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Greedy polyominoes and first-passage times on random Voronoi tilings*

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Abstract

Let $\mathcal{N}$ be distributed as a Poisson random set on $\mathbb{R}^d$, $d \geq 2$, with intensity comparable to the Lebesgue measure. Consider the Voronoi tiling of $\mathbb{R}^d$, $\{C_v\}_{v\in\mathcal{N}}$, where C_v is composed of points $\mathbf{x} \in \mathbb{R}^d$ that are closer to $v \in \mathcal{N}$ than to any other $v' \in \mathcal{N}$. A polyomino $\mathcal{P}$ of size n is a connected union (in the usual $\mathbb{R}^d$ topological sense) of n tiles, and we denote by Π_n the collection of all polyominos $\mathcal{P}$ of size n containing the origin. Assume that the weight of a Voronoi tile C_v is given by $F(C_v)$, where F is a nonnegative functional on Voronoi tiles. In this paper we investigate for some functionals F, mainly when $F(C_v)$ is a polynomial function of the number of faces of C_v, the tail behavior of the maximal weight among polyominoes in Π_n: $F_n = F_n(\mathcal{N}) := \max_{\mathcal{P}\in\Pi_n} \sum_{v\in\mathcal{P}} F(C_v)$. Next we apply our results to study self-avoiding paths, first-passage percolation models and the stabbing number on the dual graph, named the Delaunay triangulation. As the main application we show that first passage percolation has at most linear variance.

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1 Introduction

Greedy animals on $\mathbb{Z}^d$ have been studied notably in [4]. Imagine that positive weights, or awards, are placed on all vertices of $\mathbb{Z}^d$. A greedy animal of size n is a connected subset of n vertices, containing the origin, and which catches the maximum amount of awards. When these awards are random, i.i.d, it is shown in [4] that the total award collected by a greedy animal grows at most linearly in n if the tail of the award is not too thick. This is shown in a rather strong sense, giving deviations inequalities decaying at a rate $n^{-\log(n)^\alpha}$, for some α (see Proposition 1 in [4]). This result has already been proved useful for the study of percolation and (dependent) First Passage Percolation, see [5].

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The aim of the present paper is to extend the results of [4] to some *greedy polyominoes on the Poisson-Voronoi tiling*. The precise definitions will be given in section 2, but let us explain what we mean. First, in this paper, the Poisson-Voronoi tiling is the Voronoi tiling based on a Poisson random set on $\mathbb{R}^d$ with intensity comparable to the Lebesgue measure. A polyomino of size n on this tiling is a connected union of n Voronoi tiles, cf. Figure 1. Then, we let f be some fixed function from $\mathbb{N}$ to $\mathbb{R}^+$ (a *weight function*), and put down on each Voronoi tile an award depending on the number of its faces: a Voronoi tile with r faces receives an award equal to $f(r)$. Finally, a polyomino of size n is greedy if it contains the origin and catches the maximum amount of awards.

Our main result, Theorem 2.2 below, gives a deviation inequality for the total award collected by a greedy polyomino on the Poisson-Voronoi tiling. Of course, the rate of decay depends on the precise weight function f defining the award. To give an idea of our results, when $f(r) = r^k$, $k \geq 1$, we obtain the following deviation inequality.

Corollary 1.1. *Let $k \geq 0$ and denote by $F_n(k)$ the total award collected by a greedy polyomino on the Poisson-Voronoi tiling, with award on a tile being equal to the k-th power of the number of faces of the tile. Then, there are constants z_0 and K such that for any $n \geq 1$, and any $z \geq z_0$,*

$$\mathbb{P}\left(\frac{1}{n}F_n(k) \geq z\right) \leq e^{-K(nz)^{\frac{1}{k+2}}} .$$

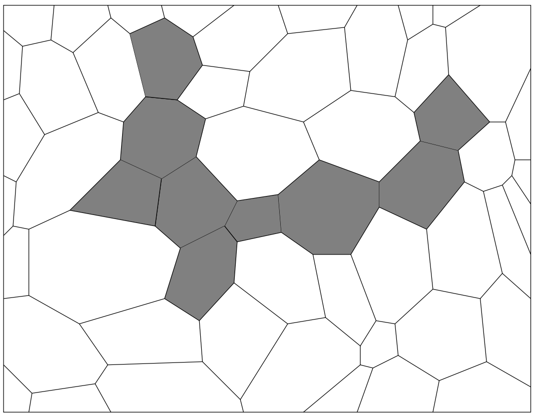

Figure 1: A two-dimensional Voronoi polyomino of size $n = 9$.

Our results may be useful to control the geometry of the Poisson-Voronoi tiling, and this may be best viewed through the facial dual of the Poisson-Voronoi tiling, which is called the *Poisson-Delaunay graph*. Notice that through this duality, a greedy polyomino on the Poisson-Voronoi tiling with weight function f is simply a greedy animal on the Delaunay graph where the award on a vertex of degree k equals $f(k)$. We shall give three applications of our results on greedy polyominoes to some geometric problems on the *Poisson-Delaunay graph*.

The first application is an estimate on the tail probability of the number of self-avoiding paths of length n starting from the origin.

The second application concerns First Passage Percolation on the Poisson-Delaunay graph. We prove that the first passage time in First Passage Percolation has (at most) linear variance on the Poisson-Delaunay graph.

Finally, our third application concerns tail estimates on the stabbing number, which is defined in [1] as the maximum number of Delaunay cells that intersect a single line in the cube $[0,n]^d$. This is an important quantity, as can be seen in [1] where it is the crucial ingredient to derive the transience or recurence properties of the simple random walk on the Poisson-Delaunay graph. We shall obtain an exponential deviation inequality for the stabbing number of the Poisson-Delaunay graph.

The rest of the paper is organized as follows. In section 2, we give the precise definitions and state our main results concerning greedy polynominoes on the Poisson-Voronoi tiling (and greedy animals on the Poisson-Delaunay graph). Section 3 is devoted to the proofs of these main results, which is based on a renormalization argument from percolation theory and an adaptation of the chaining technique of [4]. This last step being rather technical, its proof is postponed to the appendix. Finally, the three applications on the Poisson-Delaunay graph are the matter of section 4.

2 Definitions and main results

In this section, we give all the needed formal definitions and state precisely our main results on greedy polyominoes, Theorem 2.2 and Proposition 2.5.

2.1 The Poisson-Voronoi tiling and polyominoes.

In the whole paper, we suppose that $d \geq 2$. To any locally finite subset $\mathcal{N}$ of $\mathbb{R}^d$ one can associate a collection of subsets of $\mathbb{R}^d$ whose union is $\mathbb{R}^d$. To each point $v \in \mathcal{N}$ corresponds a polygonal region C_v, the *Voronoi tile* (or cell) at v, consisting of the set of points of $\mathbb{R}^d$ which are closer to v than to any other $v' \in \mathcal{N}$. Closer is understood here in the large sense, and this collection is not a partition, but the set of points which belong to more than one Voronoi tile has Lebesgue measure 0. The collection $\{C_v\}_{v\in\mathcal{N}}$ is called the *Voronoi tiling* (or tessellation) of the plane based on $\mathcal{N}$. From now on, $\mathcal{N}$ is understood to be distributed like a Poisson random set on $\mathbb{R}^d$ with intensity measure μ. We shall always suppose that μ is comparable to Lebesgue's measure on $\mathbb{R}^d$, λ_d, in the sense that there exists a positive constant c_μ such that for every Lebesgue-measurable subset A of $\mathbb{R}^d$:

$$\frac{1}{c_\mu}\lambda_d(A) \leq \mu(A) \leq c_\mu \lambda_d(A) \ . \tag{2.1}$$

For each positive integer number $n \geq 1$, a *Voronoi polyomino* $\mathcal{P}$ of size n is a connected union of n Voronoi tiles (Figure 1). Notice that with probability one, when two Voronoi tiles are connected, they share a $(d-1)$-dimensional face. We denote by Π_n the set of all polyominoes $\mathcal{P}$ of size n and such that the origin 0 belongs to $\mathcal{P}$.

Assume that the "weight" of a Voronoi tile C_v is given by $f(d_{\mathcal{N}}(v))$, where $d_{\mathcal{N}}(v)$ is the number of $(d-1)$-dimensional faces of C_v and f is a nondecreasing function from $[1,\infty)$ to $[1,\infty)$. In this way we define a random weight functional on polyominoes by

$$F(f,\mathcal{N},\mathcal{P}) := \sum_{v\in\mathcal{P}} f(d_{\mathcal{N}}(v)) \, .$$

The maximal weight among polyominoes in Π_n is:

$$F_n(f,\mathcal{N}) := \max_{\mathcal{P}\in\Pi_n} F(f,\mathcal{N},\mathcal{P}) \ . \tag{2.2}$$

A greedy Voronoi polyomino $\mathcal{P}_n$ is a Voronoi polyomino that attains the maximum in the definition of F_n: $F(f,\mathcal{N},\mathcal{P}_n) = F_n(f,\mathcal{N})$.

To state our main theorem, we require the following notation:

Definition 2.1. *Let g^{-1} denote the pseudo-inverse of any strictly increasing function g from $[1,\infty)$ to $[1,\infty)$:*

$$g^{-1}(u) = \sup\{u' \in [1,\infty) \text{ s.t. } g(u') < u\} \ .$$

A weight function f *is a nondecreasing function from* $[1,\infty)$ *to* $[1,\infty)$. *Given a weight function* f*, we define two functions* $\hat{f}$ *and* f^* *as follows:*

$$\hat{f} = (x \mapsto xf(x))^{-1} ,$$

and

$$f^* = \hat{f} \circ (y \mapsto y\hat{f}(y))^{-1} .$$

In addition, for each $c \in (0,\infty)$*, we say that* f *is* c*-nice if*

$$\liminf_{y\to\infty} \frac{\hat{f}(y)}{\log y} \geq c .$$

Examples of c-nice weight functions f, and various $\hat{f}$, f^* are given after Theorem 2.2.

Theorem 2.2. *Let* f *be a non-decreasing weight function. There exists constants* z_1, c_1, c_2 *and* c_3 *in* $(0,\infty)$ *such that if* f *is* c_1*-nice then for all* $z \geq z_1$ *and for all* $n \geq 1$*:*

$$\mathbb{P}\big(F_n(f,\mathcal{N}) > nz\big) \leq \exp\Big\{ -c_2 f^*(c_3 nz)\Big\} .$$

In particular, there also exists a constant $c_4 \in (0,\infty)$ *such that*

$$0 \leq f(3) \leq \liminf_{n\to\infty} \mathbb{E}\Big(\frac{F_n(f,\mathcal{N})}{n}\Big) \leq \limsup_{n\to\infty} \mathbb{E}\Big(\frac{F_n(f,\mathcal{N})}{n}\Big) < c_4 .$$

Example 2.3. *If* $f(x) = x^k$ *with* $k \geq 0$*, then* $\hat{f}(u) = u^{\frac{1}{k+1}}$ *and* $f^*(x) = x^{\frac{1}{k+2}}$*. This weight function* f *is* c*-nice for any* $c > 0$*, and this gives Corollary 1.1.*

Example 2.4. *Let* $f(x) = \exp\{(x/K)^\alpha\}$*, with* $K > 0$ *and* $\alpha > 0$*. Then as* x *goes to infinity,* $\hat{f}(u) \sim K(\log u)^{1/\alpha}$ *and* $f^*(x) \sim K(\log x)^{1/\alpha}$*. Thus if* $\alpha < 1$*,* f *is* c*-nice for any* $c > 0$*, and Theorem 2.2 gives*

$$\mathbb{P}\big(F_n(f,\mathcal{N}) > nz\big) \leq \exp\Big\{ -C(\log(nz))^{\frac{1}{\alpha}}\Big\} ,$$

for some constant $C > 0$ *and* z *large enough. Also, when* $\alpha = 1$*,* f *is* c*-nice for any* $c \in (0,K]$*. Thus, if* K *is large enough (at least the constant* c_1 *of Theorem 2.2), then:*

$$\mathbb{P}\big(F_n(f,\mathcal{N}) > nz\big) \leq \exp\Big\{ -C\log(nz)\Big\} ,$$

for some constant $C > 0$ *and* z *large enough.*

2.2 The Delaunay graph.

An important graph for the study of a Voronoi tiling is its facial dual, the *Delaunay graph* based on $\mathcal{N}$. This graph, denoted by $\mathcal{D}(\mathcal{N})$ is an unoriented graph embedded in $\mathbb{R}^d$ which has vertex set $\mathcal{N}$ and edges $\{u,v\}$ every time C_u and C_v share a $(d-1)$-dimensional face (Figure 2). We remark that, for our Poisson random set, almost surely no $d+1$ points are on the same hyperplane and no $d+2$ points are on the same hypersphere, which makes the Delaunay graph a well defined triangulation. This triangulation divides $\mathbb{R}^d$ into bounded simplices called *Delaunay cells*. For a Delaunay cell Δ, let $B(\Delta)$ denote the closed circumball of Δ. An important property which we shall use several times is that for each Delaunay cell Δ no point in $\mathcal{N}$ lies in the interior of $B(\Delta)$. Polyominoes on the Voronoi tiling correspond to connected (in the graph topology) subsets of the Delaunay graph. Also, the number of faces of a Voronoi tile C_v, which we denoted by $d_{\mathcal{N}}(v)$, is simply the degree of v in $\mathcal{D}(\mathcal{N})$.

