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Spectral analysis of 1D nearest–neighbor random walks and applications to subdiffusive trap and barrier models*

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Abstract

We consider a sequence $X^{(n)}$, $n \geq 1$, of continuous–time nearest–neighbor random walks on the one dimensional lattice $\mathbb{Z}$. We reduce the spectral analysis of the Markov generator of $X^{(n)}$ with Dirichlet conditions outside $(0, n)$ to the analogous problem for a suitable generalized second order differential operator $-D_{m_n} D_x$, with Dirichlet conditions outside a given interval. If the measures dm_n weakly converge to some measure dm_∞, we prove a limit theorem for the eigenvalues and eigenfunctions of $-D_{m_n} D_x$ to the corresponding spectral quantities of $-D_{m_\infty} D_x$. As second result, we prove the Dirichlet–Neumann bracketing for the operators $-D_m D_x$ and, as a consequence, we establish lower and upper bounds for the asymptotic annealed eigenvalue counting functions in the case that m is a self–similar stochastic process. Finally, we apply the above results to investigate the spectral structure of some classes of subdiffusive random trap and barrier models coming from one–dimensional physics.

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1 Introduction

Continuous–time nearest–neighbor random walks on $\mathbb{Z}$ are a basic object in probability theory with numerous applications, including the modeling of one–dimensional physical systems. A fundamental example is given by the simple symmetric random walk (SSRW) on $\mathbb{Z}$, of which we recall some standard results. It is well known that the SSRW converges to the standard Brownian motion under diffusive space–time rescaling. Moreover, the sign–inverted Markov generator with Dirichlet conditions outside $(0, n)$ has exactly $n - 1$ eigenvalues, which are all positive and simple. Labeling the eigenvalues in increasing order $\left(\lambda_k^{(n)} : 1 \leq k < n\right)$, the k–th one is given by $\lambda_k^{(n)} = 1 - \cos(\pi k / n)$

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with associated eigenfunction $f_k^{(n)}(j) = \sin(k\pi j/n)$, $j \in \mathbb{Z} \cap [0,n]$. Extending $f_k^{(n)}$ to all $[0,n]$ by linear interpolation, one observes that

$$\lim_{n\uparrow\infty} n^2 \lambda_k^{(n)} = \frac{\pi^2 k^2}{2} =: \lambda_k$$

and

$$\lim_{n\uparrow\infty} f_k^{(n)}([nx]) = \sin(k\pi x) =: f_k(x)\,,$$

where $[nx]$ denotes the integer part of nx and the last limit is in the space $C([0,1])$ endowed of the uniform norm. On the other hand, the standard Laplacian $-(1/2)\Delta$ on $[0,1]$ with Dirichlet boundary conditions has $\big(\lambda_k : k \geq 1\big)$ as family of eigenvalues and f_k as eigenfunction associated to the simple eigenvalue λ_k.

Considering this simple example it is natural to ask how general the above considerations can be. In particular, given a family of continuous–time nearest–neighbor random walks $X^{(n)}$ defined on the rescaled interval $[0,1] \cap \mathbb{Z}_n$, $\mathbb{Z}_n := \{k/n \,:\, k \in \mathbb{Z}\}$, killed when reaching the boundary, one would like very general criteria to establish (i) the convergence of $X^{(n)}$ to some stochastic process $X^{(\infty)}$, (ii) the convergence of the eigenvalues and eigenfunctions of the Dirichlet Markov generator of $X^{(n)}$ to the corresponding spectral quantities of the Dirichlet Markov generator of some stochastic process $Y^{(\infty)}$. Note that we have not imposed $X^{(\infty)} = Y^{(\infty)}$ and the reason will be clarified soon.

Criteria to establish (i) also in a more general context have been developed by C. Stone in [39]. These results have been successfully applied in order to study rigorously the asymptotic behavior of nearest–neighbor random walks on $\mathbb{Z}$ with random environment, as the random barrier model [23], [15] and the random trap model [16], [3], [4] (see below). In the first part of the paper, we focus on a general criterion to establish (ii). As well known, by an injective map $\mathbb{Z}_n \to \mathbb{R}$, one can always transform $X^{(n)}$ into a random walk $Y^{(n)}$ which can be expressed as time change of the Brownian motion B, suitably killed, with scale function given by the identity map and speed measure dm_n. This transformation reveals crucial, since the Markov generator of $Y^{(n)}$ can be defined on continuous and piecewise–linear functions and the convergence of eigenfunctions is simply in the uniform topology (otherwise one is forced to deal with rather complex functional spaces as in [15]). We point out that in order to establish the convergence (i) one often needs to consider an additional transformation (thus explaining why the above processes $X^{(\infty)}$ and $Y^{(\infty)}$ can differ). The sign–inverted Markov generator of $Y^{(n)}$ can be written as a generalized differential operator $-D_{m_n} D_x$ on a suitable interval with Dirichlet b.c. (boundary conditions). Briefly, in Theorem 2.1 we will show that the asymptotic spectral structure of $-D_{m_n} D_x$ coincides with the one of $-D_m D_x$ if the measure dm_n weakly converges to the measure dm, in particular we show the convergence of the k–th eigenvalue and the associated eigenfunction. A similar convergence result is proven by T.Uno and I. Hong in [40] for a family of differential operators on Γ_n, where Γ_n is a suitable sequence of subsets in $\mathbb{R}$ converging to the Cantor set. Some ideas in their proof have been applied to our context, while others are very model–dependent. The route followed here is more inspired by modern Sturm–Liouville theory [26], [44], where the continuity of the spectral structure is related to the continuity properties of a suitable family of entire functions. We point out that continuity theorems for the spectral structure already exist. See for example Ogura's paper [34][Section 5]. There the author proves the vague convergence (even a stronger version) of the so called spectral measure $\sigma_n(dx)$ associated to $-D_{m_n} D_x$ to the one $\sigma(dx)$ associated to $-D_m D_x$ if the measure dm_n weakly converges to the measure dm. The spectral measure comes from the Weyl–Kodaira–Titchmarsh theorem [25, 41], is a matrix–valued measure on $\mathbb{R}$

with support coinciding with the spectrum of the operator. One could work on Ogura's convergence result to deduce the convergence of the eigenvalues and the associated eigenfunctions. We did not follow this route since more elaborated, preferring a more elementary approach. The same observation holds for the continuity theorem of Kasahara [22][Theorem 1] based on Krein's correspondence.

As second step in our investigation we prove the Dirichlet–Neumann bracketing for the generalized operator $-D_m D_x$ (Theorem 6.7). This is a key result in order to get estimates on the asymptotics of eigenvalues as in the Weyl's classical theorem for the Laplacian on bounded Euclidean domains (see [42], [43], [10], [36][Chapter XIII.15]). The form of the bracketing used in our analysis goes back to G. Métivier and M.L. Lapidus (cf. [32], [28]) and has been successfully applied in [24] to establish an analogue of Weyl's classical theorem for the Laplacian on finitely ramified self–similar fractals. Finally, from the Dirichlet–Neumann bracketing we derive the behavior at ∞ of the averaged eigenvalue counting function of the operator $-D_m D_x$ on a finite interval with Dirichlet b.c. under the assumption that m is a self–similar stochastic process (see Theorem 2.2). We point out that in [17], [19], [24] [40] the authors study the asymptotics of the eigenvalues for the Laplacian defined on self–similar geometric objects. In our case, the self–similarity structure enters into the problem through the self–similarity of m.

As application of the above analysis (Theorem 2.1, Theorem 6.7 and Theorem 2.2) we investigate the small eigenvalues of some classes of subdiffusive random trap and barrier models (Theorems 2.3 and 2.5). Let $\mathcal{T} = \{\tau_x : x \in \mathbb{Z}\}$ be a family of positive i.i.d. random variables belonging to the domain of attraction of an α–stable law, $0 < \alpha < 1$. Given $\mathcal{T}$, in the random trap model the particle waits at site x an exponential time with mean τ_x and after that it jumps to $x-1$, $x+1$ with equal probability. In the random barrier model, the probability rate for a jump from $x-1$ to x equals the probability rate for a jump from x to $x-1$ and is given by $1/\tau(x)$. We consider also generalized random trap models, called *asymmetric random trap models* in [3]. Let us call $X^{(n)}$ the rescaled random walk on $\mathbb{Z}_n$ obtained by accelerating the dynamics of a factor of order $n^{1+\frac{1}{\alpha}}$ (apart a slowly varying function) and rescaling the lattice by a factor $1/n$. As investigated in [23], [16] and [3], the law of $X^{(n)}$ averaged over the environment $\mathcal{T}$ equals the law of a suitable V–dependent random walk $\tilde{X}^{(n)}$ averaged over V, V being an α–stable subordinator. To this last random walk $\tilde{X}^{(n)}$ one can apply our general results, getting at the end some annealed spectral information about $X^{(n)}$.

Random trap and random barrier walks on $\mathbb{Z}$ have been introduced in Physics in order to model 1d particle or excitation dynamics, random 1d Heisenberg ferromagnets, 1d tight–binding fermion systems, electrical lines of conductances or capacitances [1]. More recently (cf. [6], [7] and references therein) subdiffusive random walks on $\mathbb{Z}$ have been used as toy models for slowly relaxing systems as glasses and spin glasses exhibiting aging, i.e. such that the time–time correlation functions keep memory of the preparation time of the system even asymptotically. Our results contribute to the investigation of the spectral properties of aging stochastic models. This analysis and the study of the relation between aging and the spectral structure of the Markov generator has been done in [8] for the REM–like trap model on the complete graph. Estimates on the first Dirichlet eigenvalue of $X^{(n)}$ in the case of subdiffusive (also asymmetric and in $\mathbb{Z}^d$, $d \geq 1$) trap models have been derived in [33], while the spectral structure of the 1d Sinai's random walk for small eigenvalues has been investigated in [9]. The method developed in [9] is based on perturbation and capacity theory together with the property that the random environment can be approximated by a multiple–well potential. This method cannot be applied here and we have followed a different route.

2 Model and results

We consider a generic continuous–time nearest–neighbor random walk $(X_t : t \geq 0)$ on $\mathbb{Z}$. We denote by $c(x,y)$ the probability rate for a jump from x to y: $c(x,y) > 0$ if and only if $|x-y| = 1$, while the Markov generator $\mathbb{L}$ of X_t can be written as

$$\mathbb{L}f(x) = c(x,x-1)\big[f(x-1) - f(x)\big] + c(x,x+1)\big[f(x+1) - f(x)\big] \tag{2.1}$$

for any bounded function $f : \mathbb{Z} \to \mathbb{R}$. The random walk X_t can be described as follows: arrived at site $x \in \mathbb{Z}$, the particle waits an exponential time of mean $1/[c(x,x-1) + c(x,x+1)]$, after that it jumps to $x-1$ and $x+1$ with probability

$$\frac{c(x,x-1)}{c(x,x-1)+c(x,x+1)} \quad \text{and} \quad \frac{c(x,x+1)}{c(x,x-1)+c(x,x+1)}\,,$$

respectively.

By a recursive procedure, one can always determine two positive functions U and H on $\mathbb{Z}$ such that

$$c(x,y) = 1/\left[H(x)U(x \vee y)\right]\,, \qquad \forall x,y \in \mathbb{Z} : |x-y| = 1\,. \tag{2.2}$$

Moreover, the above functions U and H are univocally determined apart a positive factor c multiplying U and dividing H. Indeed, the system of equation (2.2) is equivalent to the system

$$\begin{cases} U(x+1) = U(x)\frac{c(x,x-1)}{c(x,x+1)}\,, \\ H(x) = \frac{1}{c(x,x-1)U(x)}\,, \end{cases} \qquad \forall x \in \mathbb{Z}\,. \tag{2.3}$$

We observe that U is a constant function if and only if the jump rates $c(x,y)$ depend only on the starting point x. Taking without loss of generality $U \equiv 2$, we get that after arriving at site x the random walk X_t waits an exponential time of mean $H(x)$ and then jumps with equal probability to $x-1$ and to $x+1$. This special case is known in the physics literature as *trap model* [1]. Similarly, we observe that H is a constant function if and only if the jump rates $c(x,y)$ are symmetric, that is $c(x,y) = c(y,x)$ for all $x,y \in \mathbb{Z}$. Taking without loss of generality $H \equiv 1$, we get that $c(x,x-1) = c(x-1,x) = U(x)$. This special case is known in the physics literature both as *barrier model* [1] and as random walk among *conductances*, since X_t corresponds to the random walk associated in a natural way to the linear resistor network with nodes given by the sites of $\mathbb{Z}$ and electrical filaments between nearest–neighbor nodes $x-1, x$ having conductance $c(x-1,x) = U(x)$ [14]. If the rates $\{c(x,x\pm 1)\}_{x\in\mathbb{Z}}$ are random one speaks of random trap model, random barrier model and random walk among random conductances.

In order to describe some asymptotic spectral behavior as $n \uparrow \infty$, we consider a family $X^{(n)}(t)$ of continuous–time nearest–neighbor random walks on $\mathbb{Z}_n := \{k/n : k \in \mathbb{Z}\}$ parameterized by $n \in \mathbb{N}_+ = \{1,2,\dots\}$. We call $c_n(x,y)$ the corresponding jump rates and we fix positive functions U_n, H_n satisfying the analogous of equation (2.3) (all is referred to $\mathbb{Z}_n$ instead of $\mathbb{Z}$). Below we denote by L_n the pointwise operator

$$L_n f(x) = c_n(x,x-1/n)[f(x-1/n) - f(x)] + c_n(x,x+1/n)[f(x+1/n) - f(x)] \tag{2.4}$$

defined at $x \in \mathbb{Z}_n$ for all functions f whose domain contains $x - \frac{1}{n}, x, x + \frac{1}{n}$. The Markov generator of $X_t^{(n)}$ with Dirichlet conditions outside $(0,1)$ will be denoted by $\mathbb{L}_n$. We recall that it is defined as the operator $\mathbb{L}_n : \mathcal{V}_n \to \mathcal{V}_n$, where

$$\mathcal{V}_n := \{f : [0,1] \cap \mathbb{Z}_n \to \mathbb{C},\ f(0) = f(1) = 0\}\,, \tag{2.5}$$

such that

$$\mathbb{L}_n f(x) = \begin{cases} L_n f(x) & \text{if } x \in (0,1) \cap \mathbb{Z}_n\,, \\ 0 & \text{if } x = 0, 1\,. \end{cases}$$

As discussed in Section 4, the operator $-\mathbb{L}_n$ has $n-1$ eigenvalues which are all simple and positive, while the related eigenvectors can be taken as real vectors. Below we write the eigenvalues as $\lambda_1^{(n)} < \lambda_2^{(n)} < \cdots < \lambda_{n-1}^{(n)}$.

In order to determine the suitable frame for the analysis of the eigenvalues and eigenvectors of $-\mathbb{L}_n$, we recall some basic facts from the theory of generalized second order differential operators $-D_m D_x$ (cf. [21], [11], [27][Appendix]), initially developed to analyze the behavior of a vibrating string. Let $m : \mathbb{R} \to [0,\infty)$ be a nondecreasing function with $m(x) = 0$ for all $x < 0$. Without loss of generality we can suppose that m is càdlàg. We denote by dm the Lebesgue–Stieltjes measure associated to m, i.e. the Radon measure on $\mathbb{R}$ such that $dm((a,b]) = m(b) - m(a)$ for all $a < b$. We define E_m as the support of dm, i.e. the set of points where m increases:

$$E_m := \{x \in [0,\infty) \,:\, m(x-\varepsilon) < m(x+\varepsilon)\ \forall \varepsilon > 0\}\,. \tag{2.6}$$

We suppose that $E_m \neq \emptyset$, $0 = \inf E_m$ and $\ell_m := \sup E_m < \infty$. Then, $F \in C([0,\ell_m],\mathbb{C})$ is an eigenfunction with eigenvalue λ of the generalized differential operator $-D_m D_x$ with Dirichlet b.c. if $F(0) = F(\ell_m) = 0$ and if it holds

$$F(x) = b\,x - \lambda \int_0^x dy \int_{[0,y)} dm(z) F(z)\,, \qquad \forall x \in [0,\ell_m]\,, \tag{2.7}$$

for some constant b. The number b is called *derivative number* and is denoted $F'_-(0)$ (see Section 4 for further details). As discussed in [29], [30], the operator $-D_m D_x$ with Dirichlet b.c. is the generator of the quasidiffusion on $(0,\ell_m)$ with scale function $s(x) = x$ and speed measure dm, killed when reaching the boundary points $0, \ell_m$. This quasidiffusion can be suitably defined as time change of the standard one–dimensional Brownian motion [30], [39].

The spectral analysis of $-\mathbb{L}_n$ can be reduced to the spectral analysis of a suitable generalized differential operator $-D_{m_n} D_x$ as follows. We define the function $S_n : [0,1] \cap \mathbb{Z}_n \to \mathbb{R}$ as

$$S_n(k/n) = \begin{cases} 0 & \text{if } k = 0\,, \\ \sum_{j=1}^k U_n(j/n) & \text{if } 1 \le k \le n\,. \end{cases} \tag{2.8}$$

To simplify the notation, we set

$$x_k^{(n)} := S_n(k/n)\,, \ \ \text{for } k : 0 \le k \le n\,. \tag{2.9}$$

Finally, we define the nondecreasing càdlàg function $m_n : \mathbb{R} \to [0,\infty)$ as

$$m_n(x) = \sum_{k=0}^n H_n(k/n) \mathbb{I}\big(x_k^{(n)} \le x\big) \tag{2.10}$$

where $\mathbb{I}(\cdot)$ denotes the characteristic function. Then

$$dm_n = \sum_{k=0}^n H_n(k/n) \delta_{x_k^{(n)}}\,, \qquad E_n := E_{m_n} = \{x_k^{(n)} \,:\, 1 \le k \le n\}\,, \qquad \ell_n := \ell_{m_n} = x_n^{(n)}\,.$$

We denote by $C_n[0,\ell_n]$ the set of complex continuous functions on $[0,\ell_n]$ that are linear on $[0,\ell_n] \setminus E_n$. Then, the map

$$T_n : \mathbb{C}^{[0,1]\cap \mathbb{Z}_n} \ni f \to T_n f \in C_n[0,\ell_n]\,, \tag{2.11}$$

associating to f the unique function $T_n f \in C_n[0,\ell_n]$ such that

$$T_n f(x_k^{(n)}) = f(k/n)\,, \qquad 0 \le k \le n\,,$$

is trivially bijective. As discussed in Section 4, the map T_n defines also a bijection between the eigenvectors of $-\mathbb{L}_n$ with eigenvalue λ and the eigenfunctions of the differential operator $-D_{m_n} D_x$ with Dirichlet conditions outside $(0,\ell_n)$ associated to the eigenvalue λ.

