\alpha)}$ is given in Section 2.1. In Section 2.2 we explain how to relate $\Delta^{(\alpha)}$ to an infinite supercritical continuous-time Galton–Watson tree $\Gamma^{(\alpha)}$, and we reformulate Theorem 1.2 in terms of Brownian motion with drift $1/2$ on $\Gamma^{(\alpha)}$. Properties of the law of the random conductance $\mathcal{C}^{(\alpha)}$, including the first assertion of Proposition 1.4, are discussed in Section 2.3, and Section 2.4 gives the coupling argument that allows one to derive Theorem 1.3 from formula (1.5). Section 2.5 is devoted to the proofs of Theorem 1.2 and of formula (1.5). We emphasize that our approach to Theorem 1.2 is different from the one used in [5] when $\alpha = 2$. In fact we use an invariant measure for the environment seen by Brownian motion on $\Gamma^{(\alpha)}$ at the last passage time of a node of the $n$-th generation, instead of the last passage time at a height $h$ as in [5]. We then apply the ergodic theory on Galton–Watson trees, which is a powerful tool initially developed in [12].

In Section 3 we proceed to the discrete setting concerning simple random walk on the discrete reduced tree $\mathsf{T}^{*n}$. Let us emphasize that, when the critical offspring distribution $\rho$ is in the domain of attraction of a stable distribution of index $\alpha \in (1, 2)$, the convergence of discrete reduced trees is less simple than in the special case $\alpha = 2$ where we have a.s. a binary branching structure. See Proposition 3.2 for a precise statement in our more general setting. Apart from this ingredient, we need several estimates for the discrete reduced tree $\mathsf{T}^{*n}$ to derive Theorem 1.1 from Theorem 1.2. For example, Lemma 3.1 gives a bound for the size of level sets in $\mathsf{T}^{*n}$, and Lemma 3.9 presents a moment estimate for the (discrete) conductance $\mathcal{C}_n(\mathsf{T}^{*n})$ between generations 0 and $n$ in $\mathsf{T}^{*n}$. Although the result analogous to Lemma 3.9 in [5] is a second moment estimate, we only manage to give a moment estimate of order strictly smaller than $\alpha$ if the critical offspring distribution $\rho$ satisfies (1.1) with $\alpha \in (1,2]$. Nevertheless, this is sufficient for our proof of Theorem 1.1, which is adapted from the one given in [5].

Comments and several open questions are gathered in Section 4. Following the work of Aïdékon [1], we obtain a candidate for the speed of Brownian motion with drift $1/2$ on the infinite tree $\Gamma^{(\alpha)}$, expressed by (4.1) in terms of the continuous conductance $\mathcal{C}^{(\alpha)}$. Nonetheless, the monotonicity properties of this quantity remains open. It would also be of interest to know whether or not the Hausdorff dimension $\beta_\alpha$ of the continuous harmonic measure $\mu_\alpha$ is monotone with respect to $\alpha \in (1,2]$.

## 2 The continuous setting

### 2.1 The reduced stable tree

We set

$$\mathcal{V} = \bigcup_{n=0}^{\infty} \mathbb{N}^n$$

where by convention $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}^0 = \{\varnothing\}$. If $v = (v_1, \ldots, v_n) \in \mathcal{V}$, we set $|v| = n$ (in particular, $|\varnothing| = 0$), and if $n \geq 1$, we define the parent of $v$ as $\widehat{v} = (v_1, \ldots, v_{n-1})$ and then say that $v$ is a child of $\widehat{v}$. For two elements $v = (v_1, \ldots, v_n)$ and $v' = (v'_1, \ldots, v'_m)$ belonging to $\mathcal{V}$, their concatenation is $vv' := (v_1, \ldots, v_n, v'_1, \ldots, v'_m)$. The notions of a descendant and an ancestor of an element of $\mathcal{V}$ are defined in the obvious way, with the convention that every $v \in \mathcal{V}$ is both an ancestor and a descendant of itself. If $v, w \in \mathcal{V}$, $v \wedge w$ is the unique element of $\mathcal{V}$ such that it is a common ancestor of $v$ and $w$, and $|v \wedge w|$ is maximal.

An infinite subset $\Pi$ of $\mathcal{V}$ is called an infinite discrete tree if there exists a collection of positive integers $k_v = k_v(\Pi) \in \mathbb{N}$ for every $v \in \mathcal{V}$ such that

$$\Pi = \{\varnothing\} \cup \{(v_1, \ldots, v_n) \in \mathcal{V} : v_j \leq k_{(v_1, \ldots, v_{j-1})} \text{ for every } 1 \leq j \leq n\}.$$

Recall the definition of the $\alpha$-offspring distribution $\theta_\alpha$ for $\alpha \in (1,2]$. It will also be convenient to consider the case $\alpha = 1$, where we define $\theta_1$ as the probability measure on $\mathbb{Z}_+$ given by

$$\begin{aligned} \theta_1(0) &= \theta_1(1) = 0, \\ \theta_1(k) &= \frac{1}{k(k-1)}, \quad \forall k \geq 2. \end{aligned}$$

If $\alpha \in (1,2]$, the generating function of $\theta_\alpha$ is given (see e.g. [6, p.74]) as

$$\sum_{k \geq 0} \theta_\alpha(k)\, r^k = \frac{(1-r)^\alpha - 1 + \alpha r}{\alpha - 1}, \quad \forall r \in (0,1], \tag{2.1}$$

while for $\alpha = 1$,

$$\sum_{k \geq 0} \theta_1(k)\, r^k = r + (1-r)\log(1-r), \quad \forall r \in (0,1]. \tag{2.2}$$

Notice that for $\alpha \in (1,2]$, the mean of $\theta_\alpha$ is given by

$$m_\alpha = \frac{\alpha}{\alpha - 1} \in [2, \infty),$$

whereas $\theta_1$ has infinite mean.

For fixed $\alpha \in [1,2]$, we introduce a collection $(K_\alpha(v))_{v\in\mathcal{V}}$ of independent random variables distributed according to $\theta_\alpha$ under the probability measure $\mathbb{P}$, and define a random infinite discrete tree

$$\Pi^{(\alpha)} := \{\varnothing\} \cup \{(v_1,\ldots,v_n) \in \mathcal{V} \colon v_j \le K_\alpha((v_1,\ldots,v_{j-1})) \text{ for every } 1 \le j \le n\}\,.$$

We point out that $\Pi^{(2)}$ is an infinite binary tree.

Let $(U_v)_{v\in\mathcal{V}}$ be another collection, independent of $(K_\alpha(v))_{v\in\mathcal{V}}$, consisting of independent real random variables uniformly distributed over $[0,1]$ under the same probability measure $\mathbb{P}$. We set now

$$Y_\varnothing = U_\varnothing$$

and then by induction, for every $v \in \Pi^{(\alpha)} \setminus \{\varnothing\}$,

$$Y_v = Y_{\hat{v}} + U_v(1 - Y_{\hat{v}}).$$

Note that a.s. $0 \le Y_v < 1$ for every $v \in \Pi^{(\alpha)}$. Consider then the set

$$\Delta_0^{(\alpha)} := \big(\{\varnothing\} \times [0, Y_\varnothing]\big) \cup \Bigg( \bigcup_{v\in\Pi^{(\alpha)}\setminus\{\varnothing\}} \{v\} \times (Y_{\hat{v}}, Y_v] \Bigg).$$

There is a straightforward way to define a metric $\mathbf{d}$ on $\Delta_0^{(\alpha)}$, so that $(\Delta_0^{(\alpha)}, \mathbf{d})$ is a (noncompact) $\mathbb{R}$-tree and, for every $x = (v,r) \in \Delta_0^{(\alpha)}$, we have $\mathbf{d}((\varnothing,0),x) = r$. To be specific, let $x = (v,r) \in \Delta_0^{(\alpha)}$ and $y = (w,r') \in \Delta_0^{(\alpha)}$:

- If $v$ is a descendant (or an ancestor) of $w$, we set $\mathbf{d}(x,y) = |r - r'|$.
- Otherwise, $\mathbf{d}(x,y) = \mathbf{d}((v\wedge w, Y_{v\wedge w}),x) + \mathbf{d}((v\wedge w, Y_{v\wedge w}),y) = (r - Y_{v\wedge w}) + (r' - Y_{v\wedge w})$.

See Fig. 1 for an illustration of the tree $\Delta_0^{(\alpha)}$ when $\alpha < 2$.

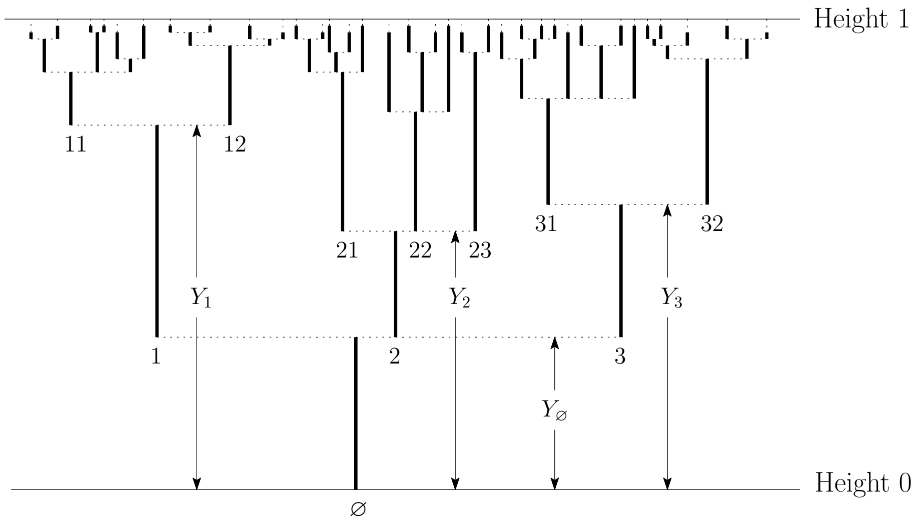


Figure 1: The random tree $\Delta_0^{(\alpha)}$ when $1 \le \alpha < 2$

We let $\Delta^{(\alpha)}$ be the completion of $\Delta_0^{(\alpha)}$ with respect to the metric $\mathbf{d}$. Then

$$\Delta^{(\alpha)} = \Delta_0^{(\alpha)} \cup \partial\Delta^{(\alpha)}$$

where by definition $\partial\Delta^{(\alpha)} := \{x \in \Delta^{(\alpha)} \colon \mathbf{d}((\varnothing,0),x) = 1\}$, which can be identified with a random subset of $\mathbb{N}^{\mathbb{N}}$. It is immediate to see that $(\Delta^{(\alpha)},\mathbf{d})$ is an a.s. compact $\mathbb{R}$-tree, which we will call the reduced stable tree of index $\alpha$.

The point $(\varnothing,0)$ is called the root of $\Delta^{(\alpha)}$. For every $x \in \Delta^{(\alpha)}$, we set $H(x) = \mathbf{d}((\varnothing,0),x)$ and call $H(x)$ the height of $x$. We can define a (non-strict) genealogical order on $\Delta^{(\alpha)}$ by setting $x \prec y$ if and only if $x$ belongs to the geodesic path from the root to $y$.

For every $\varepsilon \in (0,1)$, we set

$$\Delta^{(\alpha)}_\varepsilon := \{x \in \Delta^{(\alpha)} \colon H(x) \le 1-\varepsilon\},$$

which is also an a.s. compact $\mathbb{R}$-tree for the metric $\mathbf{d}$. The leaves of $\Delta^{(\alpha)}_\varepsilon$ are the points of the form $(v,1-\varepsilon)$ for all $v \in \mathcal{V}$ such that $Y_{\hat{v}} < 1-\varepsilon \le Y_v$. The branching points of $\Delta^{(\alpha)}_\varepsilon$ are the points of the form $(v,Y_v)$ for all $v \in \mathcal{V}$ such that $Y_v < 1-\varepsilon$.

Now conditionally on $\Delta^{(\alpha)}$, we can define Brownian motion on $\Delta^{(\alpha)}_\varepsilon$ starting from the root. Informally, this process behaves like linear Brownian motion as long as it stays on an "open interval" of the form $\{v\} \times (Y_{\hat{v}}, Y_v \wedge (1-\varepsilon))$, and it is reflected at the root $(\varnothing,0)$ and at the leaves of $\Delta^{(\alpha)}_\varepsilon$. When it arrives at a branching point of the tree, it chooses each of the possible line segments ending at this point with equal probabilities. By taking a sequence $\varepsilon_n = 2^{-n}, n \ge 1$ and then letting $n$ go to infinity, we can construct under the same probability measure $P$ a Brownian motion $B$ on $\Delta^{(\alpha)}$ starting from the root, which is defined up to its first hitting time $T$ of $\partial\Delta^{(\alpha)}$. We refer the reader to [5, Section 2.1] for the details of this construction. The harmonic measure $\mu_\alpha$ is then the distribution of $B_{T-}$ under $P$, which is a (random) probability measure on $\partial\Delta^{(\alpha)} \subseteq \mathbb{N}^{\mathbb{N}}$.

### 2.2 The continuous-time Galton–Watson tree

In this subsection, we introduce a new tree which shares the same branching structure as $\Delta^{(\alpha)}$, such that each point of $\Delta^{(\alpha)}$ at height $s \in [0,1)$ corresponds to a point of the new tree at height $-\log(1-s) \in [0,\infty)$ in a bijective way. As it turns out, this new random tree is a continuous-time Galton–Watson tree.

To define it, we take $\alpha \in [1,2]$ and start with the same random infinite tree $\Pi^{(\alpha)}$ introduced in Section 2.1. Consider now a collection $(V_v)_{v\in\mathcal{V}}$ of independent real random variables exponentially distributed with mean $1$ under the probability measure $\mathbb{P}$. We set

$$Z_\varnothing = V_\varnothing$$

and then by induction, for every $v \in \Pi^{(\alpha)} \setminus \{\varnothing\}$,

$$Z_v = Z_{\hat{v}} + V_v.$$

The continuous-time Galton–Watson tree (hereafter to be called CTGW tree for short) of stable index $\alpha$ is the set

$$\Gamma^{(\alpha)} := \big(\{\varnothing\} \times [0, Z_\varnothing]\big) \cup \bigg(\bigcup_{v\in\Pi^{(\alpha)}\setminus\{\varnothing\}} \{v\} \times (Z_{\hat{v}}, Z_v]\bigg),$$

which is equipped with the metric $d$ defined in the same way as $\mathbf{d}$ in the preceding subsection. For this metric, $\Gamma^{(\alpha)}$ is a.s. a non-compact $\mathbb{R}$-tree. For every $x = (v,r) \in \Gamma^{(\alpha)}$, we keep the notation $H(x) = r = d((\varnothing,0),x)$ for the height of the point $x$.

Observe that if $U$ is uniformly distributed over $[0,1]$, the random variable $-\log(1-U)$ is exponentially distributed with mean $1$. Hence we may and will suppose that the collection $(V_v)_{v\in\mathcal{V}}$ is constructed from the collection $(U_v)_{v\in\mathcal{V}}$ in the previous subsection via the formula $V_v = -\log(1-U_v)$ for every $v \in \mathcal{V}$. Then, the mapping $\Psi$ defined on $\Delta_0^{(\alpha)}$ by

$$\Psi(v,r) := \big(v, -\log(1-r)\big) \quad \text{for every } (v,r) \in \Delta_0^{(\alpha)},$$

is a homeomorphism from $\Delta_0^{(\alpha)}$ onto $\Gamma^{(\alpha)}$.

By stochastic analysis, we can write for every $t \in [0,T)$,

$$\Psi(B_t) = W\Big(\int_0^t (1-H(B_s))^{-2}\,\mathrm{d}s\Big) \tag{2.3}$$

where $(W(t))_{t\geq 0}$ is Brownian motion with constant drift $1/2$ towards infinity on the CTGW tree $\Gamma^{(\alpha)}$ (this process is defined in a similar way as Brownian motion on $\Delta_\varepsilon^{(\alpha)}$, except that it behaves like Brownian motion with drift $1/2$ on every "open interval" of the tree). Note that again $W$ is defined under the probability measure $P$. Since all the offspring numbers involved in the CTGW tree $\Gamma^{(\alpha)}$ are a.s. larger than 2, it is easy to see that the Brownian motion $W$ is transient. From now on, when we speak about Brownian motion on the CTGW tree or on other similar trees, we will always mean Brownian motion with drift $1/2$ towards infinity.

By definition, the boundary of $\Gamma^{(\alpha)}$ is the set of all infinite geodesics in $\Gamma^{(\alpha)}$ starting from the root $(\varnothing, 0)$ (these are called geodesic rays), and it can be canonically embedded into $\mathbb{N}^{\mathbb{N}}$. Due to the transience of Brownian motion on $\Gamma^{(\alpha)}$, there is an a.s. unique geodesic ray denoted by $W_\infty$ that is visited by $(W(t))_{t\geq 0}$ at arbitrarily large times. We say that $W_\infty$ is the exit ray of Brownian motion on $\Gamma^{(\alpha)}$. The distribution of $W_\infty$ under $P$ yields a probability measure $\nu_\alpha$ on $\mathbb{N}^{\mathbb{N}}$. Thanks to (2.3), we have in fact $\nu_\alpha = \mu_\alpha$, provided we think of both $\mu_\alpha$ and $\nu_\alpha$ as (random) probability measures on $\mathbb{N}^{\mathbb{N}}$. The statement of Theorem 1.2 is then reduced to checking that for every $1 < \alpha \leq 2$, $\mathbb{P}$-a.s., $\nu_\alpha(\mathrm{d}y)$-a.e.

$$\lim_{r\to\infty} \frac{1}{r} \log \nu_\alpha(\mathcal{B}(y,r)) = -\beta_\alpha\,, \tag{2.4}$$

where $\mathcal{B}(y,r)$ denotes the set of all geodesic rays that coincide with $y$ up to height $r$.

**Infinite continuous trees.** To prove (2.4), we will apply the tools of ergodic theory to certain transformations on a space of finite-degree rooted infinite continuous trees that we now describe. We let $\mathbb{T}$ be the set of all pairs $(\Pi, (z_v)_{v\in\Pi})$ that satisfy the following conditions:

(1) $\Pi$ is an infinite discrete tree, in the sense of Section 2.1.

(2) We have

 (i) $z_v \in [0,\infty)$ for all $v \in \Pi$;
 (ii) $z_{\hat{v}} < z_v$ for every $v \in \Pi\backslash\{\varnothing\}$;
 (iii) for every $\mathbf{v} \in \Pi_\infty := \{(v_1, v_2, \dots, v_n, \dots) \in \mathbb{N}^{\mathbb{N}} \colon (v_1, v_2, \dots, v_n) \in \Pi, \forall n \geq 1\}$,

$$\lim_{n\to\infty} z_{(v_1,\dots,v_n)} = +\infty.$$

In the preceding definition, we allow the possibility that $z_\varnothing = 0$. Notice that property (iii) implies that $\#\{v \in \Pi \colon z_v \leq r\} < \infty$ for every $r > 0$.

We equip $\mathbb{T}$ with the $\sigma$-field generated by the coordinate mappings. If $(\Pi, (z_v)_{v\in\Pi}) \in \mathbb{T}$, we can consider the associated "tree"

$$\mathcal{T} := \big(\{\varnothing\} \times [0, z_\varnothing]\big) \cup \Bigg(\bigcup_{v\in\Pi\backslash\{\varnothing\}} \{v\} \times (z_{\hat{v}}, z_v]\Bigg),$$

equipped with the distance defined as above. The set $\Pi_\infty$ is identified with the collection of all geodesic rays in $\Pi$, and will be viewed as the boundary of the tree $\mathcal{T}$. We keep the notation $H(x) = r$ for the height of a point $x = (v, r) \in \mathcal{T}$. The genealogical order on $\mathcal{T}$ is defined as previously and again is denoted by $\prec$. If $\mathbf{u} = (u_1, u_2, \ldots) \in \Pi_\infty$, and $x = (v, r) \in \mathcal{T}$, we write $x \prec \mathbf{u}$ if $v = (u_1, u_2, \ldots, u_k)$ for some integer $k \geq 0$.