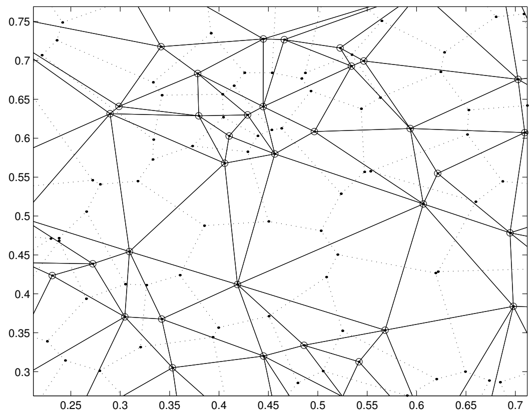

Figure 2: The Voronoi tiling (dashed lines) and the Delaunay triangulation (solid lines) in dimension $d = 2$.

We shall prove an easier variant of Theorem 2.2, which will be useful in the applications. Define Ω to be the set of locally finite subsets of $\mathbb{R}^d$. Then, for any $\omega \in \Omega$ and any subgraph ϕ of $\mathcal{D}(\omega)$, we define $\Gamma(\omega, \phi)$ to be the set of points in $\mathbb{R}^d$ whose addition to ω "perturbs" ϕ:

$$\Gamma(\omega, \phi) = \{x \in \mathbb{R}^d \text{ s.t. } \phi \not\subset \mathcal{D}(\omega \cup \{x\})\} .$$

We get the following result for the maximal size of $\Gamma(\omega, \phi)$ when ϕ belongs to $\mathcal{SA}_n$, the set of self-avoiding paths on $\mathcal{D}(\mathcal{N})$ starting from $v(0)$ and of size n, where $v(0)$ is the a.s. unique $v \in \mathcal{N}$ s.t. $0 \in C_v$:

Proposition 2.5. *There are constants z_1 and C_1 such that for every $z \geq z_1$, and for every $n \geq 0$,*

$$\mathbb{P}\Big(\max_{\phi \in \mathcal{SA}_n} \mu(\Gamma(\mathcal{N}, \phi)) > nz \Big) \leq e^{-C_1 nz} .$$

3 Proofs of the main results

There are three main issues when considering the total award of a greedy polyomino as in (2.2). The first issue lies in the nature of the weights: they are a function of the number of faces of a tile or equivalently, the degree of a vertex in the Delaunay graph. Notice that it is known how to control this quantity for the "typical" cell, in the Palm sense (cf. [3] for instance) but we are not in this "typical" case. Of course, we know really well how to count the number of points of the Poisson random set in a fixed area, but we do not know that well how far away we have to expect the neighbours of these points to lie. The idea to answer this problem is to use a renormalization trick from percolation theory. More precisely, we shall consider a box in $\mathbb{R}^d$ large enough so that it contains with a "large enough" probability some configuration of points which prevents a Delaunay cell to cross it completely. This "large enough"probability corresponds to a percolation threshold: we need that the "bad boxes" (those who can be crossed) do not percolate. This will allow us to control the degree of a vertex in a Polyomino, bounding it from above by the number of points inside a cluster of "bad boxes" containing the vertex. In section 3.1, we explain this renormalization by showing how to cover animals with boxes and clusters of boxes.

A second issue lies in the fact that the degrees of two vertices are dependent. This will be solved inside the renormalization trick, at the price of counting many times the same number of points in a cluster of "bad boxes".

The third issue is to handle the fact that the supremum in (2.2) is over a potentially high (i.e exponential in n) number of polyominoes of size n. This will be handled through the chaining technique due to [4]. We shall state the corresponding adaptation in section 3.2.

With these tools in hand, we shall prove our main results, Theorem 2.2 in section 3.3 and Proposition 2.5 in section 3.4.

3.1 The renormalization trick: comparison with site percolation and lattice animals

We denote by $\mathbb{G}_d$ the d-dimensional lattice with vertex set $\mathbb{Z}^d$ and edge set composed by pairs $(\mathbf{z}, \mathbf{z}')$ such that

$$|\mathbf{z} - \mathbf{z}'|_\infty = \max_{j=1,\ldots,d} |\mathbf{z}(j) - \mathbf{z}'(j)| = 1\,.$$

A box B in $\mathbb{R}^d$ is any set of the form

$$B = \mathbf{x} + [0, L)^d$$

for some $\mathbf{x} \in \mathbb{R}^d$ and $L \geq 0$. We shall often use boxes centered at points of some lattice embedded in $\mathbb{R}^d$, so we define, for any z in $\mathbb{R}^d$

$$B_\mathbf{z} = \mathbf{z} + [-1/2, 1/2)^d\,.$$

Now, we define the notions of nice and good boxes. These are notions depending on a set $\mathcal{N}$ of points in $\mathbb{R}^d$, which basically say that there are enough points of $\mathcal{N}$ inside the box so that a Voronoi tile of a point of $\mathcal{N}$ outside the box cannot "cross" the box. This will be given a precise sense in Lemma 3.3.

Definition 3.1. *Define the following even integer:*

$$\alpha_d = 2(4\lceil\sqrt{d}\rceil + 2)\,.$$

Let $\mathcal{N}$ be a set of points in $\mathbb{R}^d$. We say that a box $B \subseteq \mathbb{R}^d$ is $\mathcal{N}$-nice if, cutting it regularly into α_d^d sub-boxes, each one of these boxes contains at least one point of the set $\mathcal{N}$. We say that a box B is $\mathcal{N}$-good if, cutting it regularly into $(3\alpha_d)^d$ sub-boxes, each one of these boxes contains at least one point of the set $\mathcal{N}$. When $\mathcal{N}$ is understood, we shall simply say that B is nice *(resp.* good*) if it is $\mathcal{N}$-nice (resp. $\mathcal{N}$-good). We say a box is $\mathcal{N}$-*ugly *(resp. $\mathcal{N}$-*bad*) if it is not $\mathcal{N}$-nice (resp. not $\mathcal{N}$-good).*

Notice that if a box is good, then it is nice, and thus if a box is ugly, then it is bad. To count the number of points of $\mathcal{N}$ which are in a subset A of $\mathbb{R}^d$, we define:

$$|A|_\mathcal{N} = |A \cap \mathcal{N}|\,.$$

An animal $\mathbf{A}$ in $\mathbb{G}_d$ is a finite and connected subset of $\mathbb{G}_d$. To each bounded and connected subset A of $\mathbb{R}^d$, we associate an animal in $\mathbb{G}_d$:

$$\mathbf{A}(A) = \{\mathbf{z} \in \mathbb{G}_d \text{ s.t. } A \cap B_\mathbf{z} \neq \emptyset\}\,.$$

If $\mathbf{V}$ is a set of vertices of $\mathbb{Z}_d$, define the border $\partial\mathbf{V}$ of $\mathbf{V}$ as the set of vertices which are not in $\mathbf{V}$ but have a $\mathbb{G}^d$-neighbor in $\mathbf{V}$. We also define the following subsets of $\mathbb{R}^d$:

$$Bl(\mathbf{V}) := \bigcup_{\mathbf{z}\in\mathbf{V}} B_\mathbf{z}\,.$$

$$Ad(\mathbf{V}) := \left\{ x \in \mathbb{R}^d \text{ s.t. } \inf_{y \in Bl(\mathbf{V})} \|x - y\|_\infty < \frac{1}{2} \right\} .$$

A site percolation scheme on $\mathbb{G}_d$ is defined by:

$$X_{\mathbf{z}} = \mathbb{1}_{B_{\mathbf{z}} \text{ is } \mathcal{N}-bad} , \quad \text{for } \mathbf{z} \in \mathbb{Z}^d .$$

$X = (X_{\mathbf{z}})_{\mathbf{z} \in \mathbb{Z}_d}$ is a collection of independent Bernoulli random variables. A vertex $\mathbf{z}$ is said to be *bad* when $X_{\mathbf{z}} = 1$. A *bad cluster* is then a maximal connected subset $\mathbf{C}$ of bad vertices of $\mathbb{G}_d$. Similarly, we define ugly vertices and ugly clusters. For any set of vertices $\mathbf{V}$, we define by $\mathbf{Cl}(\mathbf{V})$ the collection of all the bad clusters intersecting $\mathbf{V}$. We shall make a slight abuse of notation by writing $\mathbf{Cl}(\mathbf{z})$ to be the bad cluster containing $\mathbf{z}$, if there is any (otherwise, we let $\mathbf{Cl}(\mathbf{z}) = \emptyset$).

We now want to show that the Delaunay cells cannot cross the boundaries of an ugly (or bad) cluster. Since the argument will be used at other places, we state a more general lemma first.

Lemma 3.2. *Let $\mathbf{C}$ be a non-empty connected set of vertices in $\mathbb{G}_d$. define $\partial\mathbf{C}$ to be the exterior boundary of $\mathbf{C}$:*

$$\partial\mathbf{C} = \{\mathbf{z} \in \mathbb{G}_d \textit{ s.t. } z \notin \mathbf{C} \textit{ and } \exists x \in \mathbf{C} \textit{ s.t. } |\mathbf{z} - \mathbf{z}'|_\infty = 1\} .$$

Suppose that $\partial\mathbf{C}$ is composed of nice vertices and let B' be a ball in $\mathbb{R}^d$ such that:

$$B' \cap Bl(\mathbf{C}) \neq \emptyset \quad \textit{and} \quad \overset{o}{B'} \cap \mathcal{N} = \emptyset .$$

Then,

$$B' \subset Ad(\mathbf{C}) .$$

Proof. The idea of the proof is essentially the same as in Lemma 2.1 from [13]. We proceed by contradiction. Suppose that

$$B' \cap Bl(\mathbf{C}) \neq \emptyset, \quad \overset{o}{B'} \cap \mathcal{N} = \emptyset \quad \text{and} \quad B' \not\subset Ad(\mathbf{C}) .$$

Let us define:

$$\partial^\infty \mathbf{C} = Ad(\mathbf{C}) \setminus Bl(\mathbf{C}) .$$

This implies that one may find x_1, x_2 in B' and y in $\overset{o}{B'}$, and a box B of side length $\frac{1}{2}$ such that:

$$\begin{array}{l} y \in [x_1, x_2],\ y \in \text{central}(B),\ x_1 \notin B,\ x_2 \notin B \ , \\ \text{and cutting } B \text{ regularly into } (\alpha_d/2)^d \text{ sub-boxes,} \\ \text{each sub-box contains at least one point of } \mathcal{N} \ , \end{array} \tag{3.1}$$

where central(B) is the only sub-box of B containing the center of B when one cuts B regularly into $(\alpha_d/2)^d$ sub-boxes. The ball B' necessarily has diameter greater than $\|x_1 - x_2\|_2$, which is larger than $1/2$. Thus, there is a ball B' of diameter larger than $1/2$, which contains y and whose interior does not contain any point of $\mathcal{N}$. But one may see that any ball of diameter strictly larger than $2\sqrt{d}/\alpha_d$ necessarily contains at least one of the sub-boxes of side length $1/\alpha_d$. Notice that any ball of diameter strictly smaller than $\frac{(\alpha_d - 1)}{2\alpha_d}$ which contains a point in central(B) is totally included in B. Since

$$2\sqrt{d}/\alpha_d < \frac{(\alpha_d - 1)}{2\alpha_d} ,$$

and since the sub-boxes of B of side length $1/\alpha_d$ contain a point of $\mathcal{N}$, we deduce that B' contains a sub-box of B which contains a point of $\mathcal{N}$, whence a contradiction. ☐

Lemma 3.3. *Assume that* $\mathbf{C}$ *is an ugly cluster in* $\mathbb{G}_d$*. Let* Δ *be any Delaunay cell of* $\mathcal{D}(\mathcal{N})$*. Then,*

$$\Delta \cap Bl(\mathbf{C}) \neq \emptyset \Rightarrow \Delta \subset Ad(\mathbf{C}) .$$

The same holds for bad clusters.

Proof. Suppose that $\Delta \cap Bl(\mathbf{C}) \neq \emptyset$. Notice that the exterior boundary of an ugly cluster is composed of nice vertices. Since Δ is a Delaunay cell, the circumball of Δ, $B(\Delta)$, is a ball containing Δ and whose interior does not contain any point of $\mathcal{N}$. Thus, one may apply Lemma 3.2 to deduce that $B(\Delta)$, and hence Δ, is included in $Ad(\mathbf{C})$.

The same proof holds for bad clusters, since the boundary of a bad cluster is composed of good vertices, which are nice vertices. □

The renormalization trick is essentially to transfer the problem of counting the degrees in our functional to counting the points in some boxes or clusters of bad boxes which cover the animals. However, to have that bad boxes occur with small probability we need to rescale our initial boxes. For $\mathbf{z}$ in $\mathbb{Z}^d$ and $r > 0$, we define

$$B^r_{\mathbf{z}} = r\mathbf{z} + [-r/2, r/2)^d .$$

In this rescaled setup, good and nice boxes are defined according to Definition 3.1. In order to cover random polyominoes properly we need to translate the percolation scheme as follows:

$$\forall i = 1, \dots, 3^d,\ X^{r,i}_{\mathbf{z}} = \mathbb{1}_{r\vec{f}_i/3 + B^r_{\mathbf{z}} \text{ is } \mathcal{N}-bad},\ \forall \mathbf{z} \in \mathbb{Z}^d ,$$

where $\vec{f}_1 = 0$ and $\vec{f}_2, \dots, \vec{f}_{3^d}$ are the neighbors of $\mathbf{0}$ in $\mathbb{G}_d$. For each $i = 1, \dots, 3^d$ and $r > 0$, $X^{r,i} = (X^{r,i}_{\mathbf{z}})_{\mathbf{z} \in \mathbb{Z}_d}$ is a collection of independent Bernoulli random variables. Of course, the comparison of μ to Lebesgue's measure (2.1) implies that $\mu(B^{r,i}_{\mathbf{z}})$ goes to infinity when r goes to infinity. Thus,

$$\lim_{r \to \infty} \sup_i \sup_{\mathbf{z}} \mathbb{P}(X^{r,i}_{\mathbf{z}} = 1) = 0 .$$

Every time we address to the $X^{r,i}$ percolation scheme we label a name, or a variable, with a (r, i). For instance, a vertex $\mathbf{z}$ is said to be (r, i)-*bad* (or simply *bad* when r and i are implicit) when $X^{r,i}_{\mathbf{z}} = 1$. To each bounded and connected subset A of $\mathbb{R}^d$, we also associate 3^d different animals in $\mathbb{G}_d$:

$$\forall i = 1, \dots, 3^d,\ \mathbf{A}^{r,i}(A) = \{\mathbf{z} \in \mathbb{G}_d \text{ s.t. } A \cap (r\vec{f}_i/3 + B^r_{\mathbf{z}}) \neq \emptyset\} ;$$

and 3^d subsets of $\mathbb{R}^d$:

$$\forall i = 1, \dots, 3^d,\ Bl^{r,i}(\mathbf{V}) := \bigcup_{\mathbf{z} \in \mathbf{V}} (B^r_{\mathbf{z}} + r\vec{f}_i/3) ,$$

and

$$\forall i = 1, \dots, 3^d,\ Ad^{r,i}(\mathbf{V}) := \left\{ x \in \mathbb{R}^d \text{ s.t. } \inf_{y \in Bl^{r,i}(\mathbf{V})} \|x - y\|_\infty < \frac{r}{2} \right\} .$$

Of course, Lemma 3.3 still holds for the $X^{r,i}$ setup. Let Δ be any Delaunay cell of $\mathcal{D}(\mathcal{N})$. If $\mathbf{C}$ is an (r, i)-ugly cluster in $\mathbb{G}_d$ then

$$\Delta \cap Bl^{r,i}(\mathbf{C}) \neq \emptyset \Rightarrow \Delta \subset Ad^{r,i}(\mathbf{C}) .$$

The following lemma makes precise the renormalization trick.

Lemma 3.4. *Let $\mathcal{G}$ be a finite collection of bounded and connected subsets of $\mathbb{R}^d$. Then, for any positive real number $r, t > 0$.*

$$\mathbb{P}\left[\sup_{\gamma\in\mathcal{G}} \sum_{v\in\mathcal{N}\cap\gamma} f(d_{\mathcal{N}}(v)) > t\right] \leq \sum_{i=1}^{3^d} \mathbb{P}\left[\sup_{\gamma\in\mathcal{G}} \sum_{\mathbf{C}\in\mathbf{Cl}^{3r,i}(\mathbf{A}^{3r,i}(\gamma))} |Ad^{3r,i}(\mathbf{C})|_{\mathcal{N}} f(|Ad^{3r,i}(\mathbf{C})|_{\mathcal{N}}) > \frac{t}{3^d.2}\right] + \sum_{i=1}^{3^d} \mathbb{P}\left[\sup_{\gamma\in\mathcal{G}} \sum_{\mathbf{z}\in\mathbf{A}^{3r,i}(\gamma)} |Bl^{3r,i}(\mathbf{z})|_{\mathcal{N}} f(|Bl^{3r,i}(\mathbf{z})|_{\mathcal{N}}) > \frac{t}{3^d.2}\right] .$$

Proof. Let γ be any member of $\mathcal{G}$. Recall we say a box is ugly if it is not nice, and say it is bad when it is not good. We write $\mathbf{z} \sim \mathbf{z}'$ if $\mathbf{z}$ and $\mathbf{z}'$ are two points adjacent on $\mathbb{G}_d$, and $\mathbf{z} \simeq \mathbf{z}'$ if $\mathbf{z} \sim \mathbf{z}'$ or $\mathbf{z} = \mathbf{z}'$. First, we cover γ with boxes. This leads to the animal $\mathbf{A}^{r,1}(\gamma)$ and we distinguish two kind of boxes: those all of whose neighbours are nice, and the others. Formally,

$$Bl^{r,1}(\mathbf{A}^{r,1}(\gamma)) = \bigcup_{\mathbf{z}\in\mathbf{A}^{r,1}(\gamma)} B^r_{\mathbf{z}} = U_1 \cup U_2 ,$$

where

$$U_1 = \bigcup_{\substack{\mathbf{z}\in\mathbf{A}^{r,1}(\gamma)\\ \exists\ \mathbf{z}'\simeq\mathbf{z},\ B^r_{\mathbf{z}'} \text{ is ugly}}} B^r_{\mathbf{z}} ,$$

and

$$U_2 = \bigcup_{\substack{\mathbf{z}\in\mathbf{A}^{r,1}(\gamma)\\ \forall\mathbf{z}'\simeq\mathbf{z},\ B^r_{\mathbf{z}'} \text{ is nice}}} B^r_{\mathbf{z}} .$$

Accordingly,

$$\sum_{v\in\mathcal{N}\cap\gamma} f(d_{\mathcal{N}}(v)) \quad \leq \quad S_1 + S_2 .$$

where:

$$S_1 = \sum_{v\in\mathcal{N}} f(d_{\mathcal{N}}(v))\ \mathbb{1}_{v\in U_1} ,$$

and

$$S_2 = \sum_{v\in\mathcal{N}} f(d_{\mathcal{N}}(v))\ \mathbb{1}_{v\in U_2} .$$

Now, remark that if there exists $\mathbf{z}' \simeq \mathbf{z} \in \mathbf{A}^{r,1}(\gamma)$ such that $B^r_{\mathbf{z}'}$ is ugly, then $r\mathbf{z} + B^{3r}_0$ is bad, since it contains some ugly sub-box. Notice that $\{\vec{f_i} + 3\mathbb{Z}^d,\ i \in \{1, \dots, 3^d\}\}$ is a partition of $\mathbb{Z}^d$, so there is a unique pair $(i, \mathbf{z}'')$ in $\{1, \dots, 3^d\} \times \mathbb{Z}^d$ such that $\mathbf{z} = 3\mathbf{z}'' + \vec{f_i}$. Then, $r\mathbf{z} + B^{3r}_0 = 3r\mathbf{z}'' + B^{3r}_0 + 3r\vec{f_i}/3 = Bl^{3r,i}(\mathbf{z}'')$ is a bad box containing $B^r_{\mathbf{z}}$, and $\mathbf{z}'' \in \mathbf{A}^{3r,i}(\gamma)$. This gives:

$$U_1 \subset \bigcup_{i=1}^{3^d} \bigcup_{\substack{\mathbf{z}''\in\mathbf{A}^{3r,i}(\gamma)\\ Bl^{3r,i}(\mathbf{z}'') \text{ is bad}}} Bl^{3r,i}(\mathbf{z}'') .$$

And thus,

$$S_1 \leq \sum_{i=1}^{3^d} \sum_{\substack{\mathbf{z}\in\mathbf{A}^{3r,i}(\gamma)\\ Bl^{3r,i}(\mathbf{z}) \text{ is bad}}} \sum_{v\in\mathcal{N}\cap Bl^{3r,i}(\mathbf{z})} f(d_{\mathcal{N}}(v)) . \tag{3.2}$$

Notice that for $v \in \mathcal{N}$, the degree of v in the Delaunay graph, or equivalently, the number of $(d-1)$-dimensional faces of C_v, can be expressed as follows:

$$d_{\mathcal{N}}(v) = |\{u \in \mathcal{N} \text{ s.t. there exists a Delaunay cell containg } u \text{ and } v\}| \ . \tag{3.3}$$

When v belongs to some bad box $Bl^{3r,i}(\mathbf{z})$, Lemma 3.3 shows that every Delaunay cell to which v belongs is included in $Ad^{3r,i}(\mathbf{Cl}^{3r,i}(\mathbf{z}))$. Therefore, in this case, $d_{\mathcal{N}}(v)$ is bounded from above by $|Ad^{3r,i}(\mathbf{Cl}^{3r,i}(\mathbf{z}))|$. Since f is non-decreasing, we get that for every $(3r,i)$-bad $\mathbf{z}$:

$$\begin{aligned}
\sum_{v\in\mathcal{N}\cap Bl^{3r,i}(\mathbf{z})} f(d_{\mathcal{N}}(v)) &\leq \sum_{v\in\mathcal{N}\cap Bl^{3r,i}(\mathbf{z})} f(|Ad^{3r,i}(\mathbf{Cl}^{3r,i}(\mathbf{z}))|_{\mathcal{N}}) \ , \\
&= |Bl^{3r,i}(\mathbf{z})|_{\mathcal{N}} f(|Ad^{3r,i}(\mathbf{Cl}^{3r,i}(\mathbf{z}))|_{\mathcal{N}}) \ .
\end{aligned}$$

Plugging this into (3.2) gives:

$$\begin{aligned}
S_1 &\leq \sum_{i=1}^{3^d} \sum_{\substack{\mathbf{z}\in\mathbf{A}^{3r,i}(\gamma) \\ Bl^{3r,i}(\mathbf{z}) \text{ is bad}}} |Bl^{3r,i}(\mathbf{z})|_{\mathcal{N}} f(|Ad^{3r,i}(\mathbf{Cl}^{3r,i}(\mathbf{z}))|_{\mathcal{N}}) \ , \\
&= \sum_{i=1}^{3^d} \sum_{\mathbf{C}\in\mathbf{Cl}^{3r,i}(\mathbf{A}^{3r,i}(\gamma))} f(|Ad^{3r,i}(\mathbf{C})|_{\mathcal{N}}) \sum_{\substack{\mathbf{z}\in\mathbf{A}^{3r,i}(\gamma) \\ \mathbf{Cl}^{3r,i}(\mathbf{z})=\mathbf{C}}} |Bl^{3r,i}(\mathbf{z})|_{\mathcal{N}} \ , \\
&\leq \sum_{i=1}^{3^d} \sum_{\mathbf{C}\in\mathbf{Cl}^{3r,i}(\mathbf{A}^{3r,i}(\gamma))} f(|Ad^{3r,i}(\mathbf{C})|_{\mathcal{N}}) \sum_{\substack{\mathbf{z}\in\mathbb{Z}^d \\ \mathbf{Cl}^{3r,i}(\mathbf{z})=\mathbf{C}}} |Bl^{3r,i}(\mathbf{z})|_{\mathcal{N}} \ , \\
&= \sum_{i=1}^{3^d} \sum_{\mathbf{C}\in\mathbf{Cl}^{3r,i}(\mathbf{A}^{3r,i}(\gamma))} f(|Ad^{3r,i}(\mathbf{C})|_{\mathcal{N}}) |\mathbf{Cl}^{3r,i}(\mathbf{C})|_{\mathcal{N}} \ , \\
S_1 &\leq \sum_{i=1}^{3^d} \sum_{\mathbf{C}\in\mathbf{Cl}^{3r,i}(\mathbf{A}^{3r,i}(\gamma))} |Ad^{3r,i}(\mathbf{C})|_{\mathcal{N}} f(|Ad^{3r,i}(\mathbf{C})|_{\mathcal{N}}) \ , \qquad (3.4)
\end{aligned}$$

where we used the fact that f is positive in the last inequality. The last series of inequalities allows us to bound a sum of dependent variables (the first one) by a sum of variables (the last one) that resembles a greedy animal on $\mathbb{Z}^d$. This will be made clear with Lemma 3.5. Now, let us bound S_2. If $\mathbf{z} \in \mathbb{Z}^d$ is such that for every neighbor $\mathbf{z}'$ of $\mathbf{z}$, $B^r_{\mathbf{z}'}$ is nice, then Lemma 3.2 shows that any Delaunay cell touching $B^r_{\mathbf{z}}$ is included inside $Ad^{r,1}(\mathbf{z})$, which is itself included in $\bigcup_{\mathbf{z}'\simeq\mathbf{z}} B^r_{\mathbf{z}'}$. Thus, using equality (3.3):

$$\begin{aligned}
S_2 &\leq \sum_{\mathbf{z}\in\mathbf{A}^{r,1}(\gamma)} \sum_{v\in\mathcal{N}\cap B^r_{\mathbf{z}}} f(|\bigcup_{\mathbf{z}'\simeq\mathbf{z}} B^r_{\mathbf{z}'}|_{\mathcal{N}}) \ , \\
&= \sum_{\mathbf{z}\in\mathbf{A}^{r,1}(\gamma)} |B^r_{\mathbf{z}}|_{\mathcal{N}} f(|\bigcup_{\mathbf{z}'\simeq\mathbf{z}} B^r_{\mathbf{z}'}|_{\mathcal{N}}) \ , \\
&\leq \sum_{\mathbf{z}\in\mathbf{A}^{r,1}(\gamma)} |\bigcup_{\mathbf{z}'\simeq\mathbf{z}} B^r_{\mathbf{z}'}|_{\mathcal{N}} f(|\bigcup_{\mathbf{z}'\simeq\mathbf{z}} B^r_{\mathbf{z}'}|_{\mathcal{N}}) \ .
\end{aligned}$$

Again, $\{\vec{f_i} + 3\mathbb{Z}^d, \ i \in \{1,\dots,3^d\}\}$ is a partition of $\mathbb{Z}^d$, and for every $\mathbf{z}$ there is a unique couple $(i, \mathbf{z}'')$ in $\{1,\dots,3^d\} \times \mathbb{Z}^d$ such that $\bigcup_{\mathbf{z}'\simeq\mathbf{z}} B^r_{\mathbf{z}'} = Bl^{3r,i}(\mathbf{z}'')$, in which case $\mathbf{z}'' \in \mathbf{A}^{3r,i}(\gamma)$. Thus,

$$S_2 \leq \sum_{i=1}^{3^d} \sum_{\mathbf{z}\in\mathbf{A}^{3r,i}(\gamma)} |Bl^{3r,i}(\mathbf{z})|_{\mathcal{N}} f(|Bl^{3r,i}(\mathbf{z})|_{\mathcal{N}}) \ . \tag{3.5}$$

Now, the lemma follows from inequalities (3.4) and (3.5). □

3.2 The chaining lemma

The aim of this section is to get good bounds for the probabilities appearing in the right hand side of the inequality in Lemma 3.4. The following lemma is an adaptation of the technique of [4] to control the tail of the greedy animals on $\mathbb{Z}^d$. Notice that Proposition 1 in [4] is not enough for us. This stems from the fact that in our setting, according to which weight function f we have, we may hope for better deviation inequalities, simply because the award may have thinner tails than the minimal ones considered in [4]. Unfortunately, this necessitates to go over again the whole proof, adjusting the chaining argument. Since this makes the proof quite long and technical, we defer it until section 5.1.