We can finally state the asymptotic behavior of the small eigenvalues:

Theorem 2.1. *Suppose that ℓ_n converges to some $\ell \in (0,\infty)$ and that dm_n weakly converges to a measure dm, where $m : \mathbb{R} \to [0,\infty)$ is a càdlàg function such that $m(x) = 0$ for all $x \in (-\infty, 0)$. Assume that $0 = \inf E_m$, $\ell = \sup E_m$ and that dm is not a linear combination of a finite family of delta measures.*

Then the generalized differential operator $-D_m D_x$ with Dirichlet conditions outside $(0,\ell)$ has an infinite number of eigenvalues, which are all positive and simple. List these eigenvalues in increasing order as $\{\lambda_k : k \ge 1\}$, and list the $n-1$ eigenvalues of the operator $-\mathbb{L}_n$, which are all positive and simple, as $\lambda_1^{(n)} < \cdots < \lambda_{n-1}^{(n)}$. Then for each $k \ge 1$ it holds

$$\lim_{n\uparrow\infty} \lambda_k^{(n)} = \lambda_k\,. \tag{2.12}$$

For each $k \ge 1$, fix an eigenfunction F_k with eigenvalue λ_k for the operator $-D_m D_x$ with Dirichlet conditions. Then, by suitably choosing the eigenfunction $F_k^{(n)} \in C([0,\ell_n])$ of eigenvalue $\lambda_k^{(n)}$ for the operator $-D_{m_n} D_x$ with Dirichlet conditions, it holds

$$\lim_{n\uparrow\infty} F_k^{(n)} = F_k \qquad \textit{in } C([0,\ell+1]) \ \textit{w.r.t.}\ \|\cdot\|_\infty\,, \tag{2.13}$$

where F_k and $F_k^{(n)}$ are set equal to zero on $(\ell,\ell+1]$ and $(\ell_n,\ell+1]$, respectively.

Since by hypothesis the supports of dm_n and dm are all included in a common compact subset, the above weak convergence of dm_n towards dm is equivalent to the vague convergence. The proof of the above theorem in given in Section 5.

We describe now another general result relating self–similarity to the spectrum edge, whose application will be relevant below when studying subdiffusive random walks. Recall the definition (2.6) of E_m.

Theorem 2.2. *Suppose that $m : [0,\infty) \to [0,\infty)$ is a random process such that*

(i) $m(0) = 0$,
(ii) m *is càdlàg and increasing a.s.,*
(iii) m *has stationary increments,*
(iii) m *is self–similar, namely there exists $\alpha > 0$ such that for all $\gamma > 0$ the processes $\big(m(x) : x \ge 0\big)$ and $\big(\gamma^{1/\alpha} m(x/\gamma) : x \ge 0\big)$ have the same law,*
(iv) extending m to all $\mathbb{R}$ by setting $m \equiv 0$ on $(-\infty,0)$, for any $x \in \mathbb{R}$ with probability one x is not a jump point of m.

Then, a.s. all eigenvalues of the operator $-D_m D_x$ with Dirichlet conditions outside $(0,1)$ are simple and positive, and form a diverging sequence $\big(\lambda_k(m) : k \ge 1\big)$ if labeled in increasing order. The same holds for the eigenvalues $\big(\lambda_k(m^{-1}) : k \ge 1\big)$ of the operator $-D_{m^{-1}} D_x$ with Dirichlet conditions outside $(0, m(1))$, where m^{-1} denotes the càdlàg generalized inverse of m, i.e.

$$m^{-1}(t) = \inf\{s \ge 0 : m(s) > t\}\,, \qquad t \ge 0\,. \tag{2.14}$$

Moreover, if there exists $x_0 > 0$ *such that*

$$\mathbb{E}\left[\sharp\{k \geq 1 \,:\, \lambda_k(m) \leq x_0\}\right] < \infty\,, \tag{2.15}$$

then there exist positive constants c_1, c_2 *such that*

$$c_1 x^{\frac{\alpha}{1+\alpha}} \leq \mathbb{E}\left[\sharp\{k \geq 1 \,:\, \lambda_k(m) \leq x\}\right] \leq c_2 x^{\frac{\alpha}{1+\alpha}}\,, \qquad \forall x \geq 1\,. \tag{2.16}$$

Similarly, if there exists $x_0 > 0$ *such that*

$$\mathbb{E}\left[\sharp\{k \geq 1 \,:\, \lambda_k(m^{-1}) \leq x_0\}\right] < \infty\,, \tag{2.17}$$

then there exist positive constants c_1, c_2 *such that*

$$c_1 x^{\frac{\alpha}{1+\alpha}} \leq \mathbb{E}\left[\sharp\{k \geq 1 \,:\, \lambda_k(m^{-1}) \leq x\}\right] \leq c_2 x^{\frac{\alpha}{1+\alpha}}\,, \qquad \forall x \geq 1\,. \tag{2.18}$$

Strictly speaking, in the above theorem we had to write $-D_{m_*}D_x$ and $-D_{(m^{-1})_*}D_x$ instead of $-D_m D_x$ and $-D_{m^{-1}}D_x$, respectively, where

$$m_*(x) = \begin{cases} 0 & \text{if } x \leq 0\,, \\ m(x) & \text{if } 0 \leq x \leq 1\,, \\ m(1) & \text{if } x \geq 1\,, \end{cases} \qquad (m^{-1})_*(x) = \begin{cases} 0 & \text{if } x \leq 0\,, \\ m^{-1}(x) & \text{if } 0 \leq x \leq m(1)\,, \\ m^{-1}\big(m(1)\big) & \text{if } x \geq m(1)\,. \end{cases} \tag{2.19}$$

This will be understood also below, in Theorems 2.3 and 2.5. Since m is càdlàg, it has a countable (finite or infinite) number of jumps $\{z_i\}$. For $x \geq 0$ it holds

$$m^{-1}(x) = \begin{cases} y & \text{if } y = m(x)\,,\ x \in [0,\infty) \setminus \{z_i\}\,, \\ z_i & \text{if } x \in [m(z_i-), m(z_i)] \text{ for some } i\,. \end{cases} \tag{2.20}$$

Since we have assumed $E_m = [0,\infty)$ a.s., m^{-1} must be continuous a.s. (observe that the jumps of m^{-1} correspond to the flat regions of m).

The proof of the above theorem is given in Section 7 and is based on the Dirichlet–Neumann bracketing developed in Section 6 (cf. Theorem 6.7). When m is a stable subordinator (2.15) and (2.17) are fulfilled (see the proof of Theorem 2.3 and 2.5).

As application of Theorem 2.1 and Theorem 2.2, we consider special families of subdiffusive random trap and barrier models (cf. [1], [23], [16], [3], [4], [15] and references therein). To this aim we fix a family $\mathcal{T} := \{\tau(x) \,:\, x \in \mathbb{Z}\}$ of positive i.i.d. random variables in the domain of attraction of a one–sided α–stable law, $0 < \alpha < 1$. This is equivalent to the fact that there exists some function $L_1(t)$ slowly varying as $t \to \infty$ such that

$$F(t) = \mathbb{P}(\tau(x) > t) = L_1(t) t^{-\alpha}\,, \qquad t > 0\,.$$

Let us define the function h as

$$h(t) = \inf\{s \geq 0 \,:\, 1/F(s) \geq t\}\,. \tag{2.21}$$

Then, by Proposition 0.8 (v) in [37] we know that

$$h(t) = L_2(t) t^{1/\alpha} \qquad t > 0\,, \tag{2.22}$$

for some function L_2 slowly varying as $t \to \infty$.

Finally, we denote by V the double–sided α–stable subordinator defined on some probability space $(\Xi, \mathcal{F}, \mathcal{P})$ (cf. [2] Section III.2). Namely, V has a.s. càdlàg paths, $V(0) = 0$ and V has non-negative independent increments such that for all $s < t$

$$E\Big[\exp\big\{-\lambda[V(t) - V(s)]\big\}\Big] \;=\; \exp\{-\lambda^\alpha (t-s)\}\,. \tag{2.23}$$

(Strictly speaking, inside the exponential in the r.h.s. there should be an extra positive factor c_0 that we have fixed equal to 1). The sample paths of V are strictly increasing and of pure jump type, in the sense that $V(u) = \sum_{0<v\leq u}\{V(v) - V(v-)\}$. Moreover, the random set $\{(u, V(u) - V(u-)) \,:\, u \in \mathbb{R},\ V(u) > V(u-)\}$ is a Poisson point process on $\mathbb{R} \times \mathbb{R}_+$ with intensity $cw^{-1-\alpha} du\, dw$, for some $c > 0$. Finally, we denote by V^{-1} the generalized inverse function $V^{-1}(t) = \inf\{s \in \mathbb{R} \,:\, V(s) > t\}$. Since V is strictly increasing $\mathcal{P}$–a.s., V^{-1} has continuous paths $\mathcal{P}$–a.s.

For random trap models we obtain:

Theorem 2.3. *Fix $a \geq 0$ and let $\mathcal{T} = \{\tau(x)\}_{x\in\mathbb{Z}}$ be a family of positive i.i.d. random variables in the domain of attraction of an α–stable law, $0 < \alpha < 1$. If $a > 0$, assume also that $\tau(x)$ is bounded from below by a non–random positive constant a.s.*

Given a realization of $\mathcal{T}$, consider the $\mathcal{T}$–dependent trap model $\{X(t)\}_{t\geq 0}$ on $\mathbb{Z}$ with transition rates

$$c(x, y) = \begin{cases} \tau(x)^{-1+a}\tau(y)^a & \text{if } |x - y| = 1 \\ 0 & \text{otherwise}\,. \end{cases} \tag{2.24}$$

Call $\lambda_1^{(n)}(\mathcal{T}) < \lambda_2^{(n)}(\mathcal{T}) < \cdots < \lambda_{n-1}^{(n)}(\mathcal{T})$ the (simple and positive) eigenvalues of the Markov generator of $X(t)$ with Dirichlet conditions outside $(0, n)$. Then

i) For each $k \geq 1$, the $\mathcal{T}$–dependent random vector

$$\gamma^2 L_2(n) n^{1+\frac{1}{\alpha}} \left(\lambda_1^{(n)}(\mathcal{T}), \cdots, \lambda_k^{(n)}(\mathcal{T})\right) \tag{2.25}$$

weakly converges to the V–dependent random vector

$$\left(\lambda_1(V), \ldots, \lambda_k(V)\right),$$

where $\gamma = \mathbb{E}\left(\tau(0)^{-a}\right)$, the slowly varying function L_2 has been defined in (2.22) *and $\{\lambda_k(V) \,:\, k \geq 1\}$ denotes the family of the (simple and positive) eigenvalues of the generalized differential operator $-D_V D_x$ with Dirichlet conditions outside $(0, 1)$.*

ii) If $a = 0$ and $\mathbb{E}\left(\exp\{-\lambda\tau(x)\}\right) = \exp\{-\lambda^\alpha\}$, then in (2.25) *the quantity $L_2(n)$ can be replaced by the constant 1.*

iii) There exist positive constants c_1, c_2 such that

$$c_1 x^{\frac{\alpha}{1+\alpha}} \leq \mathbb{E}\left[\sharp\{k \geq 1 \,:\, \lambda_k(V) \leq x\}\right] \leq c_2 x^{\frac{\alpha}{1+\alpha}}\,, \qquad \forall x \geq 1\,. \tag{2.26}$$

The above random walk $X(t)$ can be described as follows: after arriving at site $x \in \mathbb{Z}$ the particle waits an exponential time of mean

$$\frac{\tau(x)^{1-a}}{\tau(x-1)^a + \tau(x+1)^a}\,,$$

after that it jumps to $x - 1$ and $x + 1$ with probability given respectively by

$$\frac{\tau(x-1)^a}{\tau(x-1)^a + \tau(x+1)^a} \qquad \text{and} \qquad \frac{\tau(x+1)^a}{\tau(x-1)^a + \tau(x+1)^a}\,.$$

The random walk $X(t)$ is called random trap model following [3], although according to our initial terminology the name would be correct only when $a = 0$. Sometimes we will also refer to the case $a \in (0, 1]$ as *generalized random trap model*. The additional assumption concerning the bound from below of $\tau(x)$ when $a > 0$ can be weakened. Indeed, as pointed out in the proof, we only need the validity of strong LLN for a suitable triangular arrays of random variables.

Of course, one can consider also the diffusive case. Extending the results of [5] we get

Proposition 2.4. *Fix $a \geq 0$ and let $\mathcal{T} = \{\tau(x)\}_{x\in\mathbb{Z}}$ be a family of positive random variables, ergodic w.r.t. spatial translations and such that $\mathbb{E}(\tau(x)) < \infty$, $\mathbb{E}(\tau(x)^{-a}) < \infty$. Given a realization of $\mathcal{T}$, consider the $\mathcal{T}$–dependent trap model $\{X(t)\}_{t\geq 0}$ on $\mathbb{Z}$ with transition rates* (2.24) *and call $\lambda_1^{(n)}(\mathcal{T}) < \lambda_2^{(n)}(\mathcal{T}) < \cdots < \lambda_{n-1}^{(n)}(\mathcal{T})$ the (simple and positive) eigenvalues of the Markov generator of $X(t)$ with Dirichlet conditions outside $(0,n)$. Then for each $k \geq 1$ and for a.a. $\mathcal{T}$,*

$$n^2\mathbb{E}(\tau(x)^{-a})\mathbb{E}(\tau(x))\lambda_k^{(n)}(\mathcal{T}) \to \pi^2 k^2 \,. \tag{2.27}$$

Let us state our results concerning random barrier models:

Theorem 2.5. *Let $\mathcal{T} = \{\tau(x)\}_{x\in\mathbb{Z}}$ be a family of positive i.i.d. random variables in the domain of attraction of an α–stable law, $0 < \alpha < 1$. Given a realization of $\mathcal{T}$, consider the $\mathcal{T}$–dependent barrier model $\{X(t)\}_{t\geq 0}$ on $\mathbb{Z}$ with jump rates*

$$c(x,y) = \begin{cases} \tau(x \vee y)^{-1} & \text{if } |x-y| = 1 \\ 0 & \text{otherwise}\,. \end{cases} \tag{2.28}$$

Call $\lambda_1^{(n)}(\mathcal{T}) < \lambda_2^{(n)}(\mathcal{T}) < \cdots < \lambda_{n-1}^{(n)}(\mathcal{T})$ the eigenvalues of the Markov generator of $X(t)$ with Dirichlet conditions outside $(0,1)$. Recall the definition (2.22) *of the positive slowly varying function L_2. Then:*

i) For each $k \geq 1$, the $\mathcal{T}$–dependent random vector

$$L_2(n) n^{1+\frac{1}{\alpha}} \big(\lambda_1^{(n)}(\mathcal{T}), \dots, \lambda_k^{(n)}(\mathcal{T})\big) \tag{2.29}$$

weakly converges to the V–dependent random vector

$$\big(\lambda_1(V^{-1}), \dots, \lambda_k(V^{-1})\big)\,,$$

where $\{\lambda_k(V^{-1}) \,:\, k \geq 1\}$ denotes the family of the (simple and positive) eigenvalues of the generalized differential operator $-D_{V^{-1}}D_x$ with Dirichlet conditions outside $(0, V(1))$.

ii) If $\mathbb{E}(e^{-\lambda\tau(x)}) = e^{-\lambda^\alpha}$ then in (2.29) *the quantity $L_2(n)$ can be replaced by the constant 1.*

iii) There exist positive constants c_1, c_2 such that

$$c_1 x^{\frac{\alpha}{1+\alpha}} \leq \mathbb{E}\left[\sharp\{k \geq 1 \,:\, \lambda_k(V^{-1}) \leq x\}\right] \leq c_2 x^{\frac{\alpha}{1+\alpha}}\,, \qquad \forall x \geq 1\,. \tag{2.30}$$

Again, one can consider also the diffusive case. Extending the results of [5] we get

Proposition 2.6. *Let $\mathcal{T} = \{\tau(x)\}_{x\in\mathbb{Z}}$ be a family of positive random variables, ergodic w.r.t. spatial translations and such that $\mathbb{E}(\tau(x)) < \infty$. Given a realization of $\mathcal{T}$, consider the $\mathcal{T}$–dependent barrier model $\{X(t)\}_{t\geq 0}$ on $\mathbb{Z}$ with transition rates* (2.28) *and call $\lambda_1^{(n)}(\mathcal{T}) < \lambda_2^{(n)}(\mathcal{T}) < \cdots < \lambda_{n-1}^{(n)}(\mathcal{T})$ the (simple and positive) eigenvalues of the Markov generator of $X(t)$ with Dirichlet conditions outside $(0,n)$. Then for each $k \geq 1$ and for a.a. $\mathcal{T}$,*

$$n^2\mathbb{E}(\tau(x))\lambda_k^{(n)}(\mathcal{T}) \to \pi^2 k^2 \,. \tag{2.31}$$

Theorem 2.3 and 2.5 cannot be derived by a direct application of Theorem 2.1. Indeed, for any choice of the sequence $c(n) > 0$, fixed a realization of $\mathcal{T}$ the measures dm_n associated to the space–time rescaled random walks $X^{(n)}(t) = n^{-1}X\big(c(n)t\big)$ do not converge to dV or dV^{-1} restricted to $(0,1)$, $(0,V(1))$ respectively. On the other hand, for each n one can define a random field $\mathcal{T}_n$ in terms of the α–stable process V, i.e. $\mathcal{T}_n = F_n(V)$, having the same law of $\mathcal{T}$ [3, 16]. Calling $\tilde{X}^{(n)}$ the analogous of $X^{(n)}$ with jump rates defined in terms of $\mathcal{T}_n$, one has that the associated measures $d\tilde{m}_n$ satisfy the hypothesis of Theorem 2.1. This explains why Theorems 2.3 and 2.5 give an annealed and not quenched result. On the other hand, for the random walks $\tilde{X}^{(n)}$ the result is quenched, i.e. the convergence of the eigenvalues holds for almost all realizations of the subordinator V. We refer to Sections 8 and 9 for a more detailed discussion of the above coupling and for the proof of Theorems 2.3 and 2.5.

2.1 Outline of the paper

The paper is structured as follows. In Section 3 we explain how the spectral analysis of $-\mathbb{L}_n$ reduces to the spectral analysis of the operator $-D_{m_n}D_x$. In Section 4 we recall some basic facts of generalized second order operators. In particular, we characterize the eigenvalues of $-\mathbb{L}_n$ as zeros of a suitable entire function. In Section 5 we prove Theorem 2.1. In Section 6 we prove the Dirichlet–Neumann bracketing. This result, interesting by itself, allows us to prove Theorem 2.2 in Section 7. Finally, we move to applications: in Section 8 we prove Theorem 2.3, in Section 9 we prove Theorem 2.5, while in Section 10 we prove Propositions 2.4 and 2.6.

3 From $-\mathbb{L}_n$ to $-D_{m_n}D_x$

Recall the definition of the local operator L_n given in (2.4) and of the bijection T_n given in (2.11).

Lemma 3.1. *Given functions $f,g:[0,1]\cap\mathbb{Z}_n\to\mathbb{R}$, the system of identities*

$$L_n f(x) = g(x)\,, \qquad \forall x\in(0,1)\cap\mathbb{Z}_n\,, \tag{3.1}$$

is equivalent to the system

$$f(x) = f(0) + \sum_{j=1}^{nx} U_n(j/n)\left(\frac{f(1/n)-f(0)}{U_n(1/n)} + \sum_{k=1}^{j-1} H_n(k/n)g(k/n)\right)\,, \qquad \forall x\in(0,1]\cap\mathbb{Z}_n\,, \tag{3.2}$$

where we convey to set $\sum_{k=1}^{0} H_n(k/n)g(k/n)=0$. Setting $F=T_nf$, $G=T_ng$ and

$$b = \frac{F(x_1^{(n)})-F(0)}{U_n(1/n)} - H_n(0)G(0)\,,$$

(3.2) *is equivalent to*

$$F(x) = F(0) + bx + \int_0^x dy\int_{[0,y)} G(z)dm_n(z)\,, \qquad \forall x\in[0,\ell_n]\,. \tag{3.3}$$

In particular, $f:[0,1]\cap\mathbb{Z}_n\to\mathbb{R}$ is an eigenvector with eigenvalue λ of the operator $-\mathbb{L}_n$ if and only if T_nf is an eigenfunction with eigenvalue λ of the generalized differential operator $-D_{m_n}D_x$ with Dirichlet conditions outside $(0,\ell_n)$.

Proof. For simplicity of notation we write U,H instead of U_n,H_n. Moreover, we use the natural bijection $\mathbb{Z}\ni k\to k/n\in\mathbb{Z}_n$, denoting the point k/n of $\mathbb{Z}_n$ simply as k. Setting $\Delta f(j)=f(j)-f(j-1)$, we can rewrite (3.1) by means of the recursive identities

$$\frac{\Delta f(j+1)}{U(j+1)} = H(j)g(j) + \frac{\Delta f(j)}{U(j)}\,, \qquad \forall j\in(0,n)\cap\mathbb{Z}\,.$$

By a simple telescopic argument the above system is equivalent to

$$f(x) = f(0) + \sum_{j=1}^{x} U(j) \left(\frac{\Delta f(1)}{U(1)} + \sum_{k=1}^{j-1} H(k) g(k) \right) , \qquad \forall x \in (0, n] \cap \mathbb{Z} ,$$

with the convention that the last sum is zero if $j = 1$. This proves that (3.1) is equivalent to (3.2). Using T_n, F, G, m_n we can rewrite (3.2) as

$$F(x) = F(0) + \int_0^x dy \left(\frac{F(x_1^{(n)}) - F(0)}{U(1)} + \int_{(0,y)} G(z) dm_n(z) \right) , \qquad \forall x \in (0, \ell_n] . \tag{3.4}$$

Trivially, equation (3.4) is equivalent to (3.3). Finally, the conclusion of the lemma follows from the previous observations and the discussion about the generalized differential operator $-D_m D_x$ given in Section 2. □

4 Generalized second order differential operators

For the reader's convenience and for next applications, we recall the definition of generalized differential operator. We mainly follow [21], with some slight modifications that we will point out. We refer to [21], [11] and [31] for a detailed discussion.

Let $m : \mathbb{R} \to [0, \infty)$ be a càdlàg nondecreasing function with $m(x) = 0$ for all $x < 0$. We define m_x as the magnitude of the jump of the function m at the point x:

$$m_x = m(x) - m(x-) , \qquad x \in \mathbb{R} . \tag{4.1}$$

We define E_m as the support of dm, i.e. the set of points where m increases (see (2.6)). We suppose that $E_m \neq \emptyset$, $0 = \inf E_m$ and $\ell_m := \sup E_m < \infty$.

Given a continuous function $F(x) \in C([0, \ell_m])$ and a dm–integrable function G on $[0, \ell_m]$ we write $-D_m D_x F = G$ if there exist complex constants a, b such that

$$F(x) = a + bx - \int_0^x dy \int_{[0,y)} dm(z) G(z) , \qquad \forall x \in [0, \ell_m] . \tag{4.2}$$

We remark that the integral term in equation (4.2) can be written also as

$$\int_0^x dy \int_{[0,y)} dm(z) G(z) = \int_{[0,x]} (x - z) G(z) dm(z) = \int_0^x dy \int_{[0,y]} dm(z) G(z) .$$

We point out that equation (4.2) implies that F is linear on $[x_1, x_2]$ if m is constant on $(x_1, x_2) \subset [0, \ell_m]$.

As discussed in [21], the function G is not univocally determined from F. To get uniqueness, one can for example fix the value of b and $b - \int_{[0,\ell_m]} G(s) dm(s)$. These values are called *derivative numbers* and denoted by $F'_-(0)$ and $F'_+(\ell_m)$, respectively. Indeed, in [21] the domain $\mathcal{D}_m$ of the differential operator $-D_m D_x$ is defined as the family of complex–valued extended functions $F[x]$, given by the triple $\big(F(x),\, F'_-(0),\, F'_+(\ell_m)\big)$, while the authors set $-D_m D_x F[x] = G(x)$. We prefer to avoid the notion of extended functions here, since not necessarily.

It is simple to check that the function F satisfying (4.2) fulfills the following properties: for each $x \in [0, \ell_m)$ the function $F(x)$ has right derivative $F'_+(x)$, for each $x \in (0, \ell_m]$ the function $F(x)$ has left derivative $F'_-(x)$ and

$$F'_+(x) = b - \int_{[0,x]} G(s) dm(s) , \qquad x \in [0, \ell_m) , \tag{4.3}$$

$$F'_-(x) = b - \int_{[0,x)} G(s) dm(s) , \qquad x \in (0, \ell_m] . \tag{4.4}$$

In view of the definition of $F'_-(0)$ and $F'_+(\ell_m)$, the above identities extend to any $x \in [0,\ell_m]$. In addition, if $m_0 = 0$ then $F'_-(0) = \lim_{\varepsilon \downarrow 0} F'_+(\varepsilon)$, while if $m_{\ell_m} = 0$ then $F'_+(\ell_m) = \lim_{\varepsilon\downarrow 0} F'_-(\ell_m - \varepsilon)$.

As discussed in [21], fixed $\lambda \in \mathbb{C}$ there exists a unique function $F \in C([0,\ell_m])$ solving equation (4.2) with $G = \lambda F$ for fixed a, b. In other words, fixed $F(0)$ and $F'_-(0)$ there exists a unique solution of the homogeneous differential equation

$$-D_m D_x F = \lambda F\,. \tag{4.5}$$

Given $\lambda \in \mathbb{C}$, we define $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ as the solutions (4.5) satisfying respectively the initial conditions

$$\varphi(0,\lambda) = 1\,, \qquad \varphi'_-(0,\lambda) = 0\,, \tag{4.6}$$

$$\psi(0,\lambda) = 0\,, \qquad \psi'_-(0,\lambda) = 1\,. \tag{4.7}$$

It is known that each function $F \in C([0,\ell_m])$ satisfying (4.5) is a linear combination of the independent solutions $\varphi(\cdot,\lambda)$ and $\psi(\cdot,\lambda)$. Finally, $F \not\equiv 0$ is called an eigenfunction of the operator $-D_m D_x$ with Dirichlet [Neumann] b.c. if F solves (4.5) for some $\lambda \in \mathbb{C}$, and moreover $F(0) = F(\ell_m) = 0$ [$F'_-(0) = F'_+(\ell_m) = 0$]. By the above observations, we get that F is a Dirichlet eigenfunction if and only if F is a nonzero multiple of $\psi(x,\lambda)$ for $\lambda \in \mathbb{C}$ satisfying $\psi(\ell_m,\lambda) = 0$, while F is a Neumann eigenfunction if and only if $F(x)$ is a nonzero multiple of $\varphi(x,\lambda)$ with $\lambda \in \mathbb{C}$ satisfying

$$\int_0^\ell \varphi(s,\lambda)dm(s) = 0\,. \tag{4.8}$$

In particular, the Dirichlet and the Neumann eigenvalues are all simple.

We collect in the following lemma some known results concerning the Dirichlet eigenvalues and eigenfunctions:

Lemma 4.1. *Let $m : \mathbb{R} \to [0,\infty)$ be a nondecreasing càdlàg function such that $m(x) = 0$ for $x < 0$, $0 = \inf E_m$, $\ell_m := \sup E_m < \infty$. Then the differential operator $-D_m D_x$ with Dirichlet conditions outside $(0,\ell_m)$ has a countable (finite or infinite) family of eigenvalues, which are all positive and simple. The set of eigenvalues has no accumulation points. In particular, if there is an infinite number of eigenvalues $\{\lambda_n\}_{n\geq 1}$, listed in increasing order, it must be $\lim_{n\uparrow\infty} \lambda_n = \infty$.*

The above eigenvalues coincide with the zeros of the entire function $\mathbb{C} \ni \lambda \to \psi(\ell_m,\lambda) \in \mathbb{C}$. The eigenspace associated to the eigenvalue λ is spanned by the real function $\psi(\cdot,\lambda)$. Moreover, F is an eigenfunction of $-D_m D_x$ with Dirichlet conditions outside $(0,\ell_m)$ and associated eigenvalue λ if and only if

$$F(x) = \lambda \int_{[0,\ell_m)} G_{0,\ell_m}(x,y)F(y)dm(y)\,, \qquad \forall x \in [0,\ell_m]\,, \tag{4.9}$$

where, given an interval $[a,b]$, the Dirichlet Green function $G_{a,b} : [a,b]^2 \to \mathbb{R}$ is defined as

$$G_{a,b}(x,y) = \begin{cases} \frac{(y-a)(b-x)}{b-a} & \text{if } y \leq x\,, \\ \frac{(x-a)(b-y)}{b-a} & \text{if } x \leq y\,. \end{cases} \tag{4.10}$$

In particular, for any Dirichlet eigenvalue λ it holds

$$\lambda \geq \left[\ell_m m(\ell_m)\right]^{-1}. \tag{4.11}$$

Proof. The first part of the lemma follows easily from the analysis given in [21] concerning the entire function $\psi(\cdot, \lambda_m)$. The characterization (4.9) follows by straightforward computations from the definition of Dirichlet eigenfunctions. To conclude, we observe that (4.9) implies $\|F\|_\infty \leq \lambda \|F\|_\infty \ell_m dm([0, \ell_m))$, since trivially $0 \leq G_{0,\ell_m}(x,y) \leq \ell_m$. (4.11) then follows. □

As discussed in [21], page 29, the function φ can be written as λ–power series $\varphi(s,\lambda) = \sum_{j=0}^\infty (-\lambda)^j \varphi_j(s)$ for suitable functions φ_j. Therefore the l.h.s. of (4.8) equals $\sum_{j=0}^\infty (-\lambda)^j \int_{(0,1)} \varphi_j(s) dm(s)$. From the bounds on φ_j one derives that the l.h.s. of (4.8) is an entire function in λ, thus implying that its zeros (or equivalently the eigenvalues of the operator $-D_m D_x$ with Neumann b.c.) form a discrete subset of $[0, \infty)$. Moreover (cf. [21]) the eigenvalues are nonnegative and 0 itself is an eigenvalue.

5 Proof of Theorem 2.1

We divide the proof in subsections.

5.1 Eigenvalues as zeros of entire functions

At this point, we have reduced the analysis of the spectrum of the differential operator $-D_m D_x$ with Dirichlet conditions outside $(0, \ell_m)$ to the analysis of the zeros of the entire function $\psi(\ell, \cdot)$. As in [26] and [44] a key tool is the following result, whose proof can be found in [12], page 248:

Lemma 5.1. *Let Ξ be a metric space, $f : \Xi \times \mathbb{C} \to \mathbb{C}$ be a continuous function such that for each $\alpha \in \Xi$ the map $f(\alpha, \cdot)$ is an entire function. Let $V \subset \mathbb{C}$ be an open subset whose closure $\bar{V}$ is compact, and let $\alpha_0 \in \Xi$ be such that no zero of the function $f(\alpha_0, \cdot)$ is on the boundary of V. Then there exists a neighborhood W of α_0 in Ξ such that:*

1. *for any $\alpha \in W$, $f(\alpha, \cdot)$ has no zero on the boundary of V,*
2. *the sum of the orders of the zeros of $f(\alpha, \cdot)$ contained in V is independent of α as α varies in W.*

From now on, let m_n and m be as in Theorem 2.1. Given $\lambda \in \mathbb{C}$, define $\psi(x, \lambda)$ as the solution on the homogeneous differential equation (4.5) satisfying the initial condition (4.7). Define similarly $\psi^{(n)}(x, \lambda)$ by replacing m with m_n. The following fact will be fundamental in the application of Lemma 5.1.

Lemma 5.2. *Define $\psi(\cdot, \lambda)$, $\psi(\cdot, \ell)$ as $\psi(\ell, \lambda)$, $\psi(\ell_n, \lambda)$ on $(\ell, \ell+1]$, $(\ell_n, \ell+1]$, respectively (note that $\ell_n < \ell + 1$ eventually). Fix a sequence $\lambda_n \in \mathbb{C}$ converging to some $\lambda_\infty \in \mathbb{C}$. Then $\psi_n(\cdot, \lambda_n)$ converges to $\psi(\cdot, \lambda_\infty)$ as $n \to \infty$ uniformly in $C([0, \ell+1])$.*

Proof. The proof is similar to the first part of the proof of Theorem 1 in [22]. As discussed in [21], page 30, one can write explicitly the power expansion of the entire function $\mathbb{C} \ni \lambda \to \psi^{(n)}(x, \lambda) \in \mathbb{C}$. In particular, it holds $\psi^{(n)}(x, \lambda) = \sum_{j=0}^\infty (-\lambda)^j \psi_j^{(n)}(x)$, where $\psi_0^{(n)}(x) = x$, $\psi_{j+1}^{(n)}(x) = \int_0^x (x-s) \psi_j^{(n)}(s) dm(s)$ for $j \geq 0$ and $x \in [0, \ell_n]$. Note that in the above integrals we do not need to specify the border of the integration domain since the integrand functions vanish both at 0 and at x. By the same arguments used in [21][page 32] one gets $\psi_k^{(n)}(x) \leq \left(\frac{x}{k+1}\right)^{k+1} \frac{m_n(x)}{k!}$ for $x \in [0, \ell_n]$. These bounds imply easily that the family $\mathcal{F}$ of functions $\{\psi^{(n)}(\cdot, \lambda_n)\}_n$ is uniformly bounded in $C([0, \ell+1])$. Since

$$\psi^{(n)}(x, \lambda_n) = x - \lambda_n \int_0^x dy \int_{[0,y)} dm_n(z) \psi^{(n)}(z, \lambda_n)\,, \qquad x \in [0, \ell_n]\,, \tag{5.1}$$

the above bounds imply also that the family $\mathcal{F}$ is equicontinuous in $C([0,\ell+1])$. By Ascoli–Arzelà theorem, $\mathcal{F}$ is relatively compact. From the weak convergence of dm_n to dm and from (5.1), one gets that all limit functions $\tilde{\psi}$ satisfies

$$\tilde{\psi}(x) = x - \lambda \int_0^x dy \int_{[0,y)} dm(z)\tilde{\psi}(z,\lambda)\,, \qquad x \in [0,\ell]\,, \tag{5.2}$$

and is equal to $\tilde{\psi}(\ell)$ on $[\ell,\ell+1]$. Since the integral equation (5.1) has a unique solution, given by $\psi(\cdot,\lambda)$, we get the thesis. □

By applying Lemma 4.1, Lemma 5.1 and Lemma 5.2 we obtain:

Lemma 5.3. *Let m_n and m be as in Theorem 2.1. Fix a constant $L > 0$ different from the Dirichlet eigenvalues of $-D_m D_x$, and let $\{\lambda_i : 1 \le i \le k_0\}$ be the Dirichlet eigenvalues of $-D_m D_x$ smaller than L. Let $\varepsilon > 0$ be such (i) $\lambda_{k_0} + \varepsilon < L$ and (ii) each interval $J_i := [\lambda_i - \varepsilon, \lambda_i + \varepsilon]$ intersects $\{\lambda_i : 1 \le i \le k_0\}$ only at λ_i, for any $i : 1 \le i \le k_0$. Then there exists an integer n_0 such that:*

i) for all $n \ge n_0$, the spectrum of $-\mathbb{L}_n$ has only one eigenvalue in J_i ,

ii) for all $n \ge n_0$, $-\mathbb{L}_n$ has no eigenvalue inside $(0,L) \setminus \left(\cup_{i=1}^{k_0} J_i\right)$.

Proof. We already know that the Dirichlet eigenvalues of the operator $-D_{m_n}D_x$ $[-D_m D_x]$ are given by the zeros of the entire function $\psi^{(n)}(\ell_n,\cdot)$ $[\psi(\ell,\cdot)]$. Hence, it is natural to derive the thesis by applying Lemma 5.1 with different choices of V. More precisely, we take $\alpha_0 = \infty$ and $\Xi = \mathbb{N}_+ \cup \{\infty\}$ endowed of any metric d such that all points $n \in \mathbb{N}_+$ are isolated w.r.t. d and $\lim_{n\uparrow\infty} d(n,\infty) = 0$. We define $f : \Xi \times \mathbb{C} \to \mathbb{C}$ as

$$f(\alpha,\lambda) = \begin{cases} \psi^{(n)}(\ell_n,\lambda) & \text{if } \alpha = n\,, \\ \psi(\ell,\lambda) & \text{if } \alpha = \infty\,. \end{cases}$$

Finally, we choose $V = (\lambda_i - \varepsilon, \lambda_i + \varepsilon)$ as i varies in $\{1,\dots,k_0\}$ and after that we take $V = (0,L) \setminus \left(\cup_{r=1}^{k_0} J_r\right)$. The thesis then easily follows by applying Lemma 5.1 if we prove that f is continuous. The nontrivial part is to prove that $\lim_{n\uparrow\infty} \psi^{(n)}(\ell_n,\lambda_n) = \psi(\ell,\lambda)$ for any sequence of complex numbers $\{\lambda_n\}_{n\ge1}$ converging to some $\lambda \in \mathbb{C}$. This result follows from Lemma 5.2 and the equicontinuity of the family of functions $\{\psi^{(n)}(\cdot,\lambda_n)\}_n$ in $C([0,\ell+1])$. □

5.2 Minimum–maximum characterization of the eigenvalues

For the reader's convenience, we list some vector spaces that will be repeatedly used in what follows. We introduce the vector spaces $\mathcal{A}(n)$ and $\mathcal{B}(n)$ as

$$\mathcal{A}(n) := \{f : [0,1] \cap \mathbb{Z}_n \to \mathbb{R} \,:\, f(0) = f(1) = 0\}\,, \qquad \mathcal{B}(n) = T_n \mathcal{A}(n)\,, \tag{5.3}$$

where the map T_n has been defined in (2.11). Hence $F \in \mathcal{B}(n)$ if and only if (i) $F(0) = F(1) = 0$, (ii) F is continuous and (iii) F is linear on all subintervals $[x_{j-1}^{(n)}, x_j^{(n)}]$, $1 \le j \le n$. Since we already know that the eigenvalues and suitable associated eigenfunctions of $-\mathbb{L}_n$ are real, we can think of $-\mathbb{L}_n$ as operator defined on $\mathcal{A}(n)$. Finally, given $a < b$ we write $C_0[a,b]$ for the family of continuous functions $f : [a,b] \to \mathbb{R}$ such that $f(a) = f(b) = 0$.

Let us recall the min–max formula characterizing the k–th eigenvalue $\lambda_k^{(n)}$ of $-\mathbb{L}_n$, or equivalently of the differential operator $-D_{m_n}D_x$ with Dirichlet conditions outside

$(0,\ell_n)$. We refer to [10], [36] for more details. First we observe the validity of the detailed balance equation:

$$H_n(x)c_n\big(x,x+\frac{1}{n}\big)=\frac{1}{U_n(x+1/n)}=H_n(x+\frac{1}{n})c_n\big(x+\frac{1}{n},x\big)\qquad \forall x\in\mathbb{Z}_n\,. \tag{5.4}$$

Identifying $\mathcal{A}(n)$ with $\{f:(0,1)\cap\mathbb{Z}_n\to\mathbb{R}\}$, this implies that $-\mathbb{L}_n$ is a symmetric operator in $L^2((0,1)\cap\mathbb{Z}_n,\mu_n)$, where $\mu_n:=\sum_{x\in(0,1)\cap\mathbb{Z}_n}H_n(x)\delta_x$. Given $f\in\mathcal{A}(n)$ we write $D_n(f)$ for the Dirichlet form $D_n(f):=\mu_n(f,-\mathbb{L}_n f)$. By simple computations, we obtain

$$D_n(f)=\sum_{j=1}^{n}U_n(j/n)^{-1}\big[\,f(j/n)-f((j-1)/n)\,\big]^2\,.$$

Note that $D_n(f)=0$ with $f\in\mathcal{A}(n)$ if and only if $f\equiv 0$. The min–max characterization of $\lambda_k^{(n)}$ is given by the formula

$$\lambda_k^{(n)}=\min_{V_k}\max_{f\in V_k:f\not\equiv 0}\frac{D_n(f)}{\mu_n(f^2)}\,, \tag{5.5}$$

where V_k varies among the k–dimensional subspaces of $\mathcal{A}(n)$. Moreover, the minimum is attained at $V_k=V_k^{(n)}$, defined as the subspace spanned by the eigenvectors $f_j^{(n)}$ associated to the first k eigenvalues $\{\lambda_j^{(n)}\,:\,1\leq j\leq k\}$.

We can rewrite the above min–max principle in terms of $F=T_nf$ and dm_n. Indeed, given $f\in\mathcal{A}(n)$, the function $F=T_nf$ is linear between $x_{j-1}^{(n)}$ and $x_j^{(n)}$, thus implying that

$$U_n(j/n)^{-1}\big[\,f(j/n)-f((j-1)/n)\,\big]^2=\int_{x_{j-1}^{(n)}}^{x_j^{(n)}}D_sF(s)^2ds\,.$$

Hence, $D_n(f)=\int_0^{\ell_n}D_sF(s)^2ds$. From this identity and (5.5) one easily obtains that

$$\lambda_k^{(n)}=\min_{S_k}\max_{F\in S_k\,:\,F\not\equiv 0}\Phi_n(F)\,, \tag{5.6}$$

where S_k varies among all k–dimensional subsets of $\mathcal{B}(n)$ (recall (5.3)), while for a generic function $F\in C_0[0,\ell_n]$ we define

$$\Phi_n(F):=\frac{\int_0^{\ell_n}D_sF(s)^2ds}{\int_0^{\ell_n}F(s)^2dm_n(s)} \tag{5.7}$$

whenever the denominator is nonzero. Here and in what follows, we write $\int_0^{\ell_n}$ instead of $\int_{[0,\ell_n]}$.

The following observation will reveal very useful:

Lemma 5.4. *Let $F\in\mathcal{B}(n)$ and let $G\in C_0[0,\ell_n]$ be any function satisfying $F(x_j^{(n)})=G(x_j^{(n)})$ for all $0\leq j\leq n$. Then*

$$\int_0^{\ell_n}D_sF(s)^2ds\leq\int_0^{\ell_n}D_sG(s)^2ds\,. \tag{5.8}$$

In particular, if $F\not\equiv 0$ then $\Phi_n(F)$ and $\Phi_n(G)$ are both well defined and $\Phi_n(F)\leq\Phi_n(G)$.

Proof. In order to get (5.8) it is enough to observe that by Schwarz' inequality it holds

$$\int_{x_{j-1}^{(n)}}^{x_j^{(n)}} D_sF(s)^2 ds = \frac{\big[F\big(x_j^{(n)}\big) - F\big(x_{j-1}^{(n)}\big)\big]^2}{x_j^{(n)} - x_{j-1}^{(n)}} = \frac{\big[G\big(x_j^{(n)}\big) - G\big(x_{j-1}^{(n)}\big)\big]^2}{x_j^{(n)} - x_{j-1}^{(n)}} = \frac{\big[\int_{x_{j-1}^{(n)}}^{x_j^{(n)}} D_sG(s)ds\big]^2}{x_j^{(n)} - x_{j-1}^{(n)}} \leq \int_{x_{j-1}^{(n)}}^{x_j^{(n)}} D_sG(s)^2 ds\,.$$

From (5.8) one derives the last issue by observing that $dm_n(F^2) = dm_n(G^2)$ ($dm_n(\cdot)$ denoting the average w.r.t. dm_n). □

We have now all the tools in order to prove that the eigenvalues $\lambda_k^{(n)}$ are bounded uniformly in n:

Lemma 5.5. *For each $k \geq 1$, it holds $\sup_{n>k} \lambda_k^{(n)} =: a(k) < \infty$.*

Proof. Given a function $f \in C_0[0, \ell_n]$ and $n \geq 1$, we define $K_n f$ as the unique function in $\mathcal{B}(n)$ such that $f(x_j^{(n)}) = K_n f(x_j^{(n)})$ for all $0 \leq j \leq n$. Note that K_n commutes with linear combinations: $K_n(a_1 f_1 + \cdots + a_k f_k) = a_1 K_n f_1 + \cdots + a_k K_n f_k$.

Due to the assumption that dm is not a linear combination of a finite number of delta measures, for some $\varepsilon > 0$ we can divide the interval $[0, \ell - \varepsilon)$ in k subintervals $I_j = [a_j, b_j)$ such that $dm(\text{int}(I_j)) > 0$, $\text{int}(I_j) = (a_j, b_j)$.

Since dm_n converges to dm weakly, it must be $dm_n(\text{int}(I_j)) > 0$ for all $j : 1 \leq j \leq k$, and for n large enough. For each j we fix a piecewise–linear function $f_j : \mathbb{R} \to \mathbb{R}$, with support in I_j and strictly positive on $\text{int}(I_j)$. Since $\ell_n \to \ell > \ell - \varepsilon$, taking n large enough, all functions f_j are zero outside $(0, \ell_n)$, hence we can think of f_j as function in $C_0[0, \ell_n]$. Having disjoint supports, the functions f_1, f_2,..., f_k are independent in $C_0[0, \ell_n]$.

Trivially $K_n f_1$, $K_n f_2$,..., $K_n f_k$ are independent functions in $\mathcal{B}(n)$ for n large enough since $dm_n(\text{int}(I_j)) > 0$ for all j if n is large enough. Due to the above independence, we can apply the min–max principle (5.6). Let us write S_k for the real vector space spanned by $K_n f_1, K_n f_2, \ldots, K_n f_k$ and $\bar{S}_k$ for the real vector space spanned by $f_1, f_2, \ldots, f_k$. As already observed, $S_k = K_n(\bar{S}_k)$. Using also Lemma 5.4, we conclude that for n large enough

$$\lambda_k^{(n)} \leq \max\{\Phi_n(f) : f \in S_k\,,\, dm_n(f^2) > 0\} \leq \max\{\Phi_n(f) : f \in \bar{S}_k\,,\, dm_n(f^2) > 0\}\,.$$

Take $f = a_1 f_1 + a_2 f_2 + \cdots + a_k f_k$ such that $dm_n(f^2) > 0$. Since $\Phi_n(f) = \Phi_n(cf)$, without loss of generality we can assume that $\sum_{i=1}^k a_i^2 = 1$. Since the functions f_j have disjoint supports, it holds $(D_s f)^2 = \sum_{j=1}^k a_j^2 (D_s f_j)^2$ a.e., while $f^2 = \sum_{j=1}^k a_j^2 f_j^2$. In particular, we can write

$$\Phi_n(f) = \frac{\sum_{j=1}^k a_j^2 \int_0^{\ell_n} D_s f_j(s)^2 ds}{\sum_{j=1}^k a_j^2 \int_0^{\ell_n} f_j(s)^2 dm_n(s)}\,. \tag{5.9}$$

Hence, for n large enough, it holds

$$\lambda_k^{(n)} \leq \frac{\max\{\int_0^\ell D_s f_j(s)^2 ds : 1 \leq j \leq k\}}{\min\{\int_0^\ell f_j(s)^2 dm_n(s) : 1 \leq j \leq k\}}\,. \tag{5.10}$$

The conclusion is now trivial. □

5.3 Proof of Theorem 2.1

Most of the work necessary for the convergence of the eigenvalues has been done for proving Lemma 5.3 and Lemma 5.5. Due to Lemma 4.1, we know that the eigenvalues of $-\mathbb{L}_n$ and the eigenvalues of the differential operator $-D_m D_x$ with Dirichlet conditions outside $(0, \ell)$ are simple, positive and form a set without accumulation points. Since $-\mathbb{L}_n$ is a symmetric operator on the $(n-1)$–dimensional space $L^2((0,1)\cap\mathbb{Z}_n, \mu_n)$, where μ_n has been introduced in Section 5.2, we conclude that $-\mathbb{L}_n$ has $n-1$ eigenvalues.

Given $k \geq 1$ we take $a(k)$ as in Lemma 5.5 and we fix $L \geq a(k)$ such that L is not an eigenvalue of $-D_m D_x$ with Dirichlet conditions. Let k_0, ε and n_0 be as in Lemma 5.3. Then for $n \geq n_0$ the following holds: in each interval $J_i = [\lambda_i - \varepsilon, \lambda_i + \varepsilon]$ there is exactly one eigenvalue of $-\mathbb{L}_n$ and in $[0, L) \setminus \cup_{i=1}^{k_0} J_i$ there is no eigenvalue of $-\mathbb{L}_n$. Since we know by Lemma 5.5 that $-\mathbb{L}_n$ has at least k eigenvalues in $[0, L]$ it must be $k \leq k_0$ and $\lambda_i^{(n)} \in J_i$ for all $i : 1 \leq i \leq k$. In particular, $\limsup_{n\uparrow\infty} |\lambda_i^{(n)} - \lambda_i| \leq \varepsilon$ for all $i : 1 \leq i \leq k$. Using the arbitrariness of ε and k we conclude that the operator $-D_m D_x$ with Dirichlet conditions outside $(0, \ell)$ has infinite eigenvalues satisfying (2.12). Knowing that $\lambda_k^{(n)} \to \lambda_k$ as $n \to \infty$, the convergence from the eigenfunction $\psi(\cdot, \lambda_k^{(n)})$ to $\psi(\cdot, \lambda_k)$, as specified in the theorem, follows from Lemma 5.2.

6 Dirichlet–Neumann bracketing

Let $m : \mathbb{R} \to [0, \infty)$ be a càdlàg nondecreasing function with $m(x) = 0$ for all $x < 0$. We recall that E_m denotes the support of dm, i.e. the set of points where m increases (see (2.6)) and that m_x denotes the magnitude of the jump of the function m at the point x, i.e. $m_x := m(x+) - m(x-) = m(x) - m(x-)$. We suppose that $E_m \neq \emptyset$, $0 = \inf E_m$ and $\ell_m := \sup E_m < \infty$. We want to compare the eigenvalue counting function for the generalized operator $-D_x D_m$ with Dirichlet boundary conditions to the same function when taking Neumann boundary conditions. In order to apply the Dirichlet–Neumann bracketing as stated in Section XIII.15 of [36] and as developed by Métivier and Lapidus (cf. [32] and [28]), we need to study generalized differential operators as self–adjoint operators on suitable Hilbert spaces.

In the rest of the section we assume that $m_0 = m_{\ell_m} = 0$. The reason will become clear soon. We consider the real Hilbert space $\mathcal{H} := L^2([0, \ell_m], dm)$ and denote its scalar product as $(\cdot, \cdot)$. When writing $\int dm(y)g(y)$ we mean $\int_{[0,\ell_m]} dm(y)g(y)$.

6.1 The operator $-\mathcal{L}_D$

We define the operator $-\mathcal{L}_D : \mathcal{D}(-\mathcal{L}_D) \subset \mathcal{H} \to \mathcal{H}$ as follows. First, we set that $f \in \mathcal{D}(-\mathcal{L}_D)$ if there exists a function $g \in \mathcal{H}$ such that

$$f(x) = bx - \int_0^x dy \int_{[0,y)} dm(z)g(z) \,, \qquad b := \frac{1}{\ell_m} \int_0^{\ell_m} dy \int_{[0,y)} dm(z)g(z) \,. \tag{6.1}$$

We note that the above identity implies that f has a representative given by a continuous function in $C[0, \ell_m]$ such that $f(0) = f(\ell_m) = 0$. Moreover, by the discussion following (4.2) (cf. (4.3) and (4.4)) and the assumption $m_0 = m_{\ell_m} = 0$, we derive from identity (6.1) that the function $g \in \mathcal{H}$ satisfying (6.1) is unique. Hence, we define $-\mathcal{L}_D f = g$. Always due to (4.3) and (4.4), we know that if $f \in \mathcal{D}(-\mathcal{L}_D)$, then f has right derivative $D_x^+ f$ on $[0, \ell_m)$, f has left derivative $D_x^- f$ on $(0, \ell_m]$ and has derivative $D_x f$ on $(0, \ell_m)$ apart a countable set of points. In particular, f has derivative Lebesgue a.e. on $(0, \ell_m)$. The operator $-\mathcal{L}_D$ is simply the operator $-D_x D_m$ with Dirichlet boundary conditions thought on the space $\mathcal{H}$.

Proposition 6.1. *The following holds:*

(i) the operator $-\mathcal{L}_D : \mathcal{D}(-\mathcal{L}_D) \subset \mathcal{H} \to \mathcal{H}$ is self–adjoint;
(ii) consider the symmetric compact operator $\mathcal{K} : \mathcal{H} \to \mathcal{H}$ defined as

$$\mathcal{K}g(x) = \int K(x,y)g(y)dm(y)\,, \qquad g \in \mathcal{H}\,, \tag{6.2}$$

where the function $K(x,y) := G_{0,\ell_m}(x,y)$ is given by (4.10). Then, $Ran(\mathcal{K}) = \mathcal{D}(-\mathcal{L}_D)$ and $-\mathcal{L}_D \circ \mathcal{K} = \mathbb{I}$ on $\mathcal{H}$. In particular, the operator $-\mathcal{L}_D$ admits a complete orthonormal set of eigenfunctions and therefore $-\mathcal{L}_D$ has pure point spectrum. Moreover, the above eigenvalues and eigenfunctions coincide with the ones in Lemma 4.1.

Proof. It is trivial to check that (6.1) can be rewritten as

$$f(x) = \int K(x,y)g(y)dm(y)\,. \tag{6.3}$$

Hence, by definition $\mathcal{D}(-\mathcal{L}_D) = Ran(\mathcal{K})$ and $\mathcal{L}_D(\mathcal{K}(g)) = g$ for all $g \in \mathcal{H}$ and $\mathcal{K}$ is injective (see the discussion on the well definition of $-\mathcal{L}_D$). Since $K(x,y) = K(y,x)$, the operator $\mathcal{K}$ is symmetric. Since $K \in L^2(dm \otimes dm)$ (K is bounded and dm has finite mass), by [35][Theorem VI.23] $\mathcal{K}$ is an Hilbert–Schmidt operator and therefore is compact (cf. [35][Theorem VI.22]). In particular, $\mathcal{H}$ has an orthonormal basis $\{\psi_n\}$ such that $\mathcal{K}\psi_n = \gamma_n\psi_n$ for suitable eigenvalues γ_n (cf. Theorems VI.16 in [35]). Since $\mathcal{K}$ is injective, we conclude that $\gamma_n \neq 0$, $\psi_n = \mathcal{K}((1/\gamma_n)\psi_n) \in Ran(\mathcal{K}) = \mathcal{D}(-\mathcal{L}_D)$ and $-\mathcal{L}_D\psi_n = (1/\gamma_n)\psi_n$. It follows that $\{\psi_n\}$ is an orthonormal basis of eigenvectors of $-\mathcal{L}_D$. By (6.1), the function $\psi_n \in L^2(dm)$ must have a representative in $C[0,\ell_m]$. Taking this representative, the identity $\psi_n = -(1/\gamma_n)\mathcal{L}_D\psi_n$ simply means that ψ_n is an eigenfunction with eigenvalue $1/\gamma_n$ of the generalized differential operator $-D_xD_m$ with Dirichlet boundary conditions as defined in Section 4. Finally, since $-\mathcal{L}_D$ admits an orthonormal basis of eigenvectors, its spectrum is pure point and is given by the family of eigenvalues. This concludes the proof of point (ii).

In order to prove (i), we observe that $\mathcal{D}(-\mathcal{L}_D)$ contains the finite linear combinations of the orthonormal basis $\{\psi_n\}$ and therefore it is a dense subspace in $\mathcal{H}$. Given $f,\hat{f} \in \mathcal{D}(-\mathcal{L}_D)$, let $g,\hat{g} \in \mathcal{H}$ such that $f = \mathcal{K}g$, $\hat{f} = \mathcal{K}\hat{g}$. Then, using the symmetry of $\mathcal{K}$ and point (ii), we obtain $(-\mathcal{L}_Df,\hat{f}) = (g,\mathcal{K}\hat{g}) = (\mathcal{K}g,\hat{g}) = (f,-\mathcal{L}_D\hat{f})$. This proves that $-\mathcal{L}_D$ is symmetric. In order to prove that it is self–adjoint we need to show that, given $v,w \in \mathcal{H}$ such that $(-\mathcal{L}_Df,v) = (f,w)$ for all $f \in \mathcal{D}(-\mathcal{L}_D)$, it must be $v \in \mathcal{D}(-\mathcal{L}_D)$ and $-\mathcal{L}_Dv = w$. To this aim, we write $g = -\mathcal{L}_Df$. Then, by the symmetry of $\mathcal{K}$, it holds $(g,v) = (-\mathcal{L}_Df,v) = (f,w) = (\mathcal{K}g,w) = (g,\mathcal{K}w)$. Since this holds for any $f \in \mathcal{D}(-\mathcal{L}_D)$ and therefore for any $g \in \mathcal{H}$, it must be $v = \mathcal{K}w$. By point (ii), this is equivalent to the fact that $w \in \mathcal{D}(-\mathcal{L}_D)$ and $w = -\mathcal{L}_Dv$. This concludes the proof of (i). □

6.2 The operator $-\mathcal{L}_N$

We define the operator $-\mathcal{L}_N : \mathcal{D}(-\mathcal{L}_N) \subset \mathcal{H} \to \mathcal{H}$ as follows. First, we say that $f \in \mathcal{D}(-\mathcal{L}_N)$ if there exist a function $g \in \mathcal{H}$ and a constant $a \in \mathbb{R}$ such that

$$f(x) = a - \int_0^x dy \int_{[0,y)} dm(z)g(z) \tag{6.4}$$

and

$$\int_{[0,\ell_m)} dm(z)g(z) = 0\,. \tag{6.5}$$

We note that the above identity implies that f has a representative given by a continuous function in $C[0,\ell_m]$. Moreover, by the discussion following (4.2) (cf. (4.3) and (4.4)) and the assumption $m_0 = m_{\ell_m} = 0$, we derive from identity (6.4) that the function $g \in \mathcal{H}$ satisfying (6.4) is unique. Hence, we define $-\mathcal{L}_N f = g$. Always due to (4.3) and (4.4), we know that if $f \in \mathcal{D}(-\mathcal{L}_N)$, then f has right derivative $D_x^+ f$ on $[0, \ell_m)$, f has left derivative $D_x^- f$ on $(0, \ell_m]$ and has derivative $D_x f$ on $(0, \ell_m)$ apart a countable set of points. In addition, $D_x^+ f(0)$ and $D_x^- f(\ell_m)$ are zero due to (6.4) and (6.5). The operator $-\mathcal{L}_D$ is simply the operator $-D_x D_m$ with Neumann boundary conditions thought of on the space $\mathcal{H}$.

Proposition 6.2. *The following holds:*

(i) the operator $-\mathcal{L}_N : \mathcal{D}(-\mathcal{L}_N) \subset \mathcal{H} \to \mathcal{H}$ is self–adjoint;

(ii) the operator $-\mathcal{L}_N$ admits a complete orthonormal set of eigenfunctions and therefore $-\mathcal{L}_N$ has only pure point spectrum. The eigenvalues and eigenfunctions are the same as the ones associated to the operator $-D_x D_m$ with Neumann boundary conditions as defined in Section 4.

Proof. We start with point (i). First we prove that $-\mathcal{L}_N$ is symmetric. Take f, g, a as in (6.4) and (6.5), and take $\hat{f}, \hat{g}, \hat{a}$ similarly. Then,

$$(f, -\mathcal{L}_N \hat{f}) = \int dm(x) f(x) \hat{g}(x) = a \int dm(dx) \hat{g}(x) - \int dm(x) \hat{g}(x) \int_0^x dy \int_{[0,y)} dm(z) g(z) \,.$$

Using that $\int dm(x)\hat{g}(x) = 0$ by (6.5), we conclude that

$$(f, -\mathcal{L}_N \hat{f}) = \int dm(x) \int dm(z) \hat{g}(x) g(z) \mathbb{I}_{z \le x} (z - x) \,.$$

Since, by (6.5) and its analogous version for $\hat{g}$, it holds $\int dm(x) \int dm(z) g(x) \hat{g}(z)(z-x) = 0$, we can rewrite the above expression in the symmetric form

$$(f, -\mathcal{L}_N \hat{f}) = -\frac{1}{2} \int dm(x) \int dm(z) \hat{g}(x) g(z) |x - z| \,, \tag{6.6}$$

which immediately implies that $-\mathcal{L}_N$ is symmetric.

Let us consider the Hilbert subspace $\mathcal{W} = \{f \in \mathcal{H} : (1, f) = 0\}$, namely $\mathcal{W}$ is the family of functions in $\mathcal{H}$ having zero mean w.r.t. dm. Then we define the operator $T : \mathcal{H} \to \mathcal{H}$ as

$$Tg(x) = -\int_0^x dy \int_{[0,y)} dm(z) g(z) = \int dm(z) g(z)(z - x) \mathbb{I}_{0 \le z \le x} \,. \tag{6.7}$$

Finally, we write $P : \mathcal{H} \to \mathcal{W}$ for the orthogonal projection of $\mathcal{H}$ onto $\mathcal{W}$: $Pf = f - (1, f)/(1, 1)$. Note that $[P \circ T]g(x) = \int dm(z) g(z) H(x, z)$, where

$$H(x, z) = (z - x) \mathbb{I}_{0 \le z \le x} - \int_{(z, \ell_m)} dm(u)(z - u) \Big/ \int dm(u)$$

Since $H \in L^2(dm \otimes dm)$, due to [35][Theorem VI.23] $P \circ T$ is an Hilbert–Schmidt operator on $\mathcal{H}$, and therefore a compact operator. In particular, the operator $W : \mathcal{W} \to \mathcal{W}$ defined as the restriction of $P \circ T$ to W is again a compact operator. We claim that W is symmetric. Indeed, setting $f = Wg$ and $f' = Wg'$, due to the first identity in (6.7) we get that $f, f' \in \mathcal{D}(-\mathcal{L}_N)$ and $-\mathcal{L}_N f = g$, $-\mathcal{L}_N f' = g'$. Then, using that $\mathcal{L}_N$ is symmetric as proven above, we conclude

$$(Wg, g') = (f, -\mathcal{L}_N f') = (-\mathcal{L}_N f, f') = (g, Wg') \,.$$

Having proved that W is a symmetric compact operator, from [35][Theorem VI.16] we derive that $\mathcal{W}$ has an orthonormal basis $\{\psi_n\}_n$ of eigenvectors of W, i.e. $W\psi_n = \gamma_n\psi_n$ for suitable numbers γ_n. Since W is injective (recall the discussion on the well definition of $-\mathcal{L}_N$), it must be $\gamma_n \neq 0$. From the identity $W\psi_n = \gamma_n\psi_n$ we conclude that

$$\psi_n(x) = a_n - \frac{1}{\gamma_n}\int_0^x dy \int_{[0,y)} dm(z)\psi_n(z)$$

for some constant $a_n \in \mathbb{R}$. The above identity implies that $\psi_n \in \mathcal{D}(-\mathcal{L}_N)$ and $-\mathcal{L}_N\psi_n = (1/\gamma_n)\psi_n$. On the other hand $1 \in \mathcal{D}(-\mathcal{L}_N)$ and $-\mathcal{L}_N 1 = 0$. Since $\mathcal{H} = \{c : c \in \mathbb{R}\} \oplus \mathcal{W}$, we obtain that $\mathcal{H}$ admits an orthonormal basis of eigenvectors of $-\mathcal{L}_N$. This also implies that $-\mathcal{L}_N$ has only pure point spectrum. Trivially, all eigenvectors (as all elements in $\mathcal{D}(-\mathcal{L}_N)$) are continuous and are eigenvectors of $-D_xD_m$ with Neumann b.c. in the sense of Section 4. This concludes the proof of (i) and (ii). □

6.3 The quadratic forms q_D and q_N

Consider now the symmetric form q_N on $\mathcal{H}$ with domain $Q(q_N)$ given by the elements $f \in \mathcal{H}$ having a representative f which satisfies

(A1) f is absolutely continuous on $[0, \ell_m]$,

(A2) $\int_0^{\ell_m} D_x f(x)^2 dx < \infty$,

(A3) $D_x f$ is constant on each connected component of $(0, \ell_m) \setminus \text{supp}(dm)$, $\text{supp}(dm)$ being the support of the measure dm.

and such that $q_N(f, \hat{f}) = \int_0^{\ell_m} D_x f(x) D_x \hat{f}(x) dx$ for all $f, \hat{f} \in Q(q_N)$. In addition, we set $q_N(f) := q_N(f, f)$. We point out that one cannot apply directly the theory discussed in [18][Example 1.2.2] since the fundamental condition (1.1.7) there can be violated in our setting. Some care is necessary. First of all we need to prove that q_N is well defined:

Lemma 6.3. *The representative f satisfying the above properties (A1),(A2),(A3) is unique. In particular the form q_N is well defined.*

Proof. Take two functions $f, \hat{f}$ on $[0, \ell_m]$ satisfying the above properties (A1),(A2),(A3) and such that $f = \hat{f}$ dm–a.e. We denote by $\mathcal{C}$ the support of dm. We first show that $f = \hat{f}$ on $\mathcal{C}$. Suppose that $x \in \mathcal{C}$. Then for each $\varepsilon > 0$ the set $I_\varepsilon := (x - \varepsilon, x + \varepsilon) \cap [0, \ell_m]$ has positive dm–measure and therefore there exists $x_\varepsilon \in I_\varepsilon$ such that $f(x_\varepsilon) = \hat{f}(x_\varepsilon)$ (otherwise f and $\hat{f}$ would differ on a set having positive dm–measure). By taking the limit $\varepsilon \downarrow 0$ and by continuity (property (A1)) we conclude that $f(x) = \hat{f}(x)$ as claimed. Consider now the open set $[0, \ell_m] \setminus \mathcal{C}$ and take one of its connected components (a, b) (recall that $0, \ell_m \in \mathcal{C}$). By property (A3) it must be $f(x) - \hat{f}(x) = c_0 x + c_1$ on (a, b) for a suitable constants c_0, c_1. Since $a, b \in \mathcal{C}$ and $f = \hat{f}$ on $\mathcal{C}$ we get that $c_0 = c_1 = 0$, thus proving that $f = \hat{f}$ on (a, b). This allows to conclude. □

Below, when handling with $f \in Q(q_N)$ it will be understood that we refer to the representative satisfying the above properties (A1),(A2),(A3).

Lemma 6.4. *The form q_N is closed. Equivalently, the space $Q(q_N)$ endowed of the scalar product*

$$(f, g)_1 = q_N(f, g) + (f, g), \qquad f, g \in Q(q_N)$$

is an Hilbert space.

Proof. Take a $\|\cdot\|_1$–Cauchy sequence $(f_n)_{n\geq 0}$ in $Q(q_N)$. Since $D_x f_n$ is Cauchy in $L^2(dx)$, it converges to some function $u \in L^2(dx)$. Therefore, due top Schwarz inequality,

$$f_n(x) - f_n(0) = \int_0^x D_x f_n(z)dz \to \int_0^x u(z)dz := g(x) \tag{6.8}$$

uniformly in $x \in [0, \ell_m]$. Since $(f_n)_{n\geq 0}$ is a Cauchy sequence in $\mathcal{H}$, we have that f_n converges to some f in $\mathcal{H}$. Having $f_n - f_n(0) \to g$ uniformly and therefore in $\mathcal{H}$, and $f_n \to f$ in $\mathcal{H}$, it must be $f_n(0) \to f - g$ in $\mathcal{H}$. In particular, the sequence of numbers $f_n(0)$ converges to $\int (f-g)dm / \int dm$. This result implies that f_n converges uniformly to the absolutely continuous function $h := g + \int (f-g)dm/\int dm$ on $[0, \ell_m]$ such that $D_x h = u$. In particular, h must be linear on the connected components of $[0,\ell_m] \setminus \text{supp}(dm)$. Hence $h \in Q(q_N)$ and, due to the previous considerations, f_n converges to h w.r.t. the norm $\|\cdot\|_1$. ☐

Finally, we define another symmetric form q_D on $\mathcal{H}$ with domain

$$Q(q_D) := \{f \in Q(q_N)\,:\, f(0) = f(\ell_m) = 0\} \tag{6.9}$$

setting $q_D(f, \hat f) := q_N(f, \hat f) = \int_0^{\ell_m} D_x f(x) D_x \hat f(x) dx$.

To each closed symmetric form on $\mathcal{H}$ one associates in a canonical way a nonnegative definite self–adjoint operator on $\mathcal{H}$ (see [18][Theorem 1.3.1],[35][Chapter VIII].

Lemma 6.5. *The following holds:*

- *(i) The forms q_N, q_D are the canonical closed symmetric forms associated to $-\mathcal{L}_N, -\mathcal{L}_D$, respectively.*
- *(ii) $Q(q_D)$ is a closed subspace of the Hilbert space $\big(Q(q_N), (\cdot,\cdot)_1\big)$ with codimension* 2.
- *(iii) The inclusion map*

$$\iota : \big(Q(q_N), \|\cdot\|_1\big) \ni f \to f \in \big(\mathcal{H}, \|\cdot\|\big)$$

is a continuous compact operator.

Proof. Item (i). We first focus on $q_N, -\mathcal{L}_N$. We already know that q_N is a closed symmetric form. Trivially, $\mathcal{D}(-\mathcal{L}_N)$ is included in $Q(q_N)$. We claim that

$$(-\mathcal{L}_N f, v) = \int_0^{\ell_m} D_x f(x) D_x v(x) dx\,, \qquad \forall f \in \mathcal{D}(-\mathcal{L}_N)\,,\ v \in Q(q_N)\,. \tag{6.10}$$

By Proposition 6.2 the operator $-\mathcal{L}_N$ is self–adjoint, while by the above claim it is also symmetric and nonnegative definite. Moreover, our claim (6.10) together with [18][Corollary 1.3.1] implies that q_N is canonically associated to $-\mathcal{L}_N$.

To prove (6.10) assume (6.4) and (6.5) with $g \in \mathcal{H}$. Then $D_x f(x) = -\int_{[0,x)} dm(z) g(z)$ and

$$\begin{aligned}\int_0^{\ell_m} D_x f(x) D_x v(x) dx &= -\int_0^{\ell_m} dx D_x v(x) \int_{[0,x)} dm(z) g(z) \\ &= -\int_{[0,\ell_m)} dm(z) g(z) \int_{(z,\ell_m]} dx D_x v(x) = \int_{[0,\ell_m)} dm(z) g(z) \big(v(z) - v(\ell_m)\big) \\ &= (g, v) = (-\mathcal{L}_N f, v)\,.\end{aligned}$$

Note that in the forth identity we used (6.5).

Let us now prove the correspondence between q_D and $-\mathcal{L}_D$. First we show that q_D is closed. To this aim, take $f_n \in Q(q_D)$ such that f_n is $\|\cdot\|_1$–Cauchy. Since q_N is closed, we know that there exists $f \in Q(q_N)$ with $\|f - f_n\|_1 \to 0$ as $n \to \infty$. Reasoning as in Lemma 6.4, we deduce that f_n converges to f in the uniform norm, thus implying that $f(0) = f(\ell_m) = 0$. This proves the closeness of q_D.

Knowing that q_D is a closed symmetric form and reasoning as for $q_N, -\mathcal{L}_N$, to conclude we only need to show that

$$(-\mathcal{L}_D f, v) = \int_0^{\ell_m} D_x f(x) D_x v(x) dx \,, \qquad \forall f \in \mathcal{D}(-\mathcal{L}_D)\,,\ v \in Q(q_D)\,. \tag{6.11}$$

To this aim we assume (6.1) for some $g \in \mathcal{H}$. Then

$$\begin{aligned}\int_0^{\ell_m} D_x f(x) D_x v(x) dx &= -\int_0^{\ell_m} dx D_x v(x) \int_{[0,x)} dm(z) g(z) \\ &= -\int_{[0,\ell_m)} dm(z) g(z) \int_{(z,\ell_m]} dx D_x v(x) = (g, v) = (-\mathcal{L}_D f, v)\,.\end{aligned}$$

Note that in the first identity and in the third one we used that $v(0) = v(\ell_m) = 0$.

Item (ii). The thesis follows from item (i), the definition of $Q(q_N)$ and $Q(q_D)$.

Item (iii). Since $\|f\| \leq \|f\|_1$ for each $f \in Q(q_N)$, the inclusion map ι is trivially continuous. In order to prove compactness, we need to show that each sequence $f_n \in Q(q_N)$ with $\|f_n\|_1 \leq 1$ admits a subsequence f_{n_k} which converges in $\mathcal{H}$. Since $\|f_n\|_1 \leq 1$ it holds $|f_n(x) - f_n(y)| \leq \sqrt{y - x}$ for all $x, y \in [0, \ell_m]$. Applying Ascoli–Arzelà Theorem, we then conclude that f_n admits a subsequence f_{n_k} which converges in the space $C([0, \ell_m])$ endowed of the uniform norm. Trivially, this implies the convergence in $\mathcal{H}$. □

As a consequence of the above result we get that

$$0 \leq -\mathcal{L}_N \leq -\mathcal{L}_D \tag{6.12}$$

according to the definition on [36][page 269]. For the reader's convenience and for later use, we recall the definition given in [36][page 269]: given nonnegative self–adjoint operators A, B, where A is defined on a dense subset of a Hilbert space $\mathcal{H}'$ and B is defined on a dense subset of a Hilbert subspace $\mathcal{H}'_1 \subset \mathcal{H}'$, one says that $0 \leq A \leq B$ if (i) $Q(q_A) \supset Q(q_B)$, and (ii) $0 \leq q_A(\psi) \leq q_B(\psi)$ for all $\psi \in Q(q_B)$, where $Q(q_A)$ and $Q(q_B)$ denote the domains of the quadratic forms q_A and q_B associated to the operators A and B, respectively.

Considering the space $Q(q_N)$ endowed of the scalar product $(\cdot,\cdot)_1$, the above Lemma 6.5 implies that $\big(Q(q_N), \mathcal{H}, q_N(\cdot,\cdot)\big)$ is a variational triple (cf. [32][Section II-2]). Indeed, the following holds: (i) $Q(q_N)$ and $\mathcal{H}$ are Hilbert spaces, (ii) the inclusion map gives a continuous injection of $Q(q_N)$ into $\mathcal{H}$, (iii) $q_N(\cdot,\cdot)$ is a continuous scalar product on $Q(q_N)$ since $|q_N(f,g)| \leq \|f\|_1 \|g\|_1$ for all $f, g \in Q(q_N)$, (iv) the scalar product $q_N(\cdot,\cdot)$ is coercive with respect to $\mathcal{H}$: $\|f\|_1^2 - \|f\|^2 \leq q_N(f,f)$ for all $f \in Q(q_N)$.

We denote by $\mathcal{N}_{D,m}^{[0,\ell_m]}(x)$ the number of eigenvalues of $-\mathcal{L}_D$ not larger than x. Similarly we define $\mathcal{N}_{N,m}^{[0,\ell_m]}(x)$. By Lemma 6.5 the inclusion map $\iota : Q(q_N) \hookrightarrow \mathcal{H}$ is compact and $Q(q_D)$ is a closed subspace in $Q(q_N)$. Applying Proposition 2.9 in [32] we get the equality $\mathcal{N}_{m,N}^{[0,\ell_m]}(x) = N(x; Q(q_N), \mathcal{H}, q_N)$ and $\mathcal{N}_{m,N}^{[0,\ell_m]}(x) = N(x; Q(q_D), \mathcal{H}, q_D)$, where the functions $N(x; Q(q_N), \mathcal{H}, q_N)$ and $N(x; Q(q_D), \mathcal{H}, q_D)$ are defined in [32][Page 131]. As byproduct of Lemma 6.5, Proposition 2.7 in [32] and the arguments used in Corollary 4.7 in [24], we obtain that

$$\mathcal{N}_{D,m}^{[0,\ell_m]}(x) \leq \mathcal{N}_{N,m}^{[0,\ell_m]}(x) \leq \mathcal{N}_{D,m}^{[0,\ell_m]}(x) + 2\,, \qquad \forall x \geq 0\,. \tag{6.13}$$

We point out that the first inequality follows also from (6.12) and the lemma preceding Proposition 4 in [36][Section XIII.15].

Up to now we have defined $-\mathcal{L}_D$ and $-\mathcal{L}_N$ referring to the interval $(0,\ell_m)$, where $0=\inf E_m$, $\ell_m=\sup E_m$, $m_0=0$ and $m_{\ell_m}=0$. In general, given an open interval $I=(u,v)\subset(0,\ell_m)$, such that

$$m_u=m_v=0,\qquad dm\big((u,u+\varepsilon)\big)>0 \text{ and } dm\big((v-\varepsilon,v)\big)>0\ \forall\varepsilon>0\,, \tag{6.14}$$

we define $-\mathcal{L}_D^I,-\mathcal{L}_N^I$ as the operators $-\mathcal{L}_D$ and $-\mathcal{L}_N$ but with the measure dm replaced by its restriction to I. For simplicity, we write $L^2(I,dm)$ for the space $L^2(I,\widetilde{dm})$ where $\widetilde{dm}$ denotes the restriction of dm to the interval I. Then, $f\in\mathcal{D}(-\mathcal{L}_D^I)\subset L^2(I,dm)$ if and only if there exists $g\in L^2(I,dm)$ such that, writing $I=(u,v)$,

$$f(x)=b(x-u)-\int_u^x dy\int_{[u,y)}dm(z)g(z)\,,\qquad\forall x\in I\,,$$

where $b=(v-u)^{-1}\int_u^v dy\int_{[u,y)}dm(z)g(z)$. The above $g\in L^2(I,dm)$ is unique and one sets $-\mathcal{L}_D^I f=g$. The definition is similar for $-\mathcal{L}_N^I$. Propositions 6.1 and 6.2 extend trivially to $-\mathcal{L}_D^I$ and $-\mathcal{L}_N^I$. We write q_D^I,q_N^I for the corresponding quadratic forms. Finally, for $x\geq0$ we define

$$\mathcal{N}_{m,D}^I(x):=\sharp\{\lambda\in\mathbb{R}:\lambda\leq x,\ \lambda \text{ is eigenvalue of } -\mathcal{L}_D^I\}\,, \tag{6.15}$$

$$\mathcal{N}_{m,N}^I(x):=\sharp\{\lambda\in\mathbb{R}:\lambda\leq x,\ \lambda \text{ is eigenvalue of } -\mathcal{L}_N^I\}\,. \tag{6.16}$$

Lemma 6.6. *Let $I_1=(a_1,b_1)$,...,$I_k=(a_k,b_k)$ be a finite family of disjoint open intervals, where $a_1<b_1\leq a_2<b_2\leq a_3<\cdots\leq a_k<b_k$ and*

$$m_{a_r}=0\,,\ \ m_{b_r}=0\ \ \forall r=1,\dots,k\,,$$
$$dm\big((a_r,a_r+\varepsilon)\big)>0\,,\ \ dm\big((b_r-\varepsilon,b_r)\big)>0\ \ \forall\varepsilon>0,\forall r=1,\dots k\,.$$

Then for any $x\geq0$ it holds $\mathcal{N}_{m,D}^{(a_1,b_k)}(x)\geq\sum_{r=1}^k\mathcal{N}_{m,D}^{(a_r,b_r)}(x)$. If in addition the intervals I_r are neighboring, i.e. $b_r=a_{r+1}$ for all $r=1,\dots,k-1$, then for any $x\geq0$ it holds $\mathcal{N}_{m,N}^{(a_1,b_k)}(x)\leq\sum_{r=1}^k\mathcal{N}_{m,N}^{(a_r,b_r)}(x)$.

The above result is the analogous to Point c) in Proposition 4 in [36][Section XIII.15].

Proof. We begin with the superadditivity (w.r.t. unions of intervals) of $\mathcal{N}_{m,D}^{(\cdot)}(x)$. We consider the direct sum $\oplus_{r=1}^k L^2(I_r,dm)$. We define $A=\oplus_{r=1}^k(-\mathcal{L}_D^{I_r})$ as the operator with domain

$$\mathcal{D}(A)=\oplus_{r=1}^k\mathcal{D}\big(-\mathcal{L}_D^{I_r}\big)\subset\oplus_{r=1}^k L^2(I_r,dm)$$

such that $A\big[(f_r)_{r=1}^k\big]=\big(-\mathcal{L}_D^{I_r}f_r\big)_{r=1}^k$. Due to the properties listed in [36][page 268] and due to Proposition 6.1, the operator A is a nonnegative self–adjoint operator.

Trivially, the map $\psi:\oplus_{r=1}^k L^2(I_r,dm)\to L^2([a_1,b_k],dm)$ where

$$\psi\big[(f_r)_{r=1}^k\big](x)=\begin{cases}f_r(x) & \text{if } x\in I_r \text{ for some } r\,,\\ 0 & \text{otherwise}\,,\end{cases}$$

is injective and conserves the norm. In particular, the image of ψ is a closed (and therefore Hilbert) subspace of $L^2([a_1,b_k],dm)$. Consider, the operator

$$A':\psi(\mathcal{D}(A))\subset\psi\left[\oplus_{r=1}^k L^2(I_r,dm)\right]\to\psi\left[\oplus_{r=1}^k L^2(I_r,dm)\right]\,,$$

defined as $A'(\psi(f)) = \psi(Af)$ for all $f \in \mathcal{D}(A)$. Then, A' is a nonnegative self–adjoint operator. Due to property (3) on page 268 of [36] and the characterization of the form domain $Q(q_D)$, we get that $-\mathcal{L}_D^{(a_1,b_k)} \leq A'$. The superadditivity then follows from the lemma stated in [36][page 270] and property (5) on page 268 of [36].

In order to prove subadditivity of $\mathcal{N}_{m,N}^{(\cdot)}(x)$ under the hypothesis $b_r = a_{r+1}$ for all $r = 1, \dots, k-1$, we first observe that the above map ψ is indeed an isomorphism of Hilbert spaces (recall that $m_{a_r} = 0$ and $m_{b_r} = 0$). From the definition of q_N and Lemma 6.5 it is trivial to check that

$$0 \leq \oplus_{r=1}^k \big(-\mathcal{L}_N^{(a_1,b_k)}\big) \leq \psi^{-1} \circ \big(-\mathcal{L}_N^{(a_1,b_k)}\big) \circ \psi \,,$$

where the operator on the right is simply the self–adjoint operator on $\oplus_{r=1}^k L^2(I_r, dm)$ with domain $\big\{\psi^{-1}(f) : f \in \mathcal{D}\big(-\mathcal{L}_N^{(a_1,b_k)}\big)\big\}$, mapping $\psi^{-1}(f)$ into $\psi^{-1}\big(-\mathcal{L}_N^{(a_1,b_k)} f\big)$. At this point, the subadditivity follows from the Lemma on page 270 of [36] and property (5) on page 268 of [36]. □

6.4 Conclusion

We can now conclude stating the Dirichlet–Neumann bracketing in our context:

Theorem 6.7. *(Dirichlet–Neumann bracketing). Let $I = [a, b]$, let*

$$a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$$

be a partition of the interval I and set $I_r := [a_r, a_{r+1}]$ for $r = 0, \dots, n-1$. Suppose that $m : I \to \mathbb{R}$ is a nondecreasing function such that

- *(i) $m_{a_r} = 0$ for all $r = 0, \dots, n$,*
- *(ii) $dm([a_r, a_r + \varepsilon]) > 0$ for all $r = 0, \dots, n-1$ and $\varepsilon > 0$,*
- *(iii) $dm([a_r - \varepsilon, a_r]) > 0$ for all $r = 1, \dots, n$ and $\varepsilon > 0$.*

Then, for all $x \geq 0$ it holds

$$\mathcal{N}_{m,D}^I(x) \leq \mathcal{N}_{m,N}^I(x) \leq \mathcal{N}_{m,D}^I(x) + 2 \,, \tag{6.17}$$

$$\mathcal{N}_{m,D}^I(x) \geq \sum_{i=0}^{n-1} \mathcal{N}_{m,D}^{I_i}(x) \tag{6.18}$$

$$\mathcal{N}_{m,N}^I(x) \leq \sum_{i=0}^{n-1} \mathcal{N}_{m,N}^{I_i}(x) \,. \tag{6.19}$$

Proof. The bounds in (6.17) have been obtained in (6.13). The inequalities (6.18) and (6.19) follow from Lemma 6.6. □

As immediate consequence of (6.17) and (6.19) we get a bound which will reveal very useful to derive (2.15) and (2.17):

Corollary 6.8. *In the same setting of Theorem 6.7 it holds $\mathcal{N}_{m,D}^I(x) \leq 2n + \sum_{i=0}^{n-1} \mathcal{N}_{m,D}^{I_i}(x)$.*

7 Proof of Theorem 2.2

We first consider how the eigenvalue counting functions change under affine transformations:

Lemma 7.1. *Let $m : \mathbb{R} \to \mathbb{R}$ be a nondecreasing càdlàg function. Given the interval $I = [a,b]$, suppose that $m_a = m_b = 0$ and $dm\big((a,a+\varepsilon)\big) > 0$, $dm\big((b-\varepsilon,b)\big) > 0$ for all $\varepsilon > 0$. Given $\gamma,\beta > 0$, set $J = [\gamma a, \gamma b]$ and define the function $M : \mathbb{R} \to \mathbb{R}$ as $M(x) = \gamma^{1/\beta} m(x/\gamma)$. Then*

$$\mathcal{N}^I_{m,D/N}(x) = \mathcal{N}^J_{M,D/N}(x/\gamma^{1+1/\beta})\,. \tag{7.1}$$

Trivially, $M_{\gamma a} = M_{\gamma b} = 0$ and $dM\big((\gamma a, \gamma a+\varepsilon)\big) > 0$, $dM\big((\gamma b-\varepsilon, \gamma b)\big) > 0$ for all $\varepsilon > 0$

Proof. For simplicity of notation we take $a = 0$. Suppose that λ is an eigenvalue of the operator $-D_m D_x$ on $[0,b]$ with Dirichlet b.c. at 0 and b. This means that for a nonzero function $F \in C(I)$ with $F(b) = 0$ and a constant c it holds

$$F(x) = cx - \lambda \int_0^x dy \int_{[0,y)} dm(z) F(z)\,, \qquad \forall x \in I\,. \tag{7.2}$$

Taking $X \in J$, the above identity implies that

$$F(X/\gamma) = \frac{cX}{\gamma} - \lambda \int_0^{\frac{X}{\gamma}} dy \int_{[0,y)} dm(z)F(z) = \frac{cX}{\gamma} - \frac{\lambda}{\gamma}\int_0^X dY \int_{[0,\frac{Y}{\gamma})} dm(z)F(z) = \frac{cX}{\gamma} - \frac{\lambda}{\gamma^{1+1/\beta}} \int_0^X dY \int_{[0,Y)} dM(Z) F(Z/\gamma)\,. \tag{7.3}$$

Since trivially $F(X/\gamma) = 0$ for $X = b\gamma$, the above identity implies that $\lambda/\gamma^{1+1/\beta}$ is an eigenvalue of the operator $-D_M D_x$ on J with Dirichlet b.c. and eigenfunction $F(\cdot/\gamma)$. This implies (7.1) in the case of Dirichlet b.c. The Neumann case is similar. □

We have now all the tools in order to prove Theorem 2.2:

Proof of Theorem 2.2. Take m as in Theorem 2.2 and recall the notational convention stated after the theorem. We first prove (2.16), assuming without loss of generality that (2.15) holds with $x_0 = 1$. By assumption, with probability one, for any $n \in \mathbb{N}_+$ and any $k \in \mathbb{N} : 0 \leq k \leq n$ it holds: (i) $dm(\{k/n\}) = 0$, (ii) $dm((k/n, k/n+\varepsilon)) > 0$ for all $\varepsilon > 0$ if $k < n$, (iii) $dm((k/n-\varepsilon, k/n)) > 0$ for all $\varepsilon > 0$ if $k > 0$. Below, we assume that the realization of m satisfies (i), (ii) and (iii). This allows us to apply the Dirichlet–Neumann bracketing stated in Theorem 6.7 to the non–overlapping subintervals $I_k := [k/n, (k+1)/n]$, $k \in \{0,1,\dots,n-1\}$. Due to the superadditivity (resp. subadditivity) of the Dirichlet (resp. Neumann) eigenvalue counting functions (cf. (6.18) and (6.19) in Theorem 6.7), we get for any $x \geq 0$ that $N^{[0,1]}_{m,D}(x) \geq \sum_{k=0}^{n-1} N^{I_k}_{m,D}(x)$, while $N^{[0,1]}_{m,N}(x) \leq \sum_{k=0}^{n-1} N^{I_k}_{m,N}(x)$. By taking the average over m and using that m has stationary increments we get that $\mathbb{E}\mathcal{N}^{[0,1]}_{m,D}(x) \geq n\mathbb{E}\mathcal{N}^{[0,1/n]}_{m,D}(x)$ and $\mathbb{E}\mathcal{N}^{[0,1]}_{m,N}(x) \leq n\mathbb{E}\mathcal{N}^{[0,1/n]}_{m,N}(x)$. Using now the scaling property of Lemma 7.1 with $\gamma = n, \beta = \alpha$ and the self–similarity of m, we conclude that

$$\mathbb{E}\mathcal{N}^{[0,1]}_{m,D}(x) \geq n\mathbb{E}\mathcal{N}^{[0,1/n]}_{m,D}(x) = n\mathbb{E}\mathcal{N}^{[0,1]}_{M,D}(x/n^{1+1/\alpha}) = n\mathbb{E}\mathcal{N}^{[0,1]}_{m,D}(x/n^{1+1/\alpha})\,, \tag{7.4}$$

$$\mathbb{E}\mathcal{N}^{[0,1]}_{m,N}(x) \leq n\mathbb{E}\mathcal{N}^{[0,1/n]}_{m,N}(x) = n\mathbb{E}\mathcal{N}^{[0,1]}_{M,N}(x/n^{1+1/\alpha}) = n\mathbb{E}\mathcal{N}^{[0,1]}_{m,N}(x/n^{1+1/\alpha})\,, \tag{7.5}$$

where $M(x) := n^{1/\alpha} m(x/n)$. On the other hand, by (6.17) of Theorem 6.7

$$\mathbb{E}\mathcal{N}^{[0,1]}_{m,D}(x) \leq \mathbb{E}\mathcal{N}^{[0,1]}_{m,N}(x) \leq \mathbb{E}\mathcal{N}^{[0,1]}_{m,D}(x) + 2\,. \tag{7.6}$$

From the above estimates (7.4), (7.5) and (7.6), we conclude that

$$\mathbb{E}\mathcal{N}^{[0,1]}_{m,D}(1) \leq n^{-1}\mathbb{E}\mathcal{N}^{[0,1]}_{m,D}(n^{1+1/\alpha}) \leq n^{-1}\mathbb{E}\mathcal{N}^{[0,1]}_{m,N}(n^{1+1/\alpha}) \leq \mathbb{E}\mathcal{N}^{[0,1]}_{m,N}(1) \leq \mathbb{E}\mathcal{N}^{[0,1]}_{m,D}(1) + 2\,. \tag{7.7}$$

We remark that (2.15) with $x_0 = 1$ simply reads $\mathbb{E}\mathcal{N}_{m,D}^{[0,1]}(1) < \infty$. Since the eigenvalue counting functions are monotone, in the above estimate (7.7) we can think of n as any positive number larger than 1. Then, substituting $n^{1+1/\alpha}$ with x we get (2.16).

In order to prove (2.18), we first prove the joint self–similarity of m, m^{-1}: given $\gamma > 0$, it holds

$$\big(m(x), m^{-1}(y)\,:\, x, y \geq 0\big) \sim \big(\gamma^{1/\alpha} m(x/\gamma), \gamma m^{-1}(\gamma^{-1/\alpha} y)\,:\, x, y \geq 0\big) \sim \big(\gamma m(x/\gamma^{\alpha}), \gamma^{\alpha} m^{-1}(x/\gamma)\,:\, x, y \geq 0\big)\,. \tag{7.8}$$

To check the above claim, first we observe that for each $x \geq 0$ it holds

$$\inf\Big\{t \geq 0\,:\, \gamma^{1/\alpha} m(t/\gamma) > y\Big\} = \gamma \inf\Big\{t \geq 0\,:\, m(t) > \gamma^{-1/\alpha} y\Big\} = \gamma m^{-1}(\gamma^{-1/\alpha} y)\,. \tag{7.9}$$

On the other hand, by the self–similarity of m and by the definition of the generalized inverse function, we get

$$\Big(\gamma^{1/\alpha} m(x/\gamma), \inf\Big\{t \geq 0\,:\, \gamma^{1/\alpha} m(t/\gamma) > y\Big\}\,:\, x, y \geq 0\Big) \sim \big(m(x), m^{-1}(y)\,:\, x, y \geq 0\big)\,. \tag{7.10}$$

The first identity in (7.8) follows from (7.9) and (7.10). The second identity follows by replacing $\gamma^{1/\alpha}$ with γ. This concludes the proof of (7.8).

Recall the convention established after (2.18). We already know that dm^{-1} is a continuous function a.s., hence a.s. it holds (P1) $dm^{-1}\big(\{m(k/n)\}\big) = 0$ for all $n \in \mathbb{N}$ and $k \in \mathbb{N} : 0 \leq k \leq n$. By identity (2.20) $m^{-1}(x) = m^{-1}(y)$ if and only if $x, y \in \big[m(z_i-), m(z_i)\big]$ for some jump point z_i of m. Since by property (iv) in Theorem 2.2 $m(k/n)$ is not a jump point for m a.s. (with k, n as above), the following properties hold a.s.: (P2) $dm^{-1}\big((m(k/n), m(k/n)+\varepsilon)\big) > 0$ for all $\varepsilon > 0$ if $0 \leq k < n$ and (P3) $dm^{-1}\big((m(k/n)-\varepsilon, m(k/n))\big) > 0$ for all $\varepsilon > 0$ if $0 < k \leq n$. In what follows we assume that the realization of m satisfies the properties (P1), (P2) and (P3). This allows us to apply the Dirichlet–Neumann bracketing to the measure dm^{-1} and to the non–overlapping subintervals $I_k = [m(k/n), m((k+1)/n)]$, $k \in \{0, 1, \ldots, n-1\}$. We point out that the measure dm^{-1} restricted to each subinterval I_k is univocally determined by the values $\{m(x) - m(k/n)\,:\, x \in [k/n, (k+1)/n]$. The fact that m has stationary increments, allows to conclude that the random functions $N_{m^{-1},D/N}^{I_k}(\cdot)$ are identically distributed.

We observe now that (7.8) with $\gamma = n^{1/\alpha}$ implies that

$$\big(m(x), m^{-1}(y)\,:\, x, y \geq 0\big) \sim \big(n^{1/\alpha} m(x/n), n m^{-1}(x/n^{1/\alpha})\,:\, x, y \geq 0\big)\,. \tag{7.11}$$

Then, using the Dirichlet–Neumann, Lemma 7.1 with $\beta = 1/\alpha$ and $\gamma = n^{1/\alpha}$ and the joint self–similarity (7.11),we conclude that

$$\mathbb{E}\mathcal{N}_{m^{-1},D}^{[0,m(1)]}(x) \geq n\mathbb{E}\mathcal{N}_{m^{-1},D}^{[0,m(1/n)]}(x) = n\mathbb{E}\mathcal{N}_{M,D}^{[0,n^{1/\alpha}m(1/n)]}(x/n^{1+1/\alpha}) = n\mathbb{E}\mathcal{N}_{m^{-1},D}^{[0,m(1)]}(x/n^{1+1/\alpha})\,, \tag{7.12}$$

$$\mathbb{E}\mathcal{N}_{m^{-1},N}^{[0,m(1)]}(x) \leq n\mathbb{E}\mathcal{N}_{m^{-1},N}^{[0,1/n]}(x) = n\mathbb{E}\mathcal{N}_{M,N}^{[0,n^{1/\alpha}m(1/n)]}(x/n^{1+1/\alpha}) = n\mathbb{E}\mathcal{N}_{m^{-1},N}^{[0,m(1)]}(x/n^{1+1/\alpha})\,, \tag{7.13}$$

where now $M(x) = n m^{-1}(x/n^{1/\alpha})$. Note that (7.12) and (7.13) have the same structure of (7.4) and (7.5), respectively. The conclusion then follows the same arguments used for (2.16). □

8 Proof of Theorem 2.3

As already mentioned in the Introduction, the proof of Theorem 2.3 is based on a special coupling introduced in [16] (and very similar to the coupling of [23] for the random barrier model). If $\tau(x)$ is itself the α–stable law with Laplace transform $\mathbb{E}\big[e^{-\lambda\tau(x)}\big] = e^{-\lambda^\alpha}$, this coupling is very simple since it is enough to define, for each realization of V and for all $n \geq 1$, the random variables $\tau_n(x)$'s as

$$\tau_n(x) = n^{1/\alpha}\Big[V\big(x+\frac{1}{n}\big) - V(x)\Big], \qquad \forall x \in \mathbb{Z}_n\,. \tag{8.1}$$

Due to (2.23) and the fact that V has independent increments, one easily derives that the V–dependent random field $\{\tau_n(x)\,:\, x\in\mathbb{Z}_n\}$ has the same law of $\{\tau(nx)\,:\, x\in\mathbb{Z}_n\}$. In the general case one proceeds as follows. Define a function $G:[0,\infty)\to[0,\infty)$ such that

$$\mathcal{P}(V(1) > G(x)) = \mathbb{P}(\tau(0)>x)\,, \qquad \forall x\geq 0\,.$$

(Recall that V is defined on the probability space $(\Xi,\mathcal{F},\mathcal{P})$.) The above function G is well defined since $V(1)$ has continuous distribution, G is right continuous and nondecreasing. Then the generalized inverse function

$$G^{-1}(t) = \inf\{x\geq 0\,:\, G(x)>t\}$$

is nondecreasing and right continuous. Finally, set

$$\tau_n(x) = G^{-1}\left(n^{\frac{1}{\alpha}}\left[V\big(x+\frac{1}{n}\big)-V(x)\right]\right)\,, \qquad x\in\mathbb{Z}_n\,. \tag{8.2}$$

It is trivial to check that the V–dependent random field $\{\tau_n(x)\,:\,x\in\mathbb{Z}_n\}$ has the same law of $\{\tau(nx)\,:\,x\in\mathbb{Z}_n\}$. Indeed, since V has independent and stationary increments one obtains that the $\tau_n(x)$'s are i.i.d., while since $n^{\frac{1}{\alpha}}\left(V(x+\frac{1}{n})-V(x)\right)$ and $V(1)$ have the same law, one obtains that

$$\mathcal{P}(\tau_n(x)>t) = \mathcal{P}(G^{-1}(V(1))>t) = \mathcal{P}(V(1)>G(t)) = \mathbb{P}(\tau(nx)>t)\,, \qquad \forall t\geq 0\,.$$

We point out that the coupling obtained by this general method does not lead to (8.1) in the case that $\tau(x)$ is itself the α–stable law with Laplace transform $\mathbb{E}\big[e^{-\lambda\tau(x)}\big]=e^{-\lambda^\alpha}$.

8.1 Proof of Point (i)

Let us keep definition (8.2). For any $n\geq 1$ we introduce the generalized trap model $\{\tilde X^{(n)}(t)\}_{t\geq 0}$ on $\mathbb{Z}_n$ with jump rates

$$c_n(x,y) = \begin{cases} \gamma^2 L_2(n) n^{1+\frac{1}{\alpha}}\tau_n(x)^{-1+a}\tau_n(y)^a & \text{if } |x-y|=1/n \\ 0 & \text{otherwise}\,, \end{cases}$$

where $\gamma = \mathbb{E}(\tau(x)^{-a})$. The above jump rates can be written as $c_n(x,y) = 1/H_n(x)U_n(x\vee y)$ for $|x-y|=1/n$ by taking

$$\begin{cases} U_n(x) = \gamma^{-2}n^{-1}\tau_n(x-\frac{1}{n})^{-a}\tau_n(x)^{-a} \\ H_n(x) = L_2(n)^{-1}n^{-\frac{1}{\alpha}}\tau_n(x)\,. \end{cases}$$

Note that in all cases both U_n and H_n are functions of the α–stable subordinator V.

Then the following holds

Lemma 8.1. *Let m_n be defined as in* (2.10) *by means of the above functions U_n, H_n. Then for almost any realization of the α–stable subordinator V, $\ell_n\to 1$ and the measures dm_n weakly converge to the measure dV_* (recall definition* (2.19)*).*

Proof. Due to our definition (2.8) we have

$$S_n\Big(\frac{k}{n}\Big)=\frac{1}{n}\sum_{j=1}^{k}\gamma^{-2}\tau_n\Big(\frac{j-1}{n}\Big)^{-a}\tau_n\Big(\frac{j}{n}\Big)^{-a}, \qquad 0\leq k\leq n\,,$$

with the convention that the sum in the r.h.s. is zero if $k=0$. If $a=0$ trivially $\gamma=1$ and $S(k/n)=k/n$. If $a>0$ we can apply the strong law of large numbers for triangular arrays. Indeed, all addenda have the same law and they are independent if they are not consecutive, moreover they have bounded moments of all orders since $\tau(x)$ is bounded from below by a positive constant a.s. (this assumption is used only here and could be weakened in order to assure the validity of the strong LLN). Due to the choice of γ we have that $\gamma^{-2}\tau_n\big(\frac{j-1}{n}\big)^{-a}\tau_n\big(\frac{j}{n}\big)^{-a}$ has mean 1. By the strong law of large number we conclude that for a.a. V it holds $\lim_{n\uparrow\infty}S\big(\lfloor xn\rfloor/n\big)=x$ for all $x\geq 0$. This proves in particular that $\ell_n:=S_n(1)\to 1$. It remains to prove that for all $f\in C_c(\mathbb{R})$ it holds

$$\lim_{n\uparrow\infty}\sum_{k=0}^{n}f(S_n(k/n))H_n(k/n)=\int_0^1 f(s)dV_*(s)\,. \tag{8.3}$$

This limit can be obtained by reasoning as in the proof of Proposition 5.1 in [3], or can be derived by Proposition 5.1 in [3] itself together with the fact that $\mathcal{P}$ a.s. V has no jump at $0,1$. To this aim one has to observe that the constant c_ε (where $\varepsilon=1/n$) in [16] and [3][eq. (49)] equals our quantity $1/h(n)=1/\big(n^{1/\alpha}L_2(n)\big)$ (recall the definitions preceding Theorem 2.3). In particular, $H_n(k/n)=c_{1/n}\tau_n(k/n)$. □

Due to the above result, Point (i) in Theorem 2.3 follows easily from Theorem 2.1 and the fact that the random fields $\{\tau_n(x)\,:\,x\in\mathbb{Z}_n\}$ and $\{\tau(nx)\,:\,x\in\mathbb{Z}_n\}$ have the same law for all $n\geq 1$.

8.2 Proof of Point (ii)

Point (i) can be proved in a similar and simpler way. In this case, we define $\tau_n(x)$ as in (8.1) and we consider the generalized trap model $\{\tilde{X}^{(n)}(t)\}_{t\geq 0}$ on $\mathbb{Z}_n$ with jump rates

$$c_n(x,y)=\begin{cases} n^{1+\frac{1}{\alpha}}\tau_n(x)^{-1} & \text{if } |x-y|=1/n\\ 0 & \text{otherwise}\,,\end{cases}$$

with associated functions

$$U_n(x)=1/n\,,\qquad H_n(x)=n^{-\frac{1}{\alpha}}\tau_n(x)=V(x+1/n)-V(x)=:\Delta_n V(x)\,.$$

By this choice, $dm_n=\sum_{k=0}^{n}\delta_{k/n}\Delta_n V(k/n)$. Trivially, $\ell_n=1$ and $dm_n\to dV_*$ for all realizations of V giving zero mass to the extreme points 0 and 1. Since this event takes place $\mathcal{P}$–almost surely, the proof of part (ii) is concluded.

8.3 Proof of Point (iii)

Part (iii) of Theorem 2.3 (i.e. (2.26)) follows from Theorem 2.2 and Lemma 8.2 below. The self–similarity of V is the following: for each $\gamma>0$ it holds

$$\big(V(x)\,,\,x\in\mathbb{R}\big)\sim\big(\gamma^{\frac{1}{\alpha}}V(x/\gamma)\,:\,x\in\mathbb{R}\big)\,. \tag{8.4}$$

Indeed, both processes are càdlàg, take value 0 at the origin and have independent increments with the same law due to (2.23).

Lemma 8.2. *Taking $m=V$, the bound* (2.15) *is satisfied.*

Proof. Using the notation of Section 7, we denote by $\mathcal{N}_{V,D}^{[0,1]}(1)$ the number of eigenvalues not larger than 1 of the operator $-D_V D_x$ on $[0,1]$ with Dirichlet boundary conditions. We assume that V has no jump at $0,1$ (this happens $\mathcal{P}$–a.s.). We recall that V can be obtained by means of the identity $dV = \sum_{j\in J} x_j \delta_{v_j}$, where the random set $\xi = \{(x_j,v_j) : j \in J\}$ is the realization of a inhomogeneous Poisson point process on $\mathbb{R}\times\mathbb{R}_+$ with intensity $cv^{-1-\alpha}dxdv$, for a suitable positive constant c. In order to distinguish between the contribution of big jumps and not big jumps it is convenient to work with two independent inhomogeneous Poisson point processes $\xi^{(1)}$ and $\xi^{(2)}$ on $\mathbb{R}\times\mathbb{R}_+$ with intensity $cv^{-1-\alpha}\mathbb{I}(v\leq 1/2)dxdv$ and $cv^{-1-\alpha}\mathbb{I}(v>1/2)dxdv$. We write $\xi^{(1)} = \{(x_j,v_j) : j \in J_1\}$ and $\xi^{(2)} = \{(x_j,v_j) : j\in J_2\}$. The above point process ξ can be defined as $\xi = \xi^{(1)}\cup\xi^{(2)}$. Moreover, a.s. it holds $\xi^{(1)}\cap\xi^{(2)} = \emptyset$ (this fact will be understood in what follows). By the Master Formula (cf. Proposition (1.10) in [38]), it holds

$$\mathbb{E}\Big[\sum_{j\in J_1\,:\,x_j\in[0,1]} v_j\Big] = c\int_0^1 dx \int_0^{1/2} dv\, v^{-\alpha} < \infty\,, \tag{8.5}$$

$$\mathbb{E}\Big[\sharp\{j\in J_2\,:\,x_j\in[0,1]\}\Big] = c\int_0^1 dx\int_{1/2}^{\infty} dv\, v^{-1-\alpha} < \infty\,. \tag{8.6}$$

We label in increasing order the points in $\{x_j\,:\,j\in J_2\,,\ x_j\in[0,1]\}$ as $y_1<y_2<\cdots<y_N$ (note that the set is finite due to (8.6)).

Given $\delta\in(0,1/8)$, we take $\varepsilon\in(0,1)$ small enough that

(i) the intervals $(y_i-\varepsilon, y_i+\varepsilon)$ are included in $(0,1)$ and do not intersect as i varies from 1 to N,
(ii) for all $i : 1\leq i\leq N$, it holds $\sum_{j\in J_1 : x_j\in(y_i-\varepsilon,y_i+\varepsilon)} v_j < \delta$,
(iii) for all $i : 1\leq i\leq N$, the points $y_i-\varepsilon$ and $y_i+\varepsilon$ do not belong to $\{x_j : j\in J_1\}$.

Defining $V^{(1)}(t) = \sum_{j\in J_1\,:\,x_j\leq t} v_j$, the last condition (iii) can be stated as follows: for all $i : 1\leq i\leq N$, the points $y_i-\varepsilon$ and $y_i+\varepsilon$ are not jump points for $V^{(1)}$.

By construction the function $V^{(1)}$ has jumps not larger than $1/2$. In particular, all the intervals $A_0 = (0, y_1-\varepsilon)$, $A_1 = (y_1+\varepsilon, y_2-\varepsilon)$, $A_2 = (y_2+\varepsilon, y_3-\varepsilon)$,..., $A_{N-1} = (y_{N-1}+\varepsilon, y_N-\varepsilon)$, $A_N = (y_N+\varepsilon, 1)$ can be partitioned in subintervals such that, on each subinterval, the function $V^{(1)}$ has increment in $[1/2,1)$ and has no jump at the border (recall property (iii) above). As a consequence, the total number R of subintervals is bounded by $2V^{(1)}(1)$, which has finite expectation due to (8.5). By the bound (4.11) in Lemma 4.1, we get that the operator $-D_V D_x$ on any subinterval with Dirichlet boundary conditions has no eigenvalues smaller than 2. This observation and Corollary 6.8 imply that

$$\mathcal{N}_{D,V}^{[0,1]}(1) \leq 2R + \sum_{i=1}^{N} \mathcal{N}_{D,V}^{[y_i-\varepsilon, y_i+\varepsilon]}(1)\,. \tag{8.7}$$

Claim: *For each $i : 1\leq i\leq N$ it holds $\mathcal{N}_{D,V}^{[y_i-\varepsilon,y_i+\varepsilon]}(1)\leq 1$.*

Proof of the claim. We reason by contradiction supposing that f_1 and f_2 are eigenfunctions of the Dirichlet operator $-D_V D_x$ on $U = [y_i-\varepsilon, y_i+\varepsilon]$, whose corresponding eigenvalues λ_1 and λ_2 satisfy $0<\lambda_1<\lambda_2\leq 1$. We can take f_1 and f_2 continuous on U, satisfying $\int_U f_j^2(x)dV(x) = 1$ and

$$|f_j(x) - f_j(y_i)| \leq \sqrt{|x-y_i|}\,, \qquad x\in U \tag{8.8}$$

for $j = 1,2$. Indeed, recall that $|f(x)-f(y)|^2 \leq q_D(f)|x-y|$ for any $f\in Q(q_D)$ in Section 6. Calling $\Delta = dV(\{y_i\})$, (8.8) and property (ii) imply that

$$1 = \int_U f_j^2(x)dV(x) \leq \Delta f_j^2(y_i) + \delta(|f_j(y_i)| + \sqrt{\varepsilon})^2 \leq \Delta f_j^2(y_i) + 2\delta f_j^2(y_i) + 2\delta\varepsilon\,.$$

In particular, we get $f_j^2(y_i) \geq (1-2\delta\varepsilon)/(\Delta+2\delta)$. Due to our choice of the constants, $1-2\delta\varepsilon \geq 1-2(1/8) = 3/4$, while $\Delta + 2\delta \leq \Delta + 1/4 < (3/2)\Delta$ (recall that $\Delta > 1/2$). Hence, we get that $\Delta f_j^2(y_i) \geq 1/2$. On the other hand, using the orthogonality between f_1 and f_2, it must be

$$1/4 \leq \big|\Delta f_1(y_i) f_2(y_i)\big| = \Big|\int_{U\setminus\{y_i\}} f_1(x) f_2(x) dV(x)\Big| \leq \delta(|f_1(y_i)|+\sqrt{\varepsilon})(|f_2(y_i)|+\sqrt{\varepsilon})\,. \quad (8.9)$$

Since by construction $\varepsilon \leq 2 \leq \Delta f_j^2(y_i)$ and $\Delta > 1/2$ we can bound

$$|f_j(y_i)| + \sqrt{\varepsilon} \leq \sqrt{2}\sqrt{\Delta}|f_j(y_i)| + \sqrt{\varepsilon} \leq (1+\sqrt{2})\sqrt{\Delta}|f_j(y_i)|\,. \quad (8.10)$$

Combining (8.9) and (8.10), we conclude that

$$1/4 \leq \big|\Delta f_1(y_i) f_2(y_i)\big| \leq (1+\sqrt{2})^2 \delta \big|\Delta f_1(y_i) f_2(y_i)\big|\,,$$

in contrast with the bound $\delta < 1/8$. □

Applying the above claim to (8.7) we conclude that $\mathcal{N}_{D,V}^{[0,1]}(1) \leq 2R + N$. We have already observed that R has finite expectation. The same trivially holds also for N due to (8.6). □

9 Proof of Theorem 2.5

Recall the definition of $\mathcal{T}_n$ given in the previous section. Given a realization of V, for each $n \geq 1$ we consider the continuous–time nearest–neighbor random walk $\tilde{X}^{(n)}$ on $\mathbb{Z}_n$ with jump rates

$$c_n(x,y) = \begin{cases} L_2(n) n^{1+\frac{1}{\alpha}} \tau_n(x \vee y)^{-1} & \text{if } |x-y| = 1/n\,, \\ 0 & \text{otherwise}\,. \end{cases} \quad (9.1)$$

The rates $c_n(x,y)$ for $|x-y| = 1/n$ can be written as $c_n(x,y) = 1/\big[H_n(x,y) U_n(x \vee y)\big]$, where $H_n(x) = 1/n$ and $U_n(x) = L_2(n)^{-1} n^{-\frac{1}{\alpha}} \tau_n(x)$. To the above random walk we associate the measure dm_n defined in (2.10).

9.1 Proof of Point (i)

Let us show that dm_n weakly converges to $d(V^{-1})_*$ (recall (2.19)). We point out that in [23] a similar result is proved, but the definition given in [23] of the analogous of dm_n is different, hence that proof cannot be adapted to our case. In order to prove the weak convergence of dm_n to $d(V^{-1})_*$, we use some results and ideas developed in Section 3 of [16]. Recall that the constant c_ε of [16] equals our quantity $1/h(n) = 1/\big(n^{1/\alpha} L_2(n)\big)$ if $\varepsilon = 1/n$. Given $n \geq 1$ and $x > 0$ we define

$$g_n(x) = \big(L_2(n) n^{\frac{1}{\alpha}}\big)^{-1} G^{-1}(n^{\frac{1}{\alpha}} x)\,.$$

We point out that g_n coincides with the function g_ε defined in [16][(3.12)] if $\varepsilon = 1/n$. As stated in Lemma 3.1 of [16] it holds $g_n(x) \to x$ as $n \to \infty$ for all $x > 0$. Since g_n is nondecreasing, we conclude that

$$g_n(x_n) \to x \text{ as } n \to \infty\,, \qquad \forall x > 0,\ \forall \{x_n\}_{n\geq 1} : x_n > 0\,,\ x_n \to x\,. \quad (9.2)$$

As stated in Lemma 3.2 of [16], for any $\delta' > 0$ there exist positive constants C' and C'' such that

$$g_n(x) \leq C' x^{1-\delta'} \text{ for } n^{-\frac{1}{\alpha}} \leq x \leq 1 \text{ and } n \geq C''\,. \quad (9.3)$$

Since $U_n(x) = g_n\big(V(x+1/n) - V(x)\big)$, we can write

$$S_n\big(k/n\big) = \sum_{j=0}^{k-1} g_n\left(V\big((k+1)/n\big) - V\big(k/n\big)\right) . \tag{9.4}$$

Lemma 9.1. *For $\mathcal{P}$–almost all V it holds*

$$\lim_{n\uparrow\infty} \max_{0\leq k\leq n} \big|S_n(k/n) - V(k/n)\big| = 0 . \tag{9.5}$$

Proof. We recall that V can be obtained by means of the identity $dV = \sum_{j\in J} x_j \delta_{v_j}$, where the random set $\xi = \{(x_j, v_j) : j \in J\}$ is the realization of a inhomogeneous Poisson point process on $\mathbb{R}\times\mathbb{R}_+$ with intensity $cv^{-1-\alpha}dxdv$, for a suitable positive constant c. Given $y > 0$, let us define

$$J_{n,y} := \{r \in \{0, 1, \ldots, n-1\} \,:\, V((r+1)/n) - V(r/n) \geq y\} ,$$
$$J_y := \{j \in J \,:\, v_j \geq y \,,\ x_j \in [0, 1]\} .$$

Note that the set J_y is always finite. Reasoning as in the Proof of Proposition 3.1 in [16], and in particular using also (9.3), one obtains for $\mathcal{P}$–a.a. V that

$$\limsup_{n\uparrow\infty} \sum_{r:0\leq r<n\,, r\notin J_{n,\delta}} g_n\left(V\big((r+1)/n\big) - V\big(r/n\big)\right) = 0 , \qquad \forall \delta > 0 . \tag{9.6}$$

We claim that, given $\delta > 0$, for a.a. V it holds

$$J_{n,\delta} = \big\{r \in \{0, 1, \ldots, n-1\} \,:\, \exists j \in J_\delta \text{ such that } x_j \in (r/n, (r+1)/n]\big\} \tag{9.7}$$

eventually in n. Let us suppose that (9.7) is not satisfied. Since the set in the r.h.s. is trivially included in $J_{n,\delta}$, there exists a sequence of integers r_n with $0 \leq r_n < n$ such that $a_n := V((r_n+1)/n) - V(r_n/n) \geq \delta$ while $v_j < \delta$ for all $x_j \in (r_n, (r_n+1)/n]$. We introduce the càdlàg function $\bar V(t) = \sum_{j\in J: x_j\leq t} v_j \mathbb{I}(v_j < \delta)$ and we note that, if $\forall j \in J$ with $x_j \in (r_n/n, (r_n+1)/n]$ it holds $v_j < \delta$, then $a_n = \bar V((r_n+1)/n) - \bar V(r_n/n)$. At cost to take a subsequence, we can suppose that r_n/n converges to some point x. It follows then that $\bar V(x+) - \bar V(x-) \geq \delta$, in contradiction with the fact that $\bar V$ has only jumps smaller than δ. This concludes the proof of our claim.

Due to the above claim and due to (9.2), we conclude that a.s., given $\delta > 0$, it holds

$$\lim_{n\uparrow\infty} \sup_{1\leq k\leq n} \Big| \sum_{r\in J_{n,\delta}, r<k} g_n\left(V\big((r+1)/n\big) - V\big(r/n\big)\right) - \sum_{j\in J_\delta: x_j\leq k/n} v_j \Big| = 0 . \tag{9.8}$$

Combining (9.8) and (9.6), we conclude that for any $\varepsilon > 0$ one can fix a.s. $\delta > 0$ small enough such that

$$\max_{0\leq k\leq n} \big|S(k/n) - \sum_{j\in J_\delta: x_j\leq k/n} v_j\big| \leq \varepsilon \tag{9.9}$$

for n large enough. On the other hand, a.s. one can fix δ small enough that $\sum_{j\in J_\delta: x_j\in[0,1]} v_j$ is bounded by ε. This last bound and (9.9) imply (9.5). □

Lemma 9.2. *For $\mathcal{P}$–almost all V and for any function $f \in C_c(\mathbb{R})$ it holds*

$$\lim_{n\uparrow\infty} \frac{1}{n} \sum_{k=0}^{n} f\left(S_n(k/n)\right) = \int_{[0,V(1)]} f(x) dV^{-1}(x) . \tag{9.10}$$

Proof. Since f is uniformly continuous, by Lemma 9.1 it is enough to prove (9.10) with $S_n(k/n)$ replaced by $V(k/n)$. Approximating f by stepwise functions with jumps on rational points, it is enough to prove that, fixed $t \in \mathbb{Q}$, for $\mathcal{P}$–a.a. V the limit (9.10) holds with $S_n(k/n)$ replaced by $V(k/n)$ and with $f(x) = \mathbb{I}(x \leq t)$. This last check is immediate. □

We have now all the tools in order to prove Point (i) of Theorem 2.5. Indeed, by Lemma 9.1 $\ell_n = S_n(1) \to V(1)$ $\mathcal{P}$–a.s. Moreover, by Lemma 9.2 the measure dm_n defined in (2.10) weakly converges to the measure $d(V^{-1})_*$. In order to get Point (i) of Theorem 2.5 it is enough to apply Theorem 2.1.

9.2 Proof of Point (ii)

If $\mathbb{E}(e^{-\lambda\tau(x)}) = e^{-\lambda^\alpha}$ one can replace $L_2(n)$ with 1 in (9.1) and in the above definition of $U_n(x)$, and one can define $\tau_n(x)$ directly by means of (8.1). In this case, definition (2.8) gives $S_n(k/n) = V\big((k+1)/n\big)$ and therefore $dm_n = \frac{1}{n}\sum_{k=1}^{n+1}\delta_{V(k/n)}$. It is simple to prove that a.s. dm_n weakly converges to $dm := d(V^{-1})_*$. Hence, one gets that the assumptions of Theorem 2.1 are fulfilled with $\ell_n = V\big((n+1)/n\big)$, $\ell = V(1)$ and $dm = (V^{-1})_*$, for almost all realization of V. As a consequence, one derives Point (ii) in Theorem 2.5.

9.3 Proof of Point (iii)

The proof of point (iii) of Theorem 2.5 follows from Theorem 2.2 once we prove (2.17) with $m = V$. As in the proof of Lemma 8.2 we denote by $0 < y_1 < y_2 < \cdots < y_N < 1$ the points in $[0,1]$ where V has a jump larger than $1/2$ (note that V is continuous in 0 and 1 a.s.). We set $a_i := V(y_i-)$, $b_i = V(y_i)$ and remark that the function V^{-1} is constant on $[a_i, b_i]$. Then we fix $\varepsilon > 0$ (which is a random number) such that the following properties holds:

(i) the intervals $U_i := [a_i - \varepsilon, b_i + \varepsilon]$, $i = 1, ..., N$, are disjoint and included in $[0, V(1)]$,

(ii) V has no jump at $a_i - \varepsilon$ and $b_i + \varepsilon$, for all $i = 1, \dots, N$,

(iii) for all $i = 1, \dots, N$,

$$(b_i - a_i + 2\varepsilon)\big(V^{-1}(b_i+\varepsilon) - V^{-1}(a_i - \varepsilon)\big) \leq 1/2\,. \tag{9.11}$$

Note that, since V^{-1} is continuous a.s. and flat on U_i, condition (iii) is satisfied for ε small enough. Moreover, due to condition (ii) it holds $V^{-1}(x) < V^{-1}(y) < V^{-1}(z)$ if $y \in \{a_i - \varepsilon, b_i + \varepsilon\}$ and $x < y < z$.

Let now f be an eigenfunction of the operator $-D_{V^{-1}}D_x$ on U_i with Dirichlet boundary conditions. Writing λ for the associated eigenvalue, by equation (4.9) in Lemma 4.1 it holds

$$f(x) = \lambda \int_{U_i} G_{a_i-\varepsilon, b_i+\varepsilon}(x,y) f(y) dV^{-1}(y)\,.$$

Using that $\|G_{a_i-\varepsilon,b_i+\varepsilon}\|_\infty \leq b_i - a_i + 2\varepsilon$ we get

$$|f(x)| \leq \lambda(b_i - a_i + 2\varepsilon)\|f\|_\infty\big(V^{-1}(b_i+\varepsilon) - V^{-1}(a_i-\varepsilon)\big)\,. \tag{9.12}$$

Combining (9.11) and (9.12) we conclude that $\lambda \geq 2$. Hence $\mathcal{N}^{U_i}_{V^{-1},D}(1) = 0$. We now observe that the set $W = [0, V(1)] \setminus \cup_{i=1}^N U_i$ is the union of $N+1$ intervals and its total length is smaller than $V^{(1)}(1)$ (see the proof of Lemma 8.2 for the definition of $V^{(1)}$). It follows that we can partition W in at most $2V^{(1)}(1) + N$ subintervals A_r of length bounded by $1/2$. Since the dV^{-1}–mass of any subinterval A_r is bounded by the total

dV^{-1}–mass of $[0,V(1)]$ (which is a.s. 1), by the estimate (4.11) in Lemma 4.1 we get that all eigenvalues of the operator $-D_{V^{-1}}D_x$ restricted to any subinterval A_r (with Dirichlet b.c.) is at least 2, hence $\mathcal{N}^{A_r}_{V^{-1},D}(1)=0$. We now apply Corollary 6.8, observing that we are in the same setting on Theorem 6.7 (recall that V^{-1} is continuous a.s. and recall our condition (ii), thus leading to (i)–(iii) in Theorem 6.7). By Corollary 6.8, we conclude that $\mathcal{N}^{[0,V(1)]}_{V^{-1},D}(1)\leq V^{(1)}(1)+4N$ a.s. As already observed in the proof of Lemma 8.2, both $V^{(1)}(1)$ and N have finite expectation, thus leading to (2.17).

10 The diffusive case: Proof of Propositions 2.4 and 2.6

10.1 Proof of Proposition 2.4

We consider the diffusively rescaled random walk $X^{(n)}$on $\mathbb{Z}_n$ with jump rates

$$c_n(x,y)=\begin{cases}\mathbb{E}(\tau(0)^{-a})^2\mathbb{E}(\tau(0))n^2\tau(nx)^{-1+a}\tau(ny)^a & \text{if } |x-y|=1/n\\ 0 & \text{otherwise}\,.\end{cases}$$

The above jump rates can be written as $c_n(x,y)=1/H_n(x)U_n(x\vee y)$ for $|x-y|=1/n$ by taking

$$\begin{cases}U_n(x)=\mathbb{E}(\tau(0)^{-a})^{-2}n^{-1}\tau(nx-1)^{-a}\tau(nx)^{-a}\\ H_n(x)=\mathbb{E}(\tau(0))^{-1}n^{-1}\tau(nx)\,.\end{cases}$$

Due to our definition (2.8) we have

$$S_n\big(k/n\big)=\frac{1}{n\mathbb{E}(\tau(0)^{-a})^2}\sum_{j=1}^{k}\tau(j-1)^{-a}\tau(j)^{-a}\,,\qquad 0\leq k\leq n\,.$$

By the ergodic theorem and the assumption $\mathbb{E}\big(\tau(0)^{-a}\big)<\infty$, it holds $\lim_{n\uparrow\infty}S_n\big(\lfloor xn\rfloor/n\big)=x$ for all $x\geq 0$ (a.s.). In particular, it holds $\ell_n=S_n(1)\to 1$. Since π^2k^2 is the k–th eigenvalue of $-\Delta$ with Dirichlet conditions outside $(0,1)$, by Theorem 2.1 it remains to prove that, a.s., for all $f\in C_c([0,\infty))$ it holds

$$\lim_{n\uparrow\infty}dm_n(f)=\lim_{n\uparrow\infty}\sum_{k=0}^{n}f(S_n(k/n))H_n(k/n)=\int_0^1 f(s)ds\,. \tag{10.1}$$

By the ergodic theorem and the assumption $\mathbb{E}\big(\tau(0)\big)<\infty$, the total mass of dm_n, i.e.$\sum_{k=0}^{n}H_n(k/n)$, converges to 1 a.s. Hence, by a standard approximation argument with stepwise functions, it is enough to prove (10.1) for functions f of the form $f=\mathbb{I}([0,t))$. By the ergodic theorem a.s. it holds: for any $\varepsilon>0$ there exists a random integer n_0 such that $S_n(k/n)<t$ for all $k\leq(t-\varepsilon)n$ and $S_n(k/n)>t$ for all $k\geq(t+\varepsilon)/n$. Therefore, for f as above and $n\geq n_0$, we can bound

$$\frac{1}{n\mathbb{E}(\tau(0))}\sum_{k\in\mathbb{N}:k\leq(t-\varepsilon)n}\tau(k)\leq dm_n(f)\leq\frac{1}{n\mathbb{E}(\tau(0))}\sum_{k\in\mathbb{N}:k\leq(t+\varepsilon)n}\tau(k)\,.$$

Applying again the ergodic theorem, it is immediate to conclude.

10.2 Proof of Proposition 2.6

We sketch the proof since the technical steps are very easy and similar to the ones discussed above. We consider the diffusively rescaled random walk $X^{(n)}$on $\mathbb{Z}_n$ with jump rates

$$c_n(x,y)=\begin{cases}n^2\mathbb{E}(\tau(0))\tau(nx\vee ny)^{-1} & \text{if } |x-y|=1/n\,,\\ 0 & \text{otherwise}\,.\end{cases}$$

The rates $c_n(x,y)$ for $|x-y|=1/n$ can be written as $c_n(x,y)=1/\big[H_n(x,y)U_n(x\vee y)\big]$, where $H_n(x)=1/n$ and $U_n(x)=\tau(nx)/n\mathbb{E}(\tau(0))$. By the ergodic theorem and the assumption $\mathbb{E}(\tau(0))<\infty$, a.s. it holds $\lim_{n\uparrow\infty} S_n(\lfloor nx\rfloor)=x$ for all $x\geq 0$. In particular, a.s. $S_n(n)\to 1$ and

$$\lim_{n\uparrow\infty} dm_n(f)=\lim_{n\uparrow\infty}\frac{1}{n}\sum_{k=0}^{n} f\big(S_n(k/n)\big)=\int_0^1 f(x)dx\,,$$

for all $f\in C_c([0,\infty))$. At this point it is enough to apply Theorem 2.1.

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