We will often abuse notation and say that we consider a tree $\mathcal{T} \in \mathbb{T}$: This means that we are given a pair $(\Pi, (z_v)_{v\in\Pi})$ satisfying the above properties, and we consider the associated tree $\mathcal{T}$. In particular, $\mathcal{T}$ has an order structure (in addition to the genealogical partial order) given by the lexicographical order on $\Pi$. Elements of $\mathbb{T}$ will be called infinite continuous trees. Clearly, for every stable index $\alpha \in [1, 2]$, the CTGW tree $\Gamma^{(\alpha)}$ can be viewed as a random variable with values in $\mathbb{T}$, and we write $\Theta_\alpha(\mathrm{d}\mathcal{T})$ for its distribution.

Let us fix $\mathcal{T} = (\Pi, (z_v)_{v\in\Pi}) \in \mathbb{T}$. Under our previous notation, $k_\varnothing$ is the number of offspring at the first branching point of $\mathcal{T}$. We denote by $\mathcal{T}_{(1)}, \mathcal{T}_{(2)}, \ldots, \mathcal{T}_{(k_\varnothing)}$ the subtrees of $\mathcal{T}$ obtained at the first branching point. To be more precise, for every $1 \leq i \leq k_\varnothing$, we define the shifted discrete tree $\Pi[i] = \{v \in \mathcal{V} \colon iv \in \Pi\}$, and $\mathcal{T}_{(i)}$ is the infinite continuous tree corresponding to the pair

$$\Big(\Pi[i], (z_{iv} - z_\varnothing)_{v\in\Pi[i]}\Big).$$

Under $\Theta_\alpha(\mathrm{d}\mathcal{T})$, we know by definition that $k_\varnothing$ is distributed according to $\theta_\alpha$. Moreover, conditionally on $k_\varnothing$, the branching property of the CTGW tree states that the subtrees $\mathcal{T}_{(1)}, \ldots, \mathcal{T}_{(k_\varnothing)}$ are i.i.d. following the same law $\Theta_\alpha$.

If $r > 0$, the level set of $\mathcal{T} \in \mathbb{T}$ at height $r$ is

$$\mathcal{T}_r = \{x \in \mathcal{T} \colon H(x) = r\}.$$

For $\alpha \in (1, 2]$, we have the classical result

$$\mathbb{E}\big[\#\Gamma_r^{(\alpha)}\big] = \exp\big(\frac{r}{\alpha - 1}\big) = \exp\big((m_\alpha - 1)r\big)\,,$$

which can be derived from the following identity (see e.g. Theorem 2.7.1 in [6]) stating that for every $u > 0$,

$$\mathbb{E}\big[\exp(-u\,\#\Gamma_r^{(\alpha)})\big] = 1 - \big[1 - e^{-r}(1 - (1 - e^{-u})^{1-\alpha})\big]^{\frac{1}{1-\alpha}}\,.$$

### 2.3 The continuous conductance

Recall that, for $\alpha \in [1, 2]$, the random variable $\mathcal{C}^{(\alpha)}$ is defined as the conductance between the root and the set $\partial\Delta^{(\alpha)}$ in the continuous tree $\Delta^{(\alpha)}$ viewed as an electric network. One can also give a more probabilistic definition of the conductance. If $\mathcal{T}$ is a (deterministic) infinite continuous tree, the conductance $\mathcal{C}(\mathcal{T})$ between the root and the boundary $\partial\mathcal{T}$ can be defined in terms of excursion measures of Brownian motion with drift $1/2$ on $\mathcal{T}$. Under this definition, we can set $\mathcal{C}^{(\alpha)} = \mathcal{C}(\Gamma^{(\alpha)}) \in [1, \infty)$. For details, we refer the reader to Section 2.3 in [5].

In this subsection, we will prove for $\alpha \in (1, 2]$ that the law of $\mathcal{C}^{(\alpha)}$ is characterized by the distributional identity (1.4) in the class of all probability measures on $[1, \infty)$, and discuss some of the properties of this law. For $u \in (0, 1), n \in \mathbb{N}$ and $(x_i)_{i\geq1} \in [1, \infty)^{\mathbb{N}}$, we define

$$G(u, n, (x_i)_{i\geq1}) := \left(u + \frac{1 - u}{x_1 + x_2 + \cdots + x_n}\right)^{-1},$$

so that (1.4) can be rewritten as

$$\mathcal{C}^{(\alpha)} \overset{(\mathrm{d})}{=} G(U, N_\alpha, (\mathcal{C}_i^{(\alpha)})_{i\geq1}) \tag{2.5}$$

where $U, N_\alpha, (\mathcal{C}_i^{(\alpha)})_{i\geq 1}$ are as in (1.4). Note that (2.5) also holds for $\alpha = 1$. Let $\mathscr{M}$ be the set of all probability measures on $[1,\infty]$ and let $\Phi_\alpha \colon \mathscr{M} \to \mathscr{M}$ map a distribution $\sigma$ to

$$\Phi_\alpha(\sigma) = \mathsf{Law}\big(G(U, N_\alpha, (X_i)_{i\geq 1})\big)$$

where $(X_i)_{i\geq 1}$ are independent and identically distributed according to $\sigma$, while $U, N_\alpha$ are as in (1.4). We suppose in addition that $U, N_\alpha$ and $(X_i)_{i\geq 1}$ are independent.

We write $\gamma_\alpha$ for the distribution of $\mathcal{C}^{(\alpha)}$, and define for all $\ell \geq 0$ the Laplace transform

$$\varphi_\alpha(\ell) := \mathbb{E}\big[\exp(-\ell\,\mathcal{C}^{(\alpha)}/2)\big] = \int_1^\infty e^{-\ell r/2}\,\gamma_\alpha(\mathrm{d}r).$$

**Proposition 2.1.** *Let us fix the stable index $\alpha \in (1,2]$. The law $\gamma_\alpha$ of $\mathcal{C}^{(\alpha)}$ is the unique fixed point of the mapping $\Phi_\alpha$ on $\mathscr{M}$, and we have $\Phi_\alpha^k(\sigma) \to \gamma_\alpha$ weakly as $k \to \infty$, for every $\sigma \in \mathscr{M}$. Furthermore,*

1. *If $\alpha = 2$, all moments of $\gamma_2$ are finite, and $\gamma_2$ has a continuous density over $[1,\infty)$. The Laplace transform $\varphi_2$ solves the differential equation*

$$2\ell\,\varphi''(\ell) + \ell\varphi'(\ell) + \varphi^2(\ell) - \varphi(\ell) = 0.$$

2. *If $\alpha \in (1,2)$, only the first and the second moments of $\gamma_\alpha$ are finite. The distribution $\gamma_\alpha$ has a continuous density over $[1,\infty)$, and the Laplace transform $\varphi_\alpha$ solves the differential equation*

$$2\ell\,\varphi''(\ell) + \ell\varphi'(\ell) + \frac{(1-\varphi(\ell))^\alpha + \varphi(\ell) - 1}{\alpha - 1} = 0. \tag{2.6}$$

*Proof.* The case $\alpha = 2$ has been derived in [5, Proposition 6] and is listed above for the sake of completeness. We will prove the corresponding assertion for $\alpha \in (1,2)$ by similar methods.

Firstly, the stochastic partial order $\preceq$ on $\mathscr{M}$ is defined by saying that $\sigma \preceq \sigma'$ if and only if there exists a coupling $(X,Y)$ of $\sigma$ and $\sigma'$ such that a.s. $X \leq Y$. It is clear that for any $\alpha \in [1,2]$, the mapping $\Phi_\alpha$ is increasing for the stochastic partial order.

We endow the set $\mathscr{M}_1$ of all probability measures on $[1,\infty]$ that have a finite first moment with the 1-Wasserstein metric

$$\mathrm{d}_1(\sigma,\sigma') \quad := \quad \inf\big\{E\big[|X-Y|\big] \colon (X,Y) \text{ coupling of } (\sigma,\sigma')\big\}.$$

The metric space $(\mathscr{M}_1, \mathrm{d}_1)$ is Polish and its topology is finer than the weak topology on $\mathscr{M}_1$. From the easy bound

$$G(u, n, (x_i)_{i\geq 1}) \leq x_1 + x_2 + \cdots + x_n$$

and the fact that $\mathbb{E}N_\alpha < \infty$ when $\alpha \neq 1$, we immediately see that $\Phi_\alpha$ maps $\mathscr{M}_1$ into $\mathscr{M}_1$ when $\alpha > 1$. We then observe that the mapping $\Phi_\alpha$ is strictly contractant on $(\mathscr{M}_1, \mathrm{d}_1)$. To see this, let $(X_i, Y_i)_{i\geq 1}$ be independent copies of a coupling between $\sigma, \sigma' \in \mathscr{M}_1$ under the probability measure $\mathbb{P}$. As in (2.5), let $U$ be uniformly distributed over $[0,1]$ and $N_\alpha$ be distributed according to $\theta_\alpha$. Assume that $U, N_\alpha$ and $(X_i, Y_i)_{i\geq 1}$ are independent under $\mathbb{P}$. Then the two variables $G(U, N_\alpha, (X_i)_{i\geq 1})$ and $G(U, N_\alpha, (Y_i)_{i\geq 1})$ give a coupling of $\Phi_\alpha(\sigma)$ and $\Phi_\alpha(\sigma')$. Using the fact that $X_i, Y_i \geq 1$, we have

$$\begin{aligned}
&|G(U, N_\alpha, (X_i)_{i\geq 1}) - G(U, N_\alpha, (Y_i)_{i\geq 1})| \\
= \quad & \Big|\Big(U + \frac{1-U}{X_1 + X_2 + \cdots + X_{N_\alpha}}\Big)^{-1} - \Big(U + \frac{1-U}{Y_1 + Y_2 + \cdots + Y_{N_\alpha}}\Big)^{-1}\Big| \\
= \quad & \Big|\frac{(X_1 + X_2 + \cdots + X_{N_\alpha} - Y_1 - Y_2 - \cdots - Y_{N_\alpha})(1-U)}{(U(X_1 + X_2 + \cdots + X_{N_\alpha}) + 1 - U)(U(Y_1 + Y_2 + \cdots + Y_{N_\alpha}) + 1 - U)}\Big| \\
\leq \quad & \big(|X_1 - Y_1| + |X_2 - Y_2| + \cdots + |X_{N_\alpha} - Y_{N_\alpha}|\big)\,\frac{1-U}{(1 + (N_\alpha - 1)U)^2}.
\end{aligned}$$

Notice that for any integer $k \geq 2$,

$$\mathbb{E}\Big[\frac{k(1-U)}{(1+(k-1)U)^2}\Big] = 1 + \frac{k-1-k\log k}{(k-1)^2}.$$

Taking expected values and minimizing over the choice of the coupling between $\sigma$ and $\sigma'$, we get

$$\begin{aligned} \mathrm{d}_1(\Phi_\alpha(\sigma),\Phi_\alpha(\sigma')) &\leq \mathbb{E}\Big[\frac{N_\alpha(1-U)}{(1+(N_\alpha-1)U)^2}\Big]\mathrm{d}_1(\sigma,\sigma') \\ &= \Big(1+\mathbb{E}\Big[\frac{N_\alpha-1-N_\alpha\log N_\alpha}{(N_\alpha-1)^2}\Big]\Big)\mathrm{d}_1(\sigma,\sigma') \ = \ c_\alpha \mathrm{d}_1(\sigma,\sigma')\,, \end{aligned}$$

with $c_\alpha < 1$. So for $\alpha \in (1,2]$, the mapping $\Phi_\alpha$ is contractant on $\mathscr{M}_1$ and by completeness it has a unique fixed point $\tilde{\gamma}_\alpha$ in $\mathscr{M}_1$. Furthermore, for every $\sigma \in \mathscr{M}_1$, we have $\Phi_\alpha^k(\sigma) \to \tilde{\gamma}_\alpha$ for the metric $\mathrm{d}_1$, hence also weakly, as $k \to \infty$.

Since we know from (2.5) that $\gamma_\alpha$ is also a fixed point of $\Phi_\alpha$, the equality $\gamma_\alpha = \tilde{\gamma}_\alpha$ will follow if we can verify that $\tilde{\gamma}_\alpha$ is the unique fixed point of $\Phi_\alpha$ in $\mathscr{M}$. To this end, it will be enough to show that we have $\Phi_\alpha^k(\sigma) \to \tilde{\gamma}_\alpha$ as $k \to \infty$, for every $\sigma \in \mathscr{M}$.

For any $\alpha \in [1,2]$, we apply $\Phi_\alpha$ to the Dirac measure $\delta_\infty$ at $\infty$ to see

$$\begin{aligned} \Phi_\alpha(\delta_\infty) &= \mathsf{Law}\big(U^{-1}\big)\,, \\ \Phi_\alpha^2(\delta_\infty) &= \mathsf{Law}\bigg(\Big(U + \frac{1-U}{U_1^{-1}+U_2^{-1}+\cdots+U_{N_\alpha}^{-1}}\Big)^{-1}\bigg)\,, \end{aligned}$$

where we introduce a new sequence $(U_i)_{i\geq 1}$ consisting of i.i.d. copies of $U$, independent of $N_\alpha$ and $U$ under $\mathbb{P}$. Thus the first moment of $\Phi_\alpha^2(\delta_\infty)$ is given by

$$\begin{aligned} &\sum_{k\geq 2}\theta_\alpha(k)\int_0^1 \mathrm{d}u\int_0^1 \mathrm{d}u_1\cdots\int_0^1 \mathrm{d}u_k\Big(u+\frac{1-u}{u_1^{-1}+u_2^{-1}+\cdots+u_k^{-1}}\Big)^{-1} \\ = \ &\sum_{k\geq 2}\theta_\alpha(k)\int_0^1 \mathrm{d}u_1\cdots\int_0^1 \mathrm{d}u_k\frac{1}{1-(u_1^{-1}+u_2^{-1}+\cdots+u_k^{-1})^{-1}}\log\Big(\frac{1}{u_1}+\frac{1}{u_2}+\cdots+\frac{1}{u_k}\Big) \\ \leq \ &2\sum_{k\geq 2}\theta_\alpha(k)\int_0^1 \mathrm{d}u_1\cdots\int_0^1 \mathrm{d}u_k\log\Big(\frac{1}{u_1}+\frac{1}{u_2}+\cdots+\frac{1}{u_k}\Big)\,, \end{aligned}$$

in which the integrals can be bounded as follows,

$$\begin{aligned} &\int_0^1 \mathrm{d}u_1\cdots\int_0^1 \mathrm{d}u_k\log\Big(\frac{1}{u_1}+\frac{1}{u_2}+\cdots+\frac{1}{u_k}\Big) \\ &= k!\int_{0<u_1<u_2<\cdots<u_k<1}\mathrm{d}u_1\mathrm{d}u_2\cdots\mathrm{d}u_k\log\Big(\frac{1}{u_1}+\frac{1}{u_2}+\cdots+\frac{1}{u_k}\Big) \\ &= k!\int_{0<u_2<u_3<\cdots<u_k<1}\mathrm{d}u_2\mathrm{d}u_3\cdots\mathrm{d}u_k\bigg[u_2\log\Big(\frac{2}{u_2}+\frac{1}{u_3}+\cdots+\frac{1}{u_k}\Big)+\frac{\log\Big(2+\frac{u_2}{u_3}+\cdots+\frac{u_2}{u_k}\Big)}{u_2^{-1}+u_3^{-1}+\cdots+u_k^{-1}}\bigg] \\ &\leq k!\int_{0<u_2<u_3<\cdots<u_k<1}\mathrm{d}u_2\mathrm{d}u_3\cdots\mathrm{d}u_k\Big[u_2\log\frac{k}{u_2}+\frac{\log k}{k-1}\Big] \\ &= \ \log k+\frac{1}{2}+\cdots+\frac{1}{k}+\frac{k\log k}{k-1} \ \leq \ (2+\frac{k}{k-1})\log k\,. \end{aligned}$$

Using Stirling's formula, we know that $\theta_\alpha(k) = O(k^{-(1+\alpha)})$ as $k \to +\infty$. As

$$\sum_{k\geq 2}(2+\frac{k}{k-1})\frac{\log k}{k^{1+\alpha}} < +\infty$$

for all $\alpha \in [1,2]$, we get $\Phi_\alpha^2(\delta_\infty) \in \mathscr{M}_1$. By monotonicity, we have also $\Phi_\alpha^2(\sigma) \in \mathscr{M}_1$ for every $\sigma \in \mathscr{M}$, and from the preceding results we get $\Phi_\alpha^k(\sigma) \to \tilde{\gamma}_\alpha$ for every $\sigma \in \mathscr{M}$. This implies that $\gamma_\alpha = \tilde{\gamma}_\alpha$ is the unique fixed point of $\Phi_\alpha$ in $\mathscr{M}$.

For every $t \in \mathbb{R}$ we set $F_\alpha(t) = \gamma_\alpha([t,\infty])$. For every integer $k \geq 2$, we write $F_\alpha^{(k)}(t) = \mathbb{P}(\mathcal{C}_1^{(\alpha)} + \mathcal{C}_2^{(\alpha)} + \cdots + \mathcal{C}_k^{(\alpha)} \geq t)$, where $(\mathcal{C}_k^{(\alpha)})_{k\geq 1}$ are independent and identically distributed according to $\gamma_\alpha$. Then we have, for every $t > 1$,

$$\begin{aligned} F_\alpha(t) &= \mathbb{P}\Big(U + \frac{1-U}{\mathcal{C}_1^{(\alpha)} + \mathcal{C}_2^{(\alpha)} + \cdots + \mathcal{C}_{N_\alpha}^{(\alpha)}} \leq t^{-1}\Big) \\ &= \mathbb{P}\left(U < t^{-1} \text{ and } \frac{t-Ut}{1-Ut} \leq \mathcal{C}_1^{(\alpha)} + \mathcal{C}_2^{(\alpha)} + \cdots + \mathcal{C}_{N_\alpha}^{(\alpha)}\right) \\ &= \mathbb{E}\Big[\int_0^{1/t} \mathrm{d}u\, F_\alpha^{(N_\alpha)}\left(\frac{t-ut}{1-ut}\right)\Big] \\ &= \frac{t-1}{t}\int_t^\infty \frac{\mathrm{d}x}{(x-1)^2}\mathbb{E}\Big[F_\alpha^{(N_\alpha)}(x)\Big]. \end{aligned} \tag{2.7}$$

By definition, we have $F_\alpha^{(k)}(t) = 1$ for every $t \in [1,2]$ and $k \geq 2$. It follows from (2.7) that

$$F_\alpha(t) = \frac{D^{(\alpha)}}{t} + 1 - D^{(\alpha)}, \qquad \forall t \in [1,2], \tag{2.8}$$

where

$$D^{(\alpha)} = 2 - \int_2^\infty \frac{\mathrm{d}x}{(x-1)^2}\mathbb{E}\Big[F_\alpha^{(N_\alpha)}(x)\Big] \in [1,2].$$

We observe that the right-hand side of (2.7) is a continuous function of $t \in (1,\infty)$, so that $F_\alpha$ is continuous on $[1,\infty)$ (the right-continuity at $1$ is obvious from (2.8)). Thus $\gamma_\alpha$ has no atom and it follows that all functions $F_\alpha^{(k)}, k \geq 2$ are continuous on $[1,\infty)$. By dominated convergence the function $x \mapsto \mathbb{E}[F_\alpha^{(N_\alpha)}(x)]$ is also continuous on $[1,\infty)$. Using (2.7) again we obtain that $F_\alpha$ is continuously differentiable on $[1,\infty)$ and consequently $\gamma_\alpha$ has a continuous density $f_\alpha = -F'_\alpha$ with respect to the Lebesgue measure on $[1,\infty)$.