For each positive integer $m \geq 1$ let:

$$\Phi_m = \{\mathbf{A} \text{ animal in } \mathbb{G}_d \text{ s.t. } |\mathbf{A}| \leq m,\ \mathbf{0} \in \mathbf{A}\}\ .$$

Lemma 3.5. *Let g be a nondecreasing function from $\mathbb{R}^+$ to $[1,\infty)$, and define:*

$$l(y) := g^{-1}(y)\,,\ q(x) := \hat{l}(x)\,.$$

There exists $r_0 > 0$ such that for all $r > r_0$ there exist positive and finite constants c_5, c_6 and c_7 such that, if:

$$\liminf_{y\to\infty} \frac{l(y)}{\log y} \geq c_5,$$

then for every $n \geq m$,

$$\mathbb{P}(\sup_{\mathbf{A}\in\Phi_m} \sum_{\mathbf{C}\in\mathbf{Cl}^{r,i}(\mathbf{A})} g(|Ad^{r,i}(\mathbf{C})|_{\mathcal{N}}) > c_6 n) \leq e^{-c_7 l(q(n))}\ , \tag{3.6}$$

and

$$\mathbb{P}(\sup_{\mathbf{A}\in\Phi_m} \sum_{\mathbf{z}\in\mathbf{A}} g(|Bl^{r,i}(\mathbf{z})|_{\mathcal{N}}) > c_6 n) \leq e^{-c_7 l(q(n))}\ . \tag{3.7}$$

3.3 Proof of Theorem 2.2

We shall require the following result from [14] (Theorem 1):

Lemma 3.6. *There exist finite and positive constants z_2 and c_{13} such that for every $r \geq 1$, and every $i = 1, \ldots, 3^d$,*

$$\forall z \geq z_2,\ \mathbb{P}(\max_{\mathcal{P}\in\Pi_n} |\mathbf{A}^{r,i}(\mathcal{P})| > zn) \leq e^{-c_{13} zn}\ .$$

Let us prove Theorem 2.2. First,

$$\mathbb{E}(F_n) = \mathbb{E}[\max_{\mathcal{P}\in\Pi_n} \sum_{v\in\mathcal{P}} f(d_{\mathcal{N}}(v))] \quad = \quad n\int_0^\infty \mathbb{P}(\max_{\mathcal{P}\in\Pi_n} \sum_{v\in\mathcal{P}} f(d_{\mathcal{N}}(v)) > nz)\, dz\ .$$

Let K be a positive real number and $r \geq 1$. We then have

$$\begin{aligned}
\mathbb{P}(\max_{\mathcal{P}\in\Pi_n} \sum_{v\in\mathcal{P}} f(d_{\mathcal{N}}(v)) > nz) \quad &\leq \quad \mathbb{P}(\max_{\mathcal{P}\in\Pi_n} |\mathbf{A}_1^r(\mathcal{P})| > Knz) \\
&+ \quad \mathbb{P}(\max_{\mathcal{P}\in\Pi_n} \sum_{v\in\mathcal{P}} f(d_{\mathcal{N}}(v)) > nz \text{ and } \max_{\mathcal{P}\in\Pi_n} |\mathbf{A}_1^r(\mathcal{P})| \leq \lfloor Knz \rfloor) \\
&\leq \quad \mathbb{P}(\max_{\mathcal{P}\in\Pi_n} |\mathbf{A}_1^r(\mathcal{P})| > Knz) \\
&+ \quad \mathbb{P}(\sup_{\mathbf{A}\in\Phi_{\lfloor Knz \rfloor}} \sum_{v\in Bl^{r,1}(\mathbf{A})\cap\mathcal{N}} f(d_{\mathcal{N}}(v)) > nz)\ . \qquad (3.8)
\end{aligned}$$

Thus,

$$\begin{aligned}\mathbb{E}[\sum_{v\in\mathcal{P}} f(d_{\mathcal{N}}(v))] &\le \frac{1}{K}\mathbb{E}(\max_{\mathcal{P}\in\Pi_n}|\mathbf{A}_1^r(\mathcal{P})|)\\ &+ n\int_0^\infty \mathbb{P}(\sup_{\mathbf{A}\in\Phi_{\lfloor Knz\rfloor}}\sum_{v\in Bl^{r,1}(\mathbf{A})\cap\mathcal{N}} f(d_{\mathcal{N}}(v))>nz)\,dz\,,\\ &\le \frac{1}{K}\left(\frac{1}{c_{13}}+nz_2\right)\\ &+ n\int_0^\infty \mathbb{P}(\sup_{\mathbf{A}\in\Phi_{\lfloor Knz\rfloor}}\sum_{v\in Bl^{r,1}(\mathbf{A})\cap\mathcal{N}} f(d_{\mathcal{N}}(v))>nz)\,dz\,, \qquad (3.9)\end{aligned}$$

thanks to Lemma 3.6.

Now, let t be any positive number and m a positive integer. Using Lemma 3.4,

$$\begin{aligned}&\mathbb{P}(\sup_{\mathbf{A}\in\Phi_m}\sum_{v\in Bl^{r,1}(\mathbf{A})\cap\mathcal{N}} f(d_{\mathcal{N}}(v))>t)\\ \le\ & \sum_{i=1}^{3^d}\mathbb{P}\left[\sup_{\mathbf{A}\in\Phi_m}\sum_{\mathbf{C}\in\mathbf{Cl}^{3r,i}(\mathbf{A}^{3r,i}(Bl^{r,1}(\mathbf{A})))}|Ad^{3r,i}(\mathbf{C})|_{\mathcal{N}}f(|Ad^{3r,i}(\mathbf{C})|)>\frac{t}{3^d.2}\right]\\ &+\sum_{i=1}^{3^d}\mathbb{P}\left[\sup_{\mathbf{A}\in\Phi_m}\sum_{\mathbf{z}\in\mathbf{A}^{3r,i}(Bl^{r,1}(\mathbf{A}))}|Bl^{3r,i}(\mathbf{z})|_{\mathcal{N}}f(|Bl^{3r,i}(\mathbf{z})|_{\mathcal{N}})>\frac{t}{3^d.2}\right]\,,\end{aligned}$$

Notice that for each $\mathbf{A}$ in Φ_m, $\mathbf{A}^{3r,i}(Bl^{r,1}(\mathbf{A}))$ will be an animal containing $\mathbf{0}$ and with less vertices than $\mathbf{A}$. Thus, there is some $\mathbf{A}'$ in Φ_m such that $\mathbf{A}^{3r,i}(Bl^{r,1}(\mathbf{A}))\subset\mathbf{A}'$, which yields to

$$\begin{aligned}&\mathbb{P}(\sup_{\mathbf{A}\in\Phi_m}\sum_{v\in Bl^{r,1}(\mathbf{A})\cap\mathcal{N}} f(d_{\mathcal{N}}(v))>t)\\ \le\ & \sum_{i=1}^{3^d}\mathbb{P}\left[\sup_{\mathbf{A}'\in\Phi_m}\sum_{\mathbf{C}\in\mathbf{Cl}^{3r,i}(\mathbf{A}')}|Ad^{3r,i}(\mathbf{C})|_{\mathcal{N}}f(|Ad^{3r,i}(\mathbf{C})|)>\frac{t}{3^d.2}\right]\\ &+\sum_{i=1}^{3^d}\mathbb{P}\left[\sup_{\mathbf{A}'\in\Phi_m}\sum_{\mathbf{z}\in\mathbf{A}'}|Bl^{3r,i}(\mathbf{z})|_{\mathcal{N}}f(|Bl^{3r,i}(\mathbf{z})|_{\mathcal{N}})>\frac{t}{3^d.2}\right]\,. \qquad (3.10)\end{aligned}$$

Now, if

$$\liminf_{y\to\infty}\frac{\hat{f}(y)}{\log y}\ge c_5\,,$$

then inequality (3.10) and Lemma 3.5 imply:

$$\mathbb{P}(\sup_{\mathbf{A}\in\Phi_m}\sum_{v\in Bl^{r,1}(\mathbf{A})\cap\mathcal{N}} f(d_{\mathcal{N}}(v))>c_6 m)\le 3^d.2e^{-c_7 f^*(m)}\,.$$

Therefore, fix $K=\frac{1}{c_6}$, $c_1=\sup\{\frac{2}{c_7},c_5\}$ and suppose that

$$\liminf_{y\to\infty}\frac{\hat{f}(y)}{\log y}\ge c_1\,.$$

This implies that $e^{-c_7 f^*(x)}$ is integrable on $\mathbb{R}^+$. Thus,

$$\mathbb{P}(\sup_{\mathbf{A}\in\Phi_{\lfloor Knz\rfloor}}\sum_{v\in Bl^{r,1}(\mathbf{A})\cap\mathcal{N}} f(d_{\mathcal{N}}(v))>nz)\quad\le\quad 3^d.2e^{-c_7 f^*(Knz)}\,,$$

and then,

$$\int_0^\infty \mathbb{P}(\sup_{\mathbf{A}\in\Phi_{\lfloor Knz\rfloor}} \sum_{v\in Bl^{r,1}(\mathbf{A})\cap\mathcal{N}} f(d_\mathcal{N}(v)) > nz)\, dz = O\left(\frac{1}{n}\right) .$$

Plugging these bounds into (3.8), (3.9) leads to the desired result. □

3.4 Sketch of the proof of Proposition 2.5

It turns out that the proof of Proposition 2.5 is very similar to that of Theorem 2.2, so we shall only stress the differences. First, we need to see that the notion of a nice box is still a good one to control $\Gamma(\mathcal{N},\phi)$. To do that, we need to express $\Gamma(\mathcal{N},\phi)$ differently. Let $\mathcal{E}(\phi)$ be the set of edges of ϕ. Then,

$$\Gamma(\mathcal{N},\phi) = \bigcup_{e\in\mathcal{E}(\phi)} \Gamma(\mathcal{N},e) ,$$

where:

$$\Gamma(\mathcal{N},e) = \{x\in\mathbb{R}^d \text{ s.t. } e \not\subset \mathcal{D}(\mathcal{N}\cup\{x\}\} .$$

Recall that $B(\Delta)$ denotes the closed circumball of a Delaunay cell Δ. It may be seen that (cf. Lemma 5.6):

$$\Gamma(\mathcal{N},e) = \bigcap_{\Delta\ni e} \overset{o}{B}(\Delta) ,$$

where the intersection is taken over all Delaunay cells Δ which contain e. We get easily the following result from Lemma 3.2.

Lemma 3.7. *Fix $r>0$ and $i\in\{0,\dots,3^d\}$, assume that $\mathbf{C}$ is an (r,i)-ugly cluster in $\mathbb{G}_d$. Let ϕ be a self-avoiding path in $\mathcal{D}(\mathcal{N})$. Then, for any vertex v in ϕ, and any edge $e\in\mathcal{E}(\phi)$ such that $v\in e$,*

$$v\in Bl^{r,i}(\mathbf{C}) \Rightarrow \Gamma(\mathcal{N},e)\subset Ad^{r,i}(\mathbf{C}) .$$

The same holds for bad clusters.

Proof. We may drop r and i for simplicity. Let B' be any circumball of a Delaunay cell containing v. Then, $\mathbf{C}$ and B' satisfy the hypotheses of Lemma 3.2, and thus $B'\subset Ad(\mathbf{C})$. Thus, for any edge $e\in\mathcal{E}(\phi)$ such that $v\in e$, we have $\Gamma(\mathcal{N},e)\subset Ad(\mathbf{C})$. □

Notice that a vertex in $\mathbb{G}_d$ has 3^d-1 neighbours. Thus, the number of points in the union of a cluster C and its exterior boundary is at most $3^d|C|$. Lemma 3.7 shows that when $v\in Bl^{3r,i}(\mathbf{C})$, then, $\mu(\Gamma(\mathcal{N},e))\le\mu(Ad^{3r,i}(\mathbf{C}))\le c_\mu(3r)^d3^d|\mathbf{C}|$. So we obtain the following analogue of Lemma 3.4.

Lemma 3.8.

$$\begin{aligned}\mathbb{P}[\max_{\phi\in\mathcal{SA}_n}\mu(\Gamma(\mathcal{N},\phi))>t] \quad &\le \quad \sum_{i=1}^{3^d}\mathbb{P}\left[\max_{\phi\in\mathcal{SA}_n}\sum_{\mathbf{C}\in\mathbf{Cl}^{3r,i}(\mathbf{A}^{3r,i}(\phi))}|\mathbf{C}|>\frac{t}{3^d.2.c_\mu(9r)^d}\right]\\ &+\quad \sum_{i=1}^{3^d}\mathbb{P}\left[\max_{\phi\in\mathcal{SA}_n}|\mathbf{A}^{3r,i}(\phi)|>\frac{t}{3^d.2.c_\mu(9r)^d}\right] .\end{aligned}$$

The rest of the proof is similar to that of Theorem 2.2. Notice first that $\mathcal{SA}_n\subset\Pi_n$. Lemma 3.6 implies that for all $r\ge1$ and $z\ge z_2$,

$$\sum_{i=1}^{3^d}\mathbb{P}\left[\max_{\phi\in\mathcal{SA}_n}|\mathbf{A}^{3r,i}(\phi)|>zn\right]\le 3^de^{-c_{13}zn} . \tag{3.11}$$

Now we fix $r = r_0$ large enough so that we are in the sub-critical phase for the percolation of clusters of bad boxes (see the beginning of section 5.1 for more details). Then, there is some $\lambda > 0$ such that:

$$\mathbb{E}[e^{\lambda|\mathbf{Cl}(0)|}] =: M < \infty \ .$$

Lemma 3.6 shows that for any $z \geq z_2$ and any i:

$$\mathbb{P}\left[\max_{\phi\in\mathcal{SA}_n} \sum_{\mathbf{C}\in\mathbf{Cl}^{3r,i}(\mathbf{A}^{3r,i}(\phi))} |\mathbf{C}| > zn\right] \leq e^{-c_{13}zn} + \mathbb{P}\left[\max_{\Lambda\in\Phi_{\lfloor nz\rfloor}} \sum_{\mathbf{C}\in\mathbf{Cl}(\Lambda)} |\mathbf{C}| > zn\right] \tag{3.12}$$

Now, using Lemma 5.2, one gets

$$\begin{aligned}
\mathbb{P}\left[\max_{\Lambda\in\Phi_{\lfloor nz\rfloor}} \sum_{\mathbf{C}\in\mathbf{Cl}(\Lambda)} |\mathbf{C}| > zn\right] &\leq \sum_{\Lambda\in\Phi_{\lfloor nz\rfloor}} \mathbb{P}\left[\sum_{\mathbf{C}\in\mathbf{Cl}(\Lambda)} |\mathbf{C}| > zn\right] \ , \\
&\leq \sum_{\Lambda\in\Phi_{\lfloor nz\rfloor}} e^{-\lambda nz}\mathbb{E}\left[e^{\lambda\sum_{\mathbf{C}\in\mathbf{Cl}(\Lambda)}|\mathbf{C}|}\right] \ , \\
&\leq \sum_{\Lambda\in\Phi_{\lfloor nz\rfloor}} e^{-\lambda nz}M^{|\Lambda|} \ .
\end{aligned}$$

It is well known that there is a finite constant $K > 1$, depending on the dimension such that $|\Phi_k| \leq K^k$ for every k. Thus:

$$\mathbb{P}\left[\max_{\Lambda\in\Phi_{\lfloor nz\rfloor}} \sum_{\mathbf{C}\in\mathbf{Cl}(\Lambda)} |\mathbf{C}| > zn\right] \leq (KM)^{nz}e^{-\lambda nz} \ .$$

Thus, for any z larger that $2\log(KM)/\lambda$,

$$\mathbb{P}\left[\max_{\Lambda\in\Phi_{\lfloor nz\rfloor}} \sum_{\mathbf{C}\in\mathbf{Cl}(\Lambda)} |\mathbf{C}| > zn\right] \leq e^{-\lambda nz/2} \ . \tag{3.13}$$

Gathering equations (3.11), (3.12) and (3.13), we conlude that for z large enough, and for $c = \min\{c_{13}/(3^d.2.c_\mu(9r)^d), \lambda/2\}$, we have for any n:

$$\mathbb{P}[\max_{\phi\in\mathcal{SA}_n} \mu(\Gamma(\mathcal{N},\phi)) > t] \leq 3^{d+1}e^{-cnz} \ .$$

4 Applications

4.1 The connectivity constant on the Delaunay triangulation

Problems related to self-avoiding paths are connected with various branches of applied mathematics such as long chain polymers, percolation and ferromagnetism. One fundamental problem is the asymptotic behavior of the connective function κ_n, defined by the logarithm of the number N_n of self-avoiding paths (on a fixed graph $\mathcal{G}$) starting at vertex v and with n steps. For planar and periodic graphs subadditivity arguments yields that $n^{-1}\kappa_n$ converges, when $n \to \infty$, to some value $\kappa \in (0, \infty)$ (the connectivity constant) independent of the initial vertex v. In disordered planar graphs subadditivity is lost but, if the underline graph possess some statistical symmetries (ergodicity), we may believe that the rescaled connective function still converges to some constant. When x is in $\mathbb{R}^d$, let us denote by $v(x)$ the (a.s unique) point of $\mathcal{N}$ such that $x \in C_v$. Let N_n denote the number of self-avoiding paths of length n starting at $v(0)$ on the Delaunay triangulation of a Poisson random set $\mathcal{N}$. We recall that the intensity of the Poisson random set is bounded from above and below by a constant times the Lebesgue measure on $\mathbb{R}^d$. From Theorem 2.2 we obtain a linear upper bound for the connective function of the Delaunay triangulation:

Proposition 4.1. *Let $\kappa_n = \log N_n$. There exist positive constants z_3 and c_2 such that, for every $u \geq nz_3$,*

$$\mathbb{P}(\kappa_n \geq u) \leq e^{-c_2 u^{1/3}} \ .$$

In particular,

$$\mathbb{E}(\kappa_n) = O(n) \ .$$

Proof. We shall use the following intuitive inequality which is a consequence of Lemma 4.2 below.

$$N_n \leq \sup_{\mathcal{P} \in \Pi_{n-1}} \prod_{v \in \mathcal{N} \cap \mathcal{P}} d_{\mathcal{N}}(v) \ .$$

We now deduce the following:

$$\begin{aligned} \mathbb{P}(N_n \geq t) &\leq \mathbb{P}(\sup_{\mathcal{P} \in \Pi_{n-1}} \prod_{v \in \mathcal{N} \cap \mathcal{P}} d_{\mathcal{N}}(v) \geq t) \ , \\ &= \mathbb{P}(\sup_{\mathcal{P} \in \Pi_{n-1}} \sum_{v \in \mathcal{N} \cap \mathcal{P}} \log(d_{\mathcal{N}}(v)) \geq \log t) \ , \\ &\leq \mathbb{P}(\sup_{\mathcal{P} \in \Pi_{n-1}} \sum_{v \in \mathcal{N} \cap \mathcal{P}} d_{\mathcal{N}}(v) \geq \log t) \ . \end{aligned}$$

Now, notice that $f(x) = x$ is $c-$nice for every $c > 0$. Thus, according to Theorem 2.2, there are positive constants z_3 and c_2 such that, for all $t \geq e^{rz_3}$,

$$\mathbb{P}(N_n \geq t) \leq e^{-c_2 (\log t)^{1/3}} \ .$$

Or, equivalently, for every $u \geq rz_3$,

$$\mathbb{P}(\kappa_n \geq u) \leq e^{-c_2 u^{1/3}} \ .$$

This implies notably that:

$$\mathbb{E}(\kappa_n) = O(n) \ .$$

□

Lemma 4.2. *Let $G = (V, E)$ be a graph with set of vertices V and set of edges E. Define, for any $n \in \mathbb{N}$, any $x \in V$ and $I \subset V$:*

$$\Delta_n(x, I) = \{ \text{ s.a paths } \gamma \text{ with } n \text{ edges and s.t. } \gamma_0 = x, \ \gamma \cap I = \emptyset \} \ .$$

Define:

$$N_n(x, I) = |\Delta_n(x, I)| \ .$$

Then, for any $n \geq 1$,

$$\forall x \in V, \ \forall I \subset V, \ N_n(x, I) \leq \sup_{\gamma \in \Delta_{n-1}(x, I)} \prod_{v \in \gamma} d(v) \ ,$$

where $d(v)$ stands for the degree of the vertex v.

Proof. We shall prove the result by induction. When $n = 1$ it is obviously true. Indeed, if $x \in I$, then $N_1(x, I) = 0$ and if $x \notin I$, $N_1(x, I) \le d(x)$. Suppose now that the result is true until $n \ge 1$. Let $x \in V$ and $I \subset V$. If $x \in I$, then $N_{n+1}(x, I) = 0$. Suppose thus that $x \notin I$. Then, denoting $u \sim x$ the fact that u and x are neighbours, and using the induction hypothesis,

$$\begin{aligned} N_{n+1}(x,I) &= \sum_{u\sim x} N_n(u, I\cup\{x\}) , \\ &\le \sum_{u\sim x} \sup_{\gamma\in\Delta_{n-1}(u,I\cup\{x\})} \prod_{v\in\gamma} d(v) , \\ &\le \sum_{u\sim x} \sup_{u\sim x} \sup_{\gamma\in\Delta_{n-1}(u,I\cup\{x\})} \prod_{v\in\gamma} d(v) , \\ &= \sup_{u\sim x} \sup_{\gamma\in\Delta_{n-1}(u,I\cup\{x\})} d(x) \prod_{v\in\gamma} d(v) , \\ &= \sup_{\gamma'\in\Delta_n(x,I)} \prod_{v\in\gamma'} d(v) , \end{aligned}$$

and the induction is proved. □

4.2 First passage percolation on the Poisson-Delaunay graph

On any graph $\mathcal{G}$, one can define a First Passage Percolation model (FPP in the sequel) as follows. Assign to each edge e of $\mathcal{G}$ a random non-negative time $t(e)$ necessary for a particle to cross edge e. The First Passage time from a vertex u to a vertex v on $\mathcal{G}$ is the minimal time needed for a particle to go from u to v following a path on $\mathcal{G}$. This is of course a random time, and the understanding of the typical fluctuations of this times when u and v are far apart is of fundamental importance, notably because of its physical interpretation as a growth model. The model was first introduced by [7] on $\mathbb{Z}^d$ with i.i.d passage times. Some variations on this model have been proposed and studied, see [9] for a review. One of these variations is FPP on the Delaunay triangulation, introduced in [15], where the graph $\mathcal{G}$ is the Delaunay graph of a Lebesgue-homogeneous Poisson random set in $\mathbb{R}^d$. Often, this Delaunay triangulation is heuristically considered to behave like a perturbation of the triangular lattice. One important question is whether the fact that $\mathcal{G}$ itself is random affects the fluctuations of the first passage times. Since it is not already known exactly what is the right order of these fluctuations on the triangular lattice, one does not have an exact benchmark, but the best bound known up today is that the variance of the passage time between the origin and a vertex at distance n is of order $O(n/\log n)$. We are not able to show a similar bound for FPP on the Delaunay triangulation, but we can show the analogue of Kesten's bound of [11]: the above variance is at most of order $O(n)$. This improves upon the results in [13] and answers positively to open problem 12, p. 169 in [9]. To prove this result, we shall also need to bound from above the length of the optimal path, as was done in Proposition 5.8 in [10] for the classical model. This requires a separate argument, given in section 4.2.1. The bound on the variance will be given in section 4.2.2.

4.2.1 Linear passage weights of self-avoiding paths in percolation

When $\mathcal{D}$ is a fixed Delaunay triangulation on the plane, the bond percolation model $X = \{X_e : e \in \mathcal{D}\}$ on the Delaunay triangulation, with parameter p and probability law $\mathbb{P}_p(.|\mathcal{D})$, is constructed by choosing each edge $e \in \mathcal{D}$ to be open, or equivalently $X_e = 1$, independently with probability p and closed, equivalently $X_e = 0$, otherwise. An open path is a path composed by open edges. We denote by $\mathbb{P}_p$ the probability measure obtained when $\mathcal{D}$ is the Delaunay triangulation based on $\mathcal{N}$, which is distributed like a Poisson random set on $\mathbb{R}^d$ with intensity measure μ. We denote by $\mathcal{C}_\mathbf{0}$ the maximal connected sugraph of $\mathcal{D}$ composed by edges e that belong to some open path starting from $v_\mathbf{0}$, the point of $\mathcal{N}$ whose Voronoi tile contains the origin, and we call it the open cluster. Define the critical probability:

$$\bar{p}_c(d) = \sup\Big\{p \in [0,1] \,:\, \forall \alpha > 0 \sum_{n\geq 1} n^\alpha \mathbb{P}_p(|\mathcal{C}_\mathbf{0}| = n) < \infty\Big\}. \tag{4.1}$$

It is extremely plausible that this critical probability coincides with the classical one, i.e $\sup\{p \text{ s.t. } \mathbb{P}_p(|\mathcal{C}_\mathbf{0}| = \infty) = 0\}$, but we are not going to prove this here. By using Proposition 4.1 to count the number of self-avoiding paths of size n, and then evaluating the probability that it is an open path, there exists a positive and finite constant $B = B(d)$ such that if $p < 1/B$ then the probability of the event $\{|\mathcal{C}_\mathbf{0}| = n\}$ decays like $e^{-\alpha n^{1/(3d)}}$ for some constant α. Consequently:

Lemma 4.3. $0 < \frac{1}{B} \leq \bar{p}_c(d)$.

The main result of this section is the following estimate on the density of open edges.

Theorem 4.4. *If $(1-p) < \bar{p}_c(d)$ then there exists constants (only depending on p) $a_1, a_2 > 0$ such that*

$$\mathbb{P}_p\Big(\exists \text{ s.a. } \gamma \text{ s.t. } |\gamma| \geq m \text{ and } \sum_{e\in\gamma} X_e \leq a_1 m\Big) \leq e^{-a_2 m}\,.$$

Remark 4.5. *We could prove Theorem 4.4 directly under $(1-p) < 1/B$ (again using Proposition 4.1) but we want to push optimality as far as we can.*

To prove Theorem 4.4 we shall first obtain a control on the number of boxes intersected by a self-avoiding path in $\mathcal{D}$. Recall that, in section 3.1, for fixed $L > 0$ and for each self-avoiding path γ in $\mathcal{D}$ we have defined an animal $\mathbf{A}^r(\gamma) := \mathbf{A}^{r,1}(\gamma)$ on $\mathbb{G}^d$ by taking the vertices $\mathbf{z}$ such that γ intersects $B^r_\mathbf{z}$. By Corollary 2 in [14] we have:

Lemma 4.6. *For each $r \geq 1$ there exist finite and positive constants b_3, b_4, b_5, b_6 such that for any $x \geq 0$ (check reference)*

$$\mathbb{P}\left(\min_{\mathbf{v}_0\in\gamma\,,|\gamma|\geq r} |\mathbf{A}^r(\gamma)| < b_3 x\right) \leq e^{-b_4 x}\,, \tag{4.2}$$

and

$$\mathbb{P}\left(\max_{\mathbf{v}_0\in\gamma\,,|\gamma|\leq r} |\mathbf{A}^r(\gamma)| > b_5 x\right) \leq e^{-b_6 x}\,. \tag{4.3}$$

Proof of Theorem 4.4: Let $L > 0$, $\mathbf{z} \in \mathbb{G}^d$. In this section, we call $B^{L/2}_\mathbf{z}$ a *good box* if:

(i) For all $\mathbf{z}' \in \mathbb{G}^d$ with $|\mathbf{z} - \mathbf{z}'|_\infty = 2$, $B^{L/2}_{\mathbf{z}'}$ is $\mathcal{N}$-nice (recall Definition 3.1),

(ii) For all γ in $\mathcal{D}$ connecting $B^{L/2}_\mathbf{z}$ to $\partial B^{3L/2}_\mathbf{z}$ we have $\sum_{e\in\gamma} X_e \geq 1$.

Let

$$Y_\mathbf{z}(L) = \mathbb{1}_{B^{L/2}_\mathbf{z} \text{ is a good box}}\,.$$

Our first claim is: if $1 - p < \bar{p}_c(d)$ then

$$\lim_{L\to\infty} \mathbb{P}_p\big(Y_\mathbf{z}(L) = 1\big) = 1\,. \tag{4.4}$$

Since the intensity of the underlying Poisson random set is comparable with the Lebesgue measure (2.1), condition (i) has probability going to one as L goes to infinity. Now, denote by E_L the event that (ii) is false and (i) is true. Suppose that E_L occurs, and let γ be a path contradicting (ii). We may choose the first edge of γ as some $[v_1, v_2]$ intersecting $B_{\mathbf{z}}^{3L/2}$. Thanks to Lemma 3.3, and since (i) is true, $[v_1, v_2]$ lies in $B_{\mathbf{z}}^{5L/2}$. Also, there is a point v' such that either v_1 or v_2 lies at distance at least L from v' and γ connects v_1 and v_2 to v' with closed edges. Divide $B_{\mathbf{z}}^{5L/2}$ regularly into subcubes of side length 1. This partition $\mathcal{P}_L$ has cardinality of order L^d. For each box B, let $A(B)$ be the event that there exist a vertex $v \in \mathcal{N} \cap B$, $v' \in \mathcal{N}$ such that $|v - v'| \geq L$, and a path γ in $\mathcal{D}$ from v to v' such that $\sum_{e\in\gamma} X_e = 0$. Denote by A_L the event $A(B_{\mathbf{0}}^{1/2})$. The remarks we just made imply that:

$$\mathbb{P}_p\big(E_L\big) \leq \sum_{B\in\mathcal{P}_L} \mathbb{P}_p\big(A(B)\big) \leq cL^d \mathbb{P}_p\big(A_L\big) .$$

Now, let us bound $\mathbb{P}_p\big(A_L\big)$. If $p = 1$, $\mathbb{P}_p\big(A_L\big) = 0$, so we suppose that $p < 1$. Let $\mathbf{B}_d$ denote the ball of center $\mathbf{0}$ and radius $3\sqrt{d}$, and let $\mathcal{E}_d$ be the set of edges of $\mathcal{D}$ lying completely inside $\mathbf{B}_d$. A simple geometric lemma (see Lemma 5.5 in section 5.2) shows that if $v \in N \cap B_0^{1/2}$, there is a path from $v_{\mathbf{0}}$ to v on $\mathcal{D}$ whose edges are in $\mathcal{E}_d$. We can further restrict this path to belong to a spanning tree $\mathcal{T}_d$ of the connected component of $v_{\mathbf{0}}$ in $\mathcal{E}_d$. Thus, if A_L holds and if all the edges of $\mathcal{T}_d$ are closed, then there is a closed path from $v_{\mathbf{0}}$ to some vertex v' such that $|v'| \geq L - \sqrt{2}$. Let L be larger than $2\sqrt{2}$. Using (4.3) (with $l = 1$ and r a constant times L), there exist constants $a, b >$ such that, with probability greater than $(1 - e^{-aL})$, a path γ in the Poisson-Delaunay triangulation, that connects $v_{\mathbf{0}}$ to a point v' such that $|v'| \geq L - \sqrt{2}$, has at least bL edges. Now, using the FKG inequality, we have:

$$\mathbb{P}_p(A_L|\mathcal{D}) \leq \mathbb{P}_p(A_L|\{\forall e \in \mathcal{D},\ X_e = 0\},\ \mathcal{D}) \leq \frac{1}{(1-p)^{|\mathcal{T}_d|}} \left(\mathbb{P}_p(|\mathbf{C}_{\mathbf{0}}^{cl}| \geq bL|\mathcal{D}) + e^{-aL}\right) ,$$

where $\mathbf{C}_{\mathbf{0}}^{cl}$ is the *closed* cluster containing $\mathbf{0}$. Thus:

$$\mathbb{P}_p(A_L) \leq \mathbb{E}\left[(1-p)^{-|\mathcal{T}_d|}\left(\mathbb{P}_p(|\mathbf{C}_{\mathbf{0}}^{cl}| \geq bL|\mathcal{D}) + e^{-aL}\right)\right] .$$

Notice that for any $x > 0$,

$$\begin{aligned} \mathbb{E}\left[(1-p)^{-|\mathcal{T}_d|}\mathbb{P}_p(|\mathbf{C}_{\mathbf{0}}^{cl}| \geq bL|\mathcal{D})\right] &\leq \mathbb{E}\left[(1-p)^{-|\mathcal{T}_d|}\ \mathbb{1}_{|\mathcal{T}_d|\geq x}\right] , \\ &\quad +\mathbb{E}\left[(1-p)^{-|\mathcal{T}_d|}\ \mathbb{1}_{|\mathcal{T}_d|<x}\mathbb{P}_p(|\mathbf{C}_{\mathbf{0}}^{cl}| \geq bL|\mathcal{D})\right] , \\ &\leq \mathbb{E}\left[(1-p)^{-|\mathcal{T}_d|}\ \mathbb{1}_{|\mathcal{T}_d|\geq x}\right] + (1-p)^{-x}\mathbb{P}_p(|\mathbf{C}_{\mathbf{0}}^{cl}| \geq bL) . \end{aligned}$$

Now, $|\mathcal{T}_d|$ is less than $|\mathbf{B}_d|_{\mathcal{N}}$ and notice that $|\mathbf{B}_d|_{\mathcal{N}}$ has Poisson distribution with parameter $\mu(\mathbf{B}_d)$. Thus, one can find some positive constants C_1, C_2 and C_3, depending only on μ and d, such that, for every $L > 0$:

$$\mathbb{P}(|\mathcal{T}_d| \geq C_1 \log(L)) \leq \mathbb{P}(|\mathbf{B}_d|_{\mathcal{N}} \geq C_1 \log(L)) \leq C_2 L^{-2(d+1)} . \tag{4.5}$$

Now, using Cauchy-Schwartz inequality, and inequality (4.5),

$$\begin{aligned} \mathbb{P}_p(A_L) &\leq e^{-aL}\mathbb{E}\left[(1-p)^{-|\mathcal{T}_d|}\right] \\ &\quad +\sqrt{\mathbb{E}\left[(1-p)^{-2|\mathcal{T}_d|}\right]}.\sqrt{\mathbb{P}(|\mathcal{T}_d| \geq C_1\log(L))} + L^{C_1 \log\frac{1}{1-p}}\mathbb{P}_p(|\mathbf{C}_{\mathbf{0}}^{cl}| \geq bL) , \\ &\leq e^{-aL}C_3 + C_3\sqrt{C_2}L^{-(d+1)} + L^{C_1\log\frac{1}{1-p}}\mathbb{P}_p(|\mathbf{C}_{\mathbf{0}}^{cl}| \geq bL) , \end{aligned}$$

where C_3 is some finite bound (depending on $p<1$) on $\sqrt{\mathbb{E}\left[(1-p)^{-2|\mathcal{T}_d|}\right]}$ (all exponential moments of a Poisson random variable are finite). Now, if $0<(1-p)<\bar{p}_c(d)$,

$$\begin{aligned} L^d\mathbb{P}(A_L) &\leq L^d e^{-aL}C_3 + C_3\sqrt{C_2}L^{-1} + L^{d+C_1\log\frac{1}{1-p}}\mathbb{P}_p(|\mathbf{C}_{\mathbf{0}}^{cl}|\geq bL)\,, \\ &\leq \sum_{n\geq bL}\left(\frac{n}{b}\right)^{d+C_1\log\frac{1}{1-p}}\mathbb{P}_p(|\mathbf{C}_{\mathbf{0}}^{cl}|=n) + L^d e^{-aL}C_3 + C_3\sqrt{C_2}L^{-1}\,, \end{aligned}$$

which goes to zero as L goes to infinity. This concludes the proof of (4.4).

Our second step is: the collection $\{Y_{\mathbf{z}}(L)\,:\,\mathbf{z}\in\mathbb{G}^d\}$ is a 5-dependent Bernoulli field. To see this, first notice that $Y_z(L)=Y_{\mathbf{z}}(L,\mathcal{N},X)$. Thus, the claim will follow if we prove that $Y_{\mathbf{z}}(L,\mathcal{N},X)=Y_{\mathbf{z}}(L,\mathcal{N}\cap B_{\mathbf{z}}^{5L/2},X)$. To prove this, assume that $Y_{\mathbf{z}}(L,\mathcal{N},X)=0$. Then, either (i) does not hold, or (i) holds and (ii) does not. To see if (i) does not hold, it is clear that we only have to check $\mathcal{N}\cap B_{\mathbf{z}}^{5L/2}$. If (i) holds and (ii) not, we then use Lemma 2.2 to show that, under (i), inside $B_{\mathbf{z}}^{3L/2}$ the Delaunay triangulation based on $\mathcal{N}$ is the same as the one based on $\mathcal{N}\cap B_{\mathbf{z}}^{5L/2}$. Thus, if (ii) does not hold for $\mathcal{N}$, it will certainly not hold for $\mathcal{N}\cap B_{\mathbf{z}}^{5L/2}$. The same argument works to show that if $Y_{\mathbf{z}}(L,\mathcal{N},X)=1$ then $Y_{\mathbf{z}}(L,\mathcal{N}\cap B_{\mathbf{z}}^{5L/2},X)=1$.

With (4.4) and 5-dependence in hands, we can chose $L_0\geq 1$ large enough such that

$$\mathbb{P}_p\left(\min_{\mathbf{A}\in\Phi^r}\sum_{\mathbf{z}\in\mathbf{A}}Y_{\mathbf{z}}<c_1 r\right)\leq e^{-c_2 r}\,, \tag{4.6}$$

for c_1,c_2 only depending on L_0, where Φ^r denotes the set of lattice animals of size $\geq r$ (see Lemma 9 of [14]).

Let γ be a path in $\mathcal{D}$ with $v_0\in\gamma$ and $|\gamma|\geq m$. Let

$$\mathcal{S}_L(\gamma):=\{\mathbf{z}\in\mathbf{A}^L(\gamma)\,:\,Y_{\mathbf{z}}(L)=1\}\,.$$

Notice that there exists at least one subset $\mathcal{S}$ of $\mathcal{S}_L(\gamma)$ such that $|\mathbf{z}-\mathbf{z}'|_\infty\geq 5$ for all $\mathbf{z},\mathbf{z}'\in\mathcal{S}$ and $k=|\mathcal{S}|\geq|\mathcal{S}_L(\gamma)|/5^d$. Now, write $\mathcal{S}=\{\mathbf{z}_1,\ldots,\mathbf{z}_k\}$. By Lemma 3.3, one can find disjoint pieces of γ, say $\gamma_1,\ldots,\gamma_k$ such that, for $i=1,\ldots,k$, $\sum_{e\in\gamma_i}X_e\geq 1$. Hence,

$$\sum_{e\in\gamma}X_e\geq\sum_{i=1}^k\left(\sum_{e\in\gamma_i}X_e\right)\geq|\mathcal{S}|\geq\frac{|\mathcal{S}_L(\gamma)|}{5^d}=\frac{\sum_{\mathbf{z}\in\mathbf{A}^L(\gamma)}Y_z(L)}{5^d}\,,$$

which shows that

$$\begin{aligned} \mathbb{P}\left(\exists\text{ s.a. }\gamma\text{ s.t. }|\gamma|\geq m\text{ and }\sum_{e\in\gamma}X_e\leq a_1 m\right) &\leq \mathbb{P}\left(\min_{\mathbf{v}_0\in\gamma,|\gamma|\geq m}|\mathbf{A}^L(\gamma)|<b_3 m\right) \\ &+ \mathbb{P}\left(\min_{\mathbf{A}\in\Phi^{b_3 m}}\sum_{\mathbf{z}\in\mathbf{A}}Y_{\mathbf{z}}\leq 5^d a_1 m\right)\,. \end{aligned}$$

Combining this together with (4.2) and (4.6), we finish the proof of Theorem 4.4.

□

4.2.2 The variance of the first passage time is at most linear

Let ν be a probability measure on $\mathbb{R}^+$. Recall that Ω is the set of locally finite subsets of $\mathbb{R}^d$. Let π denote the Poisson measure on $\mathbb{R}^d$ with intensity μ, where μ is comparable to the Lebesgue measure on $\mathbb{R}^d$, in the sense of (2.1). Let $\mathcal{E}_d$ denote the set of pairs $\{x,y\}$ of points of $\mathbb{R}^d$. We endow the space $\Omega\times\mathbb{R}_+^{\mathcal{E}_d}$ with the product measure $\pi\otimes\nu^{\otimes\mathcal{E}_d}$ and each element of $\Omega \times \mathbb{R}_+^{\mathcal{E}_d}$ is denoted $(\mathcal{N}, \tau)$. This means that $\mathcal{N}$ is a Poisson random set with intensity μ, and to each edge $e \in \mathcal{D}(\mathcal{N})$ is independently assigned a non-negative random variable $\tau(e)$ from the common probability measure ν.

The passage time $T(\gamma)$ of a path γ in the Delaunay triangulation is the sum of the passage times of the edges in γ:

$$T(\gamma) = \sum_{e \in \gamma} \tau(e) \ .$$

The first-passage time between two vertices v and v' is defined by

$$T(v, v') := T(v, v', \mathcal{N}, \tau)) = \inf\{T(\gamma) \, ; \, \gamma \in \Gamma(v, v', \mathcal{N})\} \ .$$

where $\Gamma(v, v', \mathcal{N})$ is the set of all finite paths connecting v to v'. Given $x, y \in \mathbb{R}$ we define $T(x, y) := T(v(x), v(y))$.

Remark that $(\mathcal{N}, \tau) \mapsto T(x, y)$ is measurable with respect to the completion of $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)^{\otimes \mathcal{E}_d}$, where $\mathcal{B}(\mathbb{R}_+)$ denotes the Borel σ-field over $\mathbb{R}_+$ and $\mathcal{F}$ is the smallest algebra on Ω which lets the coordinate applications $\mathcal{N} \mapsto \mathbb{1}_{x \in \mathcal{N}}$ measurable. To see this, fix $x, y \in \mathbb{R}^d$ and for each $r > |x, y|$ define $T_r(x, y)$ to be the first-passage time restricted to the Delaunay graph $\mathcal{D}(\mathcal{N} \cap D_r(x))$, where $D_r(x)$ is the ball centred at x and of radius r. Then clearly T_r is measurable, since it is the infimum over a countable collection of measurable functions. On the other hand, $T_r(x, y)$ is non-increasing with r, and so $T = \lim_{r \to \infty} T_r(x, y)$ is also measurable.

We may now state the announced upper-bound on the typical fluctuations of the first passage time. We denote by $M_k(\nu)$ the $k - th$ moment of ν:

$$M_k(\nu) = \int |x|^k \ d\nu(x) \ .$$

Theorem 4.7. *For any natural number $n \geq 1$, let $\vec{n}$ denote the point $(n, 0, \ldots, 0) \in \mathbb{R}^d$. Assume that $\nu(\{0\}) < \bar{p}_c$ and that*

$$M_2(\nu) := \int x^2 \ d\nu(x) < \infty \ .$$

Then, as n tends to infinity,

$$\mathsf{Var}(T(0, \vec{n}, \tau, \mathcal{N})) = O(n) \ .$$

Before we prove Theorem 4.7, we need to state a Poincaré inequality for Poisson random sets (for a proof, see for instance [8], inequality (2.12) and Lemma 2.3).

Proposition 4.8. *Let $F : \Omega \to \mathbb{R}$ be a square integrable random variable. Then,*

$$\mathsf{Var}_\pi(F) \leq \mathbb{E}_\pi(V_-(F)) \ ,$$

where:

$$V_-(F)(\mathcal{N}) = \sum_{v \in \mathcal{N}} (F(\mathcal{N}) - F(\mathcal{N} \setminus \{v\}))_-^2 + \int (F(\mathcal{N}) - F(\mathcal{N} \cup \{x\}))_-^2 \ d\lambda(x) \ .$$

A geodesic between x and y, in the FPP model, is a path $\rho(x, y)$ connecting v_x to v_y and such that

$$T(x, y) = T\big(\rho(x, y)\big) = \sum_{e \in \rho(x,y)} \tau(e) \, .$$

When a geodesic exists, we shall denote by $\rho_n(\tau, \mathcal{N})$ a geodesic between 0 and $\vec{n}$ with minimal number of vertices. If there are more than one such geodesics, we select one according to some deterministic rule. We shall write $T_n(\tau, \mathcal{N})$ for $T(0, \vec{n}, \tau, \mathcal{N})$.

To control the length of $\rho_n(\tau, \mathcal{N})$ we use the following lemma.

Lemma 4.9. *There exist positive constants a_0, C_1 and C_2 depending only on $\nu(\{0\})$ and d, and a random variable Z_n such that if $\nu(\{0\}) < \bar{p}_c(d)$, then for every n and every m,*

$$\mathbb{P}(|\rho_n(\tau,\mathcal{N})| \geq m) \leq e^{-C_1 m} + \mathbb{P}(Z_n > a_0 m) ,$$

and

$$\mathbb{E}(Z_n) \leq C_2 n .$$

Proof. For any $a > 0$, and $m \in \mathbb{N}$,

$$\begin{aligned} \mathbb{P}(|\rho_n(\tau,\mathcal{N})| \geq m) &\leq \mathbb{P}(\exists \text{ s.a. } \gamma \text{ s.t. } |\gamma| \geq m \text{ and } \sum_{e\in\gamma} \tau(e) \leq am) \\ &+ \mathbb{P}(T_n(\tau,\mathcal{N}) > am) . \end{aligned}$$

Fix $\epsilon > 0$ such that $\mathbb{P}(\tau_e > \epsilon) < \bar{p}_c$ and let $X_e = \mathbb{1}\{\tau_e > \epsilon\}$. Then $\epsilon X_e \leq \tau_e$, and consequently

$$\mathbb{P}(\exists \text{ s.a. } \gamma \text{ s.t. } |\gamma| \geq m \text{ and } \sum_{e\in\gamma} \tau(e) \leq am) \leq \mathbb{P}(\exists \text{ s.a. } \gamma \text{ s.t. } |\gamma| \geq m \text{ and } \sum_{e\in\gamma} X_e \leq a'm) ,$$

where $a' = a/\epsilon$. Choosing $a_0 = \varepsilon a_1$ with a_1 as in Theorem 4.4, this implies that

$$\mathbb{P}(\exists \text{ s.a. } \gamma \text{ s.t. } |\gamma| \geq m \text{ and } \sum_{e\in\gamma} \tau(e) \leq a_0 m) \leq e^{-a_2 m} .$$

This is the analogue of Proposition 5.8 in [10]. Finally, in Corollary 4 of [14], it is shown that there is a particular path γ_n from 0 to $\vec{n}$, that is independent of τ and whose expected number of edges is of order n. This path is constructed by walking through neighbour cells that intersect line segment $[0,\vec{n}]$ (connecting 0 to $\vec{n}$). Hence, the size of γ_n is at most the number of cells intersecting $[0,\vec{n}]$, which turns to be of order n. Thus, denoting $Z_n = \sum_{e\in\gamma(0,\vec{n})} \tau(e)$ the passage time along this path, we have:

$$\mathbb{P}(T_n(\tau,\mathcal{N}) > a_0 m) \leq \mathbb{P}(Z_n > a_0 m) . \tag{4.7}$$

Using only the fact that the edge-times possess a finite moment of order 1, there is a constant C_2 such that:

$$\mathbb{E}(Z_n) = \mathbb{E}(\tau_e)\mathbb{E}(|\gamma_n|) \leq C_2 n .$$

□

Proof of Theorem 4.7 : Let F be a function from $\Omega \times \mathbb{R}_+^{\mathcal{E}_d}$ to $\mathbb{R}$ that is measurable with respect to the completion of $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)^{\otimes\mathcal{E}_d}$. If for π-almost all ω, the functions $\tau \mapsto F(\omega,\tau)$ are square integrable with respect to $\nu^{\otimes\mathcal{E}_d}$, we define the following random variables:

$$\begin{aligned} \mathbb{E}_\nu(F) &: \begin{cases} \Omega & \to \mathbb{R} \\ N & \mapsto \int F(\mathcal{N},\tau)\, d\nu^{\otimes\mathcal{E}_d}(\tau) \end{cases} \\ \mathsf{Var}_\nu(F) &= \mathbb{E}_\nu(F^2) - \mathbb{E}_\nu(F)^2 , \end{aligned}$$

and for any function g in $L^2(\Omega,\pi)$,

$$\mathbb{E}_\pi(g) := \int g(\mathcal{N})\, d\pi(\mathcal{N}) ,$$

$$\mathsf{Var}_\pi(g) := \mathbb{E}_\pi(g^2) - \mathbb{E}_\pi(g)^2 .$$

We shall use the following decomposition of the variance:

$$\mathsf{Var}(F) = \mathbb{E}_\pi(\mathsf{Var}_\nu(F)) + \mathsf{Var}_\pi(\mathbb{E}_\nu(F)) \ .$$

To show that $\mathbb{E}_\pi(\mathsf{Var}_\nu(T_n)) = O(n)$ is now standard. Indeed, π-almost surely (see (2.17) and (2.24) in [11]):

$$\mathsf{Var}_\nu(T_n) \leq 2M_2(\nu)\mathbb{E}_\nu(\rho_n) \ .$$

Thus,

$$\mathbb{E}_\pi(\mathsf{Var}_\nu(T_n)) \leq 2M_2(\nu)\mathbb{E}(\rho_n) = O(n) \ ,$$

according to Lemma 4.9. The harder part to bound is $\mathsf{Var}_\pi(\mathbb{E}_\nu(T_n))$. Let us denote by F_n the random variable $\mathbb{E}_\nu(T_n)$. We want to apply Proposition 4.8 to F_n. First note that F_n belongs to $L^2(\Omega,\pi)$: this follows from (4.7). We claim that:

$$\forall v \in \mathcal{N}, \ (F_n(\mathcal{N}) - F_n(\mathcal{N}\setminus\{v\}))_-^2 \leq 4M_2(\nu)d_{\mathcal{N}}(v)^2\mathbb{E}_\nu(\ 1\!\!1_{v\in\rho_n(\tau,\mathcal{N})}) \ , \qquad (4.8)$$

and

$$\forall x \notin \mathcal{N}, \ (F_n(\mathcal{N}) - F_n(\mathcal{N}\cup\{x\}))_-^2 \leq 4M_1(\nu)^2\mathbb{E}_\nu(\ 1\!\!1_{x\in\Gamma(\mathcal{N},\rho_n(\tau,\mathcal{N}))}) \ . \qquad (4.9)$$

Indeed, suppose first that v belongs to $\mathcal{N}$. If $v \notin \rho_n(\tau,\mathcal{N})$, then $\rho_n(\tau,\mathcal{N})$ is still included in $\mathcal{D}(\mathcal{N}\setminus\{v\})$. This implies that $T_n(\tau,\mathcal{N}\setminus\{v\}) \leq T_n(\tau,\mathcal{N})$. Suppose that on the contrary, $v \in \rho_n(\tau,\mathcal{N})$. Let $S_1(\mathcal{N},v)$ be a spanning tree of the (connected, since $d \geq 2$) subgraph induced by all the neighbors of v. Define a set of edges $S_2(\mathcal{N},v)$ containing all the edges of $\rho_n(\tau,\mathcal{N})$ that are still in $\mathcal{D}(\mathcal{N}\setminus\{v\})$, and all the edges of $S_1(\mathcal{N},v)$. Then, $S_2(\mathcal{N},v)$ is a set of edges in $\mathcal{D}(\mathcal{N}\setminus\{v\})$ which contains a path from 0 to $\vec{n}$. From these considerations, we deduce that for v in $\mathcal{N}$:

$$\begin{aligned}
(F_n(\mathcal{N}) - F_n(\mathcal{N}\setminus\{v\}))_- &\leq \mathbb{E}_\nu[(T_n(\tau,\mathcal{N}) - T_n(\tau,\mathcal{N}\setminus\{v\}))_-] \ , \\
&\leq \mathbb{E}_\nu\left[\left(\sum_{e\in S_2(\mathcal{N},v)}\tau(e) - \sum_{e\in\rho_n(\tau,\mathcal{N})}\tau(e)\right) \ 1\!\!1_{v\in\rho_n(\tau,\mathcal{N})}\right] \ , \\
&\leq \mathbb{E}_\nu[\sum_{e\in S_1(\mathcal{N},v)}\tau(e) \ 1\!\!1_{v\in\rho_n(\tau,\mathcal{N})}] \ , \\
&\leq \sqrt{\mathbb{E}_\nu\left[\left(\sum_{e\in S_1(\mathcal{N},v)}\tau(e)\right)^2\right]}\sqrt{\mathbb{E}_\nu(\ 1\!\!1_{v\in\rho_n(\tau,\mathcal{N})})} \ ,
\end{aligned}$$

where we used Cauchy-Schwarz inequality. Using the independence of the variables $\tau(e)$ and the fact that the number of edges in $S_1(\mathcal{N},v)$ is $d_{\mathcal{N}}(v)-1$, we obtain claim (4.8). To see that claim (4.9) is true, suppose that x does not belong to $\mathcal{N}$. If x is not in $\Gamma(\mathcal{N},\rho_n(\tau,\mathcal{N}))$, obviously $\rho_n(\tau,\mathcal{N})$ is still included in $\mathcal{D}(\mathcal{N}\cup\{x\})$. Thus, $T_n(\tau,\mathcal{N}\cup\{x\}) \leq T_n(\tau,\mathcal{N})$. On the contrary, if x is in $\Gamma(\mathcal{N},\rho_n(\tau,\mathcal{N}))$, there are two special neighbors of x, $v_{in}(x)$ and $v_{out}(x)$ such that $\rho_n(\tau,\mathcal{N})$ still connects $v_{in}(x)$ to 0 and $v_{out}(x)$ to $\vec{n}$ in $\mathcal{D}(\mathcal{N}\cup\{x\})$. Let $S_3(\mathcal{N},x)$ be the set of edges containing all the edges of $\rho_n(\tau,\mathcal{N})$ that are still in $\mathcal{D}(\mathcal{N}\cup\{x\})$, plus the two edges $(v_{in}(x),x)$ and $(x,v_{out}(x))$. $S_3(\mathcal{N},x)$ contains a path in $\mathcal{D}(\mathcal{N}\cup\{x\})$ from 0 to $\vec{n}$. Notice that $v_{in}(x)$ and $v_{out}(x)$ depend on $\mathcal{N}$ and $(\tau(e))_{e\in\mathcal{D}(\mathcal{N})}$, and that conditionnally on $\mathcal{N}$, $\tau(v_{in}(x),x)$ and $\tau(x,v_{out}(x))$ are independent from $(\tau(e))_{e\in\mathcal{D}(\mathcal{N})}$. From this, we deduce that for x not in $\mathcal{N}$,

$$\begin{aligned}(F_n(\mathcal{N}) &- F_n(\mathcal{N}\cup\{x\}))_- \\ &\leq \mathbb{E}_\nu[(T_n(\tau,\mathcal{N})-T_n(\tau,\mathcal{N}\cup\{x\}))_-] \ , \\ &\leq \mathbb{E}_\nu\Big[\{\sum_{e\in S_3(\mathcal{N},x)}\tau(e)-\sum_{e\in\rho_n(\tau,\mathcal{N})}\tau(e)\}\ \mathbb{1}_{x\in\Gamma(\mathcal{N},\rho_n(\tau,\mathcal{N}))}\Big] \ , \\ &\leq \mathbb{E}_\nu[\{\tau(v_{in}(x),x)+\tau(x,v_{out}(x))\}\ \mathbb{1}_{x\in\Gamma(\mathcal{N},\rho_n(\tau,\mathcal{N}))}] \ , \\ &\leq \mathbb{E}_\nu[\mathbb{E}_\nu\{\tau(v_{in}(x),x)+\tau(x,v_{out}(x))|(\tau(e))_{e\in\mathcal{D}(\mathcal{N})}\}\ \mathbb{1}_{x\in\Gamma(\mathcal{N},\rho_n(\tau,\mathcal{N}))}] \ , \\ &= 2M_1(\nu)\mathbb{E}_\nu(\ \mathbb{1}_{x\in\Gamma(\mathcal{N},\rho_n(\tau,\mathcal{N}))}) \ ,\end{aligned}$$

which proves claim (4.9) via Jensen's inequality. Now, these two claims together with Proposition 4.8 applied to F_n give:

$$\mathsf{Var}(F_n)\leq 4M_2(\nu)\mathbb{E}(\sum_{v\in\rho_n(\tau,\mathcal{N})}d_{\mathcal{N}}(v)^2)+4M_1(\nu)^2\mathbb{E}(\lambda(\Gamma(\mathcal{N},\rho_n(\tau,\mathcal{N})))\ . \tag{4.10}$$

We shall conclude using Lemma 4.9, Proposition 2.5 and Theorem 2.2. Let z_1 be as in Theorem 2.2, and a as in Lemma 4.9. Notice that $f:x\mapsto x^2$ is c-nice for any $c>0$.

$$\begin{aligned}\mathbb{E}[\sum_{v\in\rho_n(\tau,\mathcal{N})}f(d_{\mathcal{N}}(v))] &= n\int_0^\infty\mathbb{P}(\sum_{v\in\rho_n(\tau,\mathcal{N})}f(d_{\mathcal{N}}(v))>nz)\ dz \ , \\ &\leq nz_1+\int_{z_1}^\infty\mathbb{P}(\sum_{v\in\rho_n(\tau,\mathcal{N})}f(d_{\mathcal{N}}(v))>nz)\ dz \ ,\end{aligned}$$

From Lemma 4.9 and Theorem 2.2, we have:

$$\begin{aligned}\mathbb{P}(\sum_{v\in\rho_n}f(d_{\mathcal{N}}(v))>nz) &\leq \mathbb{P}(|\rho_n|>nz/z_1)+\sum_{k=0}^{nz/z_1}\mathbb{P}(F_k>nz) \ , \\ &\leq e^{-C_1(nz/z_1)}+\mathbb{P}(Z_n>a_0nz/z_1)+\frac{nz}{z_1}e^{-c_2(c_3nz)^{1/4}} \ ,\end{aligned}$$

and thus, using that $\mathbb{E}(Z_n)=O(n)$, we get:

$$\mathbb{E}[\sum_{v\in\rho_n(\tau,\mathcal{N})}f(d_{\mathcal{N}}(v))]=O(n)\ .$$

The proof that $\mathbb{E}(\lambda(\Gamma(\mathcal{N},\rho_n(\tau,\mathcal{N})))=O(n)$ is completely similar, using Proposition 2.5 and Lemma 4.9. Theorem 4.7 now follows from (4.10).

□

4.3 Stabbing number

The stabbing number $\mathrm{st}_n(\mathcal{D}(\mathcal{N}))$ of $\mathcal{D}(\mathcal{N})\cap[0,n]^d$ is defined in [1] as the maximum number of Delaunay cells that intersect a single line in $\mathcal{D}(\mathcal{N})\cap[0,n]^d$. In [1], the following deviation result for the stabbing number of $\mathcal{D}(N)\cap[0;n]^d$ is announced in Lemma 3, and credited to Addario-Berry, Broutin and Devroye. In fact, the precise reference is unavailable, and it seems, according to [2], that the proof worked only in dimension 2.

Lemma 4.10 (Addario-Berry, Broutin and Devroye)**.** *Fix $d\geq 1$. Then, there are constants $\kappa=\kappa(d)$, $K=K(d)$ such that:*

$$\mathbb{E}(\mathrm{st}_n(\mathcal{D}(\mathcal{N})))\leq\kappa n\ ,$$

and, for any $\alpha>0$,

$$\mathbb{P}(\mathrm{st}_n(\mathcal{D}(\mathcal{N}))>(\kappa+\alpha)n)\leq e^{-\alpha n/(K\log n)}\ .$$

The importance of Lemma 4.10 is due to the fact that it is the essential tool to prove that simple random walk on $\mathcal{D}(\mathcal{N})$ is recurrent in $\mathbb{R}^2$ and transient in $\mathbb{R}^d$ for $d \geq 3$. Here, we show that our method allows to improve Lemma 4.10 as follows.

Lemma 4.11. *Fix $d \geq 1$. Then, there are constants $\kappa = \kappa(d)$, $K = K(d)$ such that:*

$$\mathbb{E}(\mathrm{st}_n(\mathcal{D}(\mathcal{N}))) \leq \kappa n \ ,$$

and, for any $\alpha > 0$,

$$\mathbb{P}(\mathrm{st}_n(\mathcal{D}(\mathcal{N})) > (\kappa + \alpha) n) \leq e^{-\alpha n} \ .$$

Proof. Since the proof is very close to the proof of Proposition 2.5, we only sketch the main differences. We divide the boundary of $[0, n]^d$ into $2^d 2^{d-1} n^{d-1}$ $(d-1)$-dimensional cubes of $(d-1)$-volume 1, and call this collection S_n. For each pair of cubes (s_1, s_2) in S_n, we define $\mathrm{st}(s_1, s_2)$ as the maximum number of Delaunay cells that intersect a single line-segment going from a point in s_1 to a point in s_2. Let $V(s_1, s_2)$ be the union of those line-segments. Notice that $\mathrm{st}(s_1, s_2)$ is bounded from above by the number of points which belong to the union of the Delaunay cells intersecting $V(s_1, s_2)$. Lemma 3.3 allows us to control those points. We obtain the following analogue of Lemma 3.4.

Lemma 4.12.

$$\begin{aligned} \mathbb{P}[\mathrm{st}(s_1, s_2) > t] \quad \leq \quad & \sum_{i=1}^{3^d} \mathbb{P}\left[\sum_{\mathbf{C} \in \mathbf{Cl}^{3r,i}(\mathbf{A}^{3r,i}(V(s_1,s_2)))} |\mathbf{C}|_{\mathcal{N}} > \frac{t}{3^d.2} \right] \\ & + \sum_{i=1}^{3^d} \mathbb{P}\left[\sup_{\phi \in \mathcal{SA}_n} |\mathbf{A}^{3r,i}(V(s_1, s_2))|_{\mathcal{N}} > \frac{t}{3^d.2} \right] \ . \end{aligned}$$

Note that the cardinals of $\mathbf{A}^{3r,i}(V(s_1, s_2)),\ i = 1, \ldots, 3^d$ are of order $O(n)$, and the possible choices for the pair (s_1, s_2) is of order $O(n^{2(d-1)})$. Now, when r is chosen large enough (see section 5.1), Lemma 4.11 follows from Lemmas 3.6, 5.2 and 5.1.

□

5 Appendix

5.1 Proof of Lemma 3.5

It suffices to prove Lemma 3.5 for $i = 1$, so we shall omit i as a subscript or superscript. Remark first that, as already noted at the beginning of section 3.1, $p_r := \sup_i \sup_z \mathbb{P}(X^{r,i}_{\mathbf{z}} = 1)$ tends to 0 as r tends to infinity. Let us choose r_0 in such a way that for any $r \geq r_0$,

$$p_r < p_c(\mathbb{G}_d) \ ,$$

where $p_c(\mathbb{G}_d)$ is the critical probability for site percolation on $\mathbb{G}_d$. When $r \geq r_0$, we are in the so-called "subcritical" phase for percolation of clusters of $(\mathcal{N}, 6)$-bad boxes. From now on, we fix r to satisfy $r \geq r_0$ and we shall omit r as a subscript or superscript, to shorten the notations. It is well known that in this "subcritical" phase, the size of the (bad)-cluster containing a given vertex decays exponentially (see [6], Theorem (6.75)). For instance, there is a positive constant c, which depends only on r, such that,

$$\mathbb{P}(|\mathbf{Cl}(0)| \geq x) \leq e^{-cx} \ . \tag{5.1}$$

Remark that there is probably an exponential number of animals of size less than m, and therefore, if $\sum_{\mathbf{C} \in \mathbf{Cl}(\mathbf{A})} g(|Ad(\mathbf{C})|)$ had an exponential moment, where $g(x) = x f(x)$, it would be easy to bound the first summand in the right-hand term of the inequality above. When $f(x)$ is larger than x, $\sum_{\mathbf{C}\in\mathbf{Cl}(\mathbf{A})} g(|Ad(\mathbf{C})|)$ surely does not have exponential moments. Therefore, we have to refine the standard argument. This refinement is a chaining technique essentially due to [4] and we rely heavily on that paper.

We shall prove (3.6), the proof of (3.7) being similar and easier. We begin by stating and proving two lemmas on site percolation.

Lemma 5.1. *For all $r > r_0$ there is a constant c_9 (depending only on r) such that:*

$$\mathbb{P}(|Ad(\mathbf{Cl}(\mathbf{0}))|_{\mathcal{N}} > s) \leq \mathbb{P}_{p_r}(|Ad(\mathbf{Cl}(\mathbf{0}))|_{\mathcal{N}} > s) \leq 2e^{-c_9 s} \,.$$

where by $\mathbb{P}_{p_r}$, we mean that every site is open with the same probability

$$p_r = \sup_i \sup_z \mathbb{P}(X_{\mathbf{z}}^{r,i} = 1) < p_c(\mathbb{G}_d)\,.$$

Proof. First, we shall condition on the value of $X = (X_z)$. Let u be a positive real number.

$$\mathbb{P}(|Ad(\mathbf{Cl}(\mathbf{0}))|_{\mathcal{N}} > s|X) \leq e^{-us}\mathbb{E}(e^{u|Ad(\mathbf{Cl}(\mathbf{0}))|_{\mathcal{N}}}|X))\,.$$

Define:

$$\partial^\infty \mathbf{Cl}(\mathbf{0}) = Ad(\mathbf{Cl}(\mathbf{0})) \setminus Bl(\mathbf{Cl}(\mathbf{0}))\,.$$

Remark that, X being fixed, $|Ad(\mathbf{Cl}(\mathbf{0}))|_{\mathcal{N}} = |\mathbf{Cl}(\mathbf{0})|_{\mathcal{N}} + |\partial^\infty \mathbf{Cl}(\mathbf{0})|_{\mathcal{N}}$ and the two summands are independent. Through the comparison to Lebesgue's measure 2.1, the term $|\mathbf{Cl}(\mathbf{0})|_{\mathcal{N}}$ is stochastically dominated by a sum of $|\mathbf{Cl}(\mathbf{0})|$ independent random variables, each of which is a sum of independent random variables with a Poisson distribution of parameter $\beta = \beta(r)$, conditioned on the fact that one of them at least must be zero. The term $|\partial^\infty \mathbf{Cl}(\mathbf{0})|_{\mathcal{N}}$ is stochastically dominated by a sum of at most $c_d|\mathbf{Cl}(\mathbf{0})|$ random variables (c_d only depending on d), each of which is a sum of independent random variables with Poisson distribution, conditioned on the fact that all of them are greater than one. Obviously, the sum of independent random variables with a Poisson distribution, