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Subcritical contact processes seen from a typical infected site*

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Abstract

What is the long-time behavior of the law of a contact process started with a single infected site, distributed according to counting measure on the lattice? This question is related to the configuration as seen from a typical infected site and gives rise to the definition of so-called eigenmeasures, which are possibly infinite measures on the set of nonempty configurations that are preserved under the dynamics up to a time-dependent exponential factor. In this paper, we study eigenmeasures of contact processes on general countable groups in the subcritical regime. We prove that in this regime, the process has a unique spatially homogeneous eigenmeasure. As an application, we show that the law of the process as seen from a typical infected site, chosen according to a Campbell law, converges to a long-time limit. We also show that the exponential decay rate of the expected number of infected sites is continuously differentiable and strictly increasing as a function of the recovery rate, and we give a formula for the derivative in terms of the long time limit law of the process as seen from a typical infected site.

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1 Introduction and main results

1.1 Introduction

In this paper, we will be interested in contact processes whose underlying lattice can be any countable group. Our set-up includes the classical contact process on $\mathbb{Z}^d$ and many other lattices, such as regular trees. We refer to [Lig99] as a general reference to contact processes on $\mathbb{Z}^d$ and trees. An important motivation for studying general lattices is that the behavior of the process may depend on the lattice, and one wants to understand this dependency. For example, in [Swa09], it was proved that if a contact process on a nonamenable group survives, then the expected number of infected sites

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must grow exponentially fast. The present paper uses many ideas and methods from that article and builds on it. For that reason, we use the same general set-up, even though it will turn out that our results in the present paper are not lattice dependent.

In the present paper, we will be interested in subcritical processes, i.e., we focus on the parameter regime where the system dies out a.s. In this regime, we study three closely related topics: the process ‘as seen from a typical infected site’, the exponential rate associated with the growth (or decay) of the expected number of infected sites, and ‘eigenmeasures’.

Here, with the process ‘as seen from a typical infected site’ we mean the following. Starting from a single infected site, we size-bias on the number of infected sites at a given time, and then choose one ‘typical’ infected site with equal probabilities from all infected sites. Shifting the process so that this site becomes the origin then yields the process ‘as seen from a typical infected site’. In Theorem 1 below, we show that in the subcritical regime, this process has a long-time limit law. Whether such a limit law exists in general is an open problem.

An important reason why we are interested in the process as seen from a typical infected site (chosen in this particular way) is that its law is closely connected to the quantity $r := \lim_{t\to\infty} t^{-1}\log\mathbb{E}[|\eta_t^{\{0\}}|]$, where $\eta^{\{0\}}$ is the process started with only the origin infected and $|\eta_t^{\{0\}}|$ is the number of infected sites at time t. This limit is known to exist quite generally by subadditivity; we simply call it the ‘exponential growth rate’. It is known that $r < 0$ in the whole subcritical regime (see Theorem 2 (d) below), i.e., extinction, when it happens, is exponentially fast. One of our main results (Theorem 4) says that in the subcritical regime, r is continuously differentiable as a function of the recovery rate, and gives an expression for the derivative in terms of the limit laws of the process and its dual, both seen from a typical infected site.

Our final results concern processes started in possibly infinite, translation invariant initial measures. In particular, a special role will be played by the process started with a single infected site, which is distributed according to counting measure on Λ. If Λ is infinite, then such an initial ‘law’ can of course not be normalized, but it can still be used to define conditional probabilities given an event of finite measure. In particular, we will see in formula (1.22) that conditioning such a process on the origin being infected at a given time t yields in law the same as the process ‘seen from a typical infected site’ that we considered before. This new way of looking at things has a number of advantages. In particular, we will see that there are certain (possibly infinite) measures that are preserved under the dynamics of the process up to an exponential scaling that depends on time. We call such measures eigenmeasures. In Theorem 5, we prove that in the subcritical regime, the process has a unique translation invariant eigenmeasure, which is the rescaled limit ‘law’ when started from any translation invariant, possibly infinite initial ‘law’, and we use this to characterize the limit law of the process as seen from a typical infected site. In particular, Theorem 5 answers the question, asked at the start of our abstract, about the long-time behavior of the process started with a single infected site, distributed according to counting measure on the lattice.

There is a close analogy between eigenmeasures and quasi-stationary laws, as introduced in [DS67]. In the subcritical regime, this relation can be made exact: we will show that above the critical recovery rate, there is a one-to-one correspondence between eigenmeasures and quasi-stationary laws for the contact process ‘modulo shifts’. This correspondence plays an important role in our proofs but is available only in the subcritical regime, in contrast to the eigenmeasures that are known to exist quite generally (see [Swa09] and Section 1.7 below).

We discuss the process as seen from a typical infected site in Section 1.3 and eigenmeasures in Section 1.7. Quasi-invariant laws for the process modulo shifts are discussed briefly in Section 1.8 and then in more detail in Section 2.

1.2 Contact processes on groups

Before we can state our results, we need to define the class of contact processes that we will be interested in, fix notation, and recall some well-known facts. Let Λ be a finite or countably infinite group with group action $(i,j)\mapsto ij$, inverse operation $i\mapsto i^{-1}$, and unit element 0 (also refered to as the origin). Let $a:\Lambda\times\Lambda\to[0,\infty)$ be a function such that $a(i,i)=0$ $(i\in\Lambda)$ and

$$\begin{aligned} &\text{(i)}\quad a(i,j)=a(ki,kj) \qquad\qquad (i,j,k\in\Lambda),\\ &\text{(ii)}\quad |a|:=\sum_{i\in\Lambda}a(0,i)<\infty, \end{aligned} \tag{1.1}$$

and let $\delta\geq 0$. We will in general not assume that $a(i,j)=a(j,i)$. However, if this is true then we say that a is *symmetric*. By definition, the (Λ,a,δ)-*contact process* is the Markov process $\eta=(\eta_t)_{t\geq 0}$, taking values in the space $\mathcal{P}=\mathcal{P}(\Lambda):=\{A:A\subset\Lambda\}$ consisting of all subsets of Λ, with the formal generator

$$\begin{aligned} Gf(A) := &\sum_{i,j\in\Lambda}a(i,j)1_{\{i\in A\}}1_{\{j\notin A\}}\{f(A\cup\{j\})-f(A)\}\\ &+\delta\sum_{i\in\Lambda}1_{\{i\in A\}}\{f(A\backslash\{i\})-f(A)\}. \end{aligned} \tag{1.2}$$

If $i\in\eta_t$, then we say that the site i is infected at time t; otherwise it is healthy. Then (1.2) says that an infected site i infects another site j with *infection rate* $a(i,j)\geq 0$, and infected sites become healthy with *recovery rate* $\delta\geq 0$.

To see how this relates to the more traditional definition of contact processes on $\mathbb{Z}^d$, let us assume that the infection rates take only two values: there is some $\lambda>0$ ('the infection rate') such that $a(i,j)\in\{0,\lambda\}$ for all $i,j\in\Lambda$. Assume moreover that $\Delta:=\{i:a(0,i)=\lambda\}$ is a finite subset of Λ that is symmetric in the sense that $i\in\Delta$ implies $i^{-1}\in\Delta$ and that Δ generates Λ. By definition, the (left) Cayley graph $\mathcal{G}(\Lambda,\Delta)$ associated with Λ and Δ is the graph with vertex set Λ, where two vertices $i,j\in\Lambda$ are linked by an edge if and only if $j=ik$ for some $k\in\Delta$. Then $a(i,j)=\lambda$ if i and j are linked by an edge in $\mathcal{G}(\Lambda,\Delta)$ and zero otherwise, i.e., the (Λ,a,δ)-contact process is the nearest-neighbor contact process on the Cayley graph $\mathcal{G}(\Lambda,\Delta)$ with infection rate λ and recovery rate δ.

The square lattice $\mathbb{Z}^d$ with nearest neighbor edges, and regular trees, are examples of Cayley graphs, so our set-up includes these classical lattices. Processes on graphs have only two free parameters, λ and δ, and by time scaling, one may without loss of generality take one of these rates to be one. It is tradition to fix the recovery rate. In our more general set-up, it will often be more convenient to fix the infection rates and vary the recovery rate instead.

We will usually assume that the infection rates are irreducible in some sense or another. To make this precise, let us write $i\overset{a}{\hookrightarrow}j$ if the site j can be infected through a chain of infections starting from i. Then we say that a is *irreducible* if $i\overset{a}{\hookrightarrow}j$ for all $i,j\in\Lambda$. Equivalently, this says that for all $\Lambda'\subset\Lambda$ with $\Lambda'\neq\emptyset,\Lambda$, there exist $i\in\Lambda'$ and $j\in\Lambda\backslash\Lambda'$ such that $a(i,j)>0$. Similarly, we say that a is *weakly irreducible* if for all $\Lambda'\subset\Lambda$ with $\Lambda'\neq\emptyset,\Lambda$, there exist $i\in\Lambda'$ and $j\in\Lambda\backslash\Lambda'$ such that $a(i,j)\vee a(j,i)>0$. Finally, we will sometimes need the intermediate condition

$$\forall i,j\in\Lambda:\ \exists k,l\in\Lambda:\ k\overset{a}{\hookrightarrow}i,\ k\overset{a}{\hookrightarrow}j,\ i\overset{a}{\hookrightarrow}l,\ j\overset{a}{\hookrightarrow}l. \tag{1.3}$$

In words, this says that for any two sites i,j there exists a site k from which both i and j can be infected, and a site l that can be infected both from i and from j. If the rates

a are symmetric, or more generally if one has $a(i,j) > 0$ iff $a(j,i) > 0$, then all three conditions are equivalent. In general, irreducibility implies (1.3) which implies weak irreducibility, but none of the converse implications hold.

It is well-known that contact processes can be constructed by a graphical representation. Let $\omega = (\omega^{\rm r}, \omega^{\rm i})$ be a pair of independent, locally finite random subsets of $\Lambda \times \mathbb{R}$ and $\Lambda \times \Lambda \times \mathbb{R}$, respectively, produced by Poisson point processes with intensity δ and $a(i,j)$, respectively. This is usually visualized by plotting Λ horizontally and $\mathbb{R}$ vertically, marking points $(i,s) \in \omega^{\rm r}$ with a recovery symbol (e.g., $*$), and drawing an infection arrow from (i,t) to (j,t) for each $(i,j,t) \in \omega^{\rm i}$. For any $(i,s),(j,u) \in \Lambda \times \mathbb{R}$ with $s \leq u$, by definition, an *open path* from (i,s) to (j,u) is a cadlag function $\pi : [s,u] \to \Lambda$ such that $\{(\pi(t),t) : t \in [s,u]\} \cap \omega^{\rm r} = \emptyset$ and $(\pi(t-),\pi(t),t) \in \omega^{\rm i}$ whenever $\pi(t-) \neq \pi(t)$. Thus, open paths must avoid recovery symbols and may follow infection arrows. We write $(i,s) \rightsquigarrow (j,u)$ to indicate the presence of an open path from (i,s) to (j,u). Then, for any $s \in \mathbb{R}$, we can construct a (Λ,a,δ)-contact process started in an initial state $A \in \mathcal{P}$ by setting

$$\eta^{A,s}_t := \{j \in \Lambda : (i,s) \rightsquigarrow (j,s+t) \text{ for some } i \in A\} \qquad (A \in \mathcal{P},\ s \in \mathbb{R},\ t \geq 0). \tag{1.4}$$

In particular, we set $\eta^A_t := \eta^{A,0}_t$. Note that this construction defines contact processes with different initial states on the same probability space, i.e., the graphical representation provides a natural coupling between such processes. Moreover, the graphical representation shows that the contact process is essentially a sort of oriented percolation model (in continuous time but discrete space).

Since the graphical representation is also defined for negative times we can, in analogy to (1.4), define 'backward' or 'dual' processes by

$$\eta^{\dagger\,A,s}_t := \{j \in \Lambda : (j,s-t) \rightsquigarrow (i,s) \text{ for some } i \in A\} \qquad (A \in \mathcal{P},\ s \in \mathbb{R},\ t \geq 0). \tag{1.5}$$

In particular, we set $\eta^{\dagger\,A}_t := \eta^{\dagger\,A,0}_t$. It is not hard to see that $(\eta^{\dagger\,A,s}_t)_{t\geq 0}$ is a $(\Lambda, a^\dagger, \delta)$-contact process, where we define *reversed infection rates* as $a^\dagger(i,j) := a(j,i)$. Since

$$\big\{\eta^A_t \cap B \neq \emptyset\big\} = \big\{(i,0) \rightsquigarrow (j,t) \text{ for some } i \in A,\ j \in B\big\} = \big\{\eta^A_0 \cap \eta^{\dagger\,B,t}_t \neq \emptyset\big\} \qquad (0 \leq s \leq t) \tag{1.6}$$

and the process $\eta^{\dagger\,B,t}$ is equal in law with $\eta^{\dagger\,B}$, we see that the (Λ,a,δ)-contact process and $(\Lambda,a^\dagger,\delta)$-contact process are dual in the sense that

$$\mathbb{P}[\eta^A_t \cap B \neq \emptyset] = \mathbb{P}[A \cap \eta^{\dagger\,B}_t \neq \emptyset] \qquad (A,B \in \mathcal{P},\ t \geq 0). \tag{1.7}$$

We note that unless a is symmetric or the group Λ is abelian, the (Λ,a,δ)- and $(\Lambda,a^\dagger,\delta)$-contact processes have in general different dynamics and need to be distinguished. (If Λ is abelian, then the (Λ,a,δ)- and $(\Lambda,a^\dagger,\delta)$-contact processes can be mapped into each other by the transformation $i \mapsto i^{-1}$.)

We say that the (Λ,a,δ)-contact process *survives* if $\mathbb{P}[\eta^A_t \neq \emptyset\ \forall t \geq 0] > 0$ for some, and hence for all nonempty A of finite cardinality $|A|$. We call

$$\delta_{\rm c} = \delta_{\rm c}(\Lambda,a) := \sup\big\{\delta \geq 0 :\ \text{the } (\Lambda,a,\delta)\text{-contact process survives}\big\} \tag{1.8}$$

the *critical recovery rate*. It is known that $\delta_{\rm c} < \infty$. If Λ is finite, then $\delta_{\rm c} = 0$, but if Λ is infinite, then it is often the case that $\delta_{\rm c} > 0$. In particular, this is true if Λ is finitely generated and a is weakly irreducible [Swa07, Lemma 4.18]. For non-finitely generated infinite groups, irreducibility is in general not enough to guarantee $\delta_{\rm c} > 0$ [AS10]. It is well-known that

$$\mathbb{P}\big[\eta^\Lambda_t \in \cdot\,\big] \underset{t\to\infty}{\Longrightarrow} \overline{\nu}, \tag{1.9}$$

where $\overline{\nu}$ is an invariant law of the (Λ, a, δ)-contact process, known as the *upper invariant law*. Using duality, it is not hard to prove that $\overline{\nu} = \delta_\emptyset$ if the dual $(\Lambda, a^\dagger, \delta)$-contact process dies out, while $\overline{\nu}$ is concentrated on the nonempty subsets of Λ if the dual process survives. In this case, we say that $\overline{\nu}$ is *nontrivial*.

1.3 The process seen from a typical infected site

Let $(\eta_t^A)_{t\geq 0}$ be a (Λ, a, δ)-contact process, started in a finite nonempty initial state A, defined on some underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then, for each $t \geq 0$, we can define a new probability law $\hat{\mathbb{P}}_t^A$ on a suitably enriched probability space $\hat{\Omega}$ that also contains a Λ-valued random variable ι, by setting

$$\hat{\mathbb{P}}_t^A\big[\omega \in \mathcal{A},\ \iota = i\big] := \frac{\mathbb{P}[\omega \in \mathcal{A},\ i \in \eta_t^A(\omega)]}{\mathbb{E}[|\eta_t^A|]} \qquad (\mathcal{A} \in \mathcal{F},\ i \in \Lambda). \tag{1.10}$$

The law $\hat{\mathbb{P}}_t^A$ is a (normalized) Campbell law.[1] It is closely related to the so called Palm law. It is easy to check the following claims: $\hat{\mathbb{P}}_t^A$ is a probability law, $\hat{\mathbb{P}}_t^A\big[\omega \in \cdot\big]$ is the law $\mathbb{P}$ size-biased on $|\eta_t^A|$, and the conditional law $\hat{\mathbb{P}}_t^A\big[\iota \in \cdot\,|\,\omega\big]$ is the uniform distribution on η_t^A. In words, $\hat{\mathbb{P}}_t^A$ is obtained by size-biasing on the number of infected sites at time t and then choosing one site ι from η_t^A with equal probabilities. We note that if $(\eta_t^A)_{t\geq 0}$ is constructed from a graphical representation, which in turn is defined on an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then the Campbell laws $\hat{\mathbb{P}}_t^A$ in (1.10) allow us to discuss probabilities (under these laws) of events relating jointly to the typical site ι and the graphical representation ω. Campbell laws are a common tool in the study of spatial branching processes, see for example [LMW88] or [GKW99], but their use in the study of contact processes seems to be new and was initiated in [Swa09].

For $A \subset \Lambda$ and $i \in \Lambda$, let us write $iA := \{ij : j \in A\}$. Then $\hat{\mathbb{P}}_t^A[\iota^{-1}\eta_t^A \in \cdot\,]$ is the law of the contact process η_t^A with ι shifted to the origin, i.e., this is the law of the process 'as seen from' a typical infected site. Our first result, which is an easy consequence of the more general Theorem 5 below, says that for subcritical contact processes, these laws converge to a long-time limit.

Recall that $\mathcal{P} = \mathcal{P}(\Lambda)$ denotes the space of all subsets of Λ. Identifying subsets with their indicator functions, we observe that $\mathcal{P} \cong \{0,1\}^\Lambda$ and equip it with the product topology and Borel-σ-field. We let $\mathcal{P}_+ := \{A : |A| > 0\}$ and $\mathcal{P}_{\rm fin} := \{A : |A| < \infty\}$ denote the subspaces consisting of all nonempty, respectively finite subsets of Λ, and write $\mathcal{P}_{\rm fin,+} := \mathcal{P}_{\rm fin} \cap \mathcal{P}_+$. Note that $\mathcal{P}_{\rm fin}$ is countable. In view of this, apart from the product topology which it inherits from its embedding in $\mathcal{P}$, it is often natural (and important for technical reasons) to equip $\mathcal{P}_{\rm fin}$ with the discrete topology instead. If Λ is infinite, then the discrete topology on $\mathcal{P}_{\rm fin}$ is strictly stronger than the product topology on $\mathcal{P}$. For example, if $\Lambda = \mathbb{Z}$, then $\{0, n\} \to \{0\}$ as $n \to \infty$ in the product topology, but not in the discrete topology on $\mathcal{P}_{\rm fin}$.

Theorem 1 (Limit law seen from typical site, subcritical case)**.** *Assume that the infection rates satisfy the irreducibility condition (1.3) and that $\delta > \delta_{\rm c}$. Then there exists a probability law $\hat{\nu}$ on $\mathcal{P}_{\rm fin}$ such that*

$$\hat{\mathbb{P}}_t^A\big[\iota^{-1}\eta_t^A \in \cdot\,\big] \underset{t\to\infty}{\Longrightarrow} \hat{\nu} \qquad (A \in \mathcal{P}_{\rm fin,+}), \tag{1.11}$$

[1] We may view the random set η_t^A as a random measure on Λ that puts mass one on each point $i \in \eta_t^A$ (and zero mass on points outside η_t^A). According to terminology as used in, e.g., [LMW88, GKW99], the Campbell measure associated with η_t^A is then the finite measure $\mathcal{C}_t^A$ on $\mathcal{P}_{\rm fin,+} \times \Lambda$ defined by $\mathcal{C}_t^A(\{(B,i)\}) := \mathbb{P}[\eta_t^A = B]1_{\{i\in B\}}$. Normalizing this measure so that it becomes a probability measure yields the joint law of (η_t^A, ι) under $\hat{\mathbb{P}}_t^A$. Conditioning this measure on $\iota = i$ yields the Palm law relative to i. We will need to keep track of more information than just η_t^A, which is why we define $\hat{\mathbb{P}}_t^A$ on a larger space than just $\mathcal{P}_{\rm fin,+} \times \Lambda$. In branching theory one also often considers Palm measures that carry more information, for example about the genealogy of the process, which leads to Kallenberg's backward tree construction [Kal77].

where ⇒ denotes weak convergence of probability measures on $\mathcal{P}_{\rm fin}$, equipped with the discrete topology.

The proof of Theorem 1 will follow from the more general Theorem 5 and is completed in Section 2.9. We expect the limit in (1.11) to exist more generally, perhaps even for any (Λ, a, δ)-contact process, but we expect $\hat{\nu}$ to be concentrated on $\mathcal{P}_{\rm fin}$ only in the subcritical regime. In general, we expect convergence as in (1.11) only w.r.t. the product topology on $\mathcal{P}$.

1.4 The exponential growth rate

It follows from subadditivity (see [Swa09, Lemma 1.1]) that for any (Λ, a, δ)-contact process, there exists a constant $r = r(\Lambda, a, \delta)$ with $-\delta \leq r \leq |a| - \delta$ such that

$$r = \lim_{t\to\infty} \tfrac{1}{t} \log \mathbb{E}\big[|\eta_t^A|\big] \qquad (A \in \mathcal{P}_{\rm fin,+}). \tag{1.12}$$

We call r the *exponential growth rate*.

The following theorem, which we essentially cite from the literature, lists some properties of the function $r(\Lambda, a, \delta)$. In particular, part (d) says that the subcritical regime that we are interested in here coincides with the parameter regime where $r < 0$.

Theorem 2 (Properties of the exponential growth rate)**.**
For any (Λ, a, δ)-contact process:

(a) $r(\Lambda, a, \delta) = r(\Lambda, a^\dagger, \delta)$.

(b) *The function $\delta \to r(\Lambda, a, \delta)$ is nonincreasing and Lipschitz continuous on $[0, \infty)$, with Lipschitz constant 1.*

(c) *If $r(\Lambda, a, \delta) > 0$, then the (Λ, a, δ)-contact process survives.*

(d) $\{\delta \geq 0 : r(\Lambda, a, \delta) < 0\} = (\delta_{\rm c}, \infty)$.

The (easy) proofs of parts (a)–(c) can be found in [Swa09, Theorem 1.2]. The analogue of part (d) for unoriented percolation on $\mathbb{Z}^d$ was first proved by Menshikov [Men86] and Aizenman and Barsky [AB87]. Using the approach of the latter paper, Bezuidenhout and Grimmett [BG91, formula (1.13)] proved the statement in part (d) for contact processes on $\mathbb{Z}^d$. This has been generalized to processes on general transitive graphs in [AJ07]. As we point out in Appendix A, their arguments are not restricted to graphs but apply in the generality we need here.

We note that in general, if one drops the assumption that the underlying lattice is a group and the infection rates are invariant under left multiplication, then it is possible for a contact process to survive while its dual dies out.[2] Whether this can happen in our class of (Λ, a, δ)-contact processes is an open problem, but parts (a) and (d) of Theorem 2 imply that $\delta_{\rm c}(\Lambda, a) = \delta_{\rm c}(\Lambda, a^\dagger)$, so any example would have to be at $\delta = \delta_{\rm c}$, while by [Swa09, Corollary 1.3], Λ would have to be amenable. Part (a) of Theorem 2 is related to unimodularity, see the discussion in Section A.1 of the appendix.

If Λ is a finitely generated group of subexponential growth and the infection rates satisfy an exponential moment condition (for example, if $\Lambda = \mathbb{Z}^d$ and a is nearest-neighbor), then $r \leq 0$ [Swa09, Thm 1.2 (e)], but in general (e.g. on trees), it is possible that $r > 0$. Indeed, one of the main results of [Swa09] says that if Λ is nonamenable

[2] In this case, the upper invariant law of the process is trivial while the upper invariant law of the dual process is nontrivial. Consider, for example, a contact process on an infinite regular tree, where infections take place only in the direction away from a chosen end resp. towards the chosen end for the dual process. It is not hard to show that for a suitable choice of the revovery rate, such a process survives while its dual dies out.

(e.g., a regular tree), the (Λ, a, δ)-contact process survives, and the infection rates satisfy the irreducibility condition (1.3), then $r > 0$ [Swa09, Thm. 1.2 (f)].

The following theorem follows easily from results in [Swa09]; for completeness, we provide a proof in Section 2.9. Below, as before, η^A denotes the (Λ, a, δ)-contact process started in the initial state A.

Theorem 3 (Limit law seen from typical site, supercritical subexponential case)**.** *Assume that the infection rates satisfy the irreducibility condition (1.3), that $r(\Lambda, a, \delta) = 0$, and that the upper invariant law $\overline{\nu}$ of the (Λ, a, δ)-contact process is nontrivial. Let $\overline{\eta}$ be a random variable with law $\overline{\nu}$. Then, for each $A \in \mathcal{P}_{\mathrm{fin},+}$, there exist probability laws $(\rho_n)_{n\in\mathbb{N}}$ on $[0, \infty)$ such that $\lim_{n\to\infty} \rho_n([0, T]) \to 0$ for all $T < \infty$ and*

$$\int_0^\infty \rho_n(\mathrm{d}t)\, \hat{\mathbb{P}}_t^A\big[\iota^{-1}\eta_t^A \in \cdot\,\big] \underset{n\to\infty}{\Longrightarrow} \mathbb{P}\big[\overline{\eta} \in \cdot\,\big|0 \in \overline{\eta}\big], \tag{1.13}$$

where $\Rightarrow$ denotes weak convergence of probability measures on $\mathcal{P}$, equipped with the product topology.

In particular, Theorem 3 shows that for processes that grow slower than exponentially in the supercritical regime (e.g., the classical process on $\mathbb{Z}^d$), if the process as seen from a typical infected site has a long-time limit law, then this must be the upper invariant law conditioned on the origin being infected. So far, there are no results on the long-time limit law of the process as seen from a typical infected site for processes with $r > 0$.

1.5 The derivative of the exponential growth rate

As we have seen in the previous section, there is a close connection between the law of the process as seen from a typical infected site and the exponential growth rate of the process. In the present section, we elaborate on this and formulate our second main result (after Theorem 1), which says that in the subcritical regime, $r(\Lambda, a, \delta)$ is continuously differentiable as a function of the recovery rate δ, and which gives an expression of the derivative in terms of the law $\hat{\nu}$ from Theorem 1, and its analogue for the dual process.

To see how the derivative of r is connected to the law of the process as seen from a typical infected site, let $\eta_t^{\delta,\{0\}}$ denote the process with a given recovery rate δ (and (Λ, a) fixed), started with only the origin infected and constructed with the graphical representation. A version of Russo's formula (see [Swa09, formula (3.10)] and compare [Gri99, Thm 2.25]) tells us that

$$-\tfrac{\partial}{\partial\delta}\frac{1}{t}\log\mathbb{E}\big[|\eta_t^{\delta,\{0\}}|\big] = \frac{1}{t}\int_0^t \hat{\mathbb{P}}_t^{\{0\}}\big[\exists j \in \Lambda \text{ s.t. } (0,0) \rightsquigarrow_{(j,s)} (\iota, t)\big]\mathrm{d}s, \tag{1.14}$$

where $(0,0) \rightsquigarrow_{(j,s)} (\iota, t)$ denotes the event that in the graphical representation, all open paths from $(0,0)$ to (ι, t) lead through (j, s). In other words, the right-hand side of (1.14) is the fraction of time that there is a *pivotal* site on the way from $(0,0)$ to the typical site (ι, t).

The probability that at time s, there is a pivotal site on the way from $(0,0)$ to the typical site (ι, t), can be expressed in terms of two independent processes that are defined in terms of the graphical representation before and after s, respectively. To write this down, let $\hat{\eta}_s$ be a random variable with law

$$\mathbb{P}\big[\hat{\eta}_s \in \cdot\,\big] = \hat{\mathbb{P}}_s^{\{0\}}\big[\iota^{-1}\eta_s^{\{0\}} \in \cdot\,\big], \tag{1.15}$$

i.e., $\hat{\eta}_s$ is a contact process as seen from a typical site at time s. Define $\hat{\eta}_{t-s}^\dagger$ similarly, but for the dual process and with s replaced by $t - s$, and let $\hat{\eta}_s$ and $\hat{\eta}_{t-s}^\dagger$ be independent.

Then it can be shown (compare formula (2.58) below, which can be written in terms of $\hat\eta_s$ and $\hat\eta^\dagger_{t-s}$ as in formula (2.69)) that

$$\hat{\mathbb{P}}^{\{0\}}_t\big[\exists j\in\Lambda \text{ s.t. } (0,0)\rightsquigarrow_{(j,s)} (\iota,t)\big]=\frac{\mathbb{P}\big[\hat\eta_s\cap\hat\eta^\dagger_{t-s}=\{0\}\big]}{\mathbb{E}\big[|\hat\eta_s\cap\hat\eta^\dagger_{t-s}|^{-1}\big]}\qquad(0<s<t).\tag{1.16}$$

Using this, we are able to take the limit $t\to\infty$ in (1.14) and prove the following result.

Theorem 4 (Derivative of the exponential growth rate)**.** *Assume that the infection rates satisfy the irreducibility condition (1.3). For $\delta\in(\delta_{\rm c},\infty)$, let $\hat\nu_\delta$ and $\hat\nu^\dagger_\delta$ be the long-time limit laws of the (Λ,a,δ)- and $(\Lambda,a^\dagger,\delta)$-contact processes as seen from a typical infected site, respectively, as in Theorem 1. Then the map $(\delta_{\rm c},\infty)\ni\delta\mapsto\hat\nu_\delta$ is continuous with respect to weak convergence of probability measures on $\mathcal{P}_{\rm fin,+}$, equipped with the discrete topology, and similarly for $\hat\nu^\dagger_\delta$. Let $\hat\eta^\delta$ and $\hat\eta^{\dagger\,\delta}$ denote independent random variables with laws $\hat\nu_\delta$ and $\hat\nu^\dagger_\delta$, respectively. Then the function $\delta\mapsto r(\Lambda,a,\delta)$ is continuously differentiable on $(\delta_{\rm c},\infty)$ and satisfies*

$$-\tfrac{\partial}{\partial\delta}r(\Lambda,a,\delta)=\frac{\mathbb{P}\big[\hat\eta^\delta\cap\hat\eta^{\dagger\,\delta}=\{0\}\big]}{\mathbb{E}\big[|\hat\eta^\delta\cap\hat\eta^{\dagger\,\delta}|^{-1}\big]}>0\qquad\big(\delta\in(\delta_{\rm c},\infty)\big).\tag{1.17}$$

Remark The continuity of $\hat\nu_\delta$ and $\hat\nu^\dagger_\delta$ as a function of δ in the sense of weak convergence with respect to the discrete topology on $\mathcal{P}_{\rm fin,+}$ is easily seen to imply the continuity of the right-hand side of (1.17) in δ. On the other hand, no such conclusion could be drawn from weak convergence with respect to the product topology on $\mathcal{P}_{\rm fin,+}$, since the functions $A\mapsto 1_{\{A=\{0\}\}}$ and $A\mapsto|A|^{-1}1_{\{0\in A\}}$ are not continuous with respect to this topology.

We will prove Theorem 4 in the setting of eigenmeasures and to this aim restate it in this language in Theorem 6 below. The proof is completed in Section 2.9.

The differentiability of the exponential growth rate in the subcritical regime is expected. Indeed, for normal (unoriented) percolation in the subcritical regime, it is even known that the number of open clusters per vertex and the mean size of the cluster at the origin depend analytically on the percolation parameter. This result is due to Kesten [Kes81]; see also [Gri99, Section 6.4]. For oriented percolation in one plus one dimension in the *supercritical* regime, Durrett [Dur84, Section 14] has shown that the percolation probability is infinitely differentiable as a function of the percolation parameter. It is not so clear, however, if the methods in these papers can be adapted to cover the exponential growth rate. At any rate, they would not give very explicit information about the derivative such as positivity.

In principle, if for a given lattice one can show that the right-hand side of (1.17) stays positive uniformly as $\delta\downarrow\delta_{\rm c}$, then this would imply that $-r(\delta)\propto(\delta-\delta_{\rm c})^1$ as $\delta\downarrow\delta_{\rm c}$, i.e., the critical exponent associated with the function r is one. But this is probably difficult in the most interesting cases, such as $\mathbb{Z}^d$ above the upper critical dimension, which is 4 for the contact process [HS10].

1.6 Locally finite starting measures

So far, we have formulated our results in terms of the contact process as seen from a typical infected site, and its long-time limit. In the present section, we will see that the same results can alternatively be formulated in terms of a different object, namely, the contact process started in an initial 'law' that is the counting measure on the set of all translations of a finite nonempty set A. In particular, if Λ is infinite, then such an initial 'law' is an infinite measure, so we cannot use it to assign probabilities to events in the usual way. We can use it, however, to define conditional distributions given an event of finite measure. In particular, as we will see in formula (1.22) below, conditioning such a process on the origin being infected at some time t yields the law of the contact process as seen from a typical infected site.

The new language of infinite initial 'laws' takes a bit of time to get accustomed to, but it will allow us to restate our main convergence result, Theorem 1, in a much more general and stronger form, and, most importantly, to characterize the limit as a unique object with certain properties. (To be precise, in the new formulation, the limit will be the unique homogenous eigenmeasure of the process, as defined below.) One of the main advantages of the new formulation is that it preserves the translation invariance of the problem, which is broken in the original formulation because of the special role played by the origin.

To deal with contact processes started in initial 'laws' that are infinite measures, we need a bit of theory. Recall that $\mathcal{P}$, $\mathcal{P}_+$, and $\mathcal{P}_{\rm fin}$ denote the space of all subsets, nonempty subsets, and finite subsets of Λ, respectively, and that $\mathcal{P}_{\rm fin,+} := \mathcal{P}_{\rm fin} \cap \mathcal{P}_+$. As before, we observe that $\mathcal{P} \cong \{0,1\}^\Lambda$ and equip it with the product topology and Borel-σ-field. Note that since $\mathcal{P}$ is compact, $\mathcal{P}_+ = \mathcal{P}\backslash\{\emptyset\}$ is a locally compact space. Recall that a measure on a locally compact space is *locally finite* if it gives finite mass to compact sets, and that a sequence of locally finite measures converges vaguely if the integrals of all compactly supported, continuous functions converge.

Below, and throughout the rest of the paper, when we write that 'μ_n are measures', we mean that we have a sequence $(\mu_n)_{n\geq 0}$ (or $(\mu_n)_{n\geq 1}$) of measures. The same simplified notation applies to sequences λ_n of real numbers, etc. We cite the following simple facts from [Swa09, Lemmas 3.1 and 3.2].

Lemma 1.1 (Locally finite measures)**.** *Let μ be a measure on $\mathcal{P}_+$. Then the following statements are equivalent:*

1. *μ is locally finite.*
2. *$\int \mu(\mathrm{d}A) 1_{\{i\in A\}} < \infty$ for all $i \in \Lambda$.*
3. *$\int \mu(\mathrm{d}A) 1_{\{A\cap B\neq\emptyset\}} < \infty$ for all $B \in \mathcal{P}_{\rm fin,+}$.*

Moreover, if μ_n, μ are locally finite measures on $\mathcal{P}_+$, then the μ_n converge vaguely to μ if and only if

$$\int \mu_n(\mathrm{d}A) 1_{\{A\cap B\neq\emptyset\}} \underset{n\to\infty}{\longrightarrow} \int \mu(\mathrm{d}A) 1_{\{A\cap B\neq\emptyset\}} \qquad (B \in \mathcal{P}_{\rm fin,+}). \tag{1.18}$$

For $A \subset \Lambda$ and $i \in \Lambda$, we write $iA := \{ij : j \in A\}$, and for any $\mathcal{A} \subset \mathcal{P}$ we write $i\mathcal{A} := \{iA : A \in \mathcal{A}\}$. We say that a measure μ on $\mathcal{P}$ is (spatially) *homogeneous* if it is invariant under the left action of the group, i.e., if $\mu(\mathcal{A}) = \mu(i\mathcal{A})$ for each $i \in \Lambda$ and measurable $\mathcal{A} \subset \mathcal{P}$.

We now turn our attention to contact processes started in infinite initial 'laws'. For a given (Λ, a, δ)-contact process, we define subprobability kernels P_t $(t \geq 0)$ on $\mathcal{P}_+$ by

$$P_t(A, \cdot\,) := \mathbb{P}\big[\eta_t^A \in \cdot\,\big]\big|_{\mathcal{P}_+} \qquad (t \geq 0,\ A \in \mathcal{P}_+), \tag{1.19}$$

where $|_{\mathcal{P}_+}$ denotes restriction to $\mathcal{P}_+$, and we define $P_t^\dagger$ similarly for the dual $(\Lambda, a^\dagger, \delta)$-contact process. For any measure μ on $\mathcal{P}_+$, we write

$$\mu P_t := \int \mu(\mathrm{d}A) P_t(A, \cdot\,) \qquad (t \geq 0), \tag{1.20}$$

which is the restriction to $\mathcal{P}_+$ of the 'law' at time t of the (Λ, a, δ)-contact process started in the initial (possibly infinite) 'law' μ. If μ is a homogeneous, locally finite measure on

$\mathcal{P}_+$, then μP_t is a homogeneous, locally finite measure on $\mathcal{P}_+$ for each $t \geq 0$ (see [Swa09, Lemma 3.3] or Lemma 2.4 below).

For each $A \in \mathcal{P}_{\rm fin,+}$, let

$$\chi_A := \sum_{i\in\Lambda} \delta_{iA} \quad \text{and hence} \quad \chi_A P_t = \sum_{i\in\Lambda} \mathbb{P}\big[\eta_t^{\{iA\}} \in \cdot\,\big]\big|_{\mathcal{P}_+} \tag{1.21}$$

(where δ_A denotes the delta-measure at a point $A \in \mathcal{P}_+$). In particular, we may loosely interpret $\chi_{\{0\}} P_t$ as the 'law' of the process at time t, starting from a single infection at a 'uniformly chosen' site in the lattice. More generally, χ_A corresponds to a 'uniformly chosen' translation of A.

Defining conditional probabilities for infinite measures in the natural way, it is easy to verify (see Section 2.9 below) that

$$\chi_A P_t\big(\,\cdot\,\big|\,\{B : 0 \in B\}\big) := \frac{\chi_A P_t\big(\,\cdot\,\cap\{B : 0 \in B\}\big)}{\chi_A P_t\big(\{B : 0 \in B\}\big)} = \hat{\mathbb{P}}_t^A\big[\iota^{-1}\eta_t^A \in \cdot\,\big], \tag{1.22}$$

i.e., $\chi_A P_t$ conditioned on the origin being infected describes the distribution of η_t^A under the Campbell law $\hat{\mathbb{P}}_t^A$ from (1.10) with the typical infected site ι shifted to the origin. One can also show that this measure is the Palm law relative to the origin of $\chi_A P_t$.

It follows easily from Lemma 1.1 that if μ_n, μ are locally finite measures on $\mathcal{P}_+$ and $\mu_n \Rightarrow \mu$ vaguely, then $\mu_n\big(\,\cdot\,\big|\,\{B : 0 \in B\}\big)$ converges weakly to $\mu\big(\,\cdot\,\big|\,\{B : 0 \in B\}\big)$ with respect to the product topology. As in Theorem 1, we will sometimes need a stronger form of convergence.

If a locally finite measure on $\mathcal{P}_+$ is concentrated on $\mathcal{P}_{\rm fin}$, then we simply refer to such a measure as a 'locally finite measure on $\mathcal{P}_{\rm fin,+}$' (even though 'locally finite' refers to the topology on $\mathcal{P}_+$). For each $i \in \Lambda$, we define

$$\mathcal{P}_i := \{A \in \mathcal{P} : i \in A\} \quad \text{and} \quad \mathcal{P}_{{\rm fin},i} := \mathcal{P}_{\rm fin} \cap \mathcal{P}_i. \tag{1.23}$$

Note that $\mathcal{P}_{{\rm fin},i}$ is a countable set. We let $\mu|_{\mathcal{P}_{{\rm fin},i}}$ denote the restriction of a measure μ to $\mathcal{P}_{{\rm fin},i}$. If μ_n, μ are locally finite measures on $\mathcal{P}_{\rm fin,+}$, then we say that the μ_n converge to μ *locally on* $\mathcal{P}_{\rm fin,+}$, if for each $i \in \Lambda$, the $\mu_n|_{\mathcal{P}_{{\rm fin},i}}$ converge weakly to $\mu|_{\mathcal{P}_{{\rm fin},i}}$ with respect to the discrete topology on $\mathcal{P}_{{\rm fin},i}$. It can be shown that local convergence on $\mathcal{P}_{\rm fin,+}$ implies vague convergence (see Proposition 2.1 below), but the converse is not true. For example, if $\Lambda = \mathbb{Z}$, then using Lemma 1.1 it is not hard to see that we have the vague convergence $\chi_{\{0,n\}} \Rightarrow 2\chi_{\{0\}}$ as $n \to \infty$, but the μ_n do not converge locally on $\mathcal{P}_{\rm fin,+}$. If μ_n, μ are locally finite measures on $\mathcal{P}_{\rm fin,+}$ and the μ_n converge to μ locally on $\mathcal{P}_{\rm fin,+}$, then the conditioned measures $\mu_n\big(\,\cdot\,\big|\,\{B : 0 \in B\}\big)$ converge weakly to $\mu\big(\,\cdot\,\big|\,\{B : 0 \in B\}\big)$ with respect to the discrete topology on $\mathcal{P}_{\rm fin,+}$.

For processes started in homogeneous, locally finite measures, we have a useful sort of analogue of the duality formula (1.7). To formulate this, we need two more definitions. For any measure μ on $\mathcal{P}_+$, we define

$$\langle\!\langle \mu \rangle\!\rangle := \int \mu(\mathrm{d}A)|A|^{-1} 1_{\{0\in A\}}, \tag{1.24}$$

where $|A|^{-1} := 0$ if A is infinite. Note that if each set $A \in \mathcal{P}_{\rm fin,+}$ carries mass $\mu(\{A\})$, and this mass is distributed evenly among all points in A, then $\langle\!\langle \mu \rangle\!\rangle$ is the mass received at the origin.

Next, for any measures μ, ν on $\mathcal{P}_+$, we let $\mu \circledwedge \nu$ denote the restriction to $\mathcal{P}_+$ of the image of the product measure $\mu \otimes \nu$ under the map $(A, B) \mapsto A \cap B$. Note that

$$\int \mu \circledwedge \nu\,(\mathrm{d}C) f(C) := \int \mu(\mathrm{d}A) \int \nu(\mathrm{d}B) f\big(A \cap B\big) \tag{1.25}$$

for any bounded measurable $f : \mathcal{P} \to \mathbb{R}$ satisfying $f(\emptyset) = 0$. We call $\mu \boxtimes \nu$ the *intersection measure* of μ and ν. It is not hard to show (see Lemma 2.2 below) that $\mu \boxtimes \nu$ is locally finite if μ and ν are. Note that if μ and ν are probability measures, then $\mu \boxtimes \nu$ is the law of the intersection of two independent random sets with laws μ and ν, restricted to the event that this intersection is nonempty. In particular, normalizing $\mu \boxtimes \nu$ yields the conditional law given this event.

With these definitions, we have the following lemma, whose proof can be found in Section 3.2.

Lemma 1.2 (Duality for infinite initial laws)**.** *Let μ, ν be homogeneous, locally finite measures on $\mathcal{P}_+$. Then*

$$\langle\!\langle \mu P_t \boxtimes \nu \rangle\!\rangle = \langle\!\langle \mu \boxtimes \nu P_t^\dagger \rangle\!\rangle \qquad (t \geq 0), \tag{1.26}$$

and $\mu P_t \boxtimes \nu$ is concentrated on $\mathcal{P}_{\mathrm{fin},+}$ if and only if $\mu \boxtimes \nu P_t^\dagger$ is.

Remark If $|\mu| := \mu(\mathcal{P}_+)$ denotes the total mass of a finite measure on $\mathcal{P}_+$, then the duality formula (1.7) is easily seen to imply that $|\mu P_t \boxtimes \nu| = |\mu \boxtimes \nu P_t^\dagger|$ for any finite measures μ, ν on $\mathcal{P}_+$. One can think of (1.26) as an analogue of this for infinite (but homogeneous) measures.

1.7 Eigenmeasures

Following [Swa09], we say that a measure μ on $\mathcal{P}_+$ is an *eigenmeasure* of the (Λ, a, δ)-contact process if μ is nonzero, locally finite, and there exists a constant $\lambda \in \mathbb{R}$ such that

$$\mu P_t = e^{\lambda t} \mu \qquad (t \geq 0). \tag{1.27}$$

We call λ the associated *eigenvalue*.

It follows from [Swa09, Prop. 1.4] that each (Λ, a, δ)-contact process has a homogeneous eigenmeasure $\mathring{\nu}$ with eigenvalue $r = r(\Lambda, a, \delta)$. In general, it is not known if $\mathring{\nu}$ is (up to a multiplicative constant) unique. Under the irreducibility condition (1.3), it has been shown in [Swa09, Thm. 1.5] that if the upper invariant measure $\overline{\nu}$ of a (Λ, a, δ)-contact process is nontrivial and $r(\Lambda, a, \delta) = 0$, then $\mathring{\nu}$ is unique up to a multiplicative constant and in fact $\mathring{\nu} = c\,\overline{\nu}$ for some $c > 0$.

The following theorem, which is one of our main results, investigates eigenmeasures in the subcritical case $r < 0$. Its proof can be found in Section 2.6. Note that in particular, setting $\mu = \sum_{i \in \Lambda} \delta_{\{i\}}$, formula (1.28) describes the long-time behavior of the law of the process started with a single infected site, distributed according to counting measure on the lattice. More generally, setting $\mu = \chi_A$ as in (1.21) yields through (1.22) Theorem 1.

Theorem 5 (Eigenmeasures in the subcritical case)**.** *Assume that the infection rates satisfy the irreducibility condition (1.3) and that the exponential growth rate from (1.12) satisfies $r < 0$. Then there exist, up to multiplicative constants, unique homogeneous eigenmeasures $\mathring{\nu}$ and $\mathring{\nu}^\dagger$ of the (Λ, a, δ)- and $(\Lambda, a^\dagger, \delta)$-contact processes, respectively. These eigenmeasures have eigenvalue r and are concentrated on $\mathcal{P}_{\mathrm{fin},+}$. If μ is any nonzero, homogeneous, locally finite measure on $\mathcal{P}_+$, then*

$$e^{-rt} \mu P_t \underset{t \to \infty}{\Longrightarrow} c\,\mathring{\nu}, \tag{1.28}$$

where $\Rightarrow$ denotes vague convergence of locally finite measures on $\mathcal{P}_+$ and $c > 0$ is a constant, given by

$$c = \frac{\langle\!\langle \mu \boxtimes \mathring{\nu}^\dagger \rangle\!\rangle}{\langle\!\langle \mathring{\nu} \boxtimes \mathring{\nu}^\dagger \rangle\!\rangle}. \tag{1.29}$$

If μ is concentrated on $\mathcal{P}_{\mathrm{fin},+}$, then (1.28) holds in the sense of local convergence on $\mathcal{P}_{\mathrm{fin},+}$.

Remark Since $\mathring{\nu}$ and $\mathring{\nu}^\dagger$ are infinite measures, their normalizations are somewhat arbitrary. For definiteness, we will usually adopt the convention that $\int \mathring{\nu}(\mathrm{d}A)1_{\{0\in A\}} = 1 = \int \mathring{\nu}^\dagger(\mathrm{d}A)1_{\{0\in A\}}$. Theorem 5 holds regardless of the choice of normalization.

We expect the convergence in (1.28) to have analogues also for critical and supercritical processes, but in these regimes the normalizing constant e^{-rt} is probably more complicated and may depend on the initial measure μ. For processes with $r = 0$ in the supercritical regime, we expect the limit to be the upper invariant law (compare Theorem 3 and see also [Swa09, Theorem 1.5 and Corollary 3.4]), while for processes with $r > 0$ the limit probably depends on the initial law μ.

In view of formula (1.22) and our discussion of local convergence, Theorem 5 implies Theorem 1. Moreover, it identifies the limit law $\hat{\nu}$ from (1.11) as the eigenmeasure $\mathring{\nu}$ conditioned on the origin being infected. Note that Theorem 5 gives more information than Theorem 1, since it is not restricted to finite initial states and also determines the multiplicative constant c of the limit.

Using the language of eigenmeasures, we can also rephrase Theorem 4. We note that the continuity of $\mathring{\nu}_\delta$ and $\mathring{\nu}_\delta^\dagger$ as a function of δ in the sense of local convergence on $\mathcal{P}_{\mathrm{fin},+}$ (as stated below) implies the continuity of the right-hand side of (1.30) in δ, while no such conclusion could be drawn from continuity in the sense of vague convergence; compare the remark below Theorem 4. Theorem 6 will be proved in Section 2.8.

Theorem 6 (Derivative of the exponential growth rate)**.** *Assume that the infection rates satisfy the irreducibility condition (1.3). For $\delta \in (\delta_\mathrm{c}, \infty)$, let $\mathring{\nu}_\delta$ and $\mathring{\nu}_\delta^\dagger$ denote the homogeneous eigenmeasures of the (Λ, a, δ)- and $(\Lambda, a^\dagger, \delta)$-contact processes, respectively, normalized such that $\int \mathring{\nu}_\delta(\mathrm{d}A)1_{\{0\in A\}} = 1 = \int \mathring{\nu}_\delta^\dagger(\mathrm{d}A)1_{\{0\in A\}}$. Then the map $(\delta_\mathrm{c}, \infty) \ni \delta \mapsto \mathring{\nu}_\delta$ is continuous with respect to local convergence on $\mathcal{P}_{\mathrm{fin},+}$, and similarly for $\mathring{\nu}_\delta^\dagger$. Moreover, the function $\delta \mapsto r(\Lambda, a, \delta)$ is continuously differentiable on $(\delta_\mathrm{c}, \infty)$ and satisfies*

$$-\tfrac{\partial}{\partial\delta} r(\Lambda, a, \delta) = \frac{\mathring{\nu}_\delta \bowtie \mathring{\nu}_\delta^\dagger(\{0\})}{\langle\langle \mathring{\nu}_\delta \bowtie \mathring{\nu}_\delta^\dagger \rangle\rangle} > 0 \qquad \big(\delta \in (\delta_\mathrm{c}, \infty)\big). \tag{1.30}$$

Remark If Λ is finite, then we may normalize $\mathring{\nu}_\delta \bowtie \mathring{\nu}_\delta^\dagger$ so that it is a probability measure. Let ζ be a random variable with this law and conditional on ζ, choose κ uniformly from ζ. Then the nominator in (1.30) is $\mathbb{P}[\zeta = \{0\}] = |\Lambda|^{-1}\mathbb{P}[|\zeta| = 1]$ while the denominator is $\mathbb{P}[\kappa = 0] = |\Lambda|^{-1}$, so the fraction equals $\mathbb{P}[|\zeta| = 1]$.

Theorems 5 and 6 are our main results, from which our other results (in particular, Theorems 1 and 4) can easily be derived, see Section 2.9.

1.8 Discussion and outlook

In general, it is not hard (but also not very interesting) to determine the limit behavior of contact processes started from a spatially homogeneous (i.e., translation invariant) initial probability law. Indeed, provided the initial law is nontrivial, it is known that the limit is the upper invariant law. The proof, based on duality (see [Lig85, formula (VI.2.1)]), works for general lattices. On the other hand, a couple of questions relating to the process started from finite sets seem to lie much deeper. For example, the proof of complete convergence on $\mathbb{Z}^d$ [Lig99, Section I.2] is rather involved and cannot easily be adapted to other lattices. Also, if λ_c and λ'_c denote the critical infection rates associated with survival and local survival, respectively (the latter meaning that starting from a finite set, with positive probability, infections reach the origin at arbitrary late times), then it is known that $\lambda_{\rm c} < \lambda'_{\rm c}$ on trees while $\lambda_{\rm c} = \lambda'_{\rm c}$ on $\mathbb{Z}^d$, but for many other lattices the question whether survival implies local survival is open. Based on analogy with unoriented percolation, one may conjecture that $\lambda_{\rm c} = \lambda'_{\rm c}$ on any amenable transitive graph; this problem seems to be quite hard.

These problems motivated the study of contact processes as seen from a typical infected sites and of eigenmeasures in [Swa09]. The main results of that paper are concerned with processes for which the expected number of infected sites in the supercritical case grows slower than exponentially (see Theorem 3 above). The main aim of the present paper is to study analogous questions in the subcritical regime.

Our present paper treats the subcritical case fairly conclusively. Arguably, this should be the easiest regime. And indeed, our analysis is made easier by the fact that the homogeneous eigenmeasures are concentrated on finite sets. As we will see (in formula (2.24) below), such eigenmeasures are in one-to-one correspondence to quasi-invariant laws (as introduced in [DS67]) for the contact process 'modulo shifts'. More precisely, call two sets $A, B \in \mathcal{P}_{\rm fin}$ equivalent if one is a translation of the other (see (2.1) below), let $\tilde{A}$ denote the corresponding equivalence class containing A, and set $\tilde{\mathcal{P}}_{\rm fin} := \{\tilde{A} : A \in \mathcal{P}_{{\rm fin},+}\}$. Then, for any (Λ, a, δ)-contact process η started in a finite initial state, $(\tilde{\eta}_t)_{t\geq 0}$ is a Markov process with countable state space $\tilde{\mathcal{P}}_{\rm fin}$. In the subcritical regime, this process a.s. ends up in the trap $\tilde{\emptyset}$. We will show that there is a one-to-one correspondence between eigenmeasures that are concentrated on $\mathcal{P}_{{\rm fin},+}$ and quasi-invariant laws for $\tilde{\eta}$. In particular, our results imply that the law of a subcritical contact process modulo shifts, started in any finite initial state and conditioned to be alive at time t, converges as $t \to \infty$ to a quasi-invariant law (Theorem 2.12 below).

For certain discrete-time versions of the contact process on $\mathbb{Z}^d$, as well as for some other, similar Markov chains, an analogous result has been proved in [FKM96]. Our methods differ significantly from the methods used there, since we use eigenmeasures of the forward and dual process to construct positive left and right eigenfunctions of the forward process. This simplifies our proofs, but, since this approach makes essential use of the contact process duality, it is less generally applicable. The correspondence between homogeneous eigenmeasures and quasi-invariant laws of the process modulo shifts is only available in the subcritical regime. In contrast, in the critical and supercritical regimes, we expect homogeneous eigenmeasures to be concentrated on infinite sets, hence the techniques of the present paper are not applicable.

Nevertheless, our methods give some hints on what to do in some of the other regimes as well. Indeed, we expect formula (1.30) to hold more generally. The second remark below Theorem 6 interprets formula (1.30), roughly speaking, as saying that $-\frac{\partial}{\partial\delta} r(\Lambda, a, \delta)$ is the probability that two independent sets which are distributed according to the eigenmeasures $\mathring{\nu}$ and $\mathring{\nu}^\dagger$ of the forward and dual (backward) process, and which are conditioned on having nonempty intersection, intersect in a single point. In view of this, it is tempting to try to replace the fact that $\mathring{\nu}$ and $\mathring{\nu}^\dagger$ are each concentrated on finite sets, which holds only in the subcritical regime, by the weaker assumption that the intersection measure $\mathring{\nu} \boxtimes \mathring{\nu}^\dagger$ is concentrated on finite sets. In particular, one wonders if this always holds in the regime $r > 0$.

A simpler problem, which we have not pursued in the present paper, is to investigate higher-order derivatives of $r(\Lambda, a, \delta)$ with respect to δ or derivatives with respect to the infection rates $a(i, j)$. It seems likely that the latter are strictly positive in the subcritical regime and given by a formula similar to (1.30). Controlling higher-order derivatives of $r(\Lambda, a, \delta)$ might be more difficult; in particular, we do not know if the function $\delta \mapsto r(\Lambda, a, \delta)$ is concave, or (which in view of (1.17) is a somewhat similar, though different question), if the conditional laws $\mathring{\nu}_\delta(\,\cdot\,|\{B : 0 \in B\})$ are decreasing in the stochastic order, as a function of δ.

2 Main line of the proofs

In this section we give an overview of the main line of our arguments. In particular, we give the proofs of Theorems 5 and 6 in Sections 2.6 and 2.8 respectively. These proofs are based on a collection of lemmas and propositions which are stated here but whose proofs are in most cases postponed until later.

In short, the line of the arguments is as follows. We start in Section 2.1 by collecting some general facts about locally finite measures on $\mathcal{P}_+$. In particular, we discuss the relation between vague and local convergence, and we show that a homogeneous, locally finite measure on $\mathcal{P}_{\rm fin,+}$ can be seen as the 'law' of a random finite set, shifted to a 'uniformly chosen' position in the lattice.

In Section 2.2, we then prove the existence part of Theorem 5. Since existence of an eigenmeasure with eigenvalue r has already been proved in [Swa09], the main task is proving that there exists such an eigenmeasure that is moreover concentrated on $\mathcal{P}_{\rm fin,+}$. This is achieved by a covariance calculation.

Once existence is proved, we fix an eigenmeasure $\mathring{\nu}$ that is concentrated on $\mathcal{P}_{\rm fin,+}$, and likewise $\mathring{\nu}^\dagger$ for the dual process, and set out to prove the convergence in (1.28), which will then also settle uniqueness. Our main strategy will be to divide out translations and show that the resulting process modulo shifts is λ-positive, which means that the subprobability kernels P_t in (1.19) can be transformed, by a variation of Doob's h-transform, into probability kernels belonging to a positively recurrent Markov process. This part of the argument is carried out in Section 2.5.

To prepare for this, in Sections 2.3 and 2.4, we study left and right eigenvectors of the semigroup of the (Λ, a, δ)-contact process modulo shifts. In particular, in Section 2.3, we show that there is a one-to-one correspondence between eigenmeasures that are concentrated on finite sets and quasi-invariant laws (i.e., normalized positive left eigenvectors) of the process modulo shifts. In Section 2.4, we show that moreover, each eigenmeasure of the dual $(\Lambda, a^\dagger, \delta)$-contact process gives rise to a *right* eigenvector for the (Λ, a, δ)-contact process modulo shifts.

In Section 2.6, we use this to prove the convergence in (1.28), completing the proof of Theorem 5. We obtain vague convergence for general starting measures by duality, using the ergodicity of the Doob transform of the dual $(\Lambda, a^\dagger, \delta)$-contact process modulo shifts. For starting measures that are concentrated on $\mathcal{P}_{\rm fin,+}$, we moreover obtain pointwise convergence by using the ergodicity of the Doob transformed (forward) (Λ, a, δ)-contact process modulo shifts, which together with vague convergence, by a general lemma from Section 2.1, implies local convergence on $\mathcal{P}_{\rm fin,+}$.

In order to prove Theorem 6, in Section 2.7 we show continuity of the eigenmeasures $\mathring{\nu}$ in the recovery rate δ. Continuity in the sense of vague convergence follows easily from a compactness argument and uniqueness, but continuity in the sense of local convergence on $\mathcal{P}_{\rm fin,+}$ requires more work. We use a generalization of the covariance calculation from Section 2.2 to obtain 'local tightness', which together with vague convergence, by a general lemma from Section 2.1, implies local convergence on $\mathcal{P}_{\rm fin,+}$.

In Section 2.8, we use the results proved so far to take the limit $t \to \infty$ in Russo's formula (1.14) and prove formula (1.30), thereby completing the proof of Theorem 6. In Section 2.9, finally, we derive Theorems 1 and 4 from Theorems 5 and 6, respectively, and show how Theorem 3 follows from results in [Swa09].

At this point, the proofs of our main results are complete, but they depend on a number of lemmas and propositions whose proofs have for readability been postponed until later. We supply these in Section 3. The paper concludes with two appendices. In

Appendix A, we point out how the arguments in [AJ07] generalize to the class of contact processes considered in the present article. Appendix B contains background material on λ-positivity and quasi-invariant laws.

2.1 More on locally finite measures

In this section, we elaborate on the discussion in Section 1.6 of (contact processes started in) locally finite measures on $\mathcal{P}_+$ by formulating some lemmas that will be useful in what follows.

Recall from Section 1.6 the definition of vague convergence and of local convergence on $\mathcal{P}_{\mathrm{fin},+}$, and recall that $\mathcal{P}_{\mathrm{fin},i} := \{A \in \mathcal{P}_{\mathrm{fin}} : i \in A\}$. If μ_n, μ are measures on $\mathcal{P}_{\mathrm{fin},+}$, then we say the μ_n converge to μ *pointwise on* $\mathcal{P}_{\mathrm{fin},+}$ if $\mu_n(\{A\}) \to \mu(\{A\})$ for all $A \in \mathcal{P}_{\mathrm{fin},+}$. We say that the $(\mu_n)_{n\geq 1}$ are *locally tight* if for each $i \in \Lambda$ and $\varepsilon > 0$ there exists a finite $\mathcal{D} \subset \mathcal{P}_{\mathrm{fin},i}$ such that $\sup_n \mu_n(\mathcal{P}_{\mathrm{fin},i} \backslash \mathcal{D}) \leq \varepsilon$. The next proposition, the proof of which can be found in Section 3.1, connects all these definitions.

Proposition 2.1 (Local convergence)**.** *Let μ_n, μ be locally finite measures on $\mathcal{P}_+$ that are concentrated on $\mathcal{P}_{\mathrm{fin},+}$. Then the following statements are equivalent.*

1. *$\mu_n \Rightarrow \mu$ locally on $\mathcal{P}_{\mathrm{fin},+}$.*
2. *$\mu_n \to \mu$ pointwise on $\mathcal{P}_{\mathrm{fin},+}$ and the $(\mu_n)_{n\geq 1}$ are locally tight.*
3. *$\mu_n \Rightarrow \mu$ vaguely on $\mathcal{P}_+$ and the $(\mu_n)_{n\geq 1}$ are locally tight.*
4. *$\mu_n \Rightarrow \mu$ vaguely on $\mathcal{P}_+$ and $\mu_n \to \mu$ pointwise on $\mathcal{P}_{\mathrm{fin},+}$.*

Recall the definition of the intersection measure $\mu \boxtimes \nu$ in (1.25). The next lemma, whose proof can be found in Section 3.1, says that the operation $\boxtimes$ is continuous with respect to vague and local convergence.

Lemma 2.2 (Intersection measure)**.** *If μ and ν are locally finite measures on $\mathcal{P}_+$, then $\mu \boxtimes \nu$ is a locally finite measure on $\mathcal{P}_+$. If μ_n and ν_n are locally finite measures on $\mathcal{P}_+$ that converge vaguely to μ, ν, respectively, then $\mu_n \boxtimes \nu_n$ converges vaguely to $\mu \boxtimes \nu$. If moreover either the μ_n or the ν_n are concentrated on $\mathcal{P}_{\mathrm{fin},+}$ and converge locally on $\mathcal{P}_{\mathrm{fin},+}$, then the $\mu_n \boxtimes \nu_n$ are concentrated on $\mathcal{P}_{\mathrm{fin},+}$ and converge locally on $\mathcal{P}_{\mathrm{fin},+}$.*

It is often useful to view a homogeneous, locally finite measure on $\mathcal{P}_{\mathrm{fin},+}$ as the 'law' of a random finite subset of Λ, shifted to a 'uniformly chosen' position in Λ. To formulate this precisely, we define an equivalence relation on $\mathcal{P}_{\mathrm{fin}}$ by

$$A \sim B \quad \text{iff} \quad A = iB \quad \text{for some } i \in \Lambda, \tag{2.1}$$

and we let $\tilde{\mathcal{P}}_{\mathrm{fin}} := \{\tilde{A} : A \in \mathcal{P}_{\mathrm{fin}}\}$ with $\tilde{A} := \{iA : i \in \Lambda\}$ denote the set of equivalence classes. We can think of $\tilde{\mathcal{P}}_{\mathrm{fin}}$ as the space of finite subsets of the lattice 'modulo shifts'. Recall the definition of $\langle\langle \mu \rangle\rangle$ from (1.24). We have the following simple lemma, which will be proved in Section 3.1.

Lemma 2.3 (Homogeneous measures on the finite sets)**.** *Let Δ be a $\mathcal{P}_{\mathrm{fin},+}$-valued random variable and let $c > 0$. Then*

$$\mu := c \sum_{i \in \Lambda} \mathbb{P}\big[i\Delta \in \cdot\,\big] \tag{2.2}$$

defines a nonzero, homogeneous measure on $\mathcal{P}_{\mathrm{fin},+}$ such that $\langle\langle \mu \rangle\rangle = c$. The measure μ is locally finite if and only if $\mathbb{E}\big[|\Delta|\big] < \infty$. Conversely, any nonzero, homogeneous measure on $\mathcal{P}_{\mathrm{fin},+}$ such that $\langle\langle \mu \rangle\rangle < \infty$ can be written in the form (2.2) with $c = \langle\langle \mu \rangle\rangle$ for some $\mathcal{P}_{\mathrm{fin},+}$-valued random variable Δ, and the law of $\tilde{\Delta}$ is uniquely determined by μ.

We finally turn our attention to contact processes started in infinite initial 'laws'. Recall the definition of the subprobability kernels P_t in (1.19) and of the meaures μP_t in (1.20). We cite the following simple fact from [Swa09, Lemma 3.3].

Lemma 2.4 (Process started in infinite law)**.** *If μ is a homogeneous, locally finite measure on $\mathcal{P}_+$, then μP_t is a homogeneous, locally finite measure on $\mathcal{P}_+$ for each $t \geq 0$. If μ_n, μ are homogeneous, locally finite measures on $\mathcal{P}_+$ such that $\mu_n \Rightarrow \mu$, then $\mu_n P_t \Rightarrow \mu P_t$ for all $t \geq 0$, where $\Rightarrow$ denotes vague convergence.*

2.2 Existence of eigenmeasures concentrated on finite sets

The first step in the proof of Theorem 5 is to show that the condition $r < 0$ implies existence of a homogeneous eigenmeasure that is concentrated on $\mathcal{P}_{\rm fin}$.

We start by recalling how homogeneous eigenmeasures with eigenvalue r are constructed in [Swa09]. It is easy to see that the function $t \mapsto \log \mathbb{E}[|\eta_t^{\{0\}}|]$ is subadditive (see [Swa09, formula (3.4)]). By Fekete's lemma [Lig99, Theorem B.22], it follows that the limit

$$\lim_{t\to\infty} \tfrac{1}{t} \log \mathbb{E}\big[|\eta_t^{\{0\}}|\big] = \inf_{t>0} \tfrac{1}{t} \log \mathbb{E}\big[|\eta_t^{\{0\}}|\big] =: r \in [-\infty, \infty) \tag{2.3}$$

exists. In particular, since the limit equals the infimum, this implies that

$$\mathbb{E}\big[|\eta_t^A|\big] \geq \mathbb{E}\big[|\eta_t^{\{0\}}|\big] \geq e^{rt} \qquad (t \geq 0,\ A \in \mathcal{P}_{{\rm fin},+}). \tag{2.4}$$

Since moreover (see [Swa09, formula (3.3)])

$$\mathbb{E}\big[|\eta_t^A|\big] \leq |A| \mathbb{E}\big[|\eta_t^{\{0\}}|\big] \tag{2.5}$$

it follows that $\lim_{t\to\infty} t^{-1} \log \mathbb{E}[|\eta_t^A|] = r$ $(A \in \mathcal{P}_{{\rm fin},+})$.

As before (see (1.19)), let P_t denote the restriction to $\mathcal{P}_+$ of the transition probabilities of the (Λ, a, δ)-contact process. Restricting ourselves to finite initial states, let

$$\hat{P}_\lambda(A, \cdot\,) := \int_0^\infty P_t(A, \cdot\,) e^{-\lambda t} {\rm d}t \qquad (A \in \mathcal{P}_{{\rm fin},+},\ \lambda \in \mathbb{R}) \tag{2.6}$$

denote the associated resolvent, which may be infinite for some values of λ. Recalling the definition of χ_A in (1.21) and $\mathcal{P}_i$ from (1.23), we observe that

$$\pi_t(A) := \chi_A P_t(\mathcal{P}_0) = \sum_{i\in\Lambda} \mathbb{P}[0 \in \eta_t^{iA}] = \sum_{i\in\Lambda} \mathbb{P}[i^{-1} \in \eta_t^A] = \mathbb{E}\big[|\eta_t^A|\big], \tag{2.7}$$

and hence

$$\hat{\pi}_\lambda(A) := \chi_A \hat{P}_\lambda(\mathcal{P}_0) = \int_0^\infty \mathbb{E}\big[|\eta_t^A|\big] e^{-\lambda t} {\rm d}t \tag{2.8}$$

satisfies

$$\hat{\pi}_\lambda(A) < \infty \quad (\lambda > r) \quad \text{and} \quad \lim_{\lambda\downarrow r} \hat{\pi}_\lambda(A) = \infty \tag{2.9}$$

by (2.5) and (2.4).

The following result has been proved in [Swa09, Corollary 3.4] in the special case that $A = \{0\}$. As explained below, the proof in the general case is basically the same. We need the general case for Theorem 3. Proposition 2.5 implies in particular the existence of a homogeneous eigenmeasure with eigenvalue r.

Proposition 2.5 (Convergence to eigenmeasure)**.** *For each $A \in \mathcal{P}_{{\rm fin},+}$, the measures $\overline{\mu}_\lambda := \hat{\pi}_\lambda(A)^{-1} \chi_A \hat{P}_\lambda$ are relatively compact in the topology of vague convergence of locally finite measures on $\mathcal{P}_+$, and each subsequential limit as $\lambda \downarrow r$ is a homogeneous eigenmeasure of the (Λ, a, δ)-contact process, with eigenvalue $r(\Lambda, a, \delta)$.*

Proof (sketch). Since the $\overline{\mu}_\lambda$ are homogeneous measures, which are normalized such that $\overline{\mu}_\lambda(\mathcal{P}_0)=1$, relatively compactness follows exactly in the same was as in the proof of [Swa09, Corollary 3.4] from [Swa09, Lemma 3.2 and formula (3.15)]. Choose $\lambda_n\downarrow r$ such that the $\overline{\mu}_{\lambda_n}$ converge vaguely to some locally finite, homogeneous measure $\overline{\mu}_r$. Then, arguing as in [Swa09, formula (3.25)] and applying Lemma 2.4,

$$\begin{aligned}\overline{\mu}_r P_t &= \lim_{n\to\infty}\overline{\mu}_{\lambda_n}P_t=\lim_{n\to\infty}\hat{\pi}_{\lambda_n}(A)^{-1}\int_0^\infty \chi_A P_s P_t e^{-\lambda_n s}\mathrm{d}s\\ &= e^{rt}\overline{\mu}_r - e^{rt}\lim_{n\to\infty}\hat{\pi}_{\lambda_n}(A)^{-1}\int_0^t \chi_A P_s e^{-\lambda_n s}\mathrm{d}s = e^{rt}\overline{\mu}_r,\end{aligned}\tag{2.10}$$

where in the last step we have used that the integral converges to a (locally) finite limit while $\lim_{n\to\infty}\hat{\pi}_{\lambda_n}(A)=\infty$ by (2.9). □

We wish to show that for $r<0$, the approximation procedure in Proposition 2.5 yields an eigenmeasure that is concentrated on $\mathcal{P}_{\rm fin}$. The key to this is the following lemma, which will be proved in Section 3.4 using a covariance calculation. Note that this lemma still holds for general $r\in\mathbb{R}$. For simplicity, we restrict ourselves to the case $A=\{0\}$ and write

$$\hat{\pi}_\lambda := \hat{\pi}_\lambda(\{0\}) \quad\text{and}\quad \hat{\mu}_\lambda := \chi_{\{0\}}\hat{P}_\lambda \qquad (\lambda>r),\tag{2.11}$$

where $\hat{\pi}_\lambda(\{0\})$ is defined as in (2.9). Recall from (1.1) that $|a|:=\sum_i a(0,i)$.

Lemma 2.6 (Uniform moment bound)**.** *Let $\hat{\pi}_\lambda$ and $\hat{\mu}_\lambda$ be defined as in (2.11). Then, for any (Λ,a,δ)-contact process with exponential growth rate $r=r(\Lambda,a,\delta)$,*

$$\limsup_{\lambda\downarrow r}\frac{1}{\hat{\pi}_\lambda}\int\hat{\mu}_\lambda(\mathrm{d}A)1_{\{0\in A\}}|A|\le(|a|+\delta)\int_0^\infty e^{-rt}\mathrm{d}t\,\mathbb{E}\big[|\eta_t^{\{0\}}|\big]^2.\tag{2.12}$$

As a consequence, we obtain the following result that completes the existence part of Theorem 5.

Lemma 2.7 (Existence of an eigenmeasure on finite configurations)**.** *Assume that the exponential growth rate $r=r(\Lambda,a,\delta)$ of the (Λ,a,δ)-contact process satisfies $r<0$. Then there exists a homogeneous eigenmeasure $\mathring{\nu}$ with eigenvalue r of the (Λ,a,δ)-contact process such that*

$$\int\mathring{\nu}(\mathrm{d}A)|A|1_{\{0\in A\}}<\infty.\tag{2.13}$$

Proof. By Proposition 2.5, we can choose $\lambda_n\downarrow r$ such that the measures $\frac{1}{\hat{\pi}_{\lambda_n}}\hat{\mu}_{\lambda_n}$ converge vaguely to a homogeneous eigenmeasure $\mathring{\nu}$ with eigenvalue r. It follows from (1.12) that $\mathbb{E}[|\eta_t^{\{0\}}|]=e^{rt+o(t)}$ where $t\mapsto o(t)$ is a continuous function such that $o(t)/t\to 0$ as $t\to\infty$, hence, by (2.12), provided $r<0$,

$$\int_0^\infty e^{-rt}\mathrm{d}t\,\mathbb{E}\big[|\eta_t^{\{0\}}|\big]^2=\int_0^\infty e^{2rt-rt+o(t)}\mathrm{d}t<\infty.\tag{2.14}$$

Let Λ_k be finite sets such that $0\in\Lambda_k\subset\Lambda$ and $\Lambda_k\uparrow\Lambda$. It is easy to check that $A\mapsto f_k(A):=|A\cap\Lambda_k|1_{\{0\in A\}}$ is a continuous, compactly supported real function on $\mathcal{P}_+$. Therefore, by the vague convergence of $\frac{1}{\hat{\pi}_{\lambda_n}}\hat{\mu}_{\lambda_n}$ to $\mathring{\nu}$, and by (2.12),

$$\begin{aligned}&\int\mathring{\nu}(\mathrm{d}A)f_k(A)=\lim_{n\to\infty}\frac{1}{\hat{\pi}_{\lambda_n}}\int\hat{\mu}_{\lambda_n}(\mathrm{d}A)f_k(A)\\ &\quad\le\liminf_{n\to\infty}\frac{1}{\hat{\pi}_{\lambda_n}}\int\hat{\mu}_{\lambda_n}(\mathrm{d}A)|A|1_{\{0\in A\}}\le(|a|+\delta)\int_0^\infty e^{-rt}\mathrm{d}t\,\mathbb{E}\big[|\eta_t^{\{0\}}|\big]^2.\end{aligned}\tag{2.15}$$

Letting $k\uparrow\infty$, using the fact that the right-hand side is finite by (2.14), we arrive at (2.13). □

2.3 The process modulo shifts

Recall the definition of the equivalence relation in (2.1) and the associated equivalence classes $\tilde{A}$. Let $(\eta_t)_{t\geq 0}$ be a (Λ, a, δ)-contact process with $|\eta_0| < \infty$ a.s. Then $\tilde{\eta} = (\tilde{\eta}_t)_{t\geq 0}$ is a Markov process with state space $\tilde{\mathcal{P}}_{\rm fin}$. We call $\tilde{\eta}$ the (Λ, a, δ)-*contact process modulo shifts*. Clearly, the point $\tilde{\emptyset}$ is a trap for this process. We will prove that there is a one-to-one correspondence between eigenmeasures of the (Λ, a, δ)-contact process that are concentrated on $\mathcal{P}_{{\rm fin},+}$ and quasi-invariant laws for the (Λ, a, δ)-contact process modulo shifts.

To prepare for this, let P_t as in (1.19) denote the restriction to $\mathcal{P}_+$ of the transition probabilities of η. If $A \in \mathcal{P}_{{\rm fin},+}$, then $P_t(A, \cdot\,)$ is concentrated on the countable set $\mathcal{P}_{{\rm fin},+}$, and we simply write $P_t(A, B) := P_t(A, \{B\})$. Likewise, let $\tilde{P}_t$ denote the transition probabilities of $\tilde{\eta}$, i.e., $\tilde{P}_t$ is a subprobability kernel on $\tilde{\mathcal{P}}_{{\rm fin},+}$ which is related to the subprobability kernel P_t by

$$\tilde{P}_t(\tilde{A}, \tilde{B}) = \sum_{C\in\tilde{B}} P_t(A, C) = m(B)^{-1} \sum_{i\in\Lambda} P_t(A, iB) \qquad (t \geq 0,\ A, B \in \mathcal{P}_{{\rm fin},+}), \tag{2.16}$$

where we have to divide by the quantity

$$m(A) := |\{i \in \Lambda : iA = A\}| \qquad \big(A \in \mathcal{P}_{{\rm fin},+}\big) \tag{2.17}$$

to avoid double counting.[3]

By Lemma 2.3, each nonzero, homogeneous measure on $\mathcal{P}_{{\rm fin},+}$ such that $\langle\!\langle \mu \rangle\!\rangle < \infty$ can be written as

$$\mu = \langle\!\langle \mu \rangle\!\rangle \sum_{i\in\Lambda} \mathbb{P}[i\Delta \in \cdot\,] \tag{2.18}$$

for some $\mathcal{P}_{{\rm fin},+}$-valued random variable Δ. We write

$$\tilde{\mu} := \mathbb{P}[\tilde{\Delta} \in \cdot\,] \tag{2.19}$$

for the law of $\tilde{\Delta}$. By Lemma 2.3, $\tilde{\mu}$ is uniquely determined by μ, and conversely, by (2.2), $\tilde{\mu}$ determines μ up to a multiplicative constant.

We say that a function $f : \mathcal{P}_{\rm fin} \to \mathbb{R}$ is *shift-invariant* if $f(iA) = f(A)$ for all $i \in \Lambda$. For any shift-invariant function $f : \mathcal{P}_{{\rm fin},+} \to \mathbb{R}$, we let $\tilde{f} : \tilde{\mathcal{P}}_{{\rm fin},+} \to \mathbb{R}$ denote the function defined by

$$\tilde{f}(\tilde{A}) := f(A) \qquad (A \in \mathcal{P}_{{\rm fin},+}). \tag{2.20}$$

Then, clearly,

$$\tilde{P}_t \tilde{f}(\tilde{A}) = \mathbb{E}[\tilde{f}(\tilde{\eta}^A_t)] = \mathbb{E}[f(\eta^A_t)] = P_t f(A) \qquad (t \geq 0,\ A \in \mathcal{P}_{{\rm fin},+}). \tag{2.21}$$

The following simple lemma will be proved in Section 3.1.

Lemma 2.8 (Laws on equivalence classes)**.** *Let μ be a nonzero, homogeneous measure on $\mathcal{P}_{{\rm fin},+}$ such that $\langle\!\langle \mu \rangle\!\rangle < \infty$, let $\tilde{\mu}$ be as in (2.19), and let P_t and $\tilde{P}_t$ denote the transition probabilities of a (Λ, a, δ)-contact process and the latter modulo shifts, respectively. Then*

$$\mu P_t(\{A\}) = m(A) \langle\!\langle \mu \rangle\!\rangle \tilde{\mu} \tilde{P}_t(\tilde{A}) \qquad (t \geq 0,\ A \in \mathcal{P}_{{\rm fin},+}). \tag{2.22}$$

Moreover, for any shift-invariant function $f : \mathcal{P}_{{\rm fin},+} \to [0, \infty)$,

$$\sum_{\tilde{A}\in\tilde{\mathcal{P}}_{{\rm fin},+}} \tilde{\mu}(\tilde{A}) \tilde{f}(\tilde{A}) = \langle\!\langle \mu \rangle\!\rangle^{-1} \langle\!\langle f\mu \rangle\!\rangle. \tag{2.23}$$

[3] It is easy to see that the constant $m(A)$ defined in (2.17) satisfies $m(A) \leq |A|$ and that $\{i \in \Lambda : iA = A\}$ is a finite subgroup of Λ. If every element of Λ is of infinite order (as is the case, for example, for $\Lambda = \mathbb{Z}^d$), then $m(A) = 1$ for all finite $A \subset \Lambda$.

Formula (2.22) shows in particular that if μ is a homogeneous measure on $\mathcal{P}_{\text{fin},+}$ such that $\langle\!\langle\mu\rangle\!\rangle < \infty$ and $\lambda \in \mathbb{R}$, then

$$\mu P_t = e^{\lambda t}\mu \quad (t \geq 0) \quad \text{if and only if} \quad \tilde{\mu}\tilde{P}_t = e^{\lambda t}\tilde{\mu} \quad (t \geq 0). \tag{2.24}$$

Here, the relation $\tilde{\mu}\tilde{P}_t = e^{\lambda t}\tilde{\mu}$ says that the probability law $\tilde{\mu}$ is a *quasi-invariant law* for the (Λ, a, δ)-contact process modulo shifts. Since μ determines $\tilde{\mu}$ uniquely, and $\tilde{\mu}$ determines μ uniquely up to a multiplcative constant, this shows that there is a one-to-one correspondence between homogeneous eigenmeasures of the (Λ, a, δ)-contact process and quasi-invariant laws of the (Λ, a, δ)-contact process modulo shifts.

2.4 Dual functions

We have just seen that eigenmeasures of the (Λ, a, δ)-contact process give rise to quasi-invariant laws for the (Λ, a, δ)-contact process modulo shifts, i.e., normalized, positive left eigenvectors of the semigroup $(\tilde{P}_t)_{t\geq 0}$. In the present section, we will see that moreover, eigenmeasures of the *dual* $(\Lambda, a^\dagger, \delta)$-contact process give rise to positive *right* eigenvectors of the semigroup $(\tilde{P}_t)_{t\geq 0}$. This will allow us to Doob-transform the (Λ, a, δ)-contact process modulo shifts into a positively recurrent Markov chain and use ergodicity of the latter to prove, among other things, convergence to the quasi-invariant law of the process conditioned not to have died out.

Recall the definition of the subprobability kernels P_t and $P_t^\dagger$ in (1.19), which are the transition probabilities of the (Λ, a, δ)-contact process and its dual $(\Lambda, a^\dagger, \delta)$-contact process, respectively, restricted to the set $\mathcal{P}_+$ of nonempty subsets of Λ. As before, for $A, B \in \mathcal{P}_{\text{fin},+}$, we simply write $P_t(A, B) := P_t(A, \{B\})$. Let

$$\mathcal{S}(\mathcal{P}_{\text{fin},+}) := \{f : \mathcal{P}_{\text{fin},+} \to \mathbb{R} : \exists K, M, k \geq 0 \text{ s.t. } |f(A)| \leq K|A|^k + M \ \forall A \in \mathcal{P}_{\text{fin},+}\}. \tag{2.25}$$

denote the class of real functions on $\mathcal{P}_{\text{fin},+}$ of polynomial growth. For any $f \in \mathcal{S}(\mathcal{P}_{\text{fin},+})$, we define

$$P_t f(A) := \sum_{B \in \mathcal{P}_{\text{fin},+}} P_t(A, B) f(B) \qquad (t \geq 0,\ A \in \mathcal{P}_{\text{fin},+}). \tag{2.26}$$

Then [Swa09, Prop. 2.1] implies that P_t maps the space $\mathcal{S}(\mathcal{P}_{\text{fin},+})$ into itself.

For any locally finite measure μ on $\mathcal{P}_+$, we define a function $h_\mu : \mathcal{P}_{\text{fin},+} \to [0, \infty)$ by

$$h_\mu(A) := \int_{\mathcal{P}_+} \mu(\mathrm{d}B)\, 1_{\{A \cap B \neq \emptyset\}} \qquad (A \in \mathcal{P}_{\text{fin},+}), \tag{2.27}$$

which is finite by Lemma 1.1. We say that a function $f : \mathcal{P}_{\text{fin}} \to \mathbb{R}$ is *monotone* if $A \subset B$ implies $f(A) \leq f(B)$, and *subadditive* if $f(A \cup B) \leq f(A) + f(B)$, for all $A, B \in \mathcal{P}_{\text{fin}}$. Below, μP_t is defined as in (1.20).

Lemma 2.9 (Linear bounds)**.** *Let μ be a nonzero, homogeneous, locally finite measure μ on $\mathcal{P}_+$. For each nonzero, homogeneous, locally finite measure μ on $\mathcal{P}_+$, the function h_μ in (2.27) is shift-invariant, monotone, subadditive, strictly positive on $\mathcal{P}_{\text{fin},+}$, and satisfies $h_\mu(A) \leq h(\{0\})|A|$ $(A \in \mathcal{P}_{\text{fin},+})$. If μ is moreover concentrated on $\mathcal{P}_{\text{fin},+}$, then*

$$\langle\!\langle\mu\rangle\!\rangle|A| \leq h_\mu(A) \leq h(\{0\})|A| \qquad (A \in \mathcal{P}_{\text{fin},+}). \tag{2.28}$$

Proof. Shift-invariance, monotonicity, subadditivity and positivity are easy to check; see [Swa09, Lemma 3.5]. Subadditivity and shift-invariance now imply the upper bound in (2.28). If μ is concentrated on $\mathcal{P}_{\text{fin},+}$, then by Lemma 2.3, there exists a $\mathcal{P}_{\text{fin},+}$-valued random variable Δ such that μ can be written as in (2.2). Letting κ be a Λ-valued random variable such that $\kappa \in \Delta$ a.s., we observe that

$$h_\mu(A) = \langle\!\langle\mu\rangle\!\rangle \sum_{i \in \Lambda} \mathbb{P}[A \cap i\Delta \neq \emptyset] \geq \langle\!\langle\mu\rangle\!\rangle \sum_{i \in \Lambda} \mathbb{P}[A \cap \{i\kappa\} \neq \emptyset] = \langle\!\langle\mu\rangle\!\rangle|A|. \tag{2.29}$$

□

The next lemma is a simple consequence of duality.

Lemma 2.10 (Dual function). *For each nonzero, homogeneous, locally finite measure μ on $\mathcal{P}_+$, one has*

$$h_{\mu P_t} = P_t^\dagger h_\mu \qquad (t \geq 0). \tag{2.30}$$

Proof. The upper bound in (2.28) shows that $h_\mu \in \mathcal{S}(\mathcal{P}_{\text{fin},+})$, so $P_t^\dagger h_\mu$ is well-defined. Also, μP_t is homogeneous and locally finite by Lemma 2.4, so that $h_{\mu P_t}$ is well-defined. Now

$$\begin{aligned} h_{\mu P_t}(A) &= \int_{\mathcal{P}_+} \mu P_t(\mathrm{d}B)\, 1_{\{A\cap B\neq\emptyset\}} = \int_{\mathcal{P}_+} \mu(\mathrm{d}C)\mathbb{P}[A \cap \eta^C \neq \emptyset] \\ &= \int_{\mathcal{P}_+} \mu(\mathrm{d}C)\mathbb{P}[\eta_t^{\dagger\, A} \cap C \neq \emptyset] = \sum_{B\in\mathcal{P}_{\text{fin},+}} P_t^\dagger(A,B) \int_{\mathcal{P}_+} \mu(\mathrm{d}C)\, 1_{\{A\cap C\neq\emptyset\}} = P_t^\dagger h_\mu(A). \end{aligned} \tag{2.31}$$

□

In particular, if μ is a homogeneous eigenmeasure of the $(\Lambda, a^\dagger, \delta)$-contact process with eigenvalue λ, then Lemma 2.10 implies that $P_t h_\mu = e^{\lambda t} h_\mu$, i.e., h_μ is a right eigenfunction of P_t, for each $t \geq 0$. By formula (2.21), h_μ then also gives rise to a right eigenfunction $\tilde{h}_\mu$ of the semigroup of the process modulo shifts. The following lemma, whose proof can be found in Section 3.1, will be handy in what follows.

Lemma 2.11 (Intersection and weighted measures). *Let μ, ν be homogeneous locally finite measures on $\mathcal{P}_+$, assume that μ is concentrated on $\mathcal{P}_{\text{fin},+}$, and let h_ν be defined as in (2.27). Then*

$$\langle\!\langle \mu \boxtimes \nu \rangle\!\rangle = \langle\!\langle h_\nu \mu \rangle\!\rangle. \tag{2.32}$$

If moreover $\int \mu(\mathrm{d}A)|A| 1_{\{0\in A\}} < \infty$, then $h_\nu \mu$ is locally finite.

2.5 Convergence to the quasi-invariant law

By Theorem 2 (a), the (Λ, a, δ)-contact processes and its dual $(\Lambda, a^\dagger, \delta)$-contact processes have the same exponential growth rate $r = r(\Lambda, a, \delta) = r(\Lambda, a^\dagger, \delta)$. In particular, if $r < 0$, then by Lemma 2.7, there exist homogeneous eigenmeasures $\mathring{\nu}$ and $\mathring{\nu}^\dagger$ of the (Λ, a, δ)- and $(\Lambda, a^\dagger, \delta)$-contact process, respectively, both with eigenvalue r, such that

$$\int \mathring{\nu}(\mathrm{d}A)|A| 1_{\{0\in A\}} < \infty \quad \text{and} \quad \int \mathring{\nu}^\dagger(\mathrm{d}A)|A| 1_{\{0\in A\}} < \infty. \tag{2.33}$$

We normalize $\mathring{\nu}$ and $\mathring{\nu}^\dagger$ such that $\int \mathring{\nu}(\mathrm{d}A) 1_{\{0\in A\}} = 1 = \int \mathring{\nu}^\dagger(\mathrm{d}A) 1_{\{0\in A\}}$. For the moment, we do not know yet if $\mathring{\nu}$ and $\mathring{\nu}^\dagger$ are unique, but we simply fix any two such measures and we let

$$\tilde{\nu} := \tilde{\mathring{\nu}} \quad \text{and} \quad \tilde{\nu}^\dagger := \tilde{\mathring{\nu}}^\dagger \tag{2.34}$$

denote the associated probability laws on $\tilde{\mathcal{P}}_{\text{fin},+}$ as in (2.19). We let $h_{\mathring{\nu}}$ and $h_{\mathring{\nu}^\dagger}$ denote the associated dual functions as in (2.27) and let $\tilde{h}_{\mathring{\nu}}$ and $\tilde{h}_{\mathring{\nu}^\dagger}$ denote the associated functions on $\tilde{\mathcal{P}}_{\text{fin},+}$ as in (2.20). Finally, we define *Doob-transformed*[4] probability kernels on $\tilde{\mathcal{P}}_{\text{fin},+}$ by

$$Q_t(\tilde{A}, \tilde{B}) := e^{-rt} \tilde{h}_{\mathring{\nu}^\dagger}(\tilde{A})^{-1} P_t(\tilde{A}, \tilde{B}) \tilde{h}_{\mathring{\nu}^\dagger}(\tilde{B}) \qquad (t \geq 0,\ \tilde{A}, \tilde{B} \in \tilde{\mathcal{P}}_{\text{fin},+}). \tag{2.35}$$

[4] Doob's classical h-transform is based on a positive harmonic function h. In (2.35) we use a slight generalization of this where h is a positive eigenfunction of the generator. This is a special case of what is called a 'compensated h-transform' in [FS02, Lemma 3].

The results from the last two sections, together with classical results about quasi-invariant laws, then combine to give the following result.

Theorem 2.12 (Convergence to the quasi-invariant law)**.** *Assume that the infection rates satisfy the irreducibility condition (1.3) and that the exponential growth rate from (1.12) satisfies $r < 0$. Then:*

(a) *The $(Q_t)_{t\geq 0}$ from (2.35) are the transition probabilities of a positively recurrent continuous-time Markov chain with unique invariant law $\tilde{\pi}$ given by*

$$\tilde{\pi}(\tilde{A}) := \frac{\langle\!\langle \mathring{\nu} \rangle\!\rangle}{\langle\!\langle \mathring{\nu} \bowtie \mathring{\nu}^{\dagger} \rangle\!\rangle} \tilde{\nu}(\tilde{A}) \tilde{h}_{\mathring{\nu}^{\dagger}}(\tilde{A}) \qquad (\tilde{A} \in \tilde{\mathcal{P}}_{\mathrm{fin},+}). \tag{2.36}$$

(b) *The law $\tilde{\nu}$ is a quasi-invariant law for the (Λ, a, δ)-contact process modulo shifts. Moreover, for any $A \in \mathcal{P}_{\mathrm{fin},+}$, one has*

$$\mathbb{P}[\tilde{\eta}_t^A \in \cdot \,|\, \eta_t^A \neq \emptyset] \underset{t\to\infty}{\Longrightarrow} \tilde{\nu}, \tag{2.37}$$

where $\Rightarrow$ denotes weak convergence of probability laws on $\tilde{\mathcal{P}}_{\mathrm{fin},+}$.

Proof. By Lemmas 2.8 and 2.10, the measure $\tilde{\nu}$ and function $\tilde{h}_{\mathring{\nu}^{\dagger}}$ are left and right eigenvectors of the operators $\tilde{P}_t$, i.e.,

$$\tilde{\nu}\tilde{P}_t = e^{rt}\tilde{\nu} \quad \text{and} \quad \tilde{P}_t\tilde{h}_{\mathring{\nu}^{\dagger}} = e^{rt}\tilde{h}_{\mathring{\nu}^{\dagger}} \qquad (t \geq 0). \tag{2.38}$$

Moreover, by formula (2.23) and Lemma 2.11,

$$\sum_{\tilde{A}\in\tilde{\mathcal{P}}_{\mathrm{fin},+}} \tilde{\nu}(\tilde{A})\tilde{h}_{\mathring{\nu}^{\dagger}}(\tilde{A}) = \langle\!\langle \mathring{\nu} \rangle\!\rangle^{-1} \langle\!\langle h_{\mathring{\nu}^{\dagger}} \mathring{\nu} \rangle\!\rangle = \langle\!\langle \mathring{\nu} \rangle\!\rangle^{-1} \langle\!\langle \mathring{\nu} \bowtie \mathring{\nu}^{\dagger} \rangle\!\rangle < \infty. \tag{2.39}$$

Thus, we have found positive left and right eigenfunctions of $(\tilde{P}_t)_{t\geq 0}$ whose pointwise product is summable. The statements of the theorem now follow readily by well-known methods. More precisely, parts (a) and (b) follow from Lemmas B.2 and B.3 in the appendix, respectively. As explained in the proof of Lemma B.2 there, formulas (2.38) and (2.39) imply that the (Λ, a, δ)-contact process modulo shifts is λ-positive in the sense of Kingman [Kin63]. Note that the use of λ-positivity and its discrete time analogue R-positivity in the study of quasi-invariant laws is well-known, see e.g. [FKM96]. □

2.6 Convergence to the eigenmeasure

Proof of Theorem 5. The existence of $\mathring{\nu}$ and $\mathring{\nu}^{\dagger}$ has already been proved in Lemma 2.7, so uniqueness will follow once we prove the convergence in (1.28), with the $\mathring{\nu}$ that we fixed earlier. We need to prove two statements: vague convergence for general (nonzero, homogeneous, locally finite) initial measures μ and local convergence on $\mathcal{P}_{\mathrm{fin},+}$ if μ is concentrated on $\mathcal{P}_{\mathrm{fin},+}$.

We start with vague convergence. By Lemma 1.1, it suffices to show that

$$e^{-rt}\int \mu P_t(\mathrm{d}A)1_{\{A\cap B\neq\emptyset\}} \underset{n\to\infty}{\longrightarrow} c\int \mathring{\nu}(\mathrm{d}A)1_{\{A\cap B\neq\emptyset\}} \qquad (B \in \mathcal{P}_{\mathrm{fin},+}), \tag{2.40}$$

where $c > 0$ is given in (1.29). By Lemma 2.10, we observe that

$$\begin{aligned} &e^{-rt}\int \mu P_t(\mathrm{d}A)1_{\{A\cap B\neq\emptyset\}} = e^{-rt}h_{\mu P_t}(B) = e^{-rt}P_t^{\dagger}h_\mu(B) \\ &= \mathbb{E}[h_\mu(\eta_t^{\dagger\,B})] = \mathbb{E}[\tilde{h}_\mu(\tilde{\eta}_t^{\dagger\,B})] = \tilde{P}_t^{\dagger}\tilde{h}_\mu(\tilde{B}) \qquad (B \in \mathcal{P}_{\mathrm{fin},+}), \end{aligned} \tag{2.41}$$

where $(\tilde{P}^\dagger_t)_{t\geq 0}$ are defined as in (2.16) but for the $(\Lambda, a^\dagger, \delta)$-contact process modulo shifts. Applying Theorem 2.12 (a) to the dual process, we obtain that

$$Q^\dagger_t(\tilde{A},\tilde{B}) := e^{-rt}\tilde{h}_{\mathring{\nu}}(\tilde{A})^{-1}P^\dagger_t(\tilde{A},\tilde{B})\tilde{h}_{\mathring{\nu}}(\tilde{B}) \qquad (t\geq 0,\ \tilde{A},\tilde{B}\in\tilde{\mathcal{P}}_{\rm fin,+}) \tag{2.42}$$

are the transition probabilities of an irreducible, positively recurrent Markov process with state space $\tilde{\mathcal{P}}_{\rm fin,+}$ and invariant law $\pi^\dagger$ given by

$$\tilde{\pi}^\dagger(\tilde{A}) := \frac{\langle\!\langle\mathring{\nu}^\dagger\rangle\!\rangle}{\langle\!\langle\mathring{\nu}\boxtimes\mathring{\nu}^\dagger\rangle\!\rangle}\tilde{\nu}^\dagger(\tilde{A})\tilde{h}_{\mathring{\nu}}(\tilde{A}) \qquad (\tilde{A}\in\tilde{\mathcal{P}}_{\rm fin,+}). \tag{2.43}$$

The right-hand side of (2.41) can now be rewritten as

$$\tilde{P}^\dagger_t\tilde{h}_\mu(\tilde{B}) = \tilde{h}_{\mathring{\nu}}(\tilde{B})\sum_{\tilde{A}\in\tilde{\mathcal{P}}_{\rm fin,+}}Q^\dagger_t(\tilde{B},\tilde{A})\frac{\tilde{h}_\mu(\tilde{A})}{\tilde{h}_{\mathring{\nu}}(\tilde{A})} \qquad (\tilde{A}\in\tilde{\mathcal{P}}_{\rm fin,+}). \tag{2.44}$$

By formula (2.28) from Lemma 2.9, $\tilde{h}_\mu/\tilde{h}_{\mathring{\nu}}$ is a bounded function, so we may use the ergodicity of the irreducible, positively recurrent Markov process with transition probabilities $(Q^\dagger_t)_{t\geq 0}$ to conclude that

$$\begin{aligned}\tilde{P}^\dagger_t\tilde{h}_\mu(\tilde{B}) &\underset{t\to\infty}{\longrightarrow} \tilde{h}_{\mathring{\nu}}(\tilde{B})\sum_{\tilde{A}\in\tilde{\mathcal{P}}_{\rm fin,+}}\tilde{\pi}^\dagger(\tilde{A})\frac{\tilde{h}_\mu(\tilde{A})}{\tilde{h}_{\mathring{\nu}}(\tilde{A})}\\ &= \tilde{h}_{\mathring{\nu}}(\tilde{B})\frac{\langle\!\langle\mathring{\nu}^\dagger\rangle\!\rangle}{\langle\!\langle\mathring{\nu}\boxtimes\mathring{\nu}^\dagger\rangle\!\rangle}\sum_{\tilde{A}\in\tilde{\mathcal{P}}_{\rm fin,+}}\tilde{\nu}^\dagger(\tilde{A})\tilde{h}_\mu(\tilde{A}) \;=\; \frac{\langle\!\langle\mu\boxtimes\mathring{\nu}^\dagger\rangle\!\rangle}{\langle\!\langle\mathring{\nu}\boxtimes\mathring{\nu}^\dagger\rangle\!\rangle}\int\mathring{\nu}(\mathrm{d}A)1_{\{A\cap B\neq\emptyset\}},\end{aligned} \tag{2.45}$$

where in the last step we have used formula (2.23) from Lemma 2.8 as well as Lemma 2.11. Combining (2.45) with (2.40)–(2.41), this proves the vague convergence in (1.28).

It remains to show that vague convergence can be strengthened to local convergence on $\mathcal{P}_{\rm fin,+}$ if μ is concentrated on $\mathcal{P}_{\rm fin,+}$. By Proposition 2.1 (iv), it suffices to prove pointwise convergence. Let Q_t and π be given by (2.35)–(2.36). Then formula (2.22) from Lemma 2.8 tells us that

$$\mu P_t(\{B\}) = m(B)\langle\!\langle\mu\rangle\!\rangle\tilde{\mu}\tilde{P}_t(\tilde{B}) = m(B)\langle\!\langle\mu\rangle\!\rangle\, e^{rt}\sum_{\tilde{A}\in\tilde{\mathcal{P}}_{\rm fin,+}}\tilde{\mu}(\tilde{A})\tilde{h}_{\mathring{\nu}^\dagger}(\tilde{A})Q_t(\tilde{A},\tilde{B})\tilde{h}_{\mathring{\nu}^\dagger}(\tilde{B})^{-1}. \tag{2.46}$$

Here, on the right-hand side, we evolve the measure $\tilde{h}_{\mathring{\nu}^\dagger}\tilde{\mu}$ under the semigroup Q_t. Using formula (2.23) from Lemma 2.8 and Lemma 2.11 we see that this measure is finite with total mass given by

$$\sum_{\tilde{A}\in\tilde{\mathcal{P}}_{\rm fin,+}}\tilde{\mu}(\tilde{A})\tilde{h}_{\mathring{\nu}^\dagger}(\tilde{A}) = \langle\!\langle\mu\rangle\!\rangle^{-1}\langle\!\langle h_{\mathring{\nu}^\dagger}\mu\rangle\!\rangle = \langle\!\langle\mu\rangle\!\rangle^{-1}\langle\!\langle\mu\boxtimes\mathring{\nu}^\dagger\rangle\!\rangle < \infty. \tag{2.47}$$

Using this and the ergodicity of the Markov process with transition probabilities $(Q_t)_{t\geq 0}$, we find that

$$\begin{aligned}e^{-rt}\mu P_t(\{B\}) &\underset{t\to\infty}{\longrightarrow} m(B)\langle\!\langle\mu\boxtimes\mathring{\nu}^\dagger\rangle\!\rangle\pi(\tilde{B})\tilde{h}_{\mathring{\nu}^\dagger}(\tilde{B})^{-1}\\ &= \frac{\langle\!\langle\mu\boxtimes\mathring{\nu}^\dagger\rangle\!\rangle}{\langle\!\langle\mathring{\nu}\boxtimes\mathring{\nu}^\dagger\rangle\!\rangle}m(B)\langle\!\langle\mathring{\nu}\rangle\!\rangle\tilde{\nu}(\tilde{B}) = \frac{\langle\!\langle\mu\boxtimes\mathring{\nu}^\dagger\rangle\!\rangle}{\langle\!\langle\mathring{\nu}\boxtimes\mathring{\nu}^\dagger\rangle\!\rangle}\mathring{\nu}(\{B\}),\end{aligned} \tag{2.48}$$

where in the last step we have used formula (2.22) from Lemma 2.8 (with $t=0$). □

2.7 Continuity in the recovery rate

The first step in proving Theorem 6 will be to show continuity of the map $(\delta_{\rm c},\infty)\ni\delta\mapsto\mathring{\nu}_\delta$. We start by proving continuity with respect to vague convergence, which is based on the following abstract result, whose proof can be found in Section 3.2.

Lemma 2.13 (Limits of eigenmeasures)**.** *Let $(\nu_n)_{n\geq 0}$ be homogeneous eigenmeasures of (Λ,a,δ_n)-contact processes, with eigenvalues λ_n, normalized such that $\int\nu_n(\mathrm{d}A)1_{\{0\in A\}} = 1$. Assume that $\lambda_n\to\lambda$ and $\delta_n\to\delta$. Then the $(\nu_n)_{n\geq 0}$ are relatively compact in the topology of vague convergence, and each vague cluster point ν is a homogeneous eigenmeasure of the (Λ,a,δ)-contact processes, with eigenvalue λ.*

Continuity of the map $(\delta_{\rm c},\infty)\ni\delta\mapsto\mathring{\nu}_\delta$ is now a simple consequence of Theorem 5 and Lemma 2.13.

Proposition 2.14 (Vague continuity of the eigenmeasure)**.** *Assume that the infection rates satisfy the irreducibility condition (1.3). For $\delta\in(\delta_{\rm c},\infty)$, let $\mathring{\nu}_\delta$ denote the unique homogeneous eigenmeasure of the (Λ,a,δ)-contact process normalized such that $\int\mathring{\nu}_\delta(\mathrm{d}A)1_{\{0\in A\}} = 1$. Then the map $\delta\mapsto\mathring{\nu}_\delta$ is continuous on $(\delta_{\rm c},\infty)$ w.r.t. vague convergence of locally finite measures on $\mathcal{P}_+$.*

Proof. Choose $\delta_n,\delta\in(\delta_{\rm c},\infty)$ such that $\delta_n\to\delta$. Since the eigenvalue $r(\Lambda,a,\delta)$ of the homogeneous eigenmeasure $\mathring{\nu}_\delta$ is continuous in δ by Theorem 2 (b), Lemma 2.13 implies that the measures $(\mathring{\nu}_{\delta_n})_{n\geq 0}$ are relatively compact in the topology of vague convergence, and each vague cluster point is a homogeneous eigenmeasure of the (Λ,a,δ)-contact processes with eigenvalue $r(\Lambda,a,\delta)$. By Theorem 5, this implies that $\mathring{\nu}_\delta$ is the only vague cluster point, hence the $\mathring{\nu}_{\delta_n}$ converge vaguely to $\mathring{\nu}_\delta$. □

Unfortunately, continuity with respect to vague convergence is not enough to prove continuity of the right-hand side of (1.30), and hence of the derivative $\frac{\partial}{\partial\delta}r(\Lambda,a,\delta)$. As mentioned earlier, we will remedy this by proving continuity of the map $(\delta_{\rm c},\infty)\ni\delta\mapsto\mathring{\nu}_\delta$ with respect to local convergence on $\mathcal{P}_{\rm fin,+}$. Since vague convergence is already proved, by Proposition 2.1 (iii), it suffices to prove local tightness. This is the most technical part of our proofs, since it involves estimating how 'large' the finite sets can be that $\mathring{\nu}_\delta$ is concentrated on. The first step is to introduce a suitable concept of distance. The next result will be proved in Section 3.3.

Lemma 2.15 (Slowly growing metric)**.** *Let Λ be a countable group and let $a:\Lambda\times\Lambda\to[0,\infty)$ satisfy (1.1). Then there exists a metric d on Λ such that*

$$\begin{array}{lll}\text{(i)} & d(i,j)=d(ki,kj) & (i,j,k\in\Lambda),\\ \text{(ii)} & \big|\{i\in\Lambda: d(0,i)\leq M\}\big|<\infty & (0\leq M<\infty),\\ \text{(iii)} & K_\gamma(\Lambda,a):=\displaystyle\sum_i a(0,i)e^{\gamma d(0,i)}<\infty & (0\leq\gamma<\infty).\end{array}\tag{2.49}$$

Next, we fix a metric d as in (2.49) and for each $0\leq\gamma<\infty$, we define a function $e_\gamma:\mathcal{P}_{\rm fin}\to[0,\infty)$ by

$$e_\gamma(A):=\sum_{i\in A}e^{\gamma d(0,i)}\qquad(\gamma\geq 0,\ A\in\mathcal{P}_{\rm fin}).\tag{2.50}$$

We note that a similar (but not entirely identical) function has proved useful in the study of contact processes on trees, see [Lig99, formula (I.4.3)]. We have in particular $e_0(A)=|A|$. The next lemma says that there is a well-defined exponential growth rate $r_\gamma(\Lambda,a,\delta)$ associated with the function e_γ, which converges to our well-known exponential growth rate $r(\Lambda,a,\delta)$ as $\gamma\downarrow 0$. The proof can be found in Section 3.3.

Lemma 2.16 (Exponential growth rates)**.** *Let $(\eta^{\{0\}}_t)_{t\geq 0}$ be the (Λ,a,δ)-contact process started in $\eta^{\{0\}}_0=\{0\}$. Let d be a metric on Λ as in Lemma 2.15, and let e_γ be the function defined in (2.50). Then, for each $0\leq\gamma<\infty$, the limit*

$$r_\gamma=r_\gamma(\Lambda,a,\delta):=\lim_{t\to\infty}\tfrac{1}{t}\log\mathbb{E}\big[e_\gamma(\eta^{\{0\}}_t)\big]=\inf_{t>0}\tfrac{1}{t}\log\mathbb{E}\big[e_\gamma(\eta^{\{0\}}_t)\big] \tag{2.51}$$

exists. The function $\gamma\mapsto r_\gamma$ is nondecreasing, right-continuous, and satisfies

$$-\delta\leq r_\gamma(\Lambda,a,\delta)\leq K_\gamma(\Lambda,a)\qquad(\gamma\geq 0), \tag{2.52}$$

where $K_\gamma(\Lambda,a)$ is defined in (2.49).

We can generalize the proof of Lemma 2.6 to yield a more general version of that lemma (see Lemma 3.5 below), which after taking the limit (as in (2.15)) yields the following bound on the eigenmeasures $\mathring{\nu}_\delta$. (We refer to Section 3.4 for the detailed proof.)

Lemma 2.17 (Tightness estimate)**.** *Let $(\eta^{\{0\}}_t)_{t\geq 0}$ be the (Λ,a,δ)-contact process started in $\eta^{\{0\}}_0=\{0\}$, let $r(\delta)=r(\Lambda,a,\delta)$ be its exponential growth rate, let d be a metric on Λ as in Lemma 2.15, and let e_γ be the function defined in (2.50). For $\delta\in(\delta_{\rm c},\infty)$, let $\mathring{\nu}_\delta$ denote the unique homogeneous eigenmeasure of the (Λ,a,δ)-contact process normalized such that $\int\mathring{\nu}_\delta(\mathrm{d}A)1_{\{0\in A\}}=1$. Then*

$$\int\mathring{\nu}_\delta(\mathrm{d}A)1_{\{0\in A\}}e_\gamma(A)\leq(|a|+\delta)\int_0^\infty e^{-r(\delta)t}\mathrm{d}t\,\mathbb{E}\big[e_\gamma(\eta^{\delta,\{0\}}_t)\big]^2\qquad\big(\gamma\geq 0,\ \delta\in(\delta_{\rm c},\infty)\big). \tag{2.53}$$

With this preparation we are now ready to prove the desired local continuity.

Proposition 2.18 (Local continuity of the eigenmeasure)**.** *Assume that the infection rates satisfy the irreducibility condition (1.3). For each $\delta\in(\delta_{\rm c},\infty)$, let $\mathring{\nu}_\delta$ denote the unique homogeneous eigenmeasure of the (Λ,a,δ)-contact process normalized such that $\int\mathring{\nu}_\delta(\mathrm{d}A)1_{\{0\in A\}}=1$. Then the map $\delta\mapsto\mathring{\nu}_\delta$ is continuous on $(\delta_{\rm c},\infty)$ in the sense of local convergence on $\mathcal{P}_{\mathrm{fin},+}$.*

Proof. Vague continuity of the map $(\delta_{\rm c},\infty)\ni\delta\mapsto\mathring{\nu}_\delta$ has been proved in Proposition 2.14, so by Proposition 2.1 (iii), it suffices to show that for any $\delta_*\in(\delta_{\rm c},\infty)$ there exists an $\varepsilon>0$ such that the measures $(\mathring{\nu}_\delta)_{\delta\in(\delta_*-\varepsilon,\delta_*+\varepsilon)}$ are locally tight.

By property (2.49) (ii), for each $\gamma>0$ and $K<\infty$, the set $\{A\in\mathcal{P}_{\mathrm{fin},0}:e_\gamma(A)\leq K\}$ is finite. Thus, by Lemma 2.17, to prove the required local tightness, it suffices to show that for each $\delta_*\in(\delta_{\rm c},\infty)$ there exist a $\gamma>0$ and $\varepsilon>0$ such that

$$\sup_{\delta\in(\delta_*-\varepsilon,\delta_*+\varepsilon)}\int_0^\infty e^{-r(\delta)t}\mathrm{d}t\,\mathbb{E}\big[e_\gamma(\eta^{\delta,\{0\}}_t)\big]^2<\infty. \tag{2.54}$$

By the continuity of $\delta\mapsto r(\delta)$ (Theorem 2 (b)), we can choose $\varepsilon>0$ such that $\delta_{\rm c}<\delta_*-\varepsilon$ and

$$r(\delta_*-\varepsilon)\leq\frac{4}{5}r(\delta_*+\varepsilon). \tag{2.55}$$

Let $r_\gamma=r_\gamma(\delta)$ be the exponential growth rate associated with the function e_γ. By Lemma 2.16, the function $\gamma\mapsto r_\gamma$ is right-continuous, so we can choose $\gamma>0$ such that

$$r_\gamma(\delta_*-\varepsilon)\leq\frac{3}{4}r(\delta_*-\varepsilon). \tag{2.56}$$

By the fact that $r(\delta)$ is nonincreasing in δ and the law of $\eta_t^{\delta,\{0\}}$ is nonincreasing in δ with respect to the stochastic order, it follows that for all $\delta \in (\delta_* - \varepsilon, \delta_* + \varepsilon)$,

$$\begin{aligned}\int_0^\infty e^{-r(\delta)t}\mathrm{d}t\,\mathbb{E}\big[e_\gamma(\eta_t^{\delta,\{0\}})\big]^2 &\le \int_0^\infty e^{-r(\delta_*+\varepsilon)t}\mathrm{d}t\,\mathbb{E}\big[e_\gamma(\eta_t^{\delta_*-\varepsilon,\{0\}})\big]^2 \\ = \int_0^\infty \mathrm{d}t\,e^{(2r_\gamma(\delta_*-\varepsilon)-r(\delta_*+\varepsilon))t+o(t)} &\le \int_0^\infty \mathrm{d}t\,e^{\frac{1}{5}r(\delta_*+\varepsilon)t+o(t)} < \infty,\end{aligned} \tag{2.57}$$

where $t \mapsto o(t)$ is continuous, $o(t)/t \to 0$ for $t \to \infty$ by the definition of r_γ in Lemma 2.16, and we have used that $2r_\gamma(\delta_*-\varepsilon) \le 2\cdot\frac{3}{4}\cdot\frac{4}{5}r(\delta_*+\varepsilon) = \frac{6}{5}r(\delta_*+\varepsilon)$. This proves (2.54) and hence the required local tightness. □

2.8 The derivative of the exponential growth rate

Recall the definition of the homogeneous, locally finite measures χ_A in (1.21). Let $(P_t^\delta)_{t\ge0}$ and $(P_t^{\dagger\,\delta})_{t\ge0}$ be the subprobability kernels defined in (1.19) for the (Λ,a,δ)- and $(\Lambda,a^\dagger,\delta)$-contact processes, respectively, in dependence on δ. Note that $\chi_{\{0\}}P_t^\delta$ denotes the 'law' at time t of the process started with a single infected site distributed according to the counting measure on Λ. We start by rewriting Russo's formula (1.14) in terms of the objects we are working with.

Lemma 2.19 (Differential formula)**.** *For each $t \ge 0$, the function $[0,\infty) \ni \delta \mapsto \mathbb{E}[|\eta_t^{\delta,\{0\}}|]$ is continuously differentiable and satisfies*

$$-\frac{\partial}{\partial\delta}\frac{1}{t}\log\mathbb{E}\big[|\eta_t^{\delta,\{0\}}|\big] = \frac{1}{t}\int_0^t \mathrm{d}s\,\frac{\chi_{\{0\}}P_s^\delta \otimes \chi_{\{0\}}P_{t-s}^{\dagger\,\delta}(\{0\})}{\langle\!\langle \chi_{\{0\}}P_s^\delta \otimes \chi_{\{0\}}P_{t-s}^{\dagger\,\delta}\rangle\!\rangle}. \tag{2.58}$$

Proof. By (1.14) and the definition of the Campbell law $\hat{\mathbb{P}}_t^{\{0\}}$ in (1.10)

$$-\frac{\partial}{\partial\delta}\frac{1}{t}\log\mathbb{E}\big[|\eta_t^{\delta,\{0\}}|\big] = \frac{1}{t}\int_0^t \mathrm{d}s\,\frac{1}{\mathbb{E}\big[|\eta_t^{\delta,\{0\}}|\big]}\sum_{i,j}\mathbb{P}[(0,0)\rightsquigarrow_{(j,s)}(i,t)], \tag{2.59}$$

where

$$\begin{aligned}\sum_{i,j}\mathbb{P}[(0,0)\rightsquigarrow_{(j,s)}(i,t)] &= \sum_{i,j}\mathbb{P}[(j^{-1},-s)\rightsquigarrow_{(0,0)}(j^{-1}i,t-s)] \\ = \sum_{i,j}\mathbb{P}[\eta_s^{\delta,\{i\}}\cap\eta_{t-s}^{\dagger\,\delta,\{j\}} = \{0\}] &= \int\chi_{\{0\}}P_s^\delta(\mathrm{d}A)\int\chi_{\{0\}}P_{t-s}^{\dagger\,\delta}(\mathrm{d}B)\,1_{\{A\cap B=\{0\}\}}.\end{aligned} \tag{2.60}$$

Since, by Lemma 1.2 and (2.7), for $0 \le s \le t$,

$$\begin{aligned}&\langle\!\langle \chi_{\{0\}}P_s^\delta \otimes \chi_{\{0\}}P_{t-s}^{\dagger\,\delta}\rangle\!\rangle = \langle\!\langle \chi_{\{0\}}P_t^\delta \otimes \chi_{\{0\}}\rangle\!\rangle \\ &= \int\chi_{\{0\}}P_t^\delta(\mathrm{d}A)\int\chi_{\{0\}}(\mathrm{d}B)\,|A\cap B|^{-1}1_{\{0\in A\cap B\}} \\ &= \sum_i\int\chi_{\{0\}}P_t^\delta(\mathrm{d}A)\,|A\cap\{i\}|^{-1}1_{\{0\in A\cap\{i\}\}} = \int\chi_{\{0\}}P_t^\delta(\mathrm{d}A)\,1_{\{0\in A\}} = \mathbb{E}\big[|\eta_t^{\delta,\{0\}}|\big],\end{aligned} \tag{2.61}$$

we may rewrite the normalizing constant in (2.59) as in (2.58). □

We will prove Theorem 6 by taking the limit $t \to \infty$ in (2.58). To justify the interchange of limit and differentiation, we will use the following lemma.

Lemma 2.20 (Interchange of limit and differentiation)**.** *Let $I \subset \mathbb{R}$ be a compact interval and let f_n, f, f' be continuous real functions on I. Assume each f_n is continuously differentiable, that $f_n(x) \to f(x)$ and $\frac{\partial}{\partial x}f_n(x) \to f'(x)$ for each $x \in I$, and that*

$$\sup_{x\in I}\sup_n |\tfrac{\partial}{\partial x}f_n(x)| < \infty. \tag{2.62}$$

Then f is continuously differentiable and $\frac{\partial}{\partial x}f(x)=f'(x)$ $(x\in I)$.

Proof. We write $I=[x_-,x_+]$ and observe that

$$\begin{aligned} f(x) &= \lim_{n\to\infty} f_n(x_-) + \lim_{n\to\infty}\int_{x_-}^{x} \tfrac{\partial}{\partial y} f_n(y)\,\mathrm{d}y \\ &= f(x_-) + \int_{x_-}^{x} \big(\lim_{n\to\infty}\tfrac{\partial}{\partial y} f_n(y)\big)\mathrm{d}y = f(x_-) + \int_{x_-}^{x} f'(y)\,\mathrm{d}y, \end{aligned} \tag{2.63}$$

where the interchange of limit and integration is justified by dominated convergence, using (2.62). Differentiation of (2.63) now yields the statement since f' is continuous. □

Proof of Theorem 6. Continuity of the map $(\delta_{\mathrm{c}},\infty)\ni\delta\mapsto\mathring{\nu}_\delta$, and likewise for $\mathring{\nu}^\dagger_\delta$, in the sense of local convergence on $\mathcal{P}_{\mathrm{fin},+}$ has already been proved in Proposition 2.18. By Lemma 2.2, this implies local continuity of the map $(\delta_{\mathrm{c}},\infty)\ni\delta\mapsto\mathring{\nu}_\delta\boxtimes\mathring{\nu}^\dagger_\delta$. Since local convergence on $\mathcal{P}_{\mathrm{fin},+}$ implies convergence of the integral of the bounded functions $A\mapsto 1_{\{A=\{0\}\}}$ and $A\mapsto|A|^{-1}1_{\{0\in A\}}$ (which occurs in the definition of $\langle\!\langle\,\cdot\,\rangle\!\rangle$), this implies continuity of the right-hand side of (1.30).

Note that the right-hand side of (1.14) is clearly bounded between zero and one. Therefore, since

$$\frac{1}{t}\log\mathbb{E}\big[|\eta^{\delta,\{0\}}_t|\big]\underset{t\to\infty}{\longrightarrow} r(\Lambda,a,\delta)\qquad(\delta\geq 0) \tag{2.64}$$

by the definition of the exponential growth rate in (1.12), using Lemma 2.20, we see that (1.30) follows provided we show that the right-hand side of (2.58) converges for each $\delta\in(\delta_{\mathrm{c}},\infty)$ to the right-hand side of (1.30) as $t\to\infty$.

We rewrite the right-hand side of (2.58) as

$$\int_0^1 \mathrm{d}u\,\frac{e^{-rtu}\chi_{\{0\}}P^\delta_{tu}\boxtimes e^{-rt(1-u)}\chi_{\{0\}}P^{\dagger\,\delta}_{t(1-u)}\,(\{0\})}{\langle\!\langle e^{-rtu}\chi_{\{0\}}P^\delta_{tu}\boxtimes e^{-rt(1-u)}\chi_{\{0\}}P^{\dagger\,\delta}_{t(1-u)}\rangle\!\rangle}. \tag{2.65}$$

It is easy to see from the definition of $\langle\!\langle\,\cdot\,\rangle\!\rangle$ that the integrand is bounded between zero and one (in fact, this is the probability in (1.14)). By Theorem 5, for each $0<u<1$, the measures $e^{-rtu}\chi_{\{0\}}P^\delta_{tu}$ and $e^{-rt(1-u)}\chi_{\{0\}}P^{\dagger\,\delta}_{t(1-u)}$ converge locally on $\mathcal{P}_{\mathrm{fin},+}$ to constant multiples of $\mathring{\nu}_\delta$ and $\mathring{\nu}^\dagger_\delta$, respectively. By Lemma 2.2 and the fact that local convergence on $\mathcal{P}_{\mathrm{fin},+}$ implies convergence of the integral of the bounded functions $A\mapsto 1_{\{A=\{0\}\}}$ and $A\mapsto|A|^{-1}1_{\{0\in A\}}$, we see that the integrand in (2.65) converges in a bounded pointwise way with respect to u to the right-hand side of (1.30). Thus, the result follows by Lebesgue's dominated convergence theorem. □

2.9 Proof of the remaining theorems

Proof of formula (1.22). For $A=\{0\}$ this is proved in [Swa09, Lemma 4.2]. Even though the proof in the general case is the same, we give it here for completeness. For any $B\in\mathcal{P}_{\mathrm{fin}}$, one has

$$\hat{\mathbb{P}}^A_t\big[\iota^{-1}\eta^A_t=B\big]=\sum_{i\in\Lambda}\hat{\mathbb{P}}^A_t\big[i^{-1}\eta^A_t=B,\ \iota=i\big]=\mathbb{E}\big[|\eta^A_t|\big]^{-1}\sum_{i\in\Lambda}\mathbb{P}\big[i^{-1}\eta^A_t=B,\ i\in\eta^A_t\big]. \tag{2.66}$$

By (2.7), the normalizing factor of the right-hand side is given by $\mathbb{E}\big[|\eta^A_t|\big]=\chi_A P_t(\mathcal{P}_0)$ (with $\mathcal{P}_0=\{B\in\mathcal{P}:0\in B\}$), while the unnormalized expression equals

$$\sum_{i\in\Lambda}\mathbb{P}\big[i^{-1}\eta^A_t=B,\ 0\in i^{-1}\eta^A_t\big]=1_{\{0\in B\}}\sum_{i\in\Lambda}\mathbb{P}\big[\eta^{i^{-1}A}_t=B\big]=1_{\{0\in B\}}\chi_A P_t(\{B\}). \tag{2.67}$$

We conclude from this that the right-hand side of (2.66) equals $\chi_A P_t(\{B\}\,|\,\mathcal{P}_0)$. □

Proof of Theorem 1. By Theorem 2 (d), $\delta > \delta_c$ implies $r < 0$. Using notation as in (1.23), let $\mathcal{P}_{\text{fin},0} := \{A \in \mathcal{P}_{\text{fin}} : 0 \in A\}$. Fix $A \in \mathcal{P}_{\text{fin},+}$ and let χ_A be as in (1.21). Then Theorem 5 implies that as $t \to \infty$, the measures $e^{-rt}\chi_A P_t$ converge locally on $\mathcal{P}_{\text{fin},+}$ to $c\mathring{\nu}$, for some $c > 0$. By the definition of local convergence on $\mathcal{P}_{\text{fin},+}$ in Section 1.6, this implies that the restrictions of $e^{-rt}\chi_A P_t$ to $\mathcal{P}_{\text{fin},0}$ converge weakly to the restriction of $c\mathring{\nu}$ to $\mathcal{P}_{\text{fin},0}$, with respect to the discrete topology. Since $e^{-rt}\chi_A P_t(\mathcal{P}_{\text{fin},0})$ converges to $c\mathring{\nu}(\mathcal{P}_{\text{fin},0})$, formula (1.22) implies that

$$\hat{\mathbb{P}}^A_t\big[\iota^{-1}\eta^A_t \in \cdot\,\big] \underset{t\to\infty}{\Longrightarrow} \frac{c\mathring{\nu}}{c\mathring{\nu}(\mathcal{P}_{\text{fin},0})} = \mathring{\nu}\big(\,\cdot\,\big|\,\mathcal{P}_{\text{fin},0}\big), \tag{2.68}$$

where $\Rightarrow$ denotes weak convergence of probability measures on $\mathcal{P}_{\text{fin},0}$, equipped with the discrete topology. □

Proof of Theorem 4. By (2.68), the laws $\hat{\nu}_\delta$ and $\hat{\nu}^\dagger_\delta$ of the random variables $\hat{\eta}^\delta$ and $\hat{\eta}^{\dagger\,\delta}$ are just the eigenmeasures $\mathring{\nu}_\delta$ and $\mathring{\nu}^\dagger_\delta$, respectively, conditioned on the event $\{A : 0 \in A\}$. In particular, normalizing these eigenmeasures such that $\int \mathring{\nu}_\delta(\mathrm{d}A)1_{\{0\in A\}} = 1 = \int \mathring{\nu}^\dagger_\delta(\mathrm{d}A)1_{\{0\in A\}}$, we have that $\hat{\nu}_\delta = \mathring{\nu}_\delta|_{\mathcal{P}_{\text{fin},0}}$ and $\hat{\nu}^\dagger_\delta = \mathring{\nu}^\dagger_\delta|_{\mathcal{P}_{\text{fin},0}}$, so the continuity on (δ_c,∞) of $\delta \mapsto \hat{\nu}_\delta$ and $\delta \mapsto \hat{\nu}^\dagger_\delta$ in the sense of weak convergence w.r.t. the discrete topology follows from the continuity in the sense of local convergence of $\mathring{\nu}_\delta$ and $\mathring{\nu}^\dagger_\delta$. Moreover, we see that

$$\frac{\mathbb{P}\big[\hat{\eta}^\delta \cap \hat{\eta}^{\dagger\,\delta} = \{0\}\big]}{\mathbb{E}\big[|\hat{\eta}^\delta \cap \hat{\eta}^{\dagger\,\delta}|^{-1}\big]} = \frac{\sum_{A,B\in\mathcal{P}_{\text{fin}}} \mathring{\nu}_\delta(\{A\})\mathring{\nu}^\dagger_\delta(\{B\})1_{\{A\cap B=\{0\}\}}}{\sum_{A,B\in\mathcal{P}_{\text{fin}}} \mathring{\nu}_\delta(\{A\})\mathring{\nu}^\dagger_\delta(\{B\})|A\cap B|^{-1}1_{\{0\in A\cap B\}}} = \frac{\mathring{\nu}_\delta \boxtimes \mathring{\nu}^\dagger_\delta(\{0\})}{\langle\!\langle \mathring{\nu}_\delta \boxtimes \mathring{\nu}^\dagger_\delta \rangle\!\rangle}. \tag{2.69}$$

Thus, Theorem 4 is simply a reformulation of Theorem 6. □

Proof of Theorem 3. Fix $A \in \mathcal{P}_{\text{fin},+}$. By Proposition 2.5, we can choose $\lambda_n \downarrow r$ such that the measures $\hat{\pi}_{\lambda_n}(A)^{-1}\chi_A\hat{P}_{\lambda_n}$ converge vaguely to a homogeneous eigenmeasure $\mathring{\nu}$ with eigenvalue r. Since $B \mapsto 1_{\{0\in B\}}$ is a continuous, compactly supported function on $\mathcal{P}_+$ (equipped with the product topology), it follows that for any bounded continuous function $f : \mathcal{P}_+ \to \mathbb{R}$,

$$\hat{\pi}_{\lambda_n}(A)^{-1}\int \chi_A\hat{P}_{\lambda_n}(\mathrm{d}B)f(B)1_{\{0\in B\}} \underset{n\to\infty}{\longrightarrow} \int \mathring{\nu}(\mathrm{d}B)f(B)1_{\{0\in B\}}, \tag{2.70}$$

which implies that the conditioned measures $\chi_A\hat{P}_{\lambda_n}(\,\cdot\,|\,\mathcal{P}_0)$ converge weakly to $\mathring{\nu}(\,\cdot\,|\,\mathcal{P}_0)$ with respect to the product topology on $\mathcal{P}$, where we use the notation $\mathcal{P}_0 := \{B \in \mathcal{P} : 0 \in B\}$ as in (1.23).

Since we are assuming that $r = 0$ and the upper invariant law $\overline{\nu}$ is nontrivial, it follows from [Swa09, Thm 1.5] that $\mathring{\nu} = c\overline{\nu}$ for some $c > 0$ and hence

$$\chi_A\hat{P}_{\lambda_n}(\,\cdot\,|\,\mathcal{P}_0) \underset{n\to\infty}{\Longrightarrow} \overline{\nu}(\,\cdot\,|\,\mathcal{P}_0), \tag{2.71}$$

where $\Rightarrow$ denotes weak convergence. Now, for any measurable $\mathcal{A} \subset \mathcal{P}_+$,

$$\chi_A\hat{P}_{\lambda_n}(\mathcal{A}\,|\,\mathcal{P}_0) = \frac{\int_0^\infty \chi_A P_t(\mathcal{A}\cap\mathcal{P}_0)e^{-\lambda_n t}\mathrm{d}t}{\int_0^\infty \chi_A P_t(\mathcal{P}_0)e^{-\lambda_n t}\mathrm{d}t} = \hat{\pi}_{\lambda_n}(A)^{-1}\int_0^\infty \chi_A P_t(\mathcal{A}\,|\,\mathcal{P}_0)\chi_A P_t(\mathcal{P}_0)e^{-\lambda_n t}\mathrm{d}t, \tag{2.72}$$

so by grace of (1.22) we see that (1.13) holds with $\rho_n(\mathrm{d}t) := \hat{\pi}_{\lambda_n}(A)^{-1}\chi_A P_t(\mathcal{P}_0)e^{-\lambda_n t}\mathrm{d}t$. Since $\lim_{n\to\infty}\hat{\pi}_{\lambda_n}(A) = \infty$ by (2.9), we see that $\lim_{n\to\infty}\rho_n([0,T]) = 0$ for each $T < \infty$. □

3 Proof details

In this section we supply the proof of all propositions and lemmas that have not been proved yet. The organization is as follows. In Section 3.1 we prove some properties of locally finite measures and different forms of convergence, concretely Proposition 2.1 and Lemmas 2.2, 2.3, 2.11 and 2.8. In Section 3.2 we consider contact processes started in infinite initial 'laws', proving Lemmas 1.2 and 2.13. In Section 3.3 we construct a metric on Λ with properties as in Lemma 2.15 and prove Lemma 2.16 on the exponential growth rate associated with the functions e_γ defined in terms of such a metric. In Section 3.4 we do a covariance calculation leading to an estimate of which Lemma 2.6 is a special case and use this to derive Lemma 2.17.

3.1 Locally finite measures

In this section, we prove Proposition 2.1 as well as Lemmas 2.2, 2.3 and 2.11. Our first aim is Proposition 2.1. We start with two preparatory lemmas. Recall the definition of $\mathcal{P}_i$ from (1.23).

Lemma 3.1 (Compact classes)**.** *If $\mathcal{C} \subset \mathcal{P}_+$ is compact, then there exists a finite $\Delta \subset \Lambda$ such that $\mathcal{C} \subset \bigcup_{i\in\Delta} \mathcal{P}_i$.*

Proof. Choose $\Delta_n \uparrow \Lambda$ with Δ_n finite. If $\mathcal{C} \not\subset \bigcup_{i\in\Delta_n} \mathcal{P}_i$ for each n, then we can find $A_n \in \mathcal{C}$ such that $A_n \cap \Delta_n = \emptyset$. It follows that $A_n \to \emptyset \notin \mathcal{C}$ (in the product topology), hence $\mathcal{C}$ is not a closed subset of $\mathcal{P}$ and therefore not compact. □

Lemma 3.2 (Vague and weak convergence)**.** *Let μ_n, μ be locally finite measures on $\mathcal{P}_+$. Then the μ_n converge vaguely to μ if and only if for each $i \in \Lambda$, the restricted measures $\mu_n|_{\mathcal{P}_i}$ converge weakly to $\mu|_{\mathcal{P}_i}$ with respect to the product topology.*

Proof. Since $\mathcal{P}\backslash\mathcal{P}_i$ is a closed subset of $\mathcal{P}$, any continuous function $f : \mathcal{P}_i \to \mathbb{R}$ can be extended to a continuous, compactly supported function on $\mathcal{P}_+$ by putting $f(A) := 0$ for $A \in \mathcal{P}_+\backslash\mathcal{P}_i$. Therefore, if the μ_n converge vaguely to μ, it follows that the $\mu_n|_{\mathcal{P}_i}$ converge weakly to $\mu|_{\mathcal{P}_i}$. Conversely, if for each $i \in \Lambda$ the $\mu_n|_{\mathcal{P}_i}$ converge weakly to $\mu|_{\mathcal{P}_i}$, then for each $i, j \in \Lambda$ one has

$$\mu_n|_{\mathcal{P}_i\cap\mathcal{P}_j} \Rightarrow \mu|_{\mathcal{P}_i\cap\mathcal{P}_j}, \quad \mu_n|_{\mathcal{P}_i\backslash\mathcal{P}_j} \Rightarrow \mu|_{\mathcal{P}_i\backslash\mathcal{P}_j} \quad \text{and} \quad \mu_n|_{\mathcal{P}_j\backslash\mathcal{P}_i} \Rightarrow \mu|_{\mathcal{P}_j\backslash\mathcal{P}_i}, \tag{3.1}$$

where we have used that $\mathcal{P}_i \cap \mathcal{P}_j$, $\mathcal{P}_i\backslash\mathcal{P}_j$ and $\mathcal{P}_j\backslash\mathcal{P}_i$ are compact sets. Continuing this process, we see by induction that for each finite $\Delta \subset \Lambda$, the restrictions $\mu_n|_{\bigcup_{i\in\Delta}\mathcal{P}_i}$ converge weakly to $\mu|_{\bigcup_{i\in\Delta}\mathcal{P}_i}$. By Lemma 3.1, if $f : \mathcal{P}_+ \to \mathbb{R}$ is a compactly supported continuous function, then f is supported on $\bigcup_{i\in\Delta}\mathcal{P}_i$ for some finite $\Delta \subset \Lambda$. It follows that $\int \mu_n(\mathrm{d}A)f(A) \to \int \mu(\mathrm{d}A)f(A)$, proving that the μ_n converge vaguely to μ. □

Proof of Proposition 2.1. The equivalence of (i) and (ii) follows in a straightforward manner from Prohorov's theorem applied to the countable space $\mathcal{P}_{\mathrm{fin},\,i}$ with the discrete topology.

Since the discrete topology on $\mathcal{P}_{\mathrm{fin},\,i}$ is stronger than the product topology, weak convergence of the $\mu_n|_{\mathcal{P}_{\mathrm{fin},\,i}}$ with respect to the discrete topology implies weak convergence with respect to the product topology. By Lemma 3.2, this shows that local convergence on $\mathcal{P}_{\mathrm{fin},\,+}$ implies vague convergence on $\mathcal{P}_+$ and hence (i) implies also (iii).

To prove (iii)⇒(i), note that by local tightness, for each $i \in \Lambda$ the measures $\mu_n|_{\mathcal{P}_{\mathrm{fin},\,i}}$ are relatively compact in the topology of weak convergence with respect to the discrete topology. Let μ_*^i be a subsequential limit. Since weak convergence with respect to the discrete topology implies weak convergence with respect to the product topology, by Lemma 3.2, we conclude that $\mu_*^i = \mu|_{\mathcal{P}_{\mathrm{fin},\,i}}$. Since this is true for each cluster point, we conclude that the $\mu_n|_{\mathcal{P}_{\mathrm{fin},\,i}}$ converge weakly to $\mu|_{\mathcal{P}_{\mathrm{fin},\,i}}$ with respect to the discrete topology.

The implication (i)⇒(iv) follows from what we have already proved. To prove the reverse implication, it suffices to show local tightness. Since for each $i \in \Lambda$, the finite measures $\mu_n|_{\mathcal{P}_{\mathrm{fin},\,i}}$ converge pointwise to $\mu|_{\mathcal{P}_{\mathrm{fin},\,i}}$, it suffices to show that their total mass satisfies

$$\limsup_{n\to\infty} \mu_n(\{A : i \in A\}) \le \mu(\{A : i \in A\}). \tag{3.2}$$

By vague convergence (see Lemma 1.1), the limit superior is actually a limit and equals the right-hand side. □

Proof of Lemma 2.2. The local finiteness of $\mu \circledast \nu$ follows from Lemma 1.1 and the fact that

$$\begin{aligned}\int \mu \circledast \nu\,(\mathrm{d}C)1_{\{i\in C\}} &= \int \mu(\mathrm{d}A) \int \nu(\mathrm{d}B)1_{\{i\in A\cap B\}} \\ &= \Big(\int \mu(\mathrm{d}A)1_{\{i\in A\}}\Big)\Big(\int \nu(\mathrm{d}B)1_{\{i\in B\}}\Big) < \infty \quad (i \in \Lambda).\end{aligned} \tag{3.3}$$

To see that $\mu_n \circledast \nu_n$ converges vaguely to $\mu \circledast \nu$ if μ_n, ν_n converge vaguely to μ, ν, respectively, by Lemma 1.1, it suffices to check that

$$\int \mu_n \circledast \nu_n(\mathrm{d}C)1_{\{C\cap D\neq\emptyset\}} \underset{n\to\infty}{\longrightarrow} \int \mu \circledast \nu(\mathrm{d}C)1_{\{C\cap D\neq\emptyset\}} \qquad (D \in \mathcal{P}_{\mathrm{fin},+}). \tag{3.4}$$

Since

$$1_{\{C\cap D\neq\emptyset\}} = 1 - \prod_{i\in D} 1_{\{i\notin C\}} = 1 - \prod_{i\in D}(1 - 1_{\{i\in C\}}) = \sum_{\substack{D'\subset D\\ D'\neq\emptyset}} (-1)^{|D'|+1} \prod_{i\in D'} 1_{\{i\in C\}}, \tag{3.5}$$

and since $\prod_{i\in D'} 1_{\{i\in C\}} = 1_{\{D'\subset C\}}$ formula (3.4) is equivalent to

$$\int \mu_n \circledast \nu_n(\mathrm{d}C)1_{\{D\subset C\}} \underset{n\to\infty}{\longrightarrow} \int \mu \circledast \nu(\mathrm{d}C)1_{\{D\subset C\}} \qquad (D \in \mathcal{P}_{\mathrm{fin},+}). \tag{3.6}$$

Now

$$\begin{aligned}\int \mu_n \circledast \nu_n(\mathrm{d}C)1_{\{D\subset C\}} &= \int \mu_n(\mathrm{d}A) \int \nu_n(\mathrm{d}B)1_{\{D\subset(A\cap B)\}} \\ &= \Big(\int \mu_n(\mathrm{d}A)1_{\{D\subset A\}}\Big)\Big(\int \nu_n(\mathrm{d}B)1_{\{D\subset B\}}\Big),\end{aligned} \tag{3.7}$$

which, by our assumptions that $\mu_n \Rightarrow \mu$ and $\nu_n \Rightarrow \nu$, converges to the analogue formula with μ_n, ν_n replaced by μ, ν.

To see that the vague convergence of $\mu_n \circledast \nu_n$ can be strengthened to local convergence on $\mathcal{P}_{\mathrm{fin},+}$ if either μ_n or ν_n converges locally on $\mathcal{P}_{\mathrm{fin},+}$, it suffices by Proposition 2.1 (iii)⇒(i) to show that the local tightness of either μ_n or ν_n implies local tightness of $\mu_n \circledast \nu_n$. By symmetry, it suffices to consider the case when the μ_n are locally tight. Since vague convergence of the ν_n implies convergence of $\int \nu_n(\mathrm{d}A)1_{\{i\in A\}}$ for each $i \in \Lambda$, the statement now follows from the following lemma, that we formulate separately since it is of some interest on its own. □

Lemma 3.3 (Local tightness of intersection measure)**.** *Let μ_n, ν_n $(n \ge 1)$ be locally finite measures on $\mathcal{P}_+$. Assume that the μ_n $(n \ge 1)$ are concentrated on $\mathcal{P}_{\mathrm{fin},+}$ and that they are locally tight. Assume that the ν_n satisfy $\sup_{n\ge1} \int \nu_n(\mathrm{d}A)1_{\{i\in A\}} < \infty$ for all $i \in \Lambda$. Then the intersection measures $\mu_n \circledast \nu_n$ $(n \ge 1)$ are concentrated on $\mathcal{P}_{\mathrm{fin},+}$ and locally tight.*

Proof. Since $\mu_n \boxtimes \nu_n$ is concentrated on sets of the form $A \cap B$ with $A \in \mathcal{P}_{\mathrm{fin},+}$, it is clear that $\mu_n \boxtimes \nu_n$ is concentrated on $\mathcal{P}_{\mathrm{fin},+}$ for each $n \geq 1$. Fix $i \in \Lambda$ and $\varepsilon > 0$, and set $K := \sup_{n\geq 1} \int \nu_n(\mathrm{d}A) 1_{\{i\in A\}}$. By the local tightness of the μ_n, there exists a finite $\mathcal{D} \subset \mathcal{P}_{\mathrm{fin},i}$ such that $\sup_n \mu_n(\mathcal{P}_{\mathrm{fin},i} \backslash \mathcal{D}) \leq \varepsilon/K$. The same obviously holds for the larger finite set $\mathcal{D}' := \mathcal{P}(D) = \{A : A \subset D\}$, where $D := \bigcup\{A : A \in \mathcal{D}\}$. Now

$$\begin{aligned} \sup_{n\geq 1} \mu_n \boxtimes \nu_n(\mathcal{P}_{\mathrm{fin},i} \backslash \mathcal{D}') &= \sup_{n\geq 1} \int \mu_n(\mathrm{d}A) \int \nu_n(\mathrm{d}B)\, 1_{\{i\in A\cap B\}} 1_{\{A\cap B \not\subset D\}} \\ &\leq \sup_{n\geq 1} \int \mu_n(\mathrm{d}A)\, 1_{\{i\in A\}} 1_{\{A\not\subset D\}} \int \nu_n(\mathrm{d}B)\, 1_{\{i\in B\}} \leq \varepsilon. \end{aligned} \tag{3.8}$$

Since $i \in \Lambda$ and $\varepsilon > 0$ are arbitrary, the claim follows. □

Proof of Lemmas 2.3 and 2.8. Formula (2.2) obviously defines a nonzero, homogeneous measure on $\mathcal{P}_{\mathrm{fin},+}$. Since

$$\mu(\{A : 0 \in A\}) = c\sum_i \mathbb{P}[0 \in i\Delta] = c\sum_i \mathbb{P}[i^{-1} \in \Delta] = cE\big[|\Delta|\big], \tag{3.9}$$

it follows from Lemma 1.1 that μ is locally finite if and only if $\mathbb{E}\big[|\Delta|\big] < \infty$. If μ is given by (2.2), then

$$\langle\!\langle \mu \rangle\!\rangle = c\sum_{i\in\Lambda} \mathbb{E}\big[|i\Delta|^{-1} 1_{\{0\in i\Delta\}}\big] = c\mathbb{E}\big[|\Delta|^{-1}\big(\sum_{i\in\Lambda} 1_{\{i^{-1}\in\Delta\}}\big)\big] = c. \tag{3.10}$$

To see that every nonzero, homogeneous measure μ on $\mathcal{P}_{\mathrm{fin},+}$ with $\langle\!\langle \mu \rangle\!\rangle < \infty$ can be written in the form (2.2), define a probability law ρ on $\mathcal{P}_{\mathrm{fin},0}$ by

$$\rho(\{A\}) := \langle\!\langle \mu \rangle\!\rangle^{-1} \mu(\{A\}) |A|^{-1} 1_{\{0\in A\}}. \tag{3.11}$$

Let Δ be a random variable with law ρ. We claim that μ is given by (2.2) with $c = \langle\!\langle \mu \rangle\!\rangle$. To check this, we calculate, for $A \in \mathcal{P}_{\mathrm{fin},+}$:

$$\begin{aligned} \langle\!\langle \mu \rangle\!\rangle \sum_{i\in\Lambda} \mathbb{P}\big[i\Delta = A\big] &= \langle\!\langle \mu \rangle\!\rangle \sum_{i\in\Lambda} \mathbb{P}\big[\Delta = i^{-1}A\big] = \langle\!\langle \mu \rangle\!\rangle \sum_{i\in\Lambda} \rho(\{i^{-1}A\}) \\ &= \sum_{i\in\Lambda} \mu(\{i^{-1}A\}) |i^{-1}A|^{-1} 1_{\{0\in i^{-1}A\}} = \mu(\{A\})|A|^{-1} \sum_{i\in\Lambda} 1_{\{i\in A\}} = \mu(\{A\}), \end{aligned} \tag{3.12}$$

where we have used the homogeneity of μ. This completes the proof of Lemma 2.3, except for the statement that the law of $\tilde{\Delta}$ is uniquely determined by μ, which will follow by setting $t = 0$ in formula (2.22) of Lemma 2.8, which we prove next.

Indeed, letting η_0 be a $\mathcal{P}_{\mathrm{fin},+}$-valued random variable such that

$$\mu = \langle\!\langle \mu \rangle\!\rangle \sum_{i\in\Lambda} \mathbb{P}[i\eta_0 \in \cdot\,], \tag{3.13}$$

and letting $(\eta_t)_{t\geq 0}$ be a (Λ, a, δ)-contact process started in η_0, we have for any $B \in \mathcal{P}_{\mathrm{fin},+}$ and $t \geq 0$ that

$$\begin{aligned} \mu P_t(\{B\}) &= \sum_{A\in\mathcal{P}_{\mathrm{fin},+}} \mu(\{A\}) P_t(A, B) = \langle\!\langle \mu \rangle\!\rangle \sum_{A\in\mathcal{P}_{\mathrm{fin},+}} \sum_{i\in\Lambda} \mathbb{P}[i\eta_0 = A] P_t(A, B) \\ &= \langle\!\langle \mu \rangle\!\rangle \sum_{A\in\mathcal{P}_{\mathrm{fin},+}} \sum_{i\in\Lambda} \mathbb{P}[i\eta_t = B] = \langle\!\langle \mu \rangle\!\rangle m(B) \mathbb{P}[\tilde{\eta}_t = \tilde{B}], \end{aligned} \tag{3.14}$$

where $m(B)$ is defined as in (2.17). Letting $\tilde{\mu}$ denote the law of $\tilde{\eta}_0$ and $\tilde{P}_t$ the transition probabilities of the (Λ, a, δ)-contact process modulo shifts, we arrive from (3.14 at (2.22).

To complete also the proof of Lemma 2.8, we still need to prove (2.23). Let $\mathcal{P}'_{\rm fin,+}$ be a set that contains exactly one representative from each equivalence class $\tilde{A} \in \tilde{\mathcal{P}}_{\rm fin,+}$. Then (2.23) follows, using (2.22), by writing

$$\begin{aligned} \langle\!\langle \mu \rangle\!\rangle \sum_{\tilde{A}\in\tilde{\mathcal{P}}_{\rm fin,+}} \tilde{\mu}(\tilde{A})\tilde{f}(\tilde{A}) &= \sum_{A\in\mathcal{P}'_{\rm fin,+}} m(A)^{-1}\mu(\{A\})f(A)|A|^{-1}\sum_{i\in\Lambda} 1_{\{i\in A\}} \\ &= \sum_{A\in\mathcal{P}'_{\rm fin,+}} m(A)^{-1}\sum_{i\in\Lambda}(f\mu)(\{i^{-1}A\})|i^{-1}A|^{-1}1_{\{0\in i^{-1}A\}} = \langle\!\langle f\mu \rangle\!\rangle. \end{aligned} \tag{3.15}$$

□

We finish the section on locally finite measures by supplying the still outstanding:

Proof of Lemma 2.11. We will apply the mass transport principle, compare the proof of Lemma 1.2 below. Let μ, ν be homogeneous, locally finite measures on $\mathcal{P}_+$ and assume that μ is concentrated on $\mathcal{P}_{\rm fin,+}$. For $A \in \mathcal{P}_{\rm fin}$ and $B \in \mathcal{P}$ such that $A \cap B \neq \emptyset$, let us define a probability distribution $M_{A,B}$ on $\Lambda \times \Lambda$ by

$$M_{A,B}(i,j) := |A|^{-1}1_{\{i\in A\}}|A\cap B|^{-1}1_{\{j\in A\cap B\}}, \tag{3.16}$$

and let $f : \Lambda\times\Lambda \to [0,\infty]$ be defined by

$$f(i,j) := \int \mu(\mathrm{d}A) \int \nu(\mathrm{d}B)\, 1_{\{A\cap B\neq\emptyset\}} M_{A,B}(i,j). \tag{3.17}$$

Since μ and ν are homogeneous, we observe that $f(ki,kj) = f(i,j)$ $(i,j,k \in \Lambda)$. Moreover,

$$\sum_j f(0,j) = \int \mu(\mathrm{d}A) \int \nu(\mathrm{d}B)\, 1_{\{A\cap B\neq\emptyset\}} \frac{1}{|A|} 1_{\{0\in A\}} = \int \mu(\mathrm{d}A)\, h_\nu(A) \frac{1}{|A|} 1_{\{0\in A\}} = \langle\!\langle h_\nu \mu \rangle\!\rangle, \tag{3.18}$$

while

$$\sum_i f(i,0) = \int \mu(\mathrm{d}A) \int \nu(\mathrm{d}B)\, 1_{\{A\cap B\neq\emptyset\}} \frac{1}{|A\cap B|} 1_{\{0\in A\cap B\}} = \langle\!\langle \mu \otimes \nu \rangle\!\rangle. \tag{3.19}$$

Formula (2.32) now follows from the fact that $\sum_i f(i,0) = \sum_i f(0,i^{-1}) = \sum_j f(0,j)$. Note that this holds regardless of whether $h_\nu\mu$ is locally finite or not. If we have $\int \mu(\mathrm{d}A)|A|1_{\{0\in A\}} < \infty$, then by the shift-invariance and subadditivity of h_ν, we see that $h_\nu(A) \leq h_\nu(\{0\})|A|$ and hence $\int \mu(\mathrm{d}A)h_\nu(A)1_{\{0\in A\}} < \infty$, proving that $h_\nu\mu$ is locally finite. □

3.2 Infinite starting measures

In this section we prove Lemma 1.2 on contact process duality for homogeneous, infinite starting measures. We also give the proof of Lemma 2.13, which is concerned with relative compactness and cluster points of eigenmeasures for (Λ, a, δ)-contact processes with varying δ.

Proof of Lemma 1.2. Fix $t \geq 0$ and for $A, B \in \mathcal{P}_+$, consider the events

$$\mathcal{E}_{A,B} := \{|\eta_t^{A,0} \cap B| < \infty\} \quad \text{and} \quad \mathcal{E}'_{A,B} := \{|A \cap \eta_t^{\dagger\, B,t}| < \infty\}. \tag{3.20}$$

We observe that $\mu P_t \otimes \nu$ (resp. $\mu \otimes \nu P_t^\dagger$) is concentrated on $\mathcal{P}_{\rm fin,+}$ if and only if $\mathbb{P}(\mathcal{E}_{A,B}) = 1$ (resp. $\mathbb{P}(\mathcal{E}'_{A,B}) = 1$) for a.e. A w.r.t. μ and a.e. B w.r.t. ν. Set $\Delta_0 := A \cap \eta_t^{\dagger\, B,t}$ and

$\Delta_t := \eta_t^{A,0} \cap B$. Since $\eta_t^{\Delta_0,0} \supset \Delta_t$ and $\eta_t^{\dagger\,\Delta_t,t} \supset \Delta_0$, we see that the events $\mathcal{E}_{A,B}$ and $\mathcal{E}'_{A,B}$ are a.s. equal, and hence $\mu P_t \otimes \nu$ is concentrated on $\mathcal{P}_{\text{fin},+}$ if and only if $\mu \otimes \nu P_t^\dagger$ is.

We will now prove (1.26) by applying the "mass transport principle". For a given graphical representation ω and sets $A, B \in \mathcal{P}_+$ such that the events $\mathcal{E}_{A,B}$ and $\mathcal{E}'_{A,B}$ hold, we define a probability distribution $M_{A,B,\omega}$ on $\Lambda \times \Lambda$ by

$$M_{A,B,\omega}(i,j) := |\Delta_0|^{-1} 1_{\{i \in \Delta_0\}} |\Delta_t|^{-1} 1_{\{j \in \Delta_t\}}. \tag{3.21}$$

We define a function $f : \Lambda \times \Lambda \to [0,\infty]$ by

$$f(i,j) := \int \mu(\mathrm{d}A) \int \nu(\mathrm{d}B) \int \mathbb{P}(\mathrm{d}\omega)\, 1_{\mathcal{E}_{A,B}}(\omega)\, M_{A,B,\omega}(i,j). \tag{3.22}$$

Obviously, $f(ki,kj) = f(i,j)$ $(i,j,k \in \Lambda)$ due to the homogeneity of μ and ν. Moreover,

$$\begin{aligned} \sum_i f(i,0) &= \int \mu(\mathrm{d}A) \int \nu(\mathrm{d}B)\, \mathbb{E}\big[|\eta_t^{A,0} \cap B|^{-1} 1_{\{0 \in \eta_t^{A,0} \cap B\}}\big] \\ &= \int \mu P_t(\mathrm{d}A') \int \nu(\mathrm{d}B) |A' \cap B|^{-1} 1_{\{0 \in A' \cap B\}} = \langle\!\langle \mu P_t \otimes \nu \rangle\!\rangle. \end{aligned} \tag{3.23}$$

The same argument shows that $\sum_j f(0,j) = \langle\!\langle \mu \otimes \nu P_t^\dagger \rangle\!\rangle$ and hence

$$\langle\!\langle \mu P_t \otimes \nu \rangle\!\rangle = \sum_i f(i,0) = \sum_i f(0,i^{-1}) = \langle\!\langle \mu \otimes \nu P_t^\dagger \rangle\!\rangle, \tag{3.24}$$

where the middle step is a simple example of what is more generally known as the mass transport principle, see [Hag11]. ☐

Proof of Lemma 2.13. By the homogeneity and normalization of the ν_n, one has

$$\int \nu_n(\mathrm{d}A) 1_{\{A \cap B \neq \emptyset\}} \leq \sum_{i \in B} \int \nu_n(\mathrm{d}A) 1_{\{i \in A\}} = |B|. \tag{3.25}$$

Since this estimate is uniform in n, applying [Swa09, Lemma 3.2] we find that the $(\nu_n)_{n \geq 0}$ are relatively compact in the topology of vague convergence. By going to a subsequence if necessary, we may assume that the ν_n converge vaguely to a limit ν. Since the ν_n are eigenmeasures, denoting the (Λ, a, δ_n)-contact process started in A by $(\eta_t^{\delta_n,A})_{t \geq 0}$, we have

$$\int \nu_n(\mathrm{d}A) \mathbb{P}[\eta_t^{\delta_n,A} \in \cdot\,]\big|_{\mathcal{P}_+} = e^{\lambda_n t} \nu_n \qquad (t \geq 0). \tag{3.26}$$

Since $\lambda_n \to \lambda$, the right-hand side of this equation converges vaguely to $e^{\lambda t}\nu$. To prove vague convergence of the left-hand side, by Lemma 1.1, it suffices to prove that for $B \in \mathcal{P}_{\text{fin}}$,

$$\int \nu_n(\mathrm{d}A) \mathbb{P}[\eta_t^{\delta_n,A} \cap B \neq \emptyset] \to \int \nu(\mathrm{d}A) \mathbb{P}[\eta_t^{\delta,A} \cap B \neq \emptyset]. \tag{3.27}$$

We estimate

$$\Big| \int \nu_n(\mathrm{d}A) \mathbb{P}[\eta_t^{\delta_n,A} \cap B \neq \emptyset] - \int \nu(\mathrm{d}A) \mathbb{P}[\eta_t^{\delta,A} \cap B \neq \emptyset] \Big|$$

$$\leq \int \nu_n(\mathrm{d}A) \Big| \mathbb{P}[\eta_t^{\delta_n,A} \cap B \neq \emptyset] - \mathbb{P}[\eta_t^{\delta,A} \cap B \neq \emptyset] \Big| \tag{3.28}$$

$$+ \Big| \int \nu_n(\mathrm{d}A) \mathbb{P}[\eta_t^{\delta,A} \cap B \neq \emptyset] - \int \nu(\mathrm{d}A) \mathbb{P}[\eta_t^{\delta,A} \cap B \neq \emptyset] \Big|. \tag{3.29}$$

The term in (3.29) tends to zero as $n \to \infty$ by Lemmas 1.1 and 2.4. By duality, we can rewrite the term in (3.28) as

$$\int \nu_n(\mathrm{d}A)\Big|\mathbb{P}[A \cap \eta_t^{\dagger\,\delta_n,B} \neq \emptyset] - \mathbb{P}[A \cap \eta_t^{\dagger\,\delta,B} \neq \emptyset]\Big|. \tag{3.30}$$

We couple the graphical representations for processes with different recovery rates in the natural way, by constructing a Poisson point process Ω^{r} on $\Lambda\times\mathbb{R}_+\times\mathbb{R}_+$ with intensity one, and letting $\omega^{\mathrm{r}}_\delta := \{(i,t) : \exists 0 \leq r \leq \delta \text{ s.t. } (i,t,r) \in \Omega^{\mathrm{r}}\}$ be the set of recovery symbols for the process with recovery rate δ. Then, letting $\eta_t^{\dagger\,0,B}$ denote the process with zero recovery rate, the quantity in (3.30) can be estimated from above by

$$\begin{aligned}
&\int \nu_n(\mathrm{d}A)\mathbb{P}\big[A \cap \eta_t^{\dagger\,0,B} \neq \emptyset,\ \eta_t^{\dagger\,\delta_n,B} \neq \eta_t^{\dagger\,\delta,B}\big]\\
&\quad= \int \mathbb{P}\big[\eta_t^{\dagger\,0,B} \in \mathrm{d}C,\ \eta_t^{\dagger\,\delta_n,B} \neq \eta_t^{\dagger\,\delta,B}\big] \int \nu_n(\mathrm{d}A) 1_{\{A\cap C\neq\emptyset\}}\\
&\quad\leq \int \mathbb{P}\big[\eta_t^{\dagger\,0,B} \in \mathrm{d}C,\ \eta_t^{\dagger\,\delta_n,B} \neq \eta_t^{\dagger\,\delta,B}\big]\,|C| = \mathbb{E}\big[|\eta_t^{\dagger\,0,B}| 1_{\{\eta_t^{\dagger\,\delta_n,B}\neq\eta_t^{\dagger\,\delta,B}\}}\big],
\end{aligned} \tag{3.31}$$

where we have used (3.25). Since the right-hand side of (3.31) tends to zero by dominated convergence, this proves the lemma. □

3.3 Exponential moments

Recall the function $e_\gamma(A) = \sum_{i\in A} e^{\gamma d(0,i)}$ from (2.50), which measures how 'spread out' a set $A \in \mathcal{P}_{\mathrm{fin}}$ is in terms of exponential weights and a suitably slowly growing metric d as in (2.49). In this section, we provide the proof of Lemma 2.15, showing that such a metric exists. We then give the proof of Lemma 2.16, which states that the expectation of the function e_γ of a contact process has a well defined exponential growth rate, with certain bounds.

Proof of Lemma 2.15. We can find finite $\{0\} = \Delta_1 \subset \Delta_2 \subset \cdots$ such that $\sum_{i\in\Lambda\setminus\Delta_n} a(0,i) \leq |a|e^{-(n-1)}$. Making the sets Δ_n for $n \geq 2$ larger if necessary, we can moreover choose these sets such that they are symmetric, i.e., $\{i^{-1} : i \in \Delta_n\} = \Delta_n$ and such that $\Delta_\infty := \bigcup_{n\geq1}\Delta_n$ generates Λ. (In particular, we can always choose $\Delta_\infty = \Lambda$, but for nearest-neighbor processes on graphs this leads to a somewhat unnatural metric d, which is why we only assume here that Δ_∞ generates Λ.) We set $\Delta_0 := \emptyset$ and define

$$\phi(i) := \begin{cases} n & (i \in \Delta_n\backslash\Delta_{n-1},\ n \geq 1)\\ \infty & (i \in \Lambda\backslash\Delta_\infty). \end{cases} \tag{3.32}$$

Since $a(0,i) = 0$ for $i \notin \Delta_\infty$, whe have that

$$\sum_{i\in\Lambda} a(0,i)\phi(i)^\gamma = \sum_{n\geq1} n^\gamma \sum_{i\in\Delta_n\backslash\Delta_{n-1}} a(0,i) \leq |a| \sum_{n\geq1} n^\gamma e^{-(n-2)} < \infty \tag{3.33}$$

for each $0 \leq \gamma < \infty$. Set

$$d'(i,j) = d'(0,i^{-1}j) := \log(\phi(i^{-1}j)) \qquad (i,j \in \Lambda). \tag{3.34}$$

Then d' satisfies properties (2.49) (i)–(iii), $d'(i,j) = 0$ if and only if $i = j$, and $d'(i,j) = d'(j,i)$ (by the symmetry of the sets Δ_n). Since d' need not yet be a metric, we define

$$d(i,j) := \inf\big\{\sum_{k=1}^n d'(i_{k-1},i_k) : n \geq 1,\ i_0,\ldots,i_n \in \Lambda,\ i_0 = i,\ i_n = j\big\}, \tag{3.35}$$

i.e., $d(i,j)$ is a graph-style distance between i and j, defined as the shortest path from i to j where an edge from i_{k-1} to i_k has length $d'(i_{k-1},i_k)$. Note that $d(i,j)<\infty$ for each $i,j\in\Lambda$ since Δ_∞ generates Λ and $d(i,j)>0$ for each $i\neq j$ since $d'(i,j)\geq\log(2)$ for each $i\neq j$. It is now straightforward to check that d is a metric on Λ and that $d(i,j)=d(ki,kj)$ for all $i,j,k\in\Lambda$. Since $d(i,j)\leq d'(i,j)$, the metric d also enjoys property (2.49) (iii). Property (2.49) (ii), finally, follows from the fact that

$$\{i\in\Lambda: d(0,i)\leq M\}\subset\{j_1\cdots j_n: 1\leq n\leq M/\log(2),\ d'(0,j_k)\leq M\ \forall k=1,\ldots,n\}, \tag{3.36}$$

where we use that $d'(i,j)\geq\log(2)$ for all $i\neq j$, and we observe that if $d(0,i)\leq M$ $(i\neq 0)$, then there must be some $n\geq 1$ and $0=i_0,\ldots,i_n=i$ with $\sum_{k=1}^n d'(i_{k-1},i_k)\leq M$. Setting $j_k:=i_{k-1}^{-1}i_k$ we see that i must be of the form $i=j_1\cdots j_n$ with $\sum_{k=1}^n d'(0,j_k)\leq M$. □

As a preparation for Lemma 2.16, we need one more result.

Lemma 3.4 (Existence of exponential moments). *Let $(\eta_t^A)_{t\geq 0}$ be a (Λ,a,δ)-contact process started in a finite initial state $\eta_0^A=A\in\mathcal{P}_{\rm fin}$ and let d be a metric on Λ as in Lemma 2.15. Then*

$$\mathbb{E}\big[e_\gamma(\eta_t^A)\big]\leq e^{K_\gamma t}e_\gamma(A)\quad(t\geq 0)\quad\textit{where}\quad K_\gamma:=\sum_{i\in\Lambda}a(0,i)e^{\gamma d(0,i)}. \tag{3.37}$$

Proof. For $\gamma=0$ this follows from [Swa09, Prop. 2.1]. To prove the statement for $\gamma>0$, let G be the generator of the (Λ,a,δ)-contact process as defined in (1.2). Then

$$\begin{aligned}Ge_\gamma(A)&=\sum_{i\in A}\sum_{j\notin A}a(i,j)e^{\gamma d(0,j)}-\delta\sum_{i\in A}e^{-\gamma d(0,i)}\\&\leq\sum_{i\in A}\sum_{j\in\Lambda}a(i,j)e^{\gamma(d(0,i)+d(i,j))}=K_\gamma e_\gamma(A),\end{aligned} \tag{3.38}$$

where we have used that $\sum_{j\in\Lambda}a(i,j)e^{\gamma d(i,j)}=\sum_{j\in\Lambda}a(0,i^{-1}j)e^{\gamma d(0,i^{-1}j)}=K_\gamma$ $(i\in\Lambda)$.

Set $\tau_N:=\inf\{t\geq 0: e_\gamma(\eta_t^A)\geq N\}$. Since the stopped process is a Markov process with finite state space, it follows by standard arguments from (3.38) that

$$\mathbb{E}\big[e_\gamma(\eta_{t\wedge\tau_N}^A)\big]\leq e^{K_\gamma t}e_\gamma(A)\quad(t\geq 0,\ N\geq 1), \tag{3.39}$$

which in turn implies that $\mathbb{P}[e_\gamma(\eta_{t\wedge\tau_N}^A)\geq N]\to 0$ as $N\to\infty$ and hence $\tau_N\to\infty$ a.s. Therefore, letting $N\to\infty$ in (3.39), we arrive at (3.37). □

Proof of Lemma 2.16. Note that $r_0(\Lambda,a,\delta)=r(\Lambda,a,\delta)$ is the exponential growth rate from (1.12). The statement for $\gamma=0$ has been proved in [Swa09, Lemma 1.1 and formula (3.5)]. To prove the general statement, set $\pi_t^\gamma:=\mathbb{E}\big[e_\gamma(\eta_t^{\{0\}})\big]$. Formula (2.51) will follow from standard facts [Lig99, Thm B.22] if we show that $t\mapsto\log\pi_t^\gamma$ is subadditive. Recalling the graphical representation of the (Λ,a,δ)-contact process, we observe that indeed

$$\begin{aligned}\pi_{s+t}^\gamma&=\sum_i\mathbb{P}[(0,0)\rightsquigarrow(i,s+t)]e^{\gamma d(0,i)}\\&\leq\sum_{ij}\mathbb{P}[(0,0)\rightsquigarrow(j,s)\rightsquigarrow(i,s+t)]e^{\gamma(d(0,j)+d(j,i))}=\pi_s^\gamma\pi_t^\gamma,\end{aligned} \tag{3.40}$$

which implies the subadditivity of $t\mapsto\log\pi_t^\gamma$ and hence formula (2.51). Since $e_\gamma(A)\leq e_{\gamma'}(A)$ for all $\gamma\leq\gamma'$, it is clear that $\gamma\mapsto r_\gamma$ is nondecreasing. The fact that $-\delta\leq r_0$ has been proved in [Swa09, Lemma 1.1] while the estimate $r_\gamma\leq K_\gamma$ is immediate from Lemma 3.4.

To prove that the function $[0,\infty)\ni\gamma\mapsto r_\gamma$ defined in Lemma 2.16 is right-continuous, we observe that it follows from (2.51) that for any $t_n\uparrow\infty$,

$$r_\gamma=\lim_{n\to\infty}\inf_{1\leq k\leq n}\frac{1}{t_k}\log\mathbb{E}\big[e_\gamma(\eta^{\{0\}}_{t_k})\big]. \tag{3.41}$$

By dominated convergence and the finiteness of exponential moments (Lemma 3.4) we have that for each fixed $t>0$, the function $\gamma\mapsto\frac{1}{t}\log\mathbb{E}[e_\gamma(\eta^{\{0\}}_t)]$ is continuous. Therefore, being the decreasing limit of continuous functions, $\gamma\mapsto r_\gamma$ must be upper semi-continuous. Since $\gamma\mapsto r_\gamma$ is nondecreasing, this is equivalent to continuity from the right. □

3.4 Covariance estimates

The next lemma gives a uniform estimate on expectations of the functions $e_\gamma(A)$ defined in (2.50) under the measures $1_{\{0\in\cdot\}}\frac{1}{\hat\pi_\lambda}\hat\mu_\lambda$. Lemma 2.6 and Lemma 2.17, which were stated and used in Sections 2.2 and 2.7 respectively, follow as corollaries to this lemma. Their proofs are given at the end of this section.

Although this is not exactly how the proof goes, the following heuristic is perhaps useful for understanding the main strategy. Since Campbell measures change second moments into first moments, what we need to control are second moments of the original process, or more precisely (see (3.45) below), expressions of the form $\mathbb{P}[i\in\eta^{\{0\}}_t,\ j\in\eta^{\{0\}}_t]$, summed over i,j and weighted with $e^{\gamma d(i,j)}$. This leads us to consider events of the form

$$(0,0)\rightsquigarrow(i,t)\quad\text{and}\quad(0,0)\rightsquigarrow(j,t). \tag{3.42}$$

Since in the subcritical regime, long connections are unlikely, the largest contribution to the probability of such an event comes from events of the form

$$(0,0)\rightsquigarrow(k,s)\begin{cases}\rightsquigarrow(i,t)\\ \rightsquigarrow(j,t).\end{cases} \tag{3.43}$$

where $s\in[0,t]$ is close to t and $k\in\Lambda$. Indeed, if the exponential growth rate $r=r(\Lambda,a,\delta)$ is negative, then the probability of an event of the form (3.43) is of the order $e^{rs}(e^{r(t-s)})^2$, which much smaller than the probability that $(0,0)\rightsquigarrow(i,t)$, unless $t-s$ is of order one. In view of this, if we find an infection at some late time t, then all other infected sites are likely to be close to it. Although this reasoning is only heuristic, it turns out that the covariance formula (3.47) below provides a convenient way of making such arguments precise.

Lemma 3.5 (Uniform exponential moment bound)**.** *Let $\hat\mu_\lambda$ and $\hat\pi_\lambda$ be defined as in (2.11) and for $\gamma\geq0$, let e_γ be the function defined in (2.50) in terms of a metric d satisfying (2.49). Then, for any (Λ,a,δ)-contact process with exponential growth rate $r=r(\Lambda,a,\delta)$,*

$$\limsup_{\lambda\downarrow r}\frac{1}{\hat\pi_\lambda}\int\hat\mu_\lambda(\mathrm{d}A)1_{\{0\in A\}}e_\gamma(A)\leq(|a|+\delta)\int_0^\infty e^{-rt}\mathrm{d}t\,\mathbb{E}\big[e_\gamma(\eta^{\{0\}}_t)\big]^2. \tag{3.44}$$

We note that although the bound in (3.44) holds regardless of the values of γ and $r=r(\Lambda,a,\delta)$, the right-hand side will usually be infinite, unless $r<0$ and γ is small enough (see the proofs of Lemma 2.7 and Proposition 2.18).

Proof. Fix $\gamma\geq0$ and, to ease notation, set $\psi_\gamma(i,j):=e^{\gamma d(i,j)}$ $(i,j,k\in\Lambda)$. We observe that

$$\begin{aligned}\int \hat{\mu}_\lambda(\mathrm{d}A)1_{\{0\in A\}}e_\gamma(A) &= \int_0^\infty e^{-\lambda t}\mathrm{d}t \sum_{i,j}\mathbb{E}\big[1_{\{0\in\eta_t^{\{i\}}\}}1_{\{j\in\eta_t^{\{i\}}\}}\psi_\gamma(0,j)\big]\\ &= \int_0^\infty e^{-\lambda t}\mathrm{d}t \sum_{i,j}\mathbb{E}\big[1_{\{i^{-1}\in\eta_t^{\{0\}}\}}1_{\{i^{-1}j\in\eta_t^{\{0\}}\}}\psi_\gamma(i^{-1},i^{-1}j)\big]\\ &= \int_0^\infty e^{-\lambda t}\mathrm{d}t \sum_{i,j}\psi_\gamma(i,j)\mathbb{P}\big[i\in\eta_t^{\{0\}},\ j\in\eta_t^{\{0\}}\big].\end{aligned} \tag{3.45}$$

Set $f_i(A) := 1_{\{i\in A\}}$. Then

$$\mathbb{P}\big[i\in\eta_t^{\{0\}},\ j\in\eta_t^{\{0\}}\big] = \mathbb{E}\big[f_i(\eta_t^{\{0\}})\big]\,\mathbb{E}\big[f_j(\eta_t^{\{0\}})\big] + \mathrm{Cov}\big(f_i(\eta_t^{\{0\}}), f_j(\eta_t^{\{0\}})\big). \tag{3.46}$$

By a standard covariance formula (see [Swa09, Prop. 2.2]), for any functions f,g of polynomial growth (as in (2.25) above), one has

$$\mathrm{Cov}\big(f(\eta_t^{\{0\}}), g(\eta_t^{\{0\}})\big) = 2\int_0^t \mathbb{E}\big[\Gamma(P_s f, P_s g)(\eta_{t-s}^{\{0\}})\big]\mathrm{d}s \qquad (t\geq 0), \tag{3.47}$$

where $(P_t)_{t\geq 0}$ denotes the semigroup of the (Λ,a,δ)-contact process and $\Gamma(f,g) = \frac{1}{2}(G(fg) - fGg - gGf)$, with G as in (1.2). A little calculation (see [Swa09, formula (4.6)]) shows that

$$\begin{aligned}2\Gamma(P_s f, P_s g)(A) = &\sum_{k\in A}\sum_{l\notin A} a(k,l)\big(P_s f(A\cup\{l\}) - P_s f(A)\big)\big(P_s g(A\cup\{l\}) - P_s g(A)\big)\\ &+\delta\sum_{k\in A}\big(P_s f(A\backslash\{k\}) - P_s f(A)\big)\big(P_s g(A\backslash\{k\}) - P_s g(A)\big).\end{aligned} \tag{3.48}$$

Applying (3.48) to the functions $f = f_i$, $g = f_j$, using the fact that, by the graphical representation,

$$\big|P_s f_i(A\cup\{l\}) - P_s f_i(A)\big| = \big|\mathbb{P}\big[i\in\eta_s^{A\cup\{l\}}\big] - \mathbb{P}\big[i\in\eta_s^{A}\big]\big| \leq \mathbb{P}\big[i\in\eta_s^{\{l\}}\big], \tag{3.49}$$

we find that

$$2\big|\Gamma(P_s f_i, P_s f_j)(A)\big| \leq \sum_{k\in A}\sum_{l\notin A} a(k,l)\mathbb{P}\big[i\in\eta_s^{\{l\}}\big]\mathbb{P}\big[j\in\eta_s^{\{l\}}\big] + \delta\sum_{k\in A}\mathbb{P}\big[i\in\eta_s^{\{k\}}\big]\mathbb{P}\big[j\in\eta_s^{\{k\}}\big], \tag{3.50}$$

which by (3.47) implies that

$$\begin{aligned}&\big|\mathrm{Cov}\big(f_i(\eta_t^{\{0\}}), f_j(\eta_t^{\{0\}})\big)\big|\\ &\quad\leq \int_0^t \sum_{k,l} a(k,l)\mathbb{P}\big[k\in\eta_{t-s}^{\{0\}},\ l\notin\eta_{t-s}^{\{0\}}\big]\mathbb{P}\big[i\in\eta_s^{\{l\}}\big]\mathbb{P}\big[j\in\eta_s^{\{l\}}\big]\mathrm{d}s\\ &\qquad+\delta\int_0^t \sum_{k}\mathbb{P}\big[k\in\eta_{t-s}^{\{0\}}\big]\mathbb{P}\big[i\in\eta_s^{\{k\}}\big]\mathbb{P}\big[j\in\eta_s^{\{k\}}\big]\mathrm{d}s.\end{aligned} \tag{3.51}$$

Inserting this into (3.46), we obtain for the quantity in (3.45) the estimate

$$\begin{aligned}&\int_0^\infty e^{-\lambda t}\mathrm{d}t\sum_{i,j}\psi_\gamma(i,j)\mathbb{P}\big[i\in\eta_t^{\{0\}},\ j\in\eta_t^{\{0\}}\big]\\ &\quad\leq \int_0^\infty e^{-\lambda t}\mathrm{d}t\sum_{i,j}\psi_\gamma(i,j)\mathbb{P}\big[i\in\eta_t^{\{0\}}\big]\mathbb{P}\big[j\in\eta_t^{\{0\}}\big]\\ &\qquad+\int_0^\infty e^{-\lambda t}\mathrm{d}t\int_0^t\mathrm{d}s\sum_{i,j,k,l}\psi_\gamma(i,j)a(k,l)\mathbb{P}\big[k\in\eta_{t-s}^{\{0\}},\ l\notin\eta_{t-s}^{\{0\}}\big]\mathbb{P}\big[i\in\eta_s^{\{l\}}\big]\mathbb{P}\big[j\in\eta_s^{\{l\}}\big]\\ &\qquad+\delta\int_0^\infty e^{-\lambda t}\mathrm{d}t\int_0^t\mathrm{d}s\sum_{i,j,k}\psi_\gamma(i,j)\mathbb{P}\big[k\in\eta_{t-s}^{\{0\}}\big]\mathbb{P}\big[i\in\eta_s^{\{k\}}\big]\mathbb{P}\big[j\in\eta_s^{\{k\}}\big].\end{aligned} \tag{3.52}$$

Here

$$\begin{aligned}\sum_{i,j,k}&\psi_\gamma(i,j)\mathbb{P}\big[k\in\eta^{\{0\}}_{t-s}\big]\mathbb{P}\big[i\in\eta^{\{k\}}_s\big]\mathbb{P}\big[j\in\eta^{\{k\}}_s\big]\\ &=\sum_{i,j,k}\psi_\gamma(k^{-1}i,k^{-1}j)\mathbb{P}\big[k\in\eta^{\{0\}}_{t-s}\big]\mathbb{P}\big[k^{-1}i\in\eta^{\{0\}}_s\big]\mathbb{P}\big[k^{-1}j\in\eta^{\{0\}}_s\big]\\ &=\Big(\sum_k\mathbb{P}\big[k\in\eta^{\{0\}}_{t-s}\big]\Big)\Big(\sum_{i,j}\psi_\gamma(i,j)\mathbb{P}\big[i\in\eta^{\{0\}}_s\big]\mathbb{P}\big[j\in\eta^{\{0\}}_s\big]\Big)\\ &=\mathbb{E}\big[|\eta^{\{0\}}_{t-s}|\big]\sum_{i,j}\psi_\gamma(i,j)\mathbb{P}\big[i\in\eta^{\{0\}}_s\big]\mathbb{P}\big[j\in\eta^{\{0\}}_s\big]\end{aligned}\tag{3.53}$$

and similarly

$$\begin{aligned}\sum_{i,j,k,l}&\psi_\gamma(i,j)a(k,l)\mathbb{P}\big[k\in\eta^{\{0\}}_{t-s},\ l\notin\eta^{\{0\}}_{t-s}\big]\mathbb{P}\big[i\in\eta^{\{l\}}_s\big]\mathbb{P}\big[j\in\eta^{\{l\}}_s\big]\\ &\leq\sum_{i,j,k,l}\psi_\gamma(l^{-1}i,l^{-1}j)a(k,l)\mathbb{P}\big[k\in\eta^{\{0\}}_{t-s}\big]\mathbb{P}\big[l^{-1}i\in\eta^{\{0\}}_s\big]\mathbb{P}\big[l^{-1}j\in\eta^{\{0\}}_s\big]\\ &=\Big(\sum_{k,l}a(k,l)\mathbb{P}\big[k\in\eta^{\{0\}}_{t-s}\big]\Big)\Big(\sum_{i,j}\psi_\gamma(i,j)\mathbb{P}\big[i\in\eta^{\{0\}}_s\big]\mathbb{P}\big[j\in\eta^{\{0\}}_s\big]\Big)\\ &=|a|\,\mathbb{E}\big[|\eta^{\{0\}}_{t-s}|\big]\sum_{i,j}\psi_\gamma(i,j)\mathbb{P}\big[i\in\eta^{\{0\}}_s\big]\mathbb{P}\big[j\in\eta^{\{0\}}_s\big].\end{aligned}\tag{3.54}$$

Inserting this into (3.52) and recalling that this is an estimate for the quantity in (3.45) yields

$$\begin{aligned}\int&\hat\mu_\lambda(\mathrm{d}A)1_{\{0\in A\}}e_\gamma(A)\\ &\leq\int_0^\infty e^{-\lambda t}\mathrm{d}t\sum_{i,j}\psi_\gamma(i,j)\mathbb{P}\big[i\in\eta^{\{0\}}_t\big]\mathbb{P}\big[j\in\eta^{\{0\}}_t\big]\\ &\quad+(|a|+\delta)\int_0^\infty e^{-\lambda t}\mathrm{d}t\int_0^t\mathrm{d}s\,\mathbb{E}\big[|\eta^{\{0\}}_{t-s}|\big]\sum_{i,j}\psi_\gamma(i,j)\mathbb{P}\big[i\in\eta^{\{0\}}_s\big]\mathbb{P}\big[j\in\eta^{\{0\}}_s\big]\\ &=\Big(1+(|a|+\delta)\int_0^\infty e^{-\lambda t}\mathrm{d}t\,\mathbb{E}\big[|\eta^{\{0\}}_t|\big]\Big)\Big(\int_0^\infty e^{-\lambda t}\mathrm{d}t\sum_{i,j}\psi_\gamma(i,j)\mathbb{P}\big[i\in\eta^{\{0\}}_t\big]\mathbb{P}\big[j\in\eta^{\{0\}}_t\big]\Big),\end{aligned}\tag{3.55}$$

where in the last step we have changed the integration order on the set $\{(s,t):0\leq s\leq t\}$. Using the fact that $\psi_\gamma(i,j)=e^{\gamma d(i,j)}$ where d is a metric, we may further estimate the sum in the second factor on the right-hand side of (3.55) as

$$\begin{aligned}\sum_{i,j}\psi_\gamma(i,j)\mathbb{P}\big[i\in\eta^{\{0\}}_t\big]\mathbb{P}\big[j\in\eta^{\{0\}}_t\big]&=\sum_{i,j}e^{\gamma d(i,j)}\mathbb{P}\big[i\in\eta^{\{0\}}_t\big]\mathbb{P}\big[j\in\eta^{\{0\}}_t\big]\\ \leq\quad&\sum_{i,j}e^{\gamma(d(0,i)+d(0,j))}\mathbb{P}\big[i\in\eta^{\{0\}}_t\big]\mathbb{P}\big[j\in\eta^{\{0\}}_t\big]\\ =\quad&\big(\sum_i e^{\gamma d(0,i)}\mathbb{P}\big[i\in\eta^{\{0\}}_t\big]\big)^2=\mathbb{E}\big[\sum_{i\in\eta^{\{0\}}_t}e^{\gamma d(0,i)}\big]^2.\end{aligned}\tag{3.56}$$

Inserting this into (3.55) and recalling the definition of $\hat\pi_\lambda=\hat\pi_\lambda(\{0\})$ in (2.8) yields

$$\int\hat\mu_\lambda(\mathrm{d}A)1_{\{0\in A\}}e_\gamma(A)\leq\Big(1+(|a|+\delta)\hat\pi_\lambda\Big)\int_0^\infty e^{-\lambda t}\mathrm{d}t\,\mathbb{E}\big[e_\gamma(\eta^{\{0\}}_t)\big]^2.\tag{3.57}$$

Since $\lim_{\lambda\downarrow r}\hat\pi_\lambda=\infty$ by (2.9), we arrive at (3.44). □

As a direct applications we obtain:

Proof of Lemma 2.6. This is special case of Lemma 3.5, where $\gamma=0$. □

Proof of Lemma 2.17. This is very similar to the proof of Lemma 2.7. For $\delta \in (\delta_{\rm c}, \infty)$, let $(\eta_t^{\delta,\{0\}})_{t\geq 0}$ and $\mathring{\nu}_\delta$ be as in Lemma 2.17. Let Λ_k be finite sets such that $0 \in \Lambda_k \subset \Lambda$ and $\Lambda_k \uparrow \Lambda$. It is again easy to check that $A \mapsto f_k^\gamma(A) := e_\gamma(A \cap \Lambda_k)1_{\{0\in A\}}$ is a continuous, compactly supported real function on $\mathcal{P}_+$. Therefore, since (by Proposition 2.5) the $\frac{1}{\hat{\pi}_{\lambda_n}}\hat{\mu}_{\lambda_n}$ converge vaguely to $\mathring{\nu}^\delta$,

$$\begin{aligned}\int \mathring{\nu}^\delta(\mathrm{d}A) f_k^\gamma(A) = \lim_{n\to\infty} \frac{1}{\hat{\pi}_{\lambda_n}} \int \hat{\mu}_{\lambda_n}(\mathrm{d}A) f_k^\gamma(A) \quad &\leq \quad \liminf_{n\to\infty} \frac{1}{\hat{\pi}_{\lambda_n}} \int \hat{\mu}_{\lambda_n}(\mathrm{d}A) e_\gamma(A) 1_{\{0\in A\}} \\ &\leq \quad (|a| + \delta) \int_0^\infty e^{-rt} \mathrm{d}t \, \mathbb{E}\big[e_\gamma(\eta_t^{\delta,\{0\}})\big]^2.\end{aligned}$$

Letting $k \uparrow \infty$ such that $f_k^\gamma \uparrow e_\gamma(A)1_{\{0\in A\}}$ we arrive at (2.53) by the monotone convergence theorem. □

A Exponential decay in the subcritical regime

A.1 Statement of the result

The aim of this appendix is to show how the arguments in [AJ07], which are written down for contact processes on transitive graphs, can be extended to prove Theorem 2 (d) for the class of (Λ, a, δ)-contact processes considered in this article. To formulate this properly, only in this appendix, we will consider a class of contact processes that is more general than both the one defined in Section 1.2 and the one considered in [AJ07], and contains them both as subclasses. Indeed, only in this appendix, will we drop the assumptions that Λ has a group structure (as in the rest of this article) or that Λ has a graph structure (as in [AJ07]). The only structure on Λ that we will use is the structure given by the infection rates $(a(i,j))_{i,j\in\Lambda}$.

Let Λ be any countable set and let $a : \Lambda \times \Lambda \to [0, \infty)$ be a function. By definition, an *automorphism* of (Λ, a) is a bijection $g : \Lambda \to \Lambda$ such that $a(gi, gj) = a(i, j)$ for each $i, j \in \Lambda$. Let $\mathrm{Aut}(\Lambda, a)$ denote the group of automorphisms of (Λ, a). We say that a subgroup $G \subset \mathrm{Aut}(\Lambda, a)$ is *(vertex) transitive* if for each $i, j \in \Lambda$ there exists a $g \in G$ such that $gi = j$. In particular, we say that (Λ, a) is transitive if $\mathrm{Aut}(\Lambda, a)$ is transitive.

Let (Λ, a) be transitive, let $a^\dagger(i, j) := a(j, i)$, and assume that

$$|a| := \sum_{j\in\Lambda} a(i,j) < \infty \quad \text{and} \quad |a^\dagger| := \sum_{j\in\Lambda} a^\dagger(i,j) < \infty, \tag{A.1}$$

where by the transitivity of (Λ, a), these definitions do not depend on the choice of $i \in \Lambda$. Then, for each $\delta \geq 0$, there exists a well-defined contact process on Λ with generator as in (1.2) and also the dual contact process with a replaced by $a^\dagger$ is well-defined. *Only in this appendix,* we will use the term (Λ, a, δ)-contact process (resp. $(\Lambda, a^\dagger, \delta)$-contact process) in this more general sense.

For any (Λ, a, δ)-contact process, as defined in this appendix, we define the critical recovery rate $\delta_{\rm c} = \delta_{\rm c}(\Lambda, a)$ as in (1.8), which satisfies $\delta_{\rm c} < \infty$ but may be zero in the generality considered here. A straightforward extension of [Swa09, Lemma 1.1] shows that the exponential growth rate $r = r(\Lambda, a, \delta)$ in (1.12) is well-defined for the class of (Λ, a, δ)-contact processes considered here.

We will show that the arguments in [AJ07] imply the following result.

Theorem A.1 (Exponential decay in the subcritical regime)**.** *Let (Λ, a) be transitive and let a satisfy (A.1). Then $\{\delta \geq 0 : r(\Lambda, a, \delta) < 0\} = (\delta_{\rm c}, \infty)$.*

We remark that Theorem 2 (a) does not hold in general for the class of (Λ, a, δ)-contact processes considered in this appendix. This is related to unimodularity. A transitive subgroup $G \subset \mathrm{Aut}(\Lambda, a)$ is *unimodular* if [BLPS99, formula (3.3)]

$$|\{gi : g \in G,\ gj = j\}| = |\{gj : g \in G,\ gi = i\}| \qquad (i, j \in \Lambda). \tag{A.2}$$

Note that this is trivially satisfied if Λ is a group and $G = \Lambda$ acts on itself by left multiplication, in which case the sets on both sides of the equation consist of a single element. Unimodularity gives rise to the *mass transport principle* which says that for any function $f : \Lambda \times \Lambda \to [0, \infty)$ such that $f(gi, gj) = f(i, j)$ $(g \in G,\ i, j \in \Lambda)$, one has $\sum_j f(i, j) = \sum_j f(j, i)$. In particular, this implies that the constants $|a|$ and $|a^\dagger|$ from (A.1) are equal and that $r(\Lambda, a, \delta) = r(\Lambda, a^\dagger, \delta)$. In the nonunimodular case, this is in general no longer true and in fact it is not hard to construct examples where the critical recovery rates $\delta_{\rm c}(\Lambda, a)$ and $\delta_{\rm c}(\Lambda, a^\dagger)$ of a contact process and its dual are different. We remark that although in [AJ07], the authors do not always clearly distinguish between a contact process and its dual (e.g., in their formulas (1.3), (1.9) and Lemma 1.4), they do not assume that $a = a^\dagger$ and their results are valid also in the asymmetric case $a \neq a^\dagger$.

A.2 The key differential inequalities and their consequences

The main method used in [AJ07], that in its essence goes back to [AB87] and that yields Theorem A.1 and a number of related results, is the derivation of differential inequalities for certain quantities related to the process. Using the graphical representation to construct a (Λ, a, δ)-contact process and its dual, we define the *susceptibility* as

$$\chi = \chi(\Lambda, a, \delta) = \mathbb{E}\big[\int_0^\infty |\eta_t^{\{0\}}|\,\mathrm{d}t\big], \tag{A.3}$$

which may be $+\infty$. Moreover, letting $\omega^{\rm c}$ be a Poisson point process on $\Lambda \times \mathbb{R}$ with intensity $h \geq 0$, independent of the Poisson point processes $\omega^{\rm i}$ and $\omega^{\rm r}$ corresponding to infection arrows and recovery symbols, we define

$$\theta = \theta(\Lambda, a, \delta, h) := \mathbb{P}\big[C_{(0,0)} \cap \omega^{\rm c} \neq \emptyset\big] \quad \text{where} \quad C_{(i,s)} := \big\{(j, t) : t \geq s,\ (i, s) \rightsquigarrow (j, t)\big\}. \tag{A.4}$$

Then θ can be interpreted as the density of infected sites in the upper invariant law of a (dual) "$(\Lambda, a^\dagger, \delta, h)$-contact process", which in addition to the dynamics in (1.2) exhibits spontaneous infection of healthy sites with rate h, corresponding to a term in the generator of the form $h \sum_i \{f(A \cup \{i\}) - f(A)\}$.

Let Λ, a, δ be fixed and for $\lambda, h \geq 0$ let $\theta = \theta(\lambda, h) := \theta(\Lambda, \lambda a, \delta, h)$ and $\chi = \chi(\lambda) := \chi(\Lambda, \lambda a, \delta)$ be the quantities defined above. The analysis in [AJ07] centers on the derivation of the following three differential inequalities (see [AJ07, formulas (1.17), (1.19) and (1.20)])

$$\begin{array}{ll} \text{(i)} & \frac{\partial}{\partial \lambda}\chi \leq |a|\chi^2, \\ \text{(ii)} & \frac{\partial}{\partial \lambda}\theta \leq |a|\theta \frac{\partial}{\partial h}\theta, \\ \text{(iii)} & \quad \theta \leq h \frac{\partial}{\partial h}\theta + \big(2\lambda^2|a|\theta + h\lambda\big)\frac{\partial}{\partial \lambda}\theta + \theta^2. \end{array} \tag{A.5}$$

These differential inequalities, and their proofs, generalize without a change to the more general class of (Λ, a, δ)-contact processes discussed in this appendix.

Since $\theta \geq h(1+h)$, which follows by estimating the $(\Lambda, \lambda a^\dagger, \delta, h)$-contact process from below by a process with no infections, one has $h \leq \theta(1 - \theta)$. Inserting this into (A.5) (iii) yields

$$\theta \leq h \tfrac{\partial}{\partial h}\theta + \Big(2\lambda^2|a| + \frac{\lambda}{1 - \theta}\Big)\theta \tfrac{\partial}{\partial \lambda}\theta + \theta^2. \tag{A.6}$$

Abstract results of Aizenman and Barsky [AB87, Lemmas 4.1 and 5.1] allow one to draw the following conclusions from (A.5) (ii) and (A.6).

Lemma A.2 (Estimates on critical exponents)**.** *Assume that there exists some* $\lambda' > 0$ *such that* $\theta(\lambda', 0) = 0$ *and* $\lim_{h\to 0} h^{-1}\theta(\lambda', h) = \infty$*. Then there exist* $c_1, c_2 > 0$ *such that*

$$\begin{aligned} &\text{(i)} \quad \theta(\lambda', h) \geq c_1 h^{1/2} \qquad && (h \geq 0), \\ &\text{(ii)} \quad \theta(\lambda, 0) \geq c_2(\lambda - \lambda') \qquad && (\lambda \geq \lambda'). \end{aligned} \tag{A.7}$$

Note that this lemma (in particular, formula (A.7) (i), which depends on the assumption that $\lim_{h\to 0} h^{-1}\theta(\lambda', h) = \infty$) implies in particular that if for some fixed $\lambda' > 0$, one has $\theta(\lambda', h) \sim h^\alpha$ as $h \to 0$, then either $\alpha \leq \frac{1}{2}$ or $\alpha \geq 1$.

Remark Lemmas 4.1 and 5.1 of [AB87] are also cited in [AJ07, Thm. 4.1], but there the statement that $c_1, c_2 > 0$ is erroneously replaced by the (empty) statement that $c_1, c_2 < \infty$.

Proof of Theorem A.1 (sketch). Set

$$\begin{aligned} \lambda_c &:= \inf\{\lambda \geq 0 : \theta(\lambda, 0) > 0\}, \\ \lambda'_c &:= \inf\{\lambda \geq 0 : \chi(\lambda) = \infty\}. \end{aligned} \tag{A.8}$$

Since $\chi(\lambda) < \infty$ implies $\theta(\lambda, 0) = 0$, obviously $\lambda'_c \leq \lambda_c$. Our first aim is to show that they are in fact equal. We note that it is always true that $\lambda'_c > 0$. It may happen that $\lambda'_c = \infty$ but in this case also $\lambda_c = \infty$ so without loss of generality we may assume that $\lambda'_c < \infty$.

It follows from (A.5) (i) and approximation of infinite systems by finite systems (compare [AN84, Lemma 3.1], which is written down for unoriented percolation and which is cited in [AJ07, formula (1.18)]) that $\lim_{\lambda\uparrow\lambda'_c} \chi(\lambda) = \chi(\lambda'_c) = \infty$, and in fact

$$\chi(\lambda) \geq \frac{|a|^{-1}}{\lambda'_c - \lambda} \qquad (\lambda < \lambda'_c). \tag{A.9}$$

Now either $\theta(\lambda'_c, 0) > 0$, in which case we are done, or $\theta(\lambda'_c, 0) = 0$. In the latter case, since

$$\chi(\lambda) = \lim_{h\to 0} h^{-1}\theta(\lambda, h) \qquad (\lambda < \lambda'_c), \tag{A.10}$$

(see [AJ07, formula (1.11)]), using the monotonicity of θ in λ and h, it follows from (A.9) that

$$\lim_{h\to 0} h^{-1}\theta(\lambda'_c, h) = \infty \tag{A.11}$$

and therefore Lemma A.2 implies that (A.7) holds at $\lambda' = \lambda'_c$. In particular, (A.7) (ii) implies that $\theta(\lambda, 0) > 0$ for $\lambda > \lambda'_c$, hence $\lambda_c = \lambda'_c$.

Since by a trivial rescaling of time, questions about critical values for λ can always be translated into questions about critical values for δ, we learn from this that for any (Λ, a, δ)-contact process, one has $\chi(\Lambda, a, \delta) < \infty$ if $\delta > \delta_c(\Lambda, a)$, where the latter critical point is defined in (1.8). It follows from (2.4) that $\chi(\Lambda, a, \delta) = \infty$ if $r(\delta) = r(\Lambda, a, \delta) \geq 0$, hence we must have $r(\delta) < 0$ for $\delta \in (\delta_c, \infty)$. Part (b) of Theorem 2 is easily generalized to the class of (Λ, a, δ)-contact processes considered in this appendix. Moreover, it is not hard to prove that $r < 0$ implies that the process does not survive. This shows that $r(\delta) \geq 0$ on $[0, \delta_c)$ while $\delta \mapsto r(\delta)$ is continuous, which allows us to conclude that $\{\delta \geq 0 : r(\delta) < 0\} = (\delta_c, \infty)$ if $\delta_c > 0$. If $\delta_c = 0$ (which may happen for the general class of models considered here), then we may use the fact that $\theta(\Lambda, a, 0) = 1$ to conclude that $r(\Lambda, a, 0) \geq 0$, hence the conclusion of Theorem A.1 is also valid in this case. □

B Some results on quasi-invariant laws

In this appendix we collect some basic results on λ-positivity and quasi-invariant laws for which we did not find an exact reference in the literature. We will be interested in continuous-time Markov chains taking values in a countable set S, which may have a finite lifetime due to killing or explosion. To formalize this, let $\overline{S} := S \cup \{\infty\}$ be the set S with one extra point added and let $\overline{Q}$ be a *Q-matrix* on $\overline{S}$, i.e., $\overline{Q} : \overline{S}^2 \to \mathbb{R}$ is a function such that

$$\overline{Q}(i,j) \geq 0 \quad (i \neq j), \qquad \sum_{k \in \overline{S},\ k \neq i} \overline{Q}(i,k) < \infty, \qquad \sum_{k \in \overline{S}} \overline{Q}(i,k) = 0 \tag{B.1}$$

for all $i, j \in \overline{S}$. We set $\overline{Q}(i) := -\overline{Q}(i,i)$ and call $T := \{i \in \overline{S} : \overline{Q}(i) = 0\}$ the set of *traps*. We assume that ∞ is a trap, i.e, $\overline{Q}(\infty) = 0$. For any initial law on $\overline{S}$, we construct a continuous-time Markov chain $X = (X_t)_{t \geq 0}$ with Q-matrix $\overline{Q}$ in the usual way from its embedded Markov chain. More precisely, let $Y = (Y_k)_{0 \leq k < N+1}$ be a Markov chain in $\overline{S}$ with possibly finite lifetime $N = \inf\{k \geq 0 : Y_k \in T\}$ and transition probabilities

$$\mathbb{P}[Y_k = j \,|\, Y_{k-1} = i] = \overline{Q}(i,j)/\overline{Q}(i) \qquad (0 < k < N+1). \tag{B.2}$$

Conditional on Y, let σ_k be independent, exponentially distributed random variables with parameter $\overline{Q}(Y_k)$ $(0 \leq k < N+1)$ (in particular, $\sigma_N = \infty$ if $N < \infty$), set $\tau_n := \sum_{0 \leq k < n} \sigma_k$ $(0 \leq n \leq N+1)$, and let $\tau := \tau_{N+1}$, which may be finite if $N = \infty$. Then setting

$$X_t := Y_k \quad (\tau_k \leq t < \tau_{k+1}) \tag{B.3}$$

defines a continuous-time Markov chain $X = (X_t)_{0 \leq t < \tau}$ with Q-matrix $\overline{Q}$ and possible finite lifetime τ. We call $\{\tau < \infty\}$ the event of *explosion* and say that the process is *nonexplosive* if this has probability zero. By construction, each trap $i \in T$ has the property that $X_s = i$ for some $s \in [0,\tau)$ implies $X_t = i$ for all $t \in [s,\tau)$. In particular, this is true for $i = \infty$ which is a trap by assumption.

We will only be interested in the process X as long as it stays in S. If X jumps to ∞ at some point, then we say that the process gets *killed*. We call $\overline{Q}(i,\infty)$ the *killing rate* at i. If $\overline{Q}(i,\infty) = 0$ for all $i \in S$ then we say the process has *zero killing rates*. If the process explodes, then we also set $X_t := \infty$ for all $t \geq \tau$, i.e., we use the same cemetery state ∞ regardless of whether the process disappears from S due to it being killed or due to explosion.

We let

$$\overline{P}_t(i,j) := \mathbb{P}[X_{s+t} = j \,|\, X_s = i] \quad (s,t \geq 0,\ i,j \in \overline{S}) \tag{B.4}$$

denote the transition probabilities of the Markov process X in $\overline{S}$, and let P_t denote the restriction of $\overline{P}_t$ to S^2. Due to the possibility of killing or explosion, the $(P_t)_{t \geq 0}$ are in general subprobability kernels on S. Let Q denote the restriction of $\overline{Q}$ to S^2. It follows from well-known results (see e.g. [Lig10, Prop 2.30], [Nor97, Thm 2.8.4]) that the functions $t \mapsto P_t(i,j)$ are continuously differentiable for each $i,j \in S$ and that the $(P_t)_{t \geq 0}$ are given by the minimal nonnegative solution to the Kolmogorov backward equations

$$\tfrac{\partial}{\partial t} P_t(i,k) = \sum_{j \in S} Q(i,j) P_t(j,k) \quad (t \geq 0,\ i,k \in S) \tag{B.5}$$

with initial condition $P_0(i,j) = 1_{\{i=j\}}$. We say that the process is *irreducible on* S if for each $i,j \in S$ there exist $i = i_0, \ldots, i_n = j$ such that $Q(i_{k-1}, i_k) > 0$ for $k = 1, \ldots, n$. If the process is irreducible on S, then by [Kin63, Thm 1] the *decay parameter*

$$\lambda_S := -\lim_{t \to \infty} t^{-1} \log P_t(i,j) \tag{B.6}$$

exists and does not depend on $i, j \in S$. For nonexplosive processes with zero killing rates, we define transience, null-recurrence and positive recurrence in the standard way. We use the usual notation for matrices and vectors indexed by S, i.e., $AB(i,k) := \sum_j A(i,j)B(j,k)$, $gA(i) := \sum_j g(j)A(j,i)$ and $Ah(i) := \sum_j A(i,j)h(j)$. With these definitions, one has the following facts that will be proven below.

Lemma B.1 (Doob transformed process)**.** *Assume that $h : S \to (0,\infty)$ and $\lambda \in \mathbb{R}$ satisfy $P_t h = e^{-\lambda t} h$ $(t \geq 0)$. Then*

$$P^h_t(i,j) := e^{\lambda t} h(i)^{-1} P_t(i,j) h(j) \qquad (i,j \in S,\ t \geq 0) \tag{B.7}$$

are the transition probabilities of a nonexplosive continuous-time Markov chain in S with zero killing rates and with Q-matrix given by

$$Q^h(i,j) := h(i)^{-1} Q(i,j) h(j) + \lambda 1_{\{i=j\}} \qquad (i,j \in S). \tag{B.8}$$

In particular, $\sum_{j:\, j\neq i} Q^h(i,j) < \infty$ and $\sum_j Q^h(i,j) = 0$ for all $i \in S$.

Lemma B.2 (λ-positivity)**.** *Assume that Q is irreducible on S and that $g, h : S \to (0,\infty)$ and $\lambda \in \mathbb{R}$ satisfy*

$$gP_t = e^{-\lambda t} g, \quad P_t h = e^{-\lambda t} h \quad (t \geq 0) \quad \text{and} \quad c := \sum_i g(i)h(i) < \infty. \tag{B.9}$$

Then the transition probabilities $(P^h_t)_{t\geq 0}$ in (B.7) belong to a positively recurrent continuous-time Markov chain with unique invariant law given by $\pi(i) = c^{-1} g(i)h(i)$. Moreover, the conditions (B.9) determine g and h uniquely up to multiplicative constants and imply that $\lambda = \lambda_S$.

Lemma B.3 (Quasi-invariant law)**.** *In the set-up of Lemma B.2), assume moreover that $\inf_{i\in S} h(i) > 0$. Then the process X started in any deterministic initial state $i \in S$ satisfies*

$$\mathbb{P}^i[X_t \in \cdot\,|X_t \neq \infty] \underset{t\to\infty}{\Longrightarrow} \nu, \tag{B.10}$$

where ν is the probability measure on S defined by $\nu(i) := g(i)/\sum_j g(j)$ $(i \in S)$ and $\Rightarrow$ denotes weak convergence of probability measures on S.

Before we sketch the proofs of these results, we first discuss what can be found about this in the literature. For discrete-time Markov chains and more generally for countable nonnegative matrices, the concepts of R-transience, R-null recurrence, and R-positivity were introduced by Vere-Jones [Ver67] (which builds on his D.Phil. thesis from 1961). Kingman [Kin63] then treated the continuous-time case. (A good general reference to this material is [And91, Sect 5.2].) In a more general set-up than ours, Kingman proved that the limit in (B.6) exists. He then defined $(P_t)_{t\geq 0}$ to be λ-transient or λ-recurrent depending on whether

$$\int_0^\infty P_t(i,i) e^{\lambda_S t} dt < \infty \quad \text{or} \quad = \infty, \tag{B.11}$$

where as a result of irreducibility the definition does not depend on the choice of the reference point $i \in S$. In the λ-recurrent case, he called $(P_t)_{t\geq 0}$ λ-null-recurrent or λ-positive (recurrent) depending on whether

$$\lim_{t\to\infty} P_t(i,i) e^{\lambda_S t} = 0 \quad \text{or} \quad > 0, \tag{B.12}$$

where the limit is shown to exist and the definition does not depend on the choice of $i \in S$. In the λ-recurrent case, he showed [Kin63, Thm 4] that there are functions $g, h :$

$S \to (0,\infty)$, unique up to multiplicative constants, such that $gP_t = e^{-\lambda_S t}g$, $P_t h = e^{-\lambda_S t}h$ $(t \geq 0)$, and these satisfy $\sum_i g(i)h(i) < \infty$ if and only if $(P_t)_{t\geq 0}$ is λ-positive. He moreover defined $(P^h_t)_{t\geq 0}$ as in (B.7) and observed that these are the transition probabilities of a continuous-time Markov process.[5]

Some care is needed in applying Kingman's results to our setting, however, since his setting is more general than ours. He assumes that S is an irreducible subclass of some larger space $\overline{S}$ that may be more complicated than in our setting, and he only assumes that the Markov process corresponding to $(\overline{P}_t)_{t\geq 0}$ a.s. assumes values in the countable set $\overline{S}$ at deterministic times. This includes processes that are not defined by a Q-matrix and that leave each state instantaneously, such as Blackwell's example [Lig10, Sect. 2.4] or the FIN diffusion defined in [FIN02], which is a Brownian motion time-changed in such a way that at deterministic times it a.s. takes values in a countable, dense subset of the real line. While Kingman's set-up is in all respects more general than ours, his results are also weaker in some respects since he does not prove that the transformed transition probabilities $(P^h_t)_{t\geq 0}$ as in (B.7) come from a Q-matrix.

In practical situations, one often does not have a direct way of verifying that a function h satisfies $P_t h = e^{-\lambda t}h$ $(t \geq 0)$, but instead starts off from a solution to $Qh = -\lambda h$. The latter equation is in general not enough to guarantee the first one, so extra conditions are needed, see [NP93]. In our case, however, solutions to $P_t h = e^{-\lambda t}h$ can be obtained directly from the eigenmeasures, which is why the lemmas above are sufficient for our purposes.

Probability measures ν satisfying $\nu P_t = e^{-\lambda_S t}\nu$ are called *quasi-invariant laws* and (B.10) is a *ratio limit theorem*. We refer to [FKM96] and references therein for a more detailed discussion of these concepts.

To prepare for the proofs of Lemmas B.1–B.3, we prove one technical lemma.

Lemma B.4 (Cadlag processes have well-defined rates)**.** *Let S be a countable set and let $(P_t)_{t\geq 0}$ be probability kernels on S such that $P_sP_t = P_{s+t}$ $(s,t \geq 0)$ and $\lim_{t\downarrow 0} P_t(i,i) = P_0(i,i) = 1$ $(i \in S)$. Assume that for each $i \in S$, there exists a Markov process $X = (X_t)_{t\geq 0}$ in S with initial state $X_0 = i$, transition probabilities $(P_t)_{t\geq 0}$, and cadlag sample paths. Then there exists a Q-matrix on S such that $(P_t)_{t\geq 0}$ is the minimal nonnegative solution of (B.5).*

Proof. Define inductively stopping times by $\tau_0 = \tau^\varepsilon_0 = 0$ and

$$\begin{aligned} \tau_k &:= \inf\{t \geq \tau_{k-1} : X_t \neq X_{\tau_{k-1}}\} \\ \tau^\varepsilon_k &:= \inf\{\varepsilon l \geq \tau^\varepsilon_{k-1} : X_{\varepsilon l} \neq X_{\tau^\varepsilon_{k-1}},\ l \in \mathbb{N}\} \qquad (\varepsilon > 0). \end{aligned} \tag{B.13}$$

Let $N_\varepsilon := 1 + \sup\{k \geq 0 : \tau^\varepsilon_k < \infty\}$. Then, for each $\varepsilon > 0$, we may define a Markov chain $Y^\varepsilon = (Y^\varepsilon_k)_{0\leq k<N_\varepsilon+1}$ by $Y^\varepsilon_k := X_{\tau^\varepsilon_k}$ $(0 \leq k < N_\varepsilon + 1)$. Conditional on Y^ε, the holding times $(\tau^\varepsilon_{k+1} - \tau^\varepsilon_k)$ with $(0 \leq k < N_\varepsilon + 1)$ are independent and geometrically distributed. By the fact that X has cadlag sample paths, $Y^\varepsilon \to Y$ a.s. where the embedded Markov chain $Y = (Y_k)_{0\leq k<N+1}$ is defined analogously to Y^ε with τ^ε_k replaced by τ_k. Moreover, the collection of times $(\tau^\varepsilon_k)_{0\leq k<N_\varepsilon+1}$ a.s. converges to $(\tau_k)_{0\leq k<N+1}$.

In particular, for the process started in i, since τ^ε_1 is geometrically distributed and $\tau^\varepsilon_1 \to \tau_1$ a.s., we see that τ_1 is exponentially distributed and the limit

$$Q(i) = -Q(i,i) := \lim_{\varepsilon\downarrow 0} \varepsilon^{-1}\big(1 - P_\varepsilon(i,i)\big) \tag{B.14}$$

[5] In fact, Kingman defines a (right) λ-subinvariant vector to be any function $h : S \to (0,\infty)$ such that $P_t h \leq e^{-\lambda_S t}h$ $(t \geq 0)$. He proves that such λ-subinvariant vectors exist quite generally and defines $(P^h_t)_{t\geq 0}$ as in (B.7) for any such h, which may now be subprobability kernels corresponding to a process with killing.

exists, where we allow for the case $Q(i)=0$ (which corresponds to $\tau_1=\infty$). (Note that the assumption of cadlag sample paths implies $\tau_1>0$.) If $Q(i)>0$, then we observe that Y^ε_1 is distributed according to the law

$$\mathbb{P}^i\big[Y^\varepsilon_1=j\big]=\big(1-P_\varepsilon(i,i)\big)^{-1}P_\varepsilon(i,j)\qquad(j\in S,\ j\neq i).\tag{B.15}$$

Since $Y^\varepsilon_1\to Y_1$ as $\varepsilon\to 0$, we conclude that the limit

$$Q(i,j):=\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}P_\varepsilon(i,j)=Q(i)\mathbb{P}^i[Y_1=j]\qquad(i\neq j)\tag{B.16}$$

exists and satisfies $\sum_{j:\,j\neq i}Q(i,j)<\infty$ and $\sum_j Q(i,j)=0$.

It is now not hard to check that Y is a Markov chain that jumps from a state i with $Q(i)>0$ to a state j with probability $Q(i)^{-1}Q(i,j)$, and that conditional on Y, the times $(\tau_{k+1}-\tau_k)$ are independent and exponentially distributed with parameter $Q(Y_k)$. By [Nor97, Thm 2.8.4], we conclude that $(P_t)_{t\geq 0}$ is the unique minimal nonnegative solution of (B.5). □

Proof of Lemma B.1. The fact that $P_th=e^{-\lambda t}h$ implies that the $(P^h_t)_{t\geq 0}$ are probability kernels satisfying $P^h_sP^h_t=P^h_{s+t}$ $(s,t\geq 0)$ and $\lim_{t\downarrow 0}P^h_t(i,i)=P^h_0(i,i)=1$ $(i\in S)$. It is not immediately clear, however, that these are the transition probabilities of a Markov process with Q-matrix as in (B.8), or, in fact, that the latter is even well-defined.

To prove this, view $\overline{S}$ as the one-point compactification of S (with the appropriate topology). Then the process X has sample paths in the space of those cadlag functions $\omega:[0,\infty)\to\overline{S}$ for which $\omega_s=\infty$ implies $\omega_s=\infty$ for all $t\geq s$. We may construct X in the canonical way on this space, where $X_t(\omega)=\omega_t$ is the coordinate projection. Let $\mathbb{P}^i$ be the law of the process started in $i\in S$. We may consistently define a new probability law $\mathbb{P}^{h,i}$ by

$$\mathbb{P}^{h,i}\big[(X_s)_{0\leq s\leq t}\in\mathrm{d}\omega\big]:=e^{\lambda t}1_{\{\omega_s\in S\ \forall 0\leq s\leq t\}}\frac{h(\omega_t)}{h(i)}\mathbb{P}^i\big[(X_s)_{0\leq s\leq t}\in\mathrm{d}\omega\big]\qquad(t\geq 0).\tag{B.17}$$

Then $P^{h,i}$ is concentrated on cadlag paths $\omega:[0,\infty)\to S$. It is straightforward to check that under this new law, X is a Markov process with transition kernels $(P^h_t)_{t\geq 0}$. Since X has cadlag sample paths, we may invoke Lemma B.4 to conclude that X is a continuous-time Markov chain with Q-matrix given by the right-hand derivative

$$\tfrac{\partial}{\partial t}P^h_t(i,j)\big|_{t=0}=\tfrac{\partial}{\partial t}\big(e^{\lambda t}h(i)^{-1}P_t(i,j)h(j)\big)\big|_{t=0}=h(i)^{-1}Q(i,j)h(j)+\lambda 1_{\{i=j\}}=Q^h(i,j).\tag{B.18}$$

□

Proof of Lemma B.2. Since $c:=\sum_i g(i)h(i)<\infty$, we may define a probability law π on S by $\pi(i):=c^{-1}g(i)h(i)$. Then

$$\pi P^h_t(i)=\sum_j c^{-1}g(j)h(j)e^{\lambda t}h(j)^{-1}P_t(j,i)h(i)=c^{-1}g(i)h(i)=\pi(i)\qquad(i\in S,\ t\geq 0),\tag{B.19}$$

where we have used that $gP_t=e^{-\lambda t}g$. It follows that π is an invariant law for the irreducible continuous-time Markov chain with Q-matrix as in (B.8), and hence the latter is positively recurrent. In particular,

$$\lim_{t\to\infty}e^{\lambda t}P_t(i,i)=\lim_{t\to\infty}P^h_t(i,i)=\pi(i)>0\tag{B.20}$$

which shows that $\lambda=\lambda_S$ and thus also that $(P_t)_{t\geq 0}$ is λ-positive. Hence, by applying [Kin63, Thm 4] we obtain that $g,h:S\to(0,\infty)$ are unique up to multiplicative constants. □

Proof of Lemma B.3. Since $\inf_{i\in S} h(i) > 0$, (B.9) implies $\sum_j g(j) < \infty$, so ν is well-defined. Moreover, for any bounded function $f : S \to \mathbb{R}$,

$$\begin{aligned} \mathbb{E}^i\big[f(X_t)\,\big|\,X_t \neq \infty\big] &= \frac{\sum_j P_t(i,j)f(j)}{\sum_j P_t(i,j)} = \frac{e^{-\lambda t}h(i)\sum_j P^h_t(i,j)h(j)^{-1}f(j)}{e^{-\lambda t}h(i)\sum_j P^h_t(i,j)h(j)^{-1}} \\ &\underset{t\to\infty}{\longrightarrow} \frac{\sum_j \pi(j)h(j)^{-1}f(j)}{\sum_j \pi(j)h(j)^{-1}} = \sum_j \nu(j)f(j), \end{aligned} \tag{B.21}$$

where all sums run over $j \in S$ and we have used the ergodicity of the positively recurrent Markov process with transition probabilities $(P^h_t)_{t\geq 0}$ and invariant law $\pi(i) = c^{-1}g(i)h(i)$, as well as the fact that $h^{-1}f$ and h^{-1} are bounded functions by our assumption that $\inf_{i\in S} h(i) > 0$. □

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