Let us finally derive the differential equation (2.6). To this end, we first differentiate (2.7) with respect to $t$ to get that the linear differential equation

$$t(t-1)F'_\alpha(t) - F_\alpha(t) = -\mathbb{E}\big[F_\alpha^{(N_\alpha)}(t)\big] \tag{2.9}$$

holds for $t \in [1,\infty)$. Then let $g\colon [1,\infty) \to \mathbb{R}_+$ be a monotone continuously differentiable function. From the definition of $F_\alpha$ and Fubini's theorem, we have

$$\int_1^\infty \mathrm{d}t\, g'(t)F_\alpha(t) = \mathbb{E}\big[g(\mathcal{C}^{(\alpha)})\big] - g(1)$$

and similarly

$$\int_1^\infty \mathrm{d}t\, g'(t)\mathbb{E}\big[F_\alpha^{(N_\alpha)}(t)\big] = \mathbb{E}\big[g(\mathcal{C}_1^{(\alpha)} + \mathcal{C}_2^{(\alpha)} + \cdots + \mathcal{C}_{N_\alpha}^{(\alpha)})\big] - g(1).$$

We then multiply both sides of (2.9) by $g'(t)$ and integrate for $t$ running from $1$ to $\infty$ to get

$$\mathbb{E}\big[\mathcal{C}_1^{(\alpha)}(\mathcal{C}_1^{(\alpha)} - 1)g'(\mathcal{C}_1^{(\alpha)})\big] + \mathbb{E}\big[g(\mathcal{C}_1^{(\alpha)})\big] = \mathbb{E}\big[g(\mathcal{C}_1^{(\alpha)} + \mathcal{C}_2^{(\alpha)} + \cdots + \mathcal{C}_{N_\alpha}^{(\alpha)})\big]. \tag{2.10}$$

When $g(x) = \exp(-x\ell/2)$ for $\ell > 0$, we readily obtain (2.6) by using the generating function of $N_\alpha$ given in (2.1). Finally, taking $g(x) = x$ in (2.10), we get

$$\mathbb{E}\big[(\mathcal{C}^{(\alpha)})^2\big] = \mathbb{E}\big[N_\alpha\big]\mathbb{E}\big[\mathcal{C}^{(\alpha)}\big] = \frac{\alpha}{\alpha-1}\mathbb{E}\big[\mathcal{C}^{(\alpha)}\big]\,.$$

Nevertheless, by taking $g(x) = x^2$ in (2.10), we see that the third moment of $\mathcal{C}^{(\alpha)}$ is infinite since $\mathbb{E}\big[(N_\alpha)^2\big] = \infty$. $\square$

The arguments of the preceding proof also yield the following lemma in the case $\alpha = 1$.

**Lemma 2.2.** *The conductance $\mathcal{C}^{(1)}$ of the tree $\Delta^{(1)}$ satisfies the bound*

$$\mathbb{E}\big[\mathcal{C}^{(1)}\big] \leq 2\sum_{k\geq 2}(2+\frac{k}{k-1})\frac{\log k}{k(k-1)} < +\infty. \tag{2.11}$$

*Additionally, the Laplace transform $\varphi_1$ of the law of $\mathcal{C}^{(1)}$ solves the differential equation*

$$2\ell\,\varphi''(\ell) + \ell\varphi'(\ell) + (1-\varphi(\ell))\log(1-\varphi(\ell)) = 0.$$

*Proof.* The law of $\mathcal{C}^{(1)}$ is a fixed point of the mapping $\Phi_1$ defined via (2.5) with $\alpha = 1$. By the same monotonicity argument that we used above, it follows that the first moment of $\mathcal{C}^{(1)}$ is bounded above by the first moment of $\Phi_1^2(\delta_\infty)$, and the calculation of this first moment in the previous proof leads to the right-hand side of (2.11).

As an analogue to (2.10), we have

$$\mathbb{E}\big[\mathcal{C}_1^{(1)}(\mathcal{C}_1^{(1)}-1)g'(\mathcal{C}_1^{(1)})\big] + \mathbb{E}\big[g(\mathcal{C}_1^{(1)})\big] = \mathbb{E}\big[g(\mathcal{C}_1^{(1)}+\mathcal{C}_2^{(1)}+\cdots+\mathcal{C}_{N_1}^{(1)})\big].$$

By taking $g(x) = \exp(-x\ell/2)$ and using (2.2), one can then derive the differential equation satisfied by $\varphi_1$. $\square$

## 2.4 The reduced stable trees are nested

In this short subsection, we introduce a coupling argument to explain how Theorem 1.3 follows from the identity (1.5) in Proposition 1.4.

Recall the definition of the $\alpha$-offspring distribution $\theta_\alpha$. From the obvious fact

$$1 - \sum_{i=2}^{k-1}\frac{\alpha}{i-\alpha} < 0, \qquad \forall \alpha \in (1,2), k \geq 3,$$

one deduces that for all $k \geq 3$,

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\theta_\alpha(k) < 0, \qquad \forall \alpha \in (1,2).$$

This implies that for every $k \geq 3$, $\theta_\alpha([2,k])$ is a strictly increasing function of $\alpha \in (1,2)$. Using the inverse transform sampling, we can construct on a common probability space a sequence of random variables $(N_\alpha, \alpha \in [1,2])$ such that a.s.

$$N_{\alpha_2} \geq N_{\alpha_1} \quad \text{for all } 1 \leq \alpha_2 \leq \alpha_1 \leq 2.$$

Then following the same procedure explained in Section 2.1, we can construct simultaneously all reduced stable trees as a nested family. More precisely, there exists a family of compact $\mathbb{R}$-trees $(\bar{\Delta}^{(\alpha)}, \alpha \in [1,2])$ such that

$$\begin{aligned} \bar{\Delta}^{(\alpha)} &\overset{(\mathrm{d})}{=} \Delta^{(\alpha)} \quad \text{for all } 1 \leq \alpha \leq 2\,; \\ \bar{\Delta}^{(\alpha_1)} &\subseteq \bar{\Delta}^{(\alpha_2)} \quad \text{for all } 1 \leq \alpha_2 \leq \alpha_1 \leq 2\,. \end{aligned}$$

Consequently, the family of conductances $(\bar{\mathcal{C}}^{(\alpha)}, \alpha \in [1,2])$ associated with $(\bar{\Delta}^{(\alpha)}, \alpha \in [1,2])$ is decreasing with respect to $\alpha$. In particular, the mean $\mathbb{E}[\mathcal{C}^{(\alpha)}]$ is decreasing with respect to $\alpha$, and it follows from (2.11) that $(\mathbb{E}[\mathcal{C}^{(\alpha)}], \alpha \in [1,2])$ is uniformly bounded by the constant

$$C_0 := 2\sum_{k\geq 2}(2+\frac{k}{k-1})\frac{\log k}{k(k-1)} < +\infty.$$

**Proof of Theorem 1.3.** For any $\alpha \in (1,2]$, $\gamma_\alpha$ is a probability measure on $[1,\infty)$ and

$$\begin{aligned}\iint \gamma_\alpha(\mathrm{d}s)\gamma_\alpha(\mathrm{d}t)\frac{st}{s+t-1} &\geq \iint \gamma_\alpha(\mathrm{d}s)\gamma_\alpha(\mathrm{d}t)\frac{st}{s+t} \\ &\geq \iint \gamma_\alpha(\mathrm{d}s)\gamma_\alpha(\mathrm{d}t)\frac{st}{2(s\vee t)} \\ &= \frac{1}{2}\iint \gamma_\alpha(\mathrm{d}s)\gamma_\alpha(\mathrm{d}t)(s\wedge t) \;\geq\; \frac{1}{2}\,.\end{aligned}$$

So we derive from (1.5) that

$$\beta_\alpha \leq \frac{1}{2}\Big(2\big(\mathbb{E}\big[\mathcal{C}^{(\alpha)}\big]\big)^2 - 1\Big) \leq \frac{1}{2}\big(2\,C_0^2 - 1\big) < \infty.$$

□

### 2.5 Proof of Theorem 1.2

The proof of Theorem 1.2 given below will follow the approach sketched in [5, Section 5.1]. We will first establish the flow property of harmonic measure (Lemma 2.3), and then find an explicit invariant measure for the environment seen by Brownian motion on the CTGW tree $\Gamma^{(\alpha)}$ at the last visit of a vertex of the $n$-th generation (Proposition 2.4). After that, we will rely on arguments of ergodic theory to complete the proof of Theorem 1.2 and that of Proposition 1.4.

**Throughout this subsection, we fix the stable index** $\alpha \in (1,2]$ **once and for all**. For notational ease, we will omit the superscripts and subscripts concerning $\alpha$ in all the proofs involved. Recall that $\mathbb{P}$ stands for the probability measure under which the CTGW tree $\Gamma^{(\alpha)}$ is defined, whereas Brownian motion with drift $1/2$ on the CTGW tree is defined under the probability measure $P$.

#### 2.5.1 The flow property of harmonic measure

We fix an infinite continuous tree $\mathcal{T} \in \mathbb{T}$, and write as before $\mathcal{T}_{(1)}, \mathcal{T}_{(2)}, \ldots, \mathcal{T}_{(k_\varnothing)}$ for the subtrees of $\mathcal{T}$ at the first branching point. Here we slightly abuse notation by writing $W = (W(t))_{t\geq 0}$ for Brownian motion with drift $1/2$ on $\mathcal{T}$ started from the root. As in Section 2.2, $W_\infty$ stands for the exit ray of $W$, and the distribution of $W_\infty$ on the boundary of $\mathcal{T}$ is the harmonic measure of $\mathcal{T}$, denoted as $\nu_{\mathcal{T}}$. Let $K$ be the index such that $W_\infty$ "belongs to" $\mathcal{T}_{(K)}$ and we write $W'_\infty$ for the ray of $\mathcal{T}_{(K)}$ obtained by shifting $W_\infty$ at the first branching point of $\mathcal{T}$.

**Lemma 2.3.** *Let $j \in \{1,2,\ldots,k_\varnothing\}$. Conditionally on $\{K = j\}$, the law of $W'_\infty$ is the harmonic measure of $\mathcal{T}_{(j)}$.*

The proof is similar to that of [5, Lemma 7] and is therefore omitted.

#### 2.5.2 The invariant measure and ergodicity

We introduce the set

$$\mathbb{T}^* \subseteq \mathbb{T}\times\mathbb{N}^{\mathbb{N}}$$

of all pairs consisting of a tree $\mathcal{T} \in \mathbb{T}$ and a distinguished geodesic ray $\mathbf{v}$ in $\mathcal{T}$. Given a distinguished geodesic ray $\mathbf{v} = (v_1, v_2, \ldots)$ in $\mathcal{T}$, we let $S(\mathcal{T},\mathbf{v})$ be obtained by shifting $(\mathcal{T},\mathbf{v})$ at the first branching point of $\mathcal{T}$, that is

$$S(\mathcal{T},\mathbf{v}) = (\mathcal{T}_{(v_1)}, \widetilde{\mathbf{v}}),$$

where $\widetilde{\mathbf{v}} = (v_2, v_3, \ldots)$ and $\mathcal{T}_{(v_1)}$ is the subtree of $\mathcal{T}$ rooted at the first branching point that is chosen by $\mathbf{v}$.

Under the probability measure $\mathbb{P} \otimes P$, we can view $(\Gamma^{(\alpha)}, W_\infty)$ as a random variable with values in $\mathbb{T}^*$. We write $\Theta^*_\alpha(\mathrm{d}\mathcal{T}\mathrm{d}\mathbf{v})$ for the distribution of $(\Gamma^{(\alpha)}, W_\infty)$. The next proposition gives an invariant measure absolutely continuous with respect to $\Theta^*_\alpha$ under the shift $S$.

**Proposition 2.4.** *For every $r \geq 1$, set*

$$\kappa_\alpha(r) := \sum_{k=2}^{\infty} k\theta_\alpha(k) \int \gamma_\alpha(\mathrm{d}t_1) \int \gamma_\alpha(\mathrm{d}t_2) \cdots \int \gamma_\alpha(\mathrm{d}t_k)\, \frac{rt_1}{r + t_1 + t_2 + \cdots + t_k - 1}.$$

*The finite measure $\kappa_\alpha(\mathcal{C}(\mathcal{T}))\Theta^*_\alpha(\mathrm{d}\mathcal{T}\mathrm{d}\mathbf{v})$ is invariant under $S$.*

**Remark 2.5.** The preceding formula for $\kappa_\alpha$ is suggested by the analogous formula in [5, Proposition 25] for $\alpha = 2$.

*Proof.* First notice that the function $\kappa$ is bounded, since for every $r \geq 1$,

$$\kappa(r) \leq \sum_{k=2}^{\infty} k\theta(k) \int t_1 \gamma(\mathrm{d}t_1) < \infty.$$

Let us fix $\mathcal{T} \in \mathbb{T}$, then for any $1 \leq i \leq k_\varnothing$ and any bounded measurable function $g$ on $\mathbb{N}^{\mathbb{N}}$, the flow property of harmonic measure gives that

$$\int \nu_{\mathcal{T}}(\mathrm{d}\mathbf{v})\, \mathbf{1}_{\{v_1 = i\}}\, g(\widetilde{\mathbf{v}}) = \frac{\mathcal{C}(\mathcal{T}_{(i)})}{\mathcal{C}(\mathcal{T}_{(1)}) + \cdots + \mathcal{C}(\mathcal{T}_{(k_\varnothing)})} \int \nu_{\mathcal{T}_{(i)}}(\mathrm{d}\mathbf{u})\, g(\mathbf{u}).$$

Recall that $\Theta^*(\mathrm{d}\mathcal{T}\,\mathrm{d}\mathbf{v}) = \Theta(\mathrm{d}\mathcal{T})\nu_{\mathcal{T}}(\mathrm{d}\mathbf{v})$ by construction. Let $F$ be a bounded measurable function on $\mathbb{T}^*$. Using the preceding display, we have

$$\begin{aligned} &\int F \circ S(\mathcal{T}, \mathbf{v})\, \kappa(\mathcal{C}(\mathcal{T}))\, \Theta^*(\mathrm{d}\mathcal{T}\,\mathrm{d}\mathbf{v}) \qquad (2.12)\\ &= \sum_{k=2}^{\infty} \theta(k) \sum_{i=1}^{k} \int F(\mathcal{T}_{(i)}, \mathbf{u}) \kappa(\mathcal{C}(\mathcal{T})) \frac{\mathcal{C}(\mathcal{T}_{(i)})}{\mathcal{C}(\mathcal{T}_{(1)}) + \cdots + \mathcal{C}(\mathcal{T}_{(k)})}\, \Theta(\mathrm{d}\mathcal{T}\,|\, k_\varnothing = k)\, \nu_{\mathcal{T}_{(i)}}(\mathrm{d}\mathbf{u}). \end{aligned}$$

Observe that under $\Theta(\mathrm{d}\mathcal{T}\,|\,k_\varnothing = k)$, the subtrees $\mathcal{T}_{(1)}, \mathcal{T}_{(2)}, \ldots, \mathcal{T}_{(k)}$ are independent and distributed according to $\Theta$, and furthermore,

$$\mathcal{C}(\mathcal{T}) = \Big(U + \frac{1 - U}{\mathcal{C}(\mathcal{T}_{(1)}) + \cdots + \mathcal{C}(\mathcal{T}_{(k)})}\Big)^{-1},$$

where $U$ is uniformly distributed over $[0,1]$ and independent of $(\mathcal{T}_{(1)}, \mathcal{T}_{(2)}, \ldots, \mathcal{T}_{(k)})$. Using these observations, together with a simple symmetry argument, we get that the integral (2.12) is given by

$$\begin{aligned} &\sum_{k=2}^{\infty} k\theta(k) \int_0^1 \mathrm{d}x \int \Theta(\mathrm{d}\mathcal{T}_1) \cdots \int \Theta(\mathrm{d}\mathcal{T}_k) \int \nu_{\mathcal{T}_1}(\mathrm{d}\mathbf{u})\, F(\mathcal{T}_1, \mathbf{u}) \\ &\qquad\qquad \times \frac{\mathcal{C}(\mathcal{T}_1)}{\mathcal{C}(\mathcal{T}_1) + \cdots + \mathcal{C}(\mathcal{T}_k)}\, \kappa\Big(\Big(x + \frac{1 - x}{\mathcal{C}(\mathcal{T}_1) + \cdots + \mathcal{C}(\mathcal{T}_k)}\Big)^{-1}\Big) \\ &= \int \Theta^*(\mathrm{d}\mathcal{T}_1\,\mathrm{d}\mathbf{u})\, F(\mathcal{T}_1, \mathbf{u}) \Big[ \sum_{k=2}^{\infty} k\theta(k) \int_0^1 \mathrm{d}x \int \Theta(\mathrm{d}\mathcal{T}_2) \cdots \int \Theta(\mathrm{d}\mathcal{T}_k) \\ &\qquad\qquad \times \frac{\mathcal{C}(\mathcal{T}_1)}{\mathcal{C}(\mathcal{T}_1) + \cdots + \mathcal{C}(\mathcal{T}_k)}\, \kappa\Big(\Big(x + \frac{1 - x}{\mathcal{C}(\mathcal{T}_1) + \cdots + \mathcal{C}(\mathcal{T}_k)}\Big)^{-1}\Big)\Big]. \end{aligned}$$

The proof is thus reduced to checking that, for every $r \geq 1$, $\kappa(r)$ is equal to

$$\sum_{k=2}^{\infty} k\theta(k) \int_0^1 \mathrm{d}x \int \Theta(\mathrm{d}\mathcal{T}_2) \cdots \int \Theta(\mathrm{d}\mathcal{T}_k) \frac{r}{r+\mathcal{C}(\mathcal{T}_2)+\cdots+\mathcal{C}(\mathcal{T}_k)} \kappa\Big(\big(x+\frac{1-x}{r+\mathcal{C}(\mathcal{T}_2)+\cdots+\mathcal{C}(\mathcal{T}_k)}\big)^{-1}\Big). \tag{2.13}$$

To this end, we will reformulate the last expression in the following way. Under the probability measure $\mathbb{P}$, we introduce an i.i.d. sequence $(\mathcal{C}_i)_{i\geq 0}$ distributed according to $\gamma$, and a random variable $N$ distributed according to $\theta$. In addition, under the same probability measure $\mathbb{P}$, let $U$ be uniformly distributed over $[0,1]$, $(\widetilde{\mathcal{C}}_i)_{i\geq 0}$ be an independent copy of $(\mathcal{C}_i)_{i\geq 0}$, and $\widetilde{N}$ be an independent copy of $N$. We assume that all these random variables are independent. Note that by definition, for every $r \geq 1$,

$$\kappa(r) = \mathbb{E}\Big[\frac{r\widetilde{N}\widetilde{\mathcal{C}}_1}{r+\widetilde{\mathcal{C}}_1+\widetilde{\mathcal{C}}_2+\cdots+\widetilde{\mathcal{C}}_{\widetilde{N}}-1}\Big].$$

It follows that (2.13) can be written as

$$\begin{aligned}
&\sum_{k=2}^{\infty} k\theta(k)\mathbb{E}\Bigg[\frac{r}{r+\mathcal{C}_2+\cdots+\mathcal{C}_k}\,\frac{\Big(U+\frac{1-U}{r+\mathcal{C}_2+\cdots+\mathcal{C}_k}\Big)^{-1}\widetilde{N}\widetilde{\mathcal{C}}_1}{\Big(U+\frac{1-U}{r+\mathcal{C}_2+\cdots+\mathcal{C}_k}\Big)^{-1}+\widetilde{\mathcal{C}}_1+\widetilde{\mathcal{C}}_2+\cdots+\widetilde{\mathcal{C}}_{\widetilde{N}}-1}\Bigg]\\
&= r\sum_{k=2}^{\infty} k\theta(k)\mathbb{E}\Bigg[\frac{\widetilde{N}\widetilde{\mathcal{C}}_1}{(r+\mathcal{C}_2+\cdots+\mathcal{C}_k)\big(1+(\widetilde{\mathcal{C}}_1+\widetilde{\mathcal{C}}_2+\cdots+\widetilde{\mathcal{C}}_{\widetilde{N}}-1)(U+\frac{1-U}{r+\mathcal{C}_2+\cdots+\mathcal{C}_k})\big)}\Bigg]\\
&= r\sum_{k=2}^{\infty} k\theta(k)\mathbb{E}\Bigg[\frac{\widetilde{\mathcal{C}}_1+\widetilde{\mathcal{C}}_2+\cdots+\widetilde{\mathcal{C}}_{\widetilde{N}}}{(\widetilde{\mathcal{C}}_1+\widetilde{\mathcal{C}}_2+\cdots+\widetilde{\mathcal{C}}_{\widetilde{N}}-1)(U(r+\mathcal{C}_2+\cdots+\mathcal{C}_k)+1-U)+r+\mathcal{C}_2+\cdots+\mathcal{C}_k}\Bigg]\\
&= r\sum_{k=2}^{\infty} k\theta(k)\mathbb{E}\Bigg[\frac{\widetilde{\mathcal{C}}_1+\widetilde{\mathcal{C}}_2+\cdots+\widetilde{\mathcal{C}}_{\widetilde{N}}}{(\widetilde{\mathcal{C}}_1+\widetilde{\mathcal{C}}_2+\cdots+\widetilde{\mathcal{C}}_{\widetilde{N}})(U(r+\mathcal{C}_2+\cdots+\mathcal{C}_k-1)+1)+(r+\mathcal{C}_2+\cdots+\mathcal{C}_k-1)(1-U)}\Bigg]\\
&= r\sum_{k=2}^{\infty} k\theta(k)\mathbb{E}\Bigg[\frac{1}{(r+\mathcal{C}_2+\cdots+\mathcal{C}_k-1)\big(U+\frac{1-U}{\widetilde{\mathcal{C}}_1+\widetilde{\mathcal{C}}_2+\cdots+\widetilde{\mathcal{C}}_{\widetilde{N}}}\big)+1}\Bigg]\\
&= r\sum_{k=2}^{\infty} k\theta(k)\mathbb{E}\Bigg[\frac{\widetilde{\mathcal{C}}}{r+\widetilde{\mathcal{C}}+\mathcal{C}_2+\cdots+\mathcal{C}_k-1}\Bigg] \;=\; \mathbb{E}\Bigg[\frac{rN\widetilde{\mathcal{C}}}{r+\widetilde{\mathcal{C}}+\mathcal{C}_2+\cdots+\mathcal{C}_N-1}\Bigg],
\end{aligned}$$

where

$$\widetilde{\mathcal{C}} := (U+\frac{1-U}{\widetilde{\mathcal{C}}_1+\cdots+\widetilde{\mathcal{C}}_{\widetilde{N}}})^{-1}$$

is independent of $(\mathcal{C}_i)_{i\geq 0}$ and $N$. By (1.4), the random variable $\widetilde{\mathcal{C}}$ is also distributed according to $\gamma$. So the right-hand side of the last long display is equal to $\kappa(r)$, which completes the proof of the proposition. □

We normalize $\kappa_\alpha$ by setting

$$\widehat{\kappa}_\alpha(r) = \frac{\kappa_\alpha(r)}{\int \kappa_\alpha(\mathcal{C}(\mathcal{T}))\Theta^*_\alpha(\mathrm{d}\mathcal{T}\mathrm{d}\mathbf{v})} = \frac{\kappa_\alpha(r)}{\int \kappa_\alpha(\mathcal{C}(\mathcal{T}))\Theta_\alpha(\mathrm{d}\mathcal{T})}$$

for every $r \geq 1$. Then $\widehat{\kappa}_\alpha(\mathcal{C}(\mathcal{T}))\Theta^*_\alpha(\mathrm{d}\mathcal{T}\mathrm{d}\mathbf{v})$ is a probability measure on $\mathbb{T}^*$ invariant under the shift $S$. To simplify notation, we set $\Upsilon^*_\alpha(\mathrm{d}\mathcal{T}\mathrm{d}\mathbf{v}) := \widehat{\kappa}_\alpha(\mathcal{C}(\mathcal{T}))\Theta^*_\alpha(\mathrm{d}\mathcal{T}\mathrm{d}\mathbf{v})$. Let $\pi_1$ be the canonical projection from $\mathbb{T}^*$ onto $\mathbb{T}$. The image of $\Upsilon^*_\alpha$ under this projection is the probability measure $\Upsilon_\alpha(\mathrm{d}\mathcal{T}) := \widehat{\kappa}_\alpha(\mathcal{C}(\mathcal{T}))\Theta_\alpha(\mathrm{d}\mathcal{T})$.

**Proposition 2.6.** *The shift $S$ acting on the probability space $(\mathbb{T}^*, \Upsilon^*_\alpha)$ is ergodic.*

*Proof.* Our arguments proceed in a similar way as in the proof of [5, Proposition 13]. We define a transition kernel $\mathbf{p}(\mathcal{T},\mathrm{d}\mathcal{T}')$ on $\mathbb{T}$ by setting

$$\mathbf{p}(\mathcal{T},\mathrm{d}\mathcal{T}') = \sum_{i=1}^{k_\varnothing} \frac{\mathcal{C}(\mathcal{T}_{(i)})}{\mathcal{C}(\mathcal{T}_{(1)})+\cdots+\mathcal{C}(\mathcal{T}_{(k_\varnothing)})}\,\delta_{\mathcal{T}_{(i)}}(\mathrm{d}\mathcal{T}').$$

Informally, under the probability measure $\mathbf{p}(\mathcal{T},\mathrm{d}\mathcal{T}')$, we choose one of the subtrees of $\mathcal{T}$ obtained at the first branching point, with probability equal to its harmonic measure.

For every integer $n\geq 1$, we denote by $S^n$ the mapping on $\mathbb{T}^*$ obtained by iterating $n$ times the shift $S$, and then we consider the process $(Z_n)_{n\geq 0}$ on the probability space $(\mathbb{T}^*,\Upsilon^*)$ with values in $\mathbb{T}$, defined by $Z_0(\mathcal{T},\mathbf{v})=\mathcal{T}$ and

$$Z_n(\mathcal{T},\mathbf{v}) = \pi_1\big(S^n(\mathcal{T},\mathbf{v})\big)$$

for every $n\geq 1$. According to Proposition 2.4 and the flow property of harmonic measure, the process $(Z_n)_{n\geq 0}$ is a Markov chain with transition kernel $\mathbf{p}$ under its stationary measure $\Upsilon(\mathrm{d}\mathcal{T})$.

We write $\mathbb{T}^\infty$ for the set of all infinite sequences $(\mathcal{T}^0,\mathcal{T}^1,\ldots)$ of elements in $\mathbb{T}$, and let $\widehat{\mathbb{T}}^\infty$ be the set of all infinite sequences $(\mathcal{T}^0,\mathcal{T}^1,\ldots)$ in $\mathbb{T}^\infty$, such that, for every integer $j\geq 1$, $\mathcal{T}^j$ is one of the subtrees of $\mathcal{T}^{j-1}$ above the first branching point of $\mathcal{T}^{j-1}$. Note that $\widehat{\mathbb{T}}^\infty$ is a measurable subset of $\mathbb{T}^\infty$ and that $(Z_n(\mathcal{T},\mathbf{v}))_{n\geq 0}\in\widehat{\mathbb{T}}^\infty$ for every $(\mathcal{T},\mathbf{v})\in\mathbb{T}^*$. If $(\mathcal{T}^0,\mathcal{T}^1,\ldots)\in\widehat{\mathbb{T}}^\infty$, there exists a geodesic ray $\mathbf{v}$ in $\mathcal{T}^0$ such that $\mathcal{T}^j=S^j(\mathcal{T}^0,\mathbf{v})$ for every $j\geq 1$, and we set $\phi(\mathcal{T}^0,\mathcal{T}^1,\ldots):=(\mathcal{T}^0,\mathbf{v})$. Notice that $\mathbf{v}$ is a priori not unique, but to make the previous definition rigorous we can take the smallest possible $\mathbf{v}$ in lexicographical ordering (of course for the random trees that we consider later this uniqueness problem does not arise). In this way, we define a measurable mapping $\phi$ from $\widehat{\mathbb{T}}^\infty$ into $\mathbb{T}^*$ such that

$$\phi(Z_0(\mathcal{T},\mathbf{v}),Z_1(\mathcal{T},\mathbf{v}),\ldots)=(\mathcal{T},\mathbf{v}),\qquad \Upsilon^*\text{-a.s.} \tag{2.14}$$

Now given a measurable subset $A$ of $\mathbb{T}^*$ such that $S^{-1}(A)=A$, we aim at proving that $\Upsilon^*(A)\in\{0,1\}$. To this end, we consider the pre-image $B=\phi^{-1}(A)$, which is a measurable subset of $\widehat{\mathbb{T}}^\infty\subset\mathbb{T}^\infty$. Due to the previous constructions, $B$ is shift-invariant for the Markov chain $Z$ in the sense that

$$\{(Z_0,Z_1,\ldots)\in B\}=\{(Z_1,Z_2,\ldots)\in B\},\quad \text{a.s.}$$

Using Proposition 16.2 in [14], we then obtain a measurable subset $D$ of $\mathbb{T}$, such that

$$\mathbf{1}_B(Z_0,Z_1,\ldots)=\mathbf{1}_D(Z_0)\quad\text{a.s.},$$

and moreover $\mathbf{p}(\mathcal{T},D)=\mathbf{1}_D(\mathcal{T})$, $\Upsilon(\mathrm{d}\mathcal{T})$-a.s. It follows thus from (2.14) that $\Upsilon^*$-a.s. we have $(\mathcal{T},\mathbf{v})\in A$ if and only if $\mathcal{T}\in D$.

However from the property $\mathbf{p}(\mathcal{T},D)=\mathbf{1}_D(\mathcal{T})$, $\Upsilon(\mathrm{d}\mathcal{T})$-a.s., one can verify that $\Upsilon(D)\in\{0,1\}$. First note that this property also implies that $\mathbf{p}(\mathcal{T},D)=\mathbf{1}_D(\mathcal{T})$, $\Theta(\mathrm{d}\mathcal{T})$-a.s. Hence, $\Theta(\mathrm{d}\mathcal{T})$-a.s., the tree $\mathcal{T}$ belongs to $D$ if and only if each of its subtrees above the first branching point belongs to $D$ (it is clear that that the measure $\mathbf{p}(\mathcal{T},\cdot)$ assigns a positive mass to each of these subtrees). Then, the branching property of the CTGW tree shows that

$$\Theta(D)=\sum_{k=2}^{\infty}\theta(k)\,\Theta(D)^k$$

which is only possible if $\Theta(D)=0$ or $1$, or equivalently if $\Upsilon(D)=0$ or $1$. Therefore $\Upsilon^*(A)$ is either 0 or 1, which completes the proof. □

### 2.5.3 Proof of Theorem 1.2

Having established Proposition 2.4 and Proposition 2.6, we can now apply the ergodic theorem to the two functionals on $\mathbb{T}^*$ defined as follows. First let $J_n(\mathcal{T},\mathbf{v})$ denote the height of the $n$-th branching point on the geodesic ray $\mathbf{v}$. One immediately verifies that, for every $n \geq 1$,

$$J_n = \sum_{i=0}^{n-1} J_1 \circ S^i.$$

If $M = \int \kappa(\mathcal{C}(\mathcal{T}))\Theta^*(\mathrm{d}\mathcal{T}\mathrm{d}\mathbf{v})$, it follows from the ergodic theorem that $\Theta^*$-a.s.,

$$\frac{1}{n} J_n \underset{n\to\infty}{\longrightarrow} M^{-1} \int J_1(T,\mathbf{v})\kappa(\mathcal{C}(\mathcal{T}))\Theta^*(\mathrm{d}\mathcal{T}\mathrm{d}\mathbf{v}). \tag{2.15}$$

Note that the limit can be written as

$$M^{-1}\mathbb{E}\Big[|\log(1-U)|\,\kappa\Big(\Big(U + \frac{1-U}{\mathcal{C}_1+\cdots+\mathcal{C}_N}\Big)^{-1}\Big)\Big]$$

with the notation used in the proof of Proposition 2.4.

Secondly, let $\mathbf{x}_{n,\mathbf{v}}$ denote the $n$+1-st branching point on the geodesic ray $\mathbf{v}$. If $\mathbf{v} = (v_1, v_2, \ldots)$, then $\mathbf{x}_{n,\mathbf{v}} = ((v_1,\ldots,v_n), J_{n+1}(\mathcal{T},\mathbf{v}))$ with the notation of Section 2.2. We set for every $n \geq 1$,

$$F_n(\mathcal{T},\mathbf{v}) := \log \nu_{\mathcal{T}}(\{\mathbf{u} \in \partial\mathcal{T} \colon \mathbf{x}_{n,\mathbf{v}} \prec \mathbf{u}\}).$$

By the flow property of harmonic measure (Lemma 2.3), we have

$$F_n = \sum_{i=0}^{n-1} F_1 \circ S^i,$$

and by the ergodic theorem, $\Theta^*$-a.s.,

$$\frac{1}{n} F_n \underset{n\to\infty}{\longrightarrow} M^{-1} \int F_1(\mathcal{T},\mathbf{v})\kappa(\mathcal{C}(\mathcal{T}))\Theta^*(\mathrm{d}\mathcal{T}\mathrm{d}\mathbf{v}), \tag{2.16}$$

where the limit can be written as

$$M^{-1}\mathbb{E}\Big[\frac{N\mathcal{C}_1}{\mathcal{C}_1+\cdots+\mathcal{C}_N}\,\log\Big(\frac{\mathcal{C}_1}{\mathcal{C}_1+\cdots+\mathcal{C}_N}\Big)\,\kappa\Big(\Big(U + \frac{1-U}{\mathcal{C}_1+\cdots+\mathcal{C}_N}\Big)^{-1}\Big)\Big].$$

By combining (2.15) and (2.16), we obtain that the convergence (2.4) holds with limit

$$-\beta = \frac{\mathbb{E}\Big[\frac{N\mathcal{C}_1}{\mathcal{C}_1+\cdots+\mathcal{C}_N}\,\log\Big(\frac{\mathcal{C}_1}{\mathcal{C}_1+\cdots+\mathcal{C}_N}\Big)\,\kappa\Big(\Big(U + \frac{1-U}{\mathcal{C}_1+\cdots+\mathcal{C}_N}\Big)^{-1}\Big)\Big]}{\mathbb{E}\Big[|\log(1-U)|\,\kappa\Big(\Big(U + \frac{1-U}{\mathcal{C}_1+\cdots+\mathcal{C}_N}\Big)^{-1}\Big)\Big]}.$$

**Proposition 2.7.** *We have $\beta < \frac{1}{\alpha-1}$.*

*Proof.* We use the notation

$$\mathcal{W}(\mathcal{T}) = \lim_{r\to\infty} e^{-\frac{r}{\alpha-1}}\,\#\mathcal{T}_r\,,$$

which exists $\Theta(\mathrm{d}\mathcal{T})$-a.s. by a martingale argument. Since $\sum \theta(k)k\log k < \infty$, the Kesten–Stigum theorem (for CTGW trees, see e.g. [2, Theorem III.7.2]) implies that the previous convergence holds in the $L^1$-sense and $\int \mathcal{W}(\mathcal{T})\Theta(\mathrm{d}\mathcal{T}) = 1$. Moreover, $\Theta(\mathcal{W}(\mathcal{T}) = 0) = 0$ and the Laplace transform

$$\int e^{-u\mathcal{W}(\mathcal{T})}\Theta(\mathrm{d}\mathcal{T}) = 1 - \frac{u}{(1+u^{\alpha-1})^{\frac{1}{\alpha-1}}} \quad \text{for any } u \in (0,\infty)$$

can be obtained by applying Theorem III.8.3 in [2] together with (2.1). In particular, it follows from a Tauberian theorem (cf. [8, Chapter XIII.5]) that $\int |\log \mathcal{W}(\mathcal{T})|\Theta(\mathrm{d}\mathcal{T}) < \infty$.

Let $\mathcal{T}_{(1)}, \ldots, \mathcal{T}_{(k_\varnothing)}$ be the subtrees of $\mathcal{T}$ at the first branching point, and let $J(\mathcal{T}) = J_1(\mathcal{T}, \mathbf{v})$ be the height of the first branching point. Then, $\Theta(\mathrm{d}\mathcal{T})$-a.s.

$$\mathcal{W}(\mathcal{T}) = e^{-\frac{J(\mathcal{T})}{\alpha-1}}\big(\mathcal{W}(\mathcal{T}_{(1)}) + \cdots + \mathcal{W}(\mathcal{T}_{(k_\varnothing)})\big),$$

so that we can define a probability measure $w_\mathcal{T}$ on $\{1, 2, \ldots, k_\varnothing\}$ by setting

$$w_\mathcal{T}(i) = \frac{e^{-\frac{J(\mathcal{T})}{\alpha-1}}\mathcal{W}(\mathcal{T}_{(i)})}{\mathcal{W}(\mathcal{T})}, \qquad 1 \le i \le k_\varnothing.$$

On the other hand, for $1 \le i \le k_\varnothing$, let $\nu^*_\mathcal{T}(i)$ denote the mass assigned by the harmonic measure $\nu_\mathcal{T}$ to the rays "contained" in $\mathcal{T}_{(i)}$, that is,

$$\nu^*_\mathcal{T}(i) = \int \mathbf{1}_{\{v_1=i\}}\nu_\mathcal{T}(\mathrm{d}\mathbf{v}) = \frac{\mathcal{C}(\mathcal{T}_{(i)})}{\mathcal{C}(\mathcal{T}_{(1)}) + \cdots + \mathcal{C}(\mathcal{T}_{(k_\varnothing)})}.$$

By a concavity argument,

$$\sum_{i=1}^{k_\varnothing} \nu^*_\mathcal{T}(i) \log \frac{w_\mathcal{T}(i)}{\nu^*_\mathcal{T}(i)} \le 0, \tag{2.17}$$

and the inequality is strict with positive $\Theta$-probability.

Recall that $\Upsilon(\mathrm{d}\mathcal{T}) = M^{-1}\kappa(\mathcal{C}(\mathcal{T}))\Theta(\mathrm{d}\mathcal{T})$ is the image of the probability measure $\Upsilon^*(\mathrm{d}\mathcal{T}\mathrm{d}\mathbf{v})$ under the canonical projection $\pi_1$ from $\mathbb{T}^*$ to $\mathbb{T}$. According to the discussion before Proposition 2.7, we can write

$$\beta = \Big(\int \Upsilon(\mathrm{d}\mathcal{T})J(\mathcal{T})\Big)^{-1} \int \Upsilon(\mathrm{d}\mathcal{T}) \sum_{i=1}^{k_\varnothing} \nu^*_\mathcal{T}(i) \log \frac{1}{\nu^*_\mathcal{T}(i)},$$

which by (2.17) is strictly smaller than

$$\Big(\int \Upsilon(\mathrm{d}\mathcal{T})J(\mathcal{T})\Big)^{-1} \int \Upsilon(\mathrm{d}\mathcal{T}) \sum_{i=1}^{k_\varnothing} \nu^*_\mathcal{T}(i) \log \frac{1}{w_\mathcal{T}(i)}.$$

However, it follows from the definition of $w_\mathcal{T}$ that

$$\begin{aligned}\int \Upsilon(\mathrm{d}\mathcal{T}) \sum_{i=1}^{k_\varnothing} \nu^*_\mathcal{T}(i) \log \frac{1}{w_\mathcal{T}(i)} &= \frac{1}{\alpha-1}\int \Upsilon(\mathrm{d}\mathcal{T})J(\mathcal{T}) + \int \Upsilon(\mathrm{d}\mathcal{T}) \sum_{i=1}^{k_\varnothing} \nu^*_\mathcal{T}(i) \log \frac{\mathcal{W}(\mathcal{T})}{\mathcal{W}(\mathcal{T}_{(i)})} \\ &= \frac{1}{\alpha-1}\int \Upsilon(\mathrm{d}\mathcal{T})J(\mathcal{T}) + \int \Upsilon^*(\mathrm{d}\mathcal{T}\mathrm{d}\mathbf{v}) \log \frac{\mathcal{W}\circ\pi_1(\mathcal{T},\mathbf{v})}{\mathcal{W}\circ\pi_1(S(\mathcal{T},\mathbf{v}))} \\ &= \frac{1}{\alpha-1}\int \Upsilon(\mathrm{d}\mathcal{T})J(\mathcal{T}),\end{aligned}$$

where in the last equality we used the fact that $\Upsilon^*$ is invariant under the shift $S$, and that $\log \mathcal{W}(\mathcal{T})$ is integrable under $\Theta(\mathrm{d}\mathcal{T})$ hence also under $\Upsilon^*$. Therefore, we have shown $\beta < \frac{1}{\alpha-1}$ and the proof of Theorem 1.2 is completed. □

### 2.5.4 Proof of Proposition 1.4

We have seen above that

$$\beta = \frac{\mathbb{E}\Big[\frac{N\mathcal{C}_1}{\mathcal{C}_1+\cdots+\mathcal{C}_N}\ \log\Big(\frac{\mathcal{C}_1}{\mathcal{C}_1+\cdots+\mathcal{C}_N}\Big)\,\kappa\Big(\Big(U + \frac{1-U}{\mathcal{C}_1+\cdots+\mathcal{C}_N}\Big)^{-1}\Big)\Big]}{\mathbb{E}\Big[\log(1-U)\,\kappa\Big(\Big(U + \frac{1-U}{\mathcal{C}_1+\cdots+\mathcal{C}_N}\Big)^{-1}\Big)\Big]}. \tag{2.18}$$

On account of Proposition 2.1, the proof of Proposition 1.4 will be completed if we can verify that the preceding expression for $\beta$ is consistent with formula (1.5). In the following calculations, we will keep using the same notation introduced in the proof of Proposition 2.4.

Firstly, the numerator of the right-hand side of (2.18) is equal to

$$\begin{aligned}
&\mathbb{E}\Big[\frac{N\mathcal{C}_1}{\mathcal{C}_1+\cdots+\mathcal{C}_N}\ \log\Big(\frac{\mathcal{C}_1}{\mathcal{C}_1+\cdots+\mathcal{C}_N}\Big)\ \frac{\big(U+\frac{1-U}{\mathcal{C}_1+\cdots+\mathcal{C}_N}\big)^{-1}(\widetilde{\mathcal{C}}_1+\cdots+\widetilde{\mathcal{C}}_{\tilde N})}{\big(U+\frac{1-U}{\mathcal{C}_1+\cdots+\mathcal{C}_N}\big)^{-1}+\widetilde{\mathcal{C}}_1+\cdots+\widetilde{\mathcal{C}}_{\tilde N}-1}\Big]\\
&\quad=\mathbb{E}\Big[\frac{N\mathcal{C}_1(\widetilde{\mathcal{C}}_1+\cdots+\widetilde{\mathcal{C}}_{\tilde N})\log\frac{\mathcal{C}_1}{\mathcal{C}_1+\cdots+\mathcal{C}_N}}{\mathcal{C}_1+\cdots+\mathcal{C}_N+\widetilde{\mathcal{C}}_1+\cdots+\widetilde{\mathcal{C}}_{\tilde N}-1+U(\mathcal{C}_1+\cdots+\mathcal{C}_N-1)(\widetilde{\mathcal{C}}_1+\cdots+\widetilde{\mathcal{C}}_{\tilde N}-1)}\Big].
\end{aligned}$$

For every integer $k\geq 2$, we define for $x\in(1,\infty)$ the function

$$G_{c_1,\ldots,c_k,u}(x):=\frac{xc_1\log\frac{c_1}{c_1+\cdots+c_k}}{c_1+\cdots+c_k+x-1+(c_1+\cdots+c_k-1)(x-1)u}\,,$$

where $u\in(0,1)$ and $c_1,\ldots,c_k\in(1,\infty)$. We can apply (2.10) to get

$$\begin{aligned}
&\mathbb{E}\big[G_{\mathcal{C}_1,\ldots,\mathcal{C}_k,U}(\widetilde{\mathcal{C}}_1+\cdots+\widetilde{\mathcal{C}}_{\tilde N})\mid\mathcal{C}_1,\ldots,\mathcal{C}_k,U\big]\\
&\quad=\mathbb{E}\left[\frac{\mathcal{C}_0^2\mathcal{C}_1(\mathcal{C}_1+\cdots+\mathcal{C}_k)\log\frac{\mathcal{C}_1}{\mathcal{C}_1+\cdots+\mathcal{C}_k}}{\big(\mathcal{C}_0+\mathcal{C}_1+\cdots+\mathcal{C}_k-1+(\mathcal{C}_0-1)(\mathcal{C}_1+\cdots+\mathcal{C}_k-1)U\big)^2}\,\middle|\,\mathcal{C}_1,\ldots,\mathcal{C}_k,U\right].
\end{aligned}$$

With help of the last display, the numerator of the right-hand side of (2.18) becomes

$$\mathbb{E}\Big[\frac{N\mathcal{C}_0^2\mathcal{C}_1(\mathcal{C}_1+\cdots+\mathcal{C}_N)\log\frac{\mathcal{C}_1}{\mathcal{C}_1+\cdots+\mathcal{C}_N}}{\big(\mathcal{C}_0+\mathcal{C}_1+\cdots+\mathcal{C}_N-1+(\mathcal{C}_0-1)(\mathcal{C}_1+\cdots+\mathcal{C}_N-1)U\big)^2}\Big].$$

We now integrate with respect to $U$ and recall that for $a,b,c>0$, $\int_0^1\mathrm{d}u\frac{a}{(b+cu)^2}=\frac{a}{b(b+c)}$. So the numerator of the right-hand side of (2.18) coincides with

$$\mathbb{E}\Big[\frac{N\mathcal{C}_0\mathcal{C}_1\log\frac{\mathcal{C}_1}{\mathcal{C}_1+\mathcal{C}_2+\cdots+\mathcal{C}_N}}{\mathcal{C}_0+\mathcal{C}_1+\cdots+\mathcal{C}_N-1}\Big].$$

On the other hand, the denominator of the right-hand side of (2.18) is equal to

$$\begin{aligned}
&\mathbb{E}\Big[\frac{(\mathcal{C}_1+\cdots+\mathcal{C}_N)(\widetilde{\mathcal{C}}_1+\cdots+\widetilde{\mathcal{C}}_{\tilde N})\log(1-U)}{\mathcal{C}_1+\cdots+\mathcal{C}_N+(\widetilde{\mathcal{C}}_1+\cdots+\widetilde{\mathcal{C}}_{\tilde N}-1)(\mathcal{C}_1+\cdots+\mathcal{C}_N)U+(\widetilde{\mathcal{C}}_1+\cdots+\widetilde{\mathcal{C}}_{\tilde N}-1)(1-U)}\Big]\\
&\quad=\ \mathbb{E}\Big[\frac{(\mathcal{C}_1+\cdots+\mathcal{C}_N)(\widetilde{\mathcal{C}}_1+\cdots+\widetilde{\mathcal{C}}_{\tilde N})\log(1-U)}{\mathcal{C}_1+\cdots+\mathcal{C}_N+\widetilde{\mathcal{C}}_1+\cdots+\widetilde{\mathcal{C}}_{\tilde N}-1+(\mathcal{C}_1+\cdots+\mathcal{C}_N-1)(\widetilde{\mathcal{C}}_1+\cdots+\widetilde{\mathcal{C}}_{\tilde N}-1)U}\Big]\\
&\quad=\ \mathbb{E}\Big[\frac{\mathcal{C}_0^2(\mathcal{C}_1+\cdots+\mathcal{C}_N)^2\log(1-U)}{\big(\mathcal{C}_0+\mathcal{C}_1+\cdots+\mathcal{C}_N-1+(\mathcal{C}_0-1)(\mathcal{C}_1+\cdots+\mathcal{C}_N-1)U\big)^2}\Big]\\
&\quad=\ -\mathbb{E}\Big[\frac{\mathcal{C}_0^2\mathcal{C}_1^2\big(-1+\mathcal{C}_0+\mathcal{C}_1-2\mathcal{C}_0\mathcal{C}_1+(\mathcal{C}_0-1)(\mathcal{C}_1-1)U\big)\log(1-U)}{\big(\mathcal{C}_0+\mathcal{C}_1-1+(\mathcal{C}_0-1)(\mathcal{C}_1-1)U\big)^3}\Big],
\end{aligned}$$

where we have repeatedly used (2.10) in the last two equalities, the first time to replace $\widetilde{\mathcal{C}}_1+\cdots+\widetilde{\mathcal{C}}_{\tilde N}$ by $\mathcal{C}_0$, the second time to replace $\mathcal{C}_1+\cdots+\mathcal{C}_N$ by $\mathcal{C}_1$. In order to integrate with respect to $U$, we appeal to the identity that for $a,b,c>0$,

$$\int_0^1\mathrm{d}u\frac{(a+bu)\log(1-u)}{(c+bu)^3}=\frac{b(c-a)+(2b+c+a)c\log\frac{c}{b+c}}{2bc(b+c)^2}.$$

Applying this formula, we see that the denominator of the right-hand side of (2.18) coincides with

$$-\mathbb{E}\Big[\frac{\mathcal{C}_0\mathcal{C}_1}{\mathcal{C}_0+\mathcal{C}_1-1}\Big].$$

We have thus obtained the following formula

$$\beta = \frac{\mathbb{E}\Big[\frac{N\mathcal{C}_0\mathcal{C}_1}{\mathcal{C}_0+\mathcal{C}_1+\cdots+\mathcal{C}_N-1}\log\frac{\mathcal{C}_1+\mathcal{C}_2+\cdots+\mathcal{C}_N}{\mathcal{C}_1}\Big]}{\mathbb{E}\Big[\frac{\mathcal{C}_0\mathcal{C}_1}{\mathcal{C}_0+\mathcal{C}_1-1}\Big]}. \tag{2.19}$$

By a symmetry argument, the numerator of the right-hand side of (2.19) is equal to

$$\begin{aligned}
&\mathbb{E}\Big[\frac{N\mathcal{C}_0\mathcal{C}_1\log(\mathcal{C}_1+\mathcal{C}_2+\cdots+\mathcal{C}_N)}{\mathcal{C}_0+\mathcal{C}_1+\cdots+\mathcal{C}_N-1}\Big]-\mathbb{E}\Big[\frac{N\mathcal{C}_0\mathcal{C}_1\log(\mathcal{C}_1)}{\mathcal{C}_0+\mathcal{C}_1+\cdots+\mathcal{C}_N-1}\Big]\\
&\quad=\mathbb{E}\Big[\frac{\mathcal{C}_0(\mathcal{C}_1+\mathcal{C}_2+\cdots+\mathcal{C}_N)\log(\mathcal{C}_1+\mathcal{C}_2+\cdots+\mathcal{C}_N)}{\mathcal{C}_0+\mathcal{C}_1+\cdots+\mathcal{C}_N-1}\Big]-\mathbb{E}\Big[\frac{\mathcal{C}_0(\mathcal{C}_1+\mathcal{C}_2+\cdots+\mathcal{C}_N)\log(\mathcal{C}_0)}{\mathcal{C}_0+\mathcal{C}_1+\cdots+\mathcal{C}_N-1}\Big]\\
&\quad=\mathbb{E}\big[f(\mathcal{C}_1+\mathcal{C}_2+\cdots+\mathcal{C}_N)\big]-\mathbb{E}\big[g(\mathcal{C}_1+\mathcal{C}_2+\cdots+\mathcal{C}_N)\big],
\end{aligned} \tag{2.20}$$

where we have set, for every $x \geq 1$,

$$f(x)=\mathbb{E}\left[\frac{\mathcal{C}_0 x}{\mathcal{C}_0+x-1}\log x\right] \quad \text{and} \quad g(x)=\mathbb{E}\left[\frac{\mathcal{C}_0 x}{\mathcal{C}_0+x-1}\log\mathcal{C}_0\right].$$

We can replace $\mathbb{E}[f(\mathcal{C}_1+\mathcal{C}_2+\cdots+\mathcal{C}_N)]$ by $\mathbb{E}[f(\mathcal{C}_1)]+\mathbb{E}[\mathcal{C}_1(\mathcal{C}_1-1)f'(\mathcal{C}_1)]$ using (2.10), and similarly for $g$, to obtain

$$\mathbb{E}\big[f(\mathcal{C}_1+\mathcal{C}_2+\cdots+\mathcal{C}_N)\big]-\mathbb{E}\big[g(\mathcal{C}_1+\mathcal{C}_2+\cdots+\mathcal{C}_N)\big]=\frac{1}{2}\left(\mathbb{E}[\mathcal{C}_0]^2-\mathbb{E}\left[\frac{\mathcal{C}_0\mathcal{C}_1}{\mathcal{C}_0+\mathcal{C}_1-1}\right]\right).$$

Plugging this into (2.20) yields the required formula (1.5), and hence finishes the proof of Proposition 1.4.

### 2.6 A second approach to Theorem 1.2

In this section, we outline a different approach to Theorem 1.2, which contains certain intermediate results of independent interest. This approach involves an invariant measure for the environment seen by Brownian motion on the CTGW tree $\Gamma^{(\alpha)}$ at the last visit of a fixed height. This is similar to Section 3 of [5], and for this reason we will omit the proofs, which may however be found in [11].

We fix the index $\alpha \in (1,2]$, and we first introduce some additional notation. For $\mathcal{T} \in \mathbb{T}$ and $r > 0$, if $x \in \mathcal{T}_r$, let $\mathcal{T}[x]$ denote the subtree of descendants of $x$ in $\mathcal{T}$. To define it formally, we write $v_x$ for the unique element of $\mathcal{V}$ such that $x = (v_x, r)$, and define the shifted discrete tree $\Pi[v_x] = \{v \in \mathcal{V} \colon v_x v \in \Pi\}$. Then $\mathcal{T}[x]$ is the infinite continuous tree corresponding to the pair

$$\Big(\Pi[v_x], (z_{v_x v}-r)_{v\in\Pi[v_x]}\Big).$$

For a fixed $r > 0$, we know that $\Gamma^{(\alpha)}$ has a.s. no branching point at height $r$. As there is a unique point $x \in \Gamma^{(\alpha)}_r$ such that $x \prec W_\infty$, we write $\Gamma^{(\alpha)}\langle r\rangle = \Gamma^{(\alpha)}[x]$ for the subtree above level $r$ selected by harmonic measure.

To describe the distribution of $\Gamma^{(\alpha)}\langle r\rangle$, recall that for every $x \geq 0$,

$$\varphi_\alpha(x) = \mathbb{E}\big[\exp(-x\,\mathcal{C}^{(\alpha)}/2)\big] = \Theta_\alpha\big(\exp(-x\,\mathcal{C}(\mathcal{T})/2)\big).$$

**Proposition 2.8.** *The distribution under $\mathbb{P} \otimes P$ of the subtree $\Gamma^{(\alpha)}\langle r\rangle$ above level $r$ selected by harmonic measure is*

$$\Phi_r^{(\alpha)}(\mathcal{C}(\mathcal{T}))\,\Theta_\alpha(\mathrm{d}\mathcal{T}),$$

*where, for every $c > 0$,*

$$\Phi_r^{(\alpha)}(c) := E_{(c)}\Big[\exp - \int_0^r \mathrm{d}s\,\Big(m_\alpha\big(1-\varphi_\alpha(X_s)\big)^{\alpha-1} - \frac{1}{\alpha-1}\Big)\Big].$$

*Here $X = (X_s)_{0\le s\le r}$ stands for the solution of the stochastic differential equation*

$$\mathrm{d}X_s = 2\sqrt{X_s}\,\mathrm{d}\eta_s + (2 - X_s)\mathrm{d}s$$

*that starts under the probability measure $P_{(c)}$ with an exponential distribution of parameter $c/2$. In the previous SDE, $(\eta_s)_{s\ge 0}$ denotes a standard linear Brownian motion.*

Now we define shifts $(\tau_r)_{r\ge 0}$ on $\mathbb{T}^*$ in the following way. For $r = 0$, $\tau_0$ is the identity mapping of $\mathbb{T}^*$. For $r > 0$ and $(\mathcal{T}, \mathbf{v}) \in \mathbb{T}^*$, we write $\mathbf{v} = (v_1, v_2, \ldots)$ and $\mathbf{v}_n = (v_1, \ldots, v_n)$ for every $n \ge 0$ (by convention, $\mathbf{v}_0 = \varnothing$). Also let $x_{r,\mathbf{v}}$ be the unique element of $\mathcal{T}_r$ such that $x_{r,\mathbf{v}} \prec \mathbf{v}$. Then we set

$$\tau_r(\mathcal{T}, \mathbf{v}) = \Big(\mathcal{T}[x_{r,\mathbf{v}}]\,,\,(v_{k+1}, v_{k+2}, \ldots)\Big),$$

where $k = \min\{n \ge 0 \colon z_{\mathbf{v}_n} \ge r\}$. Informally, $\tau_r(\mathcal{T}, \mathbf{v})$ is obtained by taking the subtree of $\mathcal{T}$ consisting of descendants of the vertex at height $r$ on the distinguished geodesic ray, and keeping in this subtree the "same" geodesic ray. It is straightforward to verify that $\tau_r \circ \tau_s = \tau_{r+s}$ for every $r, s \ge 0$.

The next proposition gives an invariant measure absolutely continuous with respect to $\Theta_\alpha^*$ under the shifts $\tau_r$. To simplify notation, we set first

$$C_1(\alpha) := 2\int_0^\infty \mathrm{d}s\,\varphi_\alpha'(s)^2\,e^{s/2} = \iint \gamma_\alpha(\mathrm{d}\ell)\gamma_\alpha(\mathrm{d}\ell')\frac{\ell\ell'}{\ell+\ell'-1}.$$

**Proposition 2.9.** *For every $c > 0$,*

$$\lim_{r\to+\infty} \Phi_r^{(\alpha)}(c) = \Phi_\infty^{(\alpha)}(c) := \frac{1}{C_1(\alpha)}\int \gamma_\alpha(\mathrm{d}s)\frac{cs}{c+s-1}.$$

*The probability measure $\Lambda_\alpha^*$ on $\mathbb{T}^*$ defined as*

$$\Lambda_\alpha^*(\mathrm{d}\mathcal{T}\mathrm{d}\mathbf{v}) := \Phi_\infty^{(\alpha)}(\mathcal{C}(\mathcal{T}))\,\Theta_\alpha^*(\mathrm{d}\mathcal{T}\mathrm{d}\mathbf{v})$$

*is invariant under the shifts $\tau_r$, $r \ge 0$.*

Furthermore, one can easily adapt the proof of Proposition 13 in [5] to show that for every $r > 0$, the shift $\tau_r$ acting on the probability space $(\mathbb{T}^*, \Lambda_\alpha^*)$ is ergodic. Applying Birkhoff's ergodic theorem to a suitable functional (see Section 3.4 of [5]) leads to the convergence (1.3) in Theorem 1.2, with $\beta_\alpha$ given by formula (2.19). See [11] for more details.

# 3 The discrete setting

## 3.1 Galton–Watson trees

Let us first introduce discrete (finite) rooted ordered trees, which are also called plane trees in combinatorics. A plane tree t is a finite subset of $\mathcal{V}$ such that the following holds:

(i) $\varnothing \in \mathsf{t}$.

(ii) If $u = (u_1, \dots, u_n) \in \mathsf{t} \backslash \{\varnothing\}$, then $\widehat{u} = (u_1, \dots, u_{n-1}) \in \mathsf{t}$.

(iii) For every $u = (u_1, \dots, u_n) \in \mathsf{t}$, there exists an integer $k_u(\mathsf{t}) \geq 0$ such that, for every $j \in \mathbb{N}$, $(u_1, \dots, u_n, j) \in \mathsf{t}$ if and only if $1 \leq j \leq k_u(\mathsf{t})$.

In this section we will say tree instead of plane tree for short. The same notation and terminology introduced at the beginning of Section 2.1 will be used in this section: $|u|$ is the generation of $u$, $uv$ denotes the concatenation of $u$ and $v$, $\prec$ stands for the (non-strict) genealogical order and $u \wedge v$ is the maximal element of $\{w \in \mathcal{V} \colon w \prec u \text{ and } w \prec v\}$. A vertex with no child is called a leaf.

The height of a tree $\mathsf{t}$ is

$$h(\mathsf{t}) := \max\{|v| \colon v \in \mathsf{t}\}.$$

We write $\mathscr{T}$ for the set of all trees, and $\mathscr{T}_n$ for the set of all trees with height $n$.

We view a tree $\mathsf{t}$ as a graph whose vertices are the elements of $\mathsf{t}$ and whose edges are the pairs $\{\widehat{u}, u\}$ for all $u \in \mathsf{t} \backslash \{\varnothing\}$. The set $\mathsf{t}$ is equipped with the distance

$$d(u, v) := \frac{1}{2}(|u| + |v| - 2|u \wedge v|).$$

Notice that this is half the usual graph distance. We will write $B_{\mathsf{t}}(v, r)$, or simply $B(v, r)$ if there is no ambiguity, for the closed ball of radius $r$ centered at $v$, with respect to the distance $d$ in the tree $\mathsf{t}$.

The set of all vertices of $\mathsf{t}$ at generation $n$ is denoted by

$$\mathsf{t}_n := \{v \in \mathsf{t} \colon |v| = n\}.$$

If $v \in \mathsf{t}$, the subtree of descendants of $v$ is

$$\widetilde{\mathsf{t}}[v] := \{v' \in \mathsf{t} \colon v \prec v'\}.$$

Note that $\widetilde{\mathsf{t}}[v]$ is not a tree under the previous definition, but we can turn it into a tree by relabeling its vertices as

$$\mathsf{t}[v] := \{w \in \mathcal{V} \colon vw \in \mathsf{t}\}.$$

If $v \in \mathsf{t}$, then for every $i \in \{0, 1, \dots, |v|\}$ we write $\langle v \rangle_i$ for the ancestor of $v$ at generation $i$. Suppose that $|v| = n$. Then $B_{\mathsf{t}}(v, i) \cap \mathsf{t}_n = \widetilde{\mathsf{t}}\,[\langle v \rangle_{n-i}] \cap \mathsf{t}_n$, for every $i \in \{0, 1, \dots, n\}$. This simple observation will be used repeatedly below.

Let $\rho$ be a non-trivial probability measure on $\mathbb{Z}_+$ with mean one, which belongs to the domain of attraction of a stable distribution of index $\alpha \in (1, 2]$. Therefore property (1.1) holds. For every integer $n \geq 0$, we let $\mathsf{T}^{(n)}$ be a Galton–Watson tree with offspring distribution $\rho$, conditioned on non-extinction at generation $n$, viewed as a random subset of $\mathcal{V}$ (see e.g. [10] for a precise definition of Galton–Watson trees). In particular, $\mathsf{T}^{(0)}$ is just a Galton–Watson tree with offspring distribution $\rho$. We suppose that the random trees $\mathsf{T}^{(n)}$ are defined under the probability measure $\mathbb{P}$.

We let $\mathsf{T}^{*n}$ be the reduced tree associated with $\mathsf{T}^{(n)}$, which consists of all vertices of $\mathsf{T}^{(n)}$ that have (at least) one descendant at generation $n$. Note that $|v| \leq n$ for every $v \in \mathsf{T}^{*n}$. A priori $\mathsf{T}^{*n}$ is not a tree in the sense of the preceding definition. However we can relabel the vertices of $\mathsf{T}^{*n}$, preserving both the lexicographical order and the genealogical order, so that $\mathsf{T}^{*n}$ becomes a tree in the sense of our definitions. We will always assume that this relabeling has been done.

Conditionally on $\mathsf{T}^{(n)}$, the hitting distribution of generation $n$ is the same for simple random walk on $\mathsf{T}^{(n)}$ and that on the reduced tree $\mathsf{T}^{*n}$. In view of studying properties of this hitting distribution, we can consider directly a simple random walk on $\mathsf{T}^{*n}$ starting from the root $\varnothing$, which we denote by $Z^n = (Z^n_k)_{k\geq 0}$. This random walk is defined under the probability measure $P$. Let

$$H_n := \inf\{k \geq 0 \colon |Z^n_k| = n\}$$

be the first hitting time of generation $n$ by $Z^n$, and set $\Sigma_n = Z^n_{H_n}$ to be the hitting point. The discrete harmonic measure $\mu_n$ is the law of $\Sigma_n$ under $P$, which is a (random) probability measure on the level set $\mathsf{T}^{*n}_n$.

Set $q_n = \mathbb{P}\big(h(\mathsf{T}^{(0)}) \geq n\big)$. If $L$ is the slowly varying function appearing in (1.1), it has been established in [16, Lemma 2] that

$$q_n^{\alpha-1} L(q_n) \sim \frac{1}{(\alpha-1)n} \quad \text{as } n \to \infty. \tag{3.1}$$

By the asymptotic inversion property of slowly varying functions (see e.g. [4, Section 1.5.7]), it follows that

$$q_n \sim n^{-\frac{1}{\alpha-1}} \ell(n) \quad \text{as } n \to \infty, \tag{3.2}$$

for a function $\ell$ slowly varying at $\infty$. Moreover, it is shown in [16, Theorem 1] that, as $n \to \infty$, $q_n \# \mathsf{T}^{*n}_n$ converges in distribution to the positive random variable $\mathcal{W}(\Gamma^{(\alpha)})$ introduced in the proof of Proposition 2.7.

We will need to estimate the size of level sets in $\mathsf{T}^{*n}$. The following lemma is an analogue of Lemma 15 in [5].

**Lemma 3.1.** *For every $r \geq 1$, there exists a constant $C = C(r,\rho)$ depending on $r$ and the offspring distribution $\rho$ such that, for every integer $n \geq 2$ and every integer $p \in [1, n/2]$,*

$$\mathbb{E}\big[(\log \# \mathsf{T}^{*n}_{n-p})^r\big]^{\frac{1}{r}} \leq C \, \log \frac{n}{p} \qquad \textit{and} \qquad \mathbb{E}\big[(\log \# \mathsf{T}^{*n}_{n})^r\big]^{\frac{1}{r}} \leq C \, \log n.$$

*Proof.* We can find $a = a(r) > 0$ such that the function $x \mapsto (\log(a+x))^r$ is concave over $[1,\infty)$. Then as in the proof of [5, Lemma 15],

$$\mathbb{E}\big[(\log \# \mathsf{T}^{*n}_{n-p})^r\big]^{\frac{1}{r}} \leq \mathbb{E}\big[(\log(a + \# \mathsf{T}^{*n}_{n-p}))^r\big]^{\frac{1}{r}} \leq \log\big(a + \mathbb{E}[\# \mathsf{T}^{*n}_{n-p}]\big) = \log(a + \frac{q_p}{q_n}).$$

Using Potter's bounds on slowly varying function (see e.g. [4, Theorem 1.5.6]), one can deduce from (3.2) that there exists a constant $C' = C'(\rho) > 0$ such that for every $n \geq 2$ and every $p \in [1, n/2]$,

$$\log\Big(\frac{q_p}{q_n}\Big) \leq C' \log\Big(\frac{n}{p}\Big),$$

from which the first bound of the lemma easily follows. The second estimate can be shown in a similar way. □

The goal of this section is to prove Theorem 1.1. We will assume in the rest of this section that the critical offspring distribution $\rho$ satisfies (1.1) with a fixed $\alpha \in (1,2]$. Accordingly, we will omit the superscripts and subscripts concerning $\alpha$ if there is no ambiguity.

### 3.2 Convergence of discrete reduced trees

We first define truncations of the discrete reduced tree $\mathsf{T}^{*n}$. For every $s \in [0,n]$, we set

$$R_s(\mathsf{T}^{*n}) := \big\{v \in \mathsf{T}^{*n} \colon |v| \leq n - \lfloor s \rfloor\big\}.$$

Recall from Section 2.1 the definition of the continuous reduced tree $\Delta$ of index $\alpha$. For every $\varepsilon \in (0,1)$, we have set $\Delta_\varepsilon = \{x \in \Delta \colon H(x) \leq 1-\varepsilon\}$. We will implicitly use the fact that, for every fixed $\varepsilon$, there is a.s. no branching point of $\Delta$ at height $1-\varepsilon$. The skeleton of $\Delta_\varepsilon$ is defined as the following plane tree

$$\mathrm{Sk}(\Delta_\varepsilon) := \{\varnothing\} \cup \big\{v \in \Pi\backslash\{\varnothing\} \colon Y_{\hat{v}} \leq 1-\varepsilon\big\} = \{\varnothing\} \cup \big\{v \in \Pi\backslash\{\varnothing\} \colon (\widehat{v}, Y_{\hat{v}}) \in \Delta_\varepsilon\big\}.$$

A vertex $v$ of $\mathrm{Sk}(\Delta_\varepsilon)$ is a leaf of $\mathrm{Sk}(\Delta_\varepsilon)$ if and only if $Y_v > 1-\varepsilon$.

Let $\mathrm{t}$ be a tree. We write $\mathcal{S}(\mathrm{t})$ for the set of all vertices of $\mathrm{t}$ whose number of children is different from 1. Then we can find a unique tree $[\mathrm{t}] \in \mathscr{T}$ such that there exists a bijection from $[\mathrm{t}]$ onto $\mathcal{S}(\mathrm{t})$ that preserves the genealogical order and the lexicographical order of vertices. Denote the inverse of this canonical bijection by $u \in \mathcal{S}(\mathrm{t}) \mapsto [u] \in [\mathrm{t}]$. In a less formal way, $[\mathrm{t}]$ is just the tree obtained from $\mathrm{t}$ by removing all vertices that have exactly one child.

**Proposition 3.2.** *We can construct the reduced trees $\mathsf{T}^{*n}$ and the (continuous) reduced stable tree $\Delta$ on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, so that the following assertions hold for every fixed $\varepsilon \in (0,1)$ with $\mathbb{P}$-probability one.*

(1) *For every sufficiently large integer $n$, there exists an injective mapping $\Psi_n^\varepsilon \colon u \mapsto w_u^{n,\varepsilon}$ from $\mathrm{Sk}(\Delta_\varepsilon)$ into $\mathcal{S}(R_{\varepsilon n}(\mathsf{T}^{*n}))$ satisfying the following properties.*

  (1.a) *The mapping $\Psi_n^\varepsilon$ preserves both the lexicographical order and the genealogical order.*

  (1.b) *If $u$ is a leaf of $\mathrm{Sk}(\Delta_\varepsilon)$, $[w_u^{n,\varepsilon}]$ is a leaf of $[R_{\varepsilon n}(\mathsf{T}^{*n})]$ and $|w_u^{n,\varepsilon}| = n - \lfloor \varepsilon n \rfloor$. The restricted mapping*

$$\Psi_n^\varepsilon \restriction_{\mathsf{Leaves}} : \textit{Leaves of } \mathrm{Sk}(\Delta_\varepsilon) \longrightarrow \big\{v \in \mathcal{S}(R_{\varepsilon n}(\mathsf{T}^{*n})) \colon [v] \textit{ is a leaf of } [R_{\varepsilon n}(\mathsf{T}^{*n})]\big\}$$

  *is bijective.*

  (1.c) *For every vertex $u$ of $\mathrm{Sk}(\Delta_\varepsilon)$,*

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n}|w_u^{n,\varepsilon}| &= Y_u \wedge (1-\varepsilon)\,, \\ \lim_{n\to\infty} \frac{1}{n}|\overline{w}_u^{n,\varepsilon}| &= Y_{\hat{u}}\,, \end{aligned}$$

  *where $\widehat{u}$ denotes the parent of $u$ in $\mathrm{Sk}(\Delta_\varepsilon)$, and $\overline{w}_u^{n,\varepsilon}$ stands for the vertex in $\mathcal{S}(R_{\varepsilon n}(\mathsf{T}^{*n}))$ such that $[\overline{w}_u^{n,\varepsilon}]$ is the parent of $[w_u^{n,\varepsilon}]$ in $[R_{\varepsilon n}(\mathsf{T}^{*n})]$. (Notice that $\overline{w}_u^{n,\varepsilon}$ does not necessarily coincide with $w_{\hat{u}}^{n,\varepsilon}$.)*

(2) *The mapping $\Psi_n^\varepsilon$ is asymptotically unique in the sense that, if $\widetilde{\Psi}_n^\varepsilon$ is another mapping such that the preceding properties hold, then for $n$ sufficiently large,*

$$\Psi_n^\varepsilon(u) = \widetilde{\Psi}_n^\varepsilon(u) \quad \textit{for every } u \in \mathrm{Sk}(\Delta_\varepsilon).$$

Proposition 3.2 (see Fig. 2 for an illustration) essentially results from the convergence in distribution of the rescaled contour functions associated with the trees $\mathsf{T}^{(n)}$ towards the excursion of the stable height process with height greater than 1 (see [6, Section 2.5]). By using the Skorokhod representation theorem, one may assume that the trees $\mathsf{T}^{(n)}$ and the excursion of the stable height process are constructed so that the latter convergence holds almost surely. The various assertions of Proposition 3.2 then easily follow (cf. [6, Section 2.6]), using the relation between the excursion of the stable height process with height greater than 1 and the limiting reduced tree $\Delta$, which can be found in [6, Section 2.7].

**Remark 3.3.** Let us take $0<\delta<\varepsilon$. If $u$ is not a leaf of $\mathrm{Sk}(\Delta_\varepsilon)$, we must have $w_u^{n,\varepsilon} = w_u^{n,\delta}$ for sufficiently large $n$. On the other hand, if $u$ is a leaf of $\mathrm{Sk}(\Delta_\varepsilon)$, then for large $n$, $w_u^{n,\varepsilon}$ must be an ancestor of $w_u^{n,\delta}$.

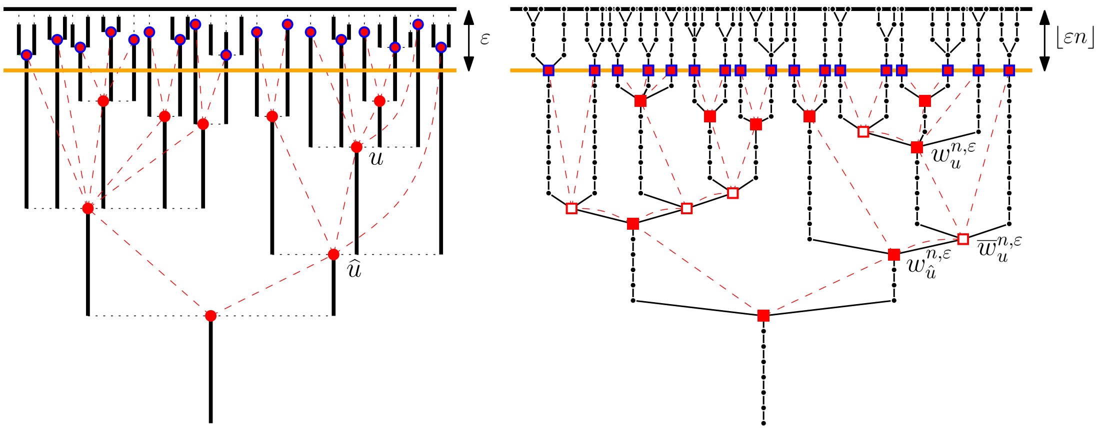


Figure 2: On the left, the tree $\Delta$, its truncation $\Delta_\varepsilon$ and its skeleton $\mathrm{Sk}(\Delta_\varepsilon)$. On the right, a large reduced tree $\mathsf{T}^{*n}$ of height $n$, its truncation $R_{\varepsilon n}(\mathsf{T}^{*n})$ and the associated tree $[R_{\varepsilon n}(\mathsf{T}^{*n})]$. The vertices depicted as filled red disks on the left correspond to the vertices depicted as filled red squares on the right, via the mapping $\Psi_n^\varepsilon$.

**Remark 3.4.** We expect that a result more precise than Proposition 3.2 should hold. For all sufficiently large $n$, the mapping $\Psi_n^\varepsilon$ should be a bijection, and the equality $\overline{w}_u^{n,\varepsilon} = w_{\hat{u}}^{n,\varepsilon}$ should hold for all $u \in \mathrm{Sk}(\Delta_\varepsilon)$ (in other words, there should be no white square in the right part of Fig. 2). However this refinement does not easily follow from the results of [6], and we will omit it since it is not needed for our purposes.

### 3.3 Convergence of harmonic measures

Recall that $\mu$ is the continuous harmonic measure on the boundary $\partial\Delta$ of the reduced stable tree, and that $\mu_n$ is the discrete harmonic measure on $\mathsf{T}_n^{*n}$. For every $x \in \partial\Delta_\varepsilon$, we set

$$\mu^\varepsilon(x) = \mu(\{y \in \partial\Delta \colon x \prec y\}) = P(x \prec B_{T_-}).$$

Similarly, we define a probability measure $\mu_n^\varepsilon$ on $\mathsf{T}_{n-\lfloor \varepsilon n\rfloor}^{*n}$ by setting

$$\mu_n^\varepsilon(u) = \mu_n(\{v \in \mathsf{T}_n^{*n} \colon u \prec v\}),$$

for every $u \in \mathsf{T}_{n-\lfloor \varepsilon n\rfloor}^{*n}$. Clearly, $\mu_n^\varepsilon$ is the distribution of $\langle \Sigma_n \rangle_{n-\lfloor \varepsilon n\rfloor}$.

**Proposition 3.5.** *Suppose that the reduced trees $\mathsf{T}^{*n}$ and the (continuous) tree $\Delta$ have been constructed so that the properties of Proposition 3.2 hold, and recall the notation $(w_u^{n,\varepsilon})_{u\in\mathrm{Sk}(\Delta_\varepsilon)}$ introduced therein. Then $\mathbb{P}$-a.s. for every $x = (u, 1-\varepsilon) \in \partial\Delta_\varepsilon$,*

$$\lim_{n\to\infty} \mu_n^\varepsilon(w_u^{n,\varepsilon}) = \mu^\varepsilon(x).$$

*Proof.* Let $\delta \in (0,\varepsilon)$ and set $T_\delta = \inf\{t \geq 0 : H(B_t) = 1-\delta\} < T$. Define a probability measure $\mu^{\varepsilon,(\delta)}$ on $\partial\Delta_\varepsilon$ by setting for every $x \in \partial\Delta_\varepsilon$,

$$\mu^{\varepsilon,(\delta)}(x) = P(x \prec B_{T_\delta}).$$

Similarly, we write $\mu_n^{(\delta)}$ for the distribution of the hitting point of generation $n - \lfloor \delta n \rfloor$ by random walk on $\mathsf{T}^{*n}$ started from $\varnothing$. Then we define a probability measure $\mu_n^{\varepsilon,(\delta)}$ on $\mathsf{T}_{n-\lfloor \varepsilon n\rfloor}^{*n}$ by setting

$$\mu_n^{\varepsilon,(\delta)}(v) = \mu_n^{(\delta)}(\{w \in \mathsf{T}_{n-\lfloor \delta n\rfloor}^{*n} \colon v \prec w\}),$$

for every $v \in \mathsf{T}^{*n}_{n-\lfloor \varepsilon n \rfloor}$.

As in the proof of [5, Proposition 18], we have $\mathbb{P}$-a.s.

$$\begin{aligned} \lim_{\delta\to 0}\Big(\sup_{x\in\partial\Delta_\varepsilon}\big|\mu^{\varepsilon,(\delta)}(x)-\mu^{\varepsilon}(x)\big|\Big) &= 0,\\ \lim_{\delta\to 0}\bigg(\limsup_{n\to\infty}\Big(\sup_{v\in\mathsf{T}^{*n}_{n-\lfloor\varepsilon n\rfloor}}\big|\mu_n^{\varepsilon,(\delta)}(v)-\mu_n^{\varepsilon}(v)\big|\Big)\bigg) &= 0.\end{aligned}$$

So the convergence of the proposition will follow if we can verify that for every fixed $\delta \in (0,\varepsilon)$, we have $\mathbb{P}$-a.s. for every $x = (u, 1-\varepsilon) \in \partial\Delta_\varepsilon$,

$$\lim_{n\to\infty} \mu_n^{\varepsilon,(\delta)}(w_u^{n,\varepsilon}) = \mu^{\varepsilon,(\delta)}(x). \tag{3.3}$$

To this end, we may and will assume that the reduced trees $\mathsf{T}^{*n}$ and the (continuous) tree $\Delta$ have been constructed so that the properties of Proposition 3.2 hold simultaneously for $\varepsilon$ and for $\delta$.

Firstly, by considering the successive passage times of Brownian motion stopped at time $T_\delta$ in the set $\{(u, Y_u \wedge (1-\delta))\colon u \in \mathrm{Sk}(\Delta_\delta)\}$, we get a Markov chain $X^{(\delta)}$, which is absorbed in the set $\{(v, 1-\delta)\colon v \text{ is a leaf of } \mathrm{Sk}(\Delta_\delta)\}$, and whose transition kernels are explicitly described in terms of the quantities $Y_u, u \in \mathrm{Sk}(\Delta_\delta)$ by series and parallel circuits calculation.

Secondly, let $n$ be sufficiently large so that assertions (1) and (2) of Proposition 3.2 hold with $\varepsilon$ as well as with $\delta$, and consider simple random walk on $\mathsf{T}^{*n}$ started from $\varnothing$ and stopped at the first hitting time of generation $n - \lfloor \delta n \rfloor$. By considering the successive passage times of this random walk in the set $\mathcal{S}(R_{\delta n}(\mathsf{T}^{*n}))$, we again get a Markov chain $X^{(\delta),n}$, which is absorbed in the set

$$\big\{v \in \mathcal{S}(R_{\delta n}(\mathsf{T}^{*n}))\colon [v] \text{ is a leaf of } [R_{\delta n}(\mathsf{T}^{*n})]\big\}.$$

By property (1.b) of Proposition 3.2, this set is exactly $\{w_v^{n,\delta}\colon v \text{ is a leaf of } \mathrm{Sk}(\Delta_\delta)\}$. As previously, the transition kernels of this Markov chain $X^{(\delta),n}$ can be written explicitly in terms of the quantities $|v|, v \in \mathcal{S}(R_{\delta n}(\mathsf{T}^{*n}))$.

Recall that by Proposition 3.2,

$$\Psi_n^\delta(\mathrm{Sk}(\Delta_\delta)) = \{w_u^{n,\delta}\colon u \in \mathrm{Sk}(\Delta_\delta)\}$$

is a subset of $\mathcal{S}(R_{\delta n}(\mathsf{T}^{*n}))$, and that the mapping $\Psi_n^\delta$ is injective. If we let $\widetilde{X}^{(\delta),n}$ be the Markov chain restricted to the subset $\Psi_n^\delta(\mathrm{Sk}(\Delta_\delta))$, then after identifying both sets $\Psi_n^\delta(\mathrm{Sk}(\Delta_\delta))$ and $\{(u, Y_u \wedge (1-\delta))\colon u \in \mathrm{Sk}(\Delta_\delta)\}$ with $\mathrm{Sk}(\Delta_\delta)$, we can view both $\widetilde{X}^{(\delta),n}$ and $X^{(\delta)}$ as Markov chains with values in the set $\mathrm{Sk}(\Delta_\delta)$. Using property (1.c) of Proposition 3.2, we see that the transition kernels of $\widetilde{X}^{(\delta),n}$ converge to those of $X^{(\delta)}$.

Write $X_\infty^{(\delta)}$ for the absorption point of $X^{(\delta)}$, and similarly write $X_\infty^{(\delta),n}$ for that of $X^{(\delta),n}$. Notice that $X_\infty^{(\delta),n}$ is also the absorption point of the restricted Markov chain $\widetilde{X}^{(\delta),n}$. We thus obtain that the distribution of $X_\infty^{(\delta),n}$ converges to that of $X_\infty^{(\delta)}$ (recall that both $X_\infty^{(\delta),n}$ and $X_\infty^{(\delta)}$ are viewed as taking values in the set of leaves of $\mathrm{Sk}(\Delta_\delta)$). Consequently, for every $u \in \mathcal{V}$ such that $x = (u, 1-\varepsilon) \in \partial\Delta_\varepsilon$, we have

$$\lim_{n\to\infty} P(u \prec X_\infty^{(\delta),n}) = P(u \prec X_\infty^{(\delta)}).$$

However, from our definitions, we have

$$P(u \prec X_\infty^{(\delta)}) = \mu^{\varepsilon,(\delta)}(x),$$

and, for $n$ sufficiently large, since $w_u^{n,\varepsilon}$ coincides with the ancestor of $w_u^{n,\delta}$ at generation $n - \lfloor \varepsilon n \rfloor$ (see Remark 3.3 after Proposition 3.2),

$$P(u \prec X_\infty^{(\delta),n}) = \mu_n^{\varepsilon,(\delta)}(w_u^{n,\varepsilon}).$$

This completes the proof of (3.3) and of the proposition. □

Recall that $\beta$ is the Hausdorff dimension of the continuous harmonic meaure $\mu$.

**Proposition 3.6.** *Let $r \geq 1$ and $\xi \in (0,1)$. We can find $\varepsilon_0 \in (0,1/2)$ such that the following holds. For every $\varepsilon \in (0,\varepsilon_0)$, there exists $n_0 \geq 0$ such that for every $n \geq n_0$,*

$$\mathbb{E} \otimes E\Big[\Big|\log \mu_n^\varepsilon\big(\langle \Sigma_n \rangle_{n-\lfloor \varepsilon n \rfloor}\big) - \beta \log \varepsilon\Big|^r\Big] \quad \leq \quad \xi\, |\log \varepsilon|^r.$$

*Proof.* Recall our notation $\mathcal{B}_{\mathbf{d}}(x,r)$ for the closed ball of radius $r$ centered at $x \in \Delta$. Fix $\eta \in (0,1)$. Since $B_{T_-}$ is distributed according to $\mu$, it follows from Theorem 1.2 that there exists $\varepsilon_0 \in (0,1/2)$ such that for every $\varepsilon \in (0,\varepsilon_0)$ we have

$$\mathbb{P} \otimes P\left(\Big|\log \mu(\mathcal{B}_{\mathbf{d}}(B_{T_-}, 2\varepsilon)) - \beta \log \varepsilon\Big| > (\eta/2)|\log \varepsilon|\right) < \eta/2. \tag{3.4}$$

Let us fix $\varepsilon \in (0,\varepsilon_0)$ and assume that the reduced trees $\mathsf{T}^{*n}$ and the (continuous) tree $\Delta$ have been constructed so that the properties of Proposition 3.2 hold. We now claim that, under $\mathbb{P} \otimes P$,

$$\mu_n^\varepsilon(\langle \Sigma_n \rangle_{n-\lfloor \varepsilon n \rfloor}) \xrightarrow[n\to\infty]{\text{(d)}} \mu(\mathcal{B}_{\mathbf{d}}(B_{T_-}, 2\varepsilon)). \tag{3.5}$$

To see this, let $f$ be a continuous function on $[0,1]$. Since the distribution of $\langle \Sigma_n \rangle_{n-\lfloor \varepsilon n \rfloor}$ under $P$ is $\mu_n^\varepsilon$, we have

$$\mathbb{E} \otimes E\Big[f(\mu_n^\varepsilon(\langle \Sigma_n \rangle_{n-\lfloor \varepsilon n \rfloor}))\Big] = \mathbb{E}\Big[\sum_{v \in \mathsf{T}^{*n}_{n-\lfloor \varepsilon n \rfloor}} \mu_n^\varepsilon(v)\, f(\mu_n^\varepsilon(v))\Big].$$

By property (1.b) of Proposition 3.2, we know that $\mathbb{P}$-a.s. for $n$ sufficiently large,

$$\sum_{v \in \mathsf{T}^{*n}_{n-\lfloor \varepsilon n \rfloor}} \mu_n^\varepsilon(v)\, f(\mu_n^\varepsilon(v)) = \sum_{x=(u,1-\varepsilon)\in \partial \Delta_\varepsilon} \mu_n^\varepsilon(w_u^{n,\varepsilon})\, f(\mu_n^\varepsilon(w_u^{n,\varepsilon})),$$

and by Proposition 3.5 the latter quantities converge as $n \to \infty$ towards

$$\sum_{x \in \partial \Delta_\varepsilon} \mu^\varepsilon(x)\, f(\mu^\varepsilon(x)) = E\big[f(\mu(\mathcal{B}_{\mathbf{d}}(B_{T_-}, 2\varepsilon)))\big],$$

which establishes the convergence (3.5) as claimed.

By (3.4) and (3.5), we can find $n_0 = n_0(\varepsilon) \geq \varepsilon^{-1}$ such that for $n \geq n_0$,

$$\mathbb{P} \otimes P\left(\left|\log \mu_n^\varepsilon\big(\langle \Sigma_n \rangle_{n-\lfloor \varepsilon n \rfloor}\big) - \beta \log \varepsilon\right| > \eta\,|\log \varepsilon|\right) < \eta.$$

Using the Cauchy–Schwarz inequality, we have then

$$\begin{aligned}
&\mathbb{E} \otimes E\left[\left|\log \mu_n^\varepsilon\big(\langle \Sigma_n \rangle_{n-\lfloor \varepsilon n \rfloor}\big) - \beta \log \varepsilon\right|^r\right] \\
&\quad \leq \eta^r |\log \varepsilon|^r + \eta^{\frac{1}{2}}\, \mathbb{E} \otimes E\Big[\left|\log \mu_n^\varepsilon\big(\langle \Sigma_n \rangle_{n-\lfloor \varepsilon n \rfloor}\big) - \beta \log \varepsilon\right|^{2r}\Big]^{1/2} \\
&\quad \leq \eta^r |\log \varepsilon|^r + 2^r \eta^{\frac{1}{2}} |\beta \log \varepsilon|^r + 2^r \eta^{\frac{1}{2}}\, \mathbb{E} \otimes E\Big[\left|\log \mu_n^\varepsilon\big(\langle \Sigma_n \rangle_{n-\lfloor \varepsilon n \rfloor}\big)\right|^{2r}\Big]^{1/2}.
\end{aligned} \tag{3.6}$$

Since $r \geq 1$, the function

$$g(x) := (x \wedge e^{-2r})\, |\log(x \wedge e^{-2r})|^{2r}$$

is nondecreasing and concave over $[0,1]$. Thus, we obtain

$$\begin{aligned}E\Big[\,\big|\log\mu_n^\varepsilon\big(\langle\Sigma_n\rangle_{n-\lfloor\varepsilon n\rfloor}\big)\big|^{2r}\,\Big] &= \sum_{v\in\mathsf{T}^{*n}_{n-\lfloor\varepsilon n\rfloor}}\mu_n^\varepsilon(v)|\log\mu_n^\varepsilon(v)|^{2r}\\ &\leq \sum_{v\in\mathsf{T}^{*n}_{n-\lfloor\varepsilon n\rfloor}} g(\mu_n^\varepsilon(v))+(2r)^{2r}\\ &\leq \#\mathsf{T}^{*n}_{n-\lfloor\varepsilon n\rfloor}\times g\Big((\#\mathsf{T}^{*n}_{n-\lfloor\varepsilon n\rfloor})^{-1}\Big)+(2r)^{2r}\\ &\leq \Big|\log\#\mathsf{T}^{*n}_{n-\lfloor\varepsilon n\rfloor}\Big|^{2r}+2(2r)^{2r}.\end{aligned}$$

We now use Lemma 3.1 to see

$$\mathbb{E}\otimes E\Big[\,\big|\log\mu_n^\varepsilon\big(\langle\Sigma_n\rangle_{n-\lfloor\varepsilon n\rfloor}\big)\big|^{2r}\Big]\leq\mathbb{E}\Big[\big|\log\#\mathsf{T}^{*n}_{n-\lfloor\varepsilon n\rfloor}\big|^{2r}\Big]+2(2r)^{2r}\leq C^{2r}\Big(\log\frac{n}{\lfloor\varepsilon n\rfloor}\Big)^{2r}+2(2r)^{2r}.$$

By combining the last estimate with (3.6), we get that, for every $n\geq n_0(\varepsilon)$,

$$\mathbb{E}\otimes E\left[\big|\log\mu_n^\varepsilon\big(\langle\Sigma_n\rangle_{n-\lfloor\varepsilon n\rfloor}\big)-\beta\log\varepsilon\big|^r\right]\leq(\eta^r+2^r\eta^{\frac12}\beta^r)|\log\varepsilon|^r+2^{r+1}\eta^{\frac12}\big((2r)^r+C^r|\log\varepsilon|^r\big).$$

The statement of the proposition follows since $\eta$ was arbitrary. □

### 3.4 The flow property of discrete harmonic measure

We briefly recall the flow property of the discrete harmonic measure $\mu_n$ presented in [5, Section 4.3.1]. Let $\mathsf{t}\in\mathscr{T}_n$ be a plane tree of height $n$ and $Z^{(\mathsf{t})}=(Z_k^{(\mathsf{t})})_{k\geq0}$ be simple random walk on $\mathsf{t}$ starting from $\varnothing$. We set

$$H_n^{(\mathsf{t})}:=\inf\{k\geq0\colon|Z_k^{(\mathsf{t})}|=n\}\quad\text{ and }\quad\Sigma_n^{(\mathsf{t})}:=Z^{(\mathsf{t})}_{H_n^{(\mathsf{t})}}.$$

We write $\mu_n^{(\mathsf{t})}$ for the distribution of $\Sigma_n^{(\mathsf{t})}$, considered as a measure on $\mathsf{t}$ supported by $\mathsf{t}_n$.

For $0\leq p\leq n$, we set

$$L_p^{(\mathsf{t})}:=\sup\{k\leq H_n^{(\mathsf{t})}\colon|Z_k^{(\mathsf{t})}|=p\}.$$

Clearly $\Sigma_n^{(\mathsf{t})}\in\widetilde{\mathsf{t}}[Z^{(\mathsf{t})}_{L_p^{(\mathsf{t})}}]$, and therefore $Z^{(\mathsf{t})}_{L_p^{(\mathsf{t})}}=\langle\Sigma_n^{(\mathsf{t})}\rangle_p$.

**Lemma 3.7** (Lemma 20 in [5])**.** *Let $p\in\{0,1,\dots,n-1\}$ and $z\in\mathsf{t}_p$. Then, conditionally on $\langle\Sigma_n^{(\mathsf{t})}\rangle_p=z$, the process*

$$\Big(Z^{(\mathsf{t})}_{(L_p^{(\mathsf{t})}+k)\wedge H_n^{(\mathsf{t})}}\Big)_{k\geq0}$$

*is distributed as simple random walk on $\widetilde{\mathsf{t}}[z]$ starting from $z$ and conditioned to hit $\widetilde{\mathsf{t}}[z]\cap\mathsf{t}_n$ before returning to $z$, and stopped at this hitting time. Consequently, for every integer $q\in\{0,1,\dots,n-p\}$, the conditional distribution of*

$$\frac{\mu_n^{(\mathsf{t})}\big(B_{\mathsf{t}}(\Sigma_n^{(\mathsf{t})},q)\big)}{\mu_n^{(\mathsf{t})}\big(B_{\mathsf{t}}(\Sigma_n^{(\mathsf{t})},n-p)\big)}$$

*knowing that $\langle\Sigma_n^{(\mathsf{t})}\rangle_p=z$ is equal to the distribution of*

$$\mu_{n-p}^{(\mathsf{t}[z])}\big(B_{\mathsf{t}[z]}(\Sigma_{n-p}^{(\mathsf{t}[z])},q)\big).$$

### 3.5 The subtree selected by the discrete harmonic measure

We begin by introducing the conductance of discrete trees. Let $i$ be a positive integer and let $\mathsf{t} \in \mathscr{T}$ be a tree such that $h(\mathsf{t}) \geq i$. Consider the new graph $\mathsf{t}'$ obtained by adding to the graph $\mathsf{t}$ an edge between the root $\varnothing$ and an extra vertex $\partial$. We denote by $\mathcal{C}_i(\mathsf{t})$ the effective conductance between $\partial$ and generation $i$ of $\mathsf{t}$ in the graph $\mathsf{t}'$. In probabilistic terms, it is equal to the probability that simple random walk on $\mathsf{t}'$ starting from $\varnothing$ hits generation $i$ of $\mathsf{t}$ before hitting the vertex $\partial$.

Recall that for $i \in \{1, \ldots, n-1\}$, $\widetilde{\mathsf{T}}^{*n}[\langle \Sigma_n \rangle_{n-i}]$ is the subtree of $\mathsf{T}^{*n}$ above generation $n-i$ that is selected by harmonic measure, and $\mathsf{T}^{*n}[\langle \Sigma_n \rangle_{n-i}]$ is the tree obtained by relabeling the vertices of $\widetilde{\mathsf{T}}^{*n}[\langle \Sigma_n \rangle_{n-i}]$ as explained above.

**Lemma 3.8.** *For every integer $i \in \{1, \ldots, n-1\}$ and every nonnegative function $F$ on $\mathscr{T}$,*

$$\mathbb{E} \otimes E\big[F\big(\mathsf{T}^{*n}[\langle \Sigma_n \rangle_{n-i}]\big)\big] \leq (i+1)\, \mathbb{E}\big[\mathcal{C}_i(\mathsf{T}^{*i})\, F(\mathsf{T}^{*i})\big].$$

This lemma is proved in [5] under the assumption that $\rho$ has finite variance. Actually the proof uses only the branching property of Galton–Watson trees and remains valid under our assumptions on $\rho$.

Meanwhile, we have the following moment estimate for the conductance $\mathcal{C}_n(\mathsf{T}^{*n})$.

**Lemma 3.9.** *For every $r \in (0, \alpha)$, there exists a constant $K = K(r, \rho) \geq 1$ depending on $r$ and the offspring distribution $\rho$ such that, for every integer $n \geq 1$,*

$$\mathbb{E}\big[\mathcal{C}_n(\mathsf{T}^{*n})^r\big] \leq \frac{K}{(n+1)^r}\,.$$

*Proof.* We can assume $n \geq 2$, and set $j = \lfloor n/2 \rfloor \geq 1$. An application of the Nash–Williams inequality [14, Chapter 2] gives

$$\mathcal{C}_n(\mathsf{T}^{*n}) \leq \frac{\#\mathsf{T}^{*n}_j}{j}.$$

On the other hand,

$$\begin{aligned}
\mathbb{E}\big[(\#\mathsf{T}^{*n}_j)^r\big] &= \mathbb{E}\big[(\#\{v \in \mathsf{T}^{(0)}_j : h(\mathsf{T}^{(0)}[v]) \geq n-j\})^r \mid h(\mathsf{T}^{(0)}) \geq n\big] \\
&= q_n^{-1}\, \mathbb{E}\big[(\#\{v \in \mathsf{T}^{(0)}_j : h(\mathsf{T}^{(0)}[v]) \geq n-j\})^r\big].
\end{aligned}$$

Notice that given $\#\mathsf{T}^{(0)}_j = k$, the conditional distribution of

$$\#\{v \in \mathsf{T}^{(0)}_j : h(\mathsf{T}^{(0)}[v]) \geq n-j\}$$

is the binomial distribution $\mathcal{B}(k, q_{n-j})$. Using Jensen's inequality, we get

$$\begin{aligned}
\mathbb{E}\Big[(\#\{v \in \mathsf{T}^{(0)}_j : h(\mathsf{T}^{(0)}[v]) \geq n-j\})^r\Big] &\leq \mathbb{E}\Big[\mathbb{E}\Big[(\#\{v \in \mathsf{T}^{(0)}_j : h(\mathsf{T}^{(0)}[v]) \geq n-j\})^2 \mid \#\mathsf{T}^{(0)}_j\Big]^{\frac{r}{2}}\Big] \\
&= \mathbb{E}\Big[\Big(q^2_{n-j}(\#\mathsf{T}^{(0)}_j)^2 + (q_{n-j} - q^2_{n-j})\#\mathsf{T}^{(0)}_j\Big)^{\frac{r}{2}}\Big] \\
&\leq q^r_{n-j}\, \mathbb{E}\big[(\#\mathsf{T}^{(0)}_j)^r\big] + (q_{n-j} - q^2_{n-j})^{\frac{r}{2}}\, \mathbb{E}\big[(\#\mathsf{T}^{(0)}_j)^{\frac{r}{2}}\big].
\end{aligned}$$

At this point, we need the following result proved in [9, Lemma 11] for the unconditioned Galton–Watson tree. For any $\gamma \in (0, \alpha)$, there is a finite constant $C(\gamma)$ such that for every $m \geq 1$,

$$\mathbb{E}\big[(\#\mathsf{T}^{(0)}_m)^\gamma\big] \leq C(\gamma)\, q_m^{1-\gamma}\,. \tag{3.7}$$

The original statement of the latter bound in [9] was given for any $\gamma \in [1, \alpha)$, while the case $\gamma \in (0,1)$ follows from the (trivial) case $\gamma = 1$ by applying the Hölder inequality to

$$\mathbb{E}\big[\mathbf{1}_{\{\mathsf{T}^{(0)}_m \neq \emptyset\}}(\#\mathsf{T}^{(0)}_m)^\gamma\big]$$

(we can in fact take $C(\gamma) = 1$ for any $\gamma \in (0,1)$).

With the help of (3.7), we conclude that

$$\mathbb{E}\big[\mathcal{C}_n(\mathsf{T}^{*n})^r\big] \leq j^{-r} q_n^{-1} \left( C(r) q_{n-j}^r q_j^{1-r} + C(r/2)(q_{n-j} - q_{n-j}^2)^{\frac{r}{2}} q_j^{1-\frac{r}{2}} \right),$$

and the statement of the lemma readily follows from (3.2). □

### 3.6 Proof of Theorem 1.1

Following [5], we will show

$$\mathbb{E} \otimes E\big[ | \log \mu_n(\Sigma_n) + \beta \log n | \big] = o(\log n) \qquad \text{as } n \to \infty, \tag{3.8}$$

which is sufficient for establishing Theorem 1.1. The proof given below is adapted from [5, Section 4.3.2]. For later convenience, we introduce the notation

$$\bar{\alpha} := \frac{\alpha+1}{2} \in (1, \frac{3}{2})$$

and its Hölder conjugate

$$\alpha^* := \frac{\bar{\alpha}}{\bar{\alpha}-1} \in (3, \infty).$$

Fix $\xi > 0$. Let $\varepsilon > 0$ and $n_0 \geq 1$ be such that the conclusion of Proposition 3.6 holds for every $n \geq n_0$ with the exponent $r = \alpha^*$. Without loss of generality, we may and will assume that $\varepsilon = 1/N$, for some integer $N \geq 4$, which is fixed throughout the proof. We also fix a constant $\gamma > 0$, such that $\gamma \log N < 1/2$.

Let $n > N$ be sufficiently large so that $N^{\lfloor \gamma \log n \rfloor} \geq n_0$. Then we let $\ell \geq 1$ be the unique integer such that $N^\ell < n \leq N^{\ell+1}$, and write

$$\log \mu_n(\Sigma_n) = \log \frac{\mu_n(\Sigma_n)}{\mu_n(B(\Sigma_n, N))} + \sum_{j=2}^{\ell} \log \frac{\mu_n(B(\Sigma_n, N^{j-1}))}{\mu_n(B(\Sigma_n, N^j))} + \log \mu_n(B(\Sigma_n, N^\ell)). \tag{3.9}$$

To simplify notation, we set

$$\begin{aligned}
A_1^n &:= \log \frac{\mu_n(\Sigma_n)}{\mu_n(B(\Sigma_n, N))} + \beta \log N, \\
A_j^n &:= \log \frac{\mu_n(B(\Sigma_n, N^{j-1}))}{\mu_n(B(\Sigma_n, N^j))} + \beta \log N \quad \text{for every } j \in \{2, \ldots, \ell\}, \\
A_{\ell+1}^n &:= \log \mu_n(B(\Sigma_n, N^\ell)) + \beta \log(n/N^\ell)\,.
\end{aligned}$$

From (3.9), we see that

$$\mathbb{E} \otimes E\Big[ \big| \log \mu_n(\Sigma_n) + \beta \log n \big| \Big] = \mathbb{E} \otimes E\Big[ \Big| \sum_{j=1}^{\ell+1} A_j^n \Big| \Big] \leq \sum_{i=1}^{\ell+1} \mathbb{E} \otimes E[|A_j^n|]. \tag{3.10}$$

We will bound each term in the sum of the right-hand side.

First step: A priori bounds. We verify that, for every $j \in \{1, 2, \ldots, \ell+1\}$,

$$\mathbb{E} \otimes E\big[|A_j^n|\big] \leq (CK^{1/\bar{\alpha}} + \beta) \log N, \tag{3.11}$$

where $C = C(\alpha^*, \rho)$ is the constant in Lemma 3.1 for the exponent $r = \alpha^*$, and $K = K(\bar{\alpha}, \rho)$ is the constant in Lemma 3.9 for the exponent $r = \bar{\alpha}$.

Suppose first that $2 \leq j \leq \ell$. Using the second assertion of Lemma 3.7, with $p = n - N^j$ and $q = N^{j-1}$, we obtain that, for every $z \in \mathsf{T}^{*n}_{n-N^j}$, the conditional distribution of $A_j^n$ under $P$, knowing that $\langle \Sigma_n \rangle_{n-N^j} = z$, is the same as the distribution of

$$\log \mu_{N^j}^{(\mathsf{T}^{*n}[z])}(B(\Sigma_{N^j}^{(\mathsf{T}^{*n}[z])}, N^{j-1})) + \beta \log N.$$

Recalling that $\mu_{N^j}^{(\mathsf{T}^{*n}[z])}$ is the distribution of $\Sigma_{N^j}^{(\mathsf{T}^{*n}[z])}$ under $P$, we get

$$\begin{aligned} E\big[|A_j^n| \mid \langle\Sigma_n\rangle_{n-N^j}=z\big] &\le E\Big[\big|\log\mu_{N^j}^{(\mathsf{T}^{*n}[z])}\big(B(\Sigma_{N^j}^{(\mathsf{T}^{*n}[z])},N^{j-1})\big)\big|\Big]+\beta\log N \\ &= G_j(\mathsf{T}^{*n}[z])+\beta\log N, \end{aligned} \tag{3.12}$$

where for any tree $\mathsf{t}\in\mathscr{T}_{N^j}$,

$$G_j(\mathsf{t}):=\int\mu_{N^j}^{(\mathsf{t})}(\mathrm{d}y)\,\big|\log\mu_{N^j}^{(\mathsf{t})}(B_{\mathsf{t}}(y,N^{j-1}))\big|=\sum_{z\in\mathsf{t}_{N^j-N^{j-1}}}\mu_{N^j}^{(\mathsf{t})}(\widetilde{\mathsf{t}}[z])\,\big|\log\mu_{N^j}^{(\mathsf{t})}(\widetilde{\mathsf{t}}[z])\big|.$$

As explained in [5], we have the entropy bound $G_j(\mathsf{t})\le\log\#\mathsf{t}_{N^j-N^{j-1}}$ for any tree $\mathsf{t}\in\mathscr{T}_{N^j}$. So we get from (3.12) that

$$\begin{aligned} \mathbb{E}\otimes E\big[|A_j^n|\big] &\le \mathbb{E}\otimes E\big[\log\#\mathsf{T}^{*n}_{N^j-N^{j-1}}[\langle\Sigma_n\rangle_{n-N^j}]\big]+\beta\log N \\ &\le (N^j+1)\,\mathbb{E}\Big[\mathcal{C}_{N^j}(\mathsf{T}^{*N^j})\,\log\#\mathsf{T}^{*N^j}_{N^j-N^{j-1}}\Big]+\beta\log N \\ &\le (N^j+1)\,\mathbb{E}\Big[\big(\mathcal{C}_{N^j}(\mathsf{T}^{*N^j})\big)^{\bar\alpha}\Big]^{1/\bar\alpha}\,\mathbb{E}\Big[\big(\log\#\mathsf{T}^{*N^j}_{N^j-N^{j-1}}\big)^{\alpha^*}\Big]^{1/\alpha^*}+\beta\log N \\ &\le K^{1/\bar\alpha}\,\mathbb{E}\Big[\big(\log\#\mathsf{T}^{*N^j}_{N^j-N^{j-1}}\big)^{\alpha^*}\Big]^{1/\alpha^*}+\beta\log N, \end{aligned}$$

using successively Lemma 3.8, the Hölder inequality and Lemma 3.9. Finally, Lemma 3.1 gives

$$\mathbb{E}\Big[\big(\log\#\mathsf{T}^{*N^j}_{N^j-N^{j-1}}\big)^{\alpha^*}\Big]^{1/\alpha^*}\le C\,\log N,$$

and this completes the proof of (3.11) when $2\le j\le\ell$. The cases $j=1$ and $j=\ell+1$ can be treated in a similar manner. For details we refer the reader to [5, Section 4.3.2].

Second step: Refined bounds. Let us prove that, if $\lfloor\gamma\log n\rfloor\le j\le\ell$,

$$\mathbb{E}\otimes E\big[|A_j^n|\big]\le K^{1/\bar\alpha}\xi^{1/\alpha^*}\log N. \tag{3.13}$$

Recall that for $j\in\{\lfloor\gamma\log n\rfloor,\dots,\ell\}$ we have $N^j\ge n_0$. From (3.12), we have

$$E\big[|A_j^n|\big]=E\big[F_j(\mathsf{T}^{*n}[\langle\Sigma_n\rangle_{n-N^j}])\big],$$

where, if $\mathsf{t}\in\mathscr{T}_{N^j}$,

$$F_j(\mathsf{t}):=\big|\beta\log N-G_j(\mathsf{t})\big|=\left|\int\mu_{N^j}^{(\mathsf{t})}(\mathrm{d}y)\,\Big(\log\mu_{N^j}^{(\mathsf{t})}(B_{\mathsf{t}}(y,N^{j-1}))+\beta\log N\Big)\right|.$$

Using Lemma 3.8 as in the first step, we have

$$\mathbb{E}\otimes E\big[|A_j^n|\big]=\mathbb{E}\otimes E\big[F_j(\mathsf{T}^{*n}[\langle\Sigma_n\rangle_{n-N^j}])\big]\le(N^j+1)\mathbb{E}\Big[\mathcal{C}_{N^j}(\mathsf{T}^{*N^j})\,F_j(\mathsf{T}^{*N^j})\Big].$$

We then apply the Hölder inequality together with the bound of Lemma 3.9 for $r=\bar\alpha$ to get

$$\begin{aligned} \mathbb{E}\otimes E\left[|A_j^n|\right] &\le K^{1/\bar\alpha}\,\mathbb{E}\big[F_j(\mathsf{T}^{*N^j})^{\alpha^*}\big]^{1/\alpha^*} \\ &\le K^{1/\bar\alpha}\,\mathbb{E}\bigg[\bigg(\int\mu_{N^j}(\mathrm{d}y)\,\big|\log\mu_{N^j}(B(y,N^{j-1}))+\beta\log N\big|\bigg)^{\alpha^*}\bigg]^{1/\alpha^*} \\ &\le K^{1/\bar\alpha}\,\mathbb{E}\bigg[\int\mu_{N^j}(\mathrm{d}y)\,\big|\log\mu_{N^j}(B(y,N^{j-1}))+\beta\log N\big|^{\alpha^*}\bigg]^{1/\alpha^*} \\ &= K^{1/\bar\alpha}\cdot\mathbb{E}\otimes E\bigg[\Big|\log\mu_{N^j}^{1/N}\Big(\langle\Sigma_{N^j}\rangle_{N^j-N^{j-1}}\Big)+\beta\log N\Big|^{\alpha^*}\bigg]^{1/\alpha^*}, \end{aligned}$$

where the last equality follows from the definition of the measure $\mu_n^\varepsilon$ at the beginning of Section 3.3. Now recall that $1/N = \varepsilon$ and note that $N^j - N^{j-1} = N^j - \varepsilon N^j$. Since we have $N^j \geq n_0$, we can apply Proposition 3.6 with $r = \alpha^*$ and get that the right-hand side of the preceding display is bounded above by $K^{1/\bar{\alpha}}\xi^{1/\alpha^*}\log N$, which finishes the proof of (3.13).

By combining (3.11) and (3.13), and using (3.10), we arrive at the bound

$$\begin{aligned}\mathbb{E}\otimes E\left[\left|\log\mu_n(\Sigma_n)+\beta\log n\right|\right] &\leq \lfloor\gamma\log n\rfloor(K^{1/\bar{\alpha}}C+\beta)\log N+\ell\,K^{1/\bar{\alpha}}\xi^{1/\alpha^*}\log N\\ &\leq \left(\gamma(K^{1/\bar{\alpha}}C+\beta)\log N+K^{1/\bar{\alpha}}\xi^{1/\alpha^*}\right)\log n,\end{aligned}$$

which holds for every sufficiently large $n$. By choosing $\xi$ and then $\gamma$ arbitrarily small, we see that our claim (3.8) follows from the last bound, and this completes the proof of Theorem 1.1.

## 4 Comments and questions

Following [5, Section 5.2], let us consider the supercritical offspring distribution $\theta_\alpha^{(n)}$ of index $\alpha \in (1,2]$, defined as $\theta_\alpha^{(n)}(1) = 1 - \frac{1}{n}$ and

$$\theta_\alpha^{(n)}(k) = \frac{1}{n}\theta_\alpha(k) \quad \text{for every } k \geq 2.$$

We let $\mathrm{T}_\alpha^{(n)}$ be an infinite Galton–Watson tree with offspring distribution $\theta_\alpha^{(n)}$, then $n^{-1}\mathrm{T}_\alpha^{(n)}$ viewed as a metric space with the graph distance rescaled by the factor $n^{-1}$, converges in distribution in an appropriate sense (e.g. for the local Gromov–Hausdorff topology) to the CTGW tree $\Gamma^{(\alpha)}$, as $n \to \infty$.

Consider then the biased random walk $(Z_k^{(n)})_{k\geq 0}$ on $\mathrm{T}_\alpha^{(n)}$ with bias parameter $\lambda^{(n)} = 1 - \frac{1}{n}$ towards the root (see [13] or [1] for a precise definition of this process). Then heuristically the rescaled process

$$\left(n^{-1}Z^{(n)}_{\lfloor n^2 t\rfloor}\right)_{t\geq 0}$$

converge in distribution in some sense, as $n \to \infty$, to Brownian motion $(W(t))_{t\geq 0}$ with drift $1/2$ on the CTGW tree $\Gamma^{(\alpha)}$. Furthermore, the rescaled "conductance" $n\,\mathcal{C}(\mathrm{T}_\alpha^{(n)}, \lambda^{(n)})$ converges in distribution to the continuous conductance $\mathcal{C}^{(\alpha)} = \mathcal{C}(\Gamma^{(\alpha)})$.

Following this informal passage to the limit, we can find a candidate for the limit of $n\mathbf{V}_\alpha^{(n)}$ as $n \to \infty$, where $\mathbf{V}_\alpha^{(n)}$ stands for the speed of the biased random walk $Z^{(n)}$ on $\mathrm{T}_\alpha^{(n)}$. One can either directly employ an explicit formula of $\mathbf{V}_\alpha^{(n)}$ stated in [1, Theorem 1.1], or use the invariant measure for the environment seen from the random walker ([1, Theorem 4.1]) to calculate the speed as the proportion of last-exit points. Both methods give rise to the following quantity which should be interpreted as the speed of Brownian motion $W$ with drift $1/2$ on $\Gamma^{(\alpha)}$,

$$\mathbf{V}_\alpha := \frac{\mathbb{E}\left[\frac{\mathcal{C}_0^{(\alpha)}\mathcal{C}_1^{(\alpha)}}{\mathcal{C}_0^{(\alpha)}+\mathcal{C}_1^{(\alpha)}-1}\right]}{\mathbb{E}\left[\frac{2\mathcal{C}_0^{(\alpha)}}{\mathcal{C}_0^{(\alpha)}+\mathcal{C}_1^{(\alpha)}-1}\right]}, \tag{4.1}$$

where $\mathcal{C}_0^{(\alpha)}$ and $\mathcal{C}_1^{(\alpha)}$ are two independent copies of $\mathcal{C}^{(\alpha)}$ under the probability measure $\mathbb{P}$.

Since the conductance $\mathcal{C}^{(\alpha)}$ is a.s. strictly larger than 1, we see immediately from (4.1) that $\mathbf{V}_\alpha < \frac{1}{2}$ for any $\alpha \in (1,2]$. On the other hand, according to the coupling explained in Section 2.4, the denominator of the right-hand side of (4.1)

$$\mathbb{E}\left[\frac{2\mathcal{C}_0^{(\alpha)}}{\mathcal{C}_0^{(\alpha)}+\mathcal{C}_1^{(\alpha)}-1}\right] = \mathbb{E}\left[\frac{\mathcal{C}_0^{(\alpha)}+\mathcal{C}_1^{(\alpha)}}{\mathcal{C}_0^{(\alpha)}+\mathcal{C}_1^{(\alpha)}-1}\right] = 1+\mathbb{E}\left[\frac{1}{\mathcal{C}_0^{(\alpha)}+\mathcal{C}_1^{(\alpha)}-1}\right]$$

is increasing with respect to $\alpha$.

**Question 4.1.** If we apply the coupling explained in Section 2.4, does the derivative $\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathcal{C}^{(\alpha)}$ of the conductance with respect to $\alpha$ exist almost surely?

An affirmative answer to Question 4.1 would allow us to take the derivative of the numerator in (4.1) with respect to $\alpha$, and to see that

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathbb{E}\Big[\frac{\mathcal{C}_0^{(\alpha)}\mathcal{C}_1^{(\alpha)}}{\mathcal{C}_0^{(\alpha)}+\mathcal{C}_1^{(\alpha)}-1}\Big]=\mathbb{E}\Bigg[\frac{\mathcal{C}_0^{(\alpha)}(\mathcal{C}_0^{(\alpha)}-1)\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathcal{C}_1^{(\alpha)}+\mathcal{C}_1^{(\alpha)}(\mathcal{C}_1^{(\alpha)}-1)\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathcal{C}_0^{(\alpha)}}{\big(\mathcal{C}_0^{(\alpha)}+\mathcal{C}_1^{(\alpha)}-1\big)^2}\Bigg]\leq 0$$

because a.s. $\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathcal{C}^{(\alpha)}\leq 0$. Hence, the numerator in the right-hand side of (4.1) would be decreasing with respect to $\alpha$, and so would be the speed $\mathbf{V}_\alpha$.

**Question 4.2.** Does the speed $\mathbf{V}_\alpha$ decrease with respect to $\alpha$?

A similar question was raised in [3], concerning the monotonicity of the speed with respect to the offspring distribution for biased random walk on Galton–Watson trees with no leaves. It has been proved in [15] that this monotonicity holds for high values of bias.

We also want to ask the same question of monotonicity for the Hausdorff dimension of the continuous harmonic measure $\mu_\alpha$.

**Question 4.3.** Does the Hausdorff dimension $\beta_\alpha$ decrease with respect to $\alpha$?

Finally, it is interesting to figure out the Hausdorff dimension of the harmonic measure on $\partial\Delta^{(1)}$. Due to the fact that $\theta_1$ has infinite mean, it may require different methods to treat the case $\alpha=1$ in the continuous setting.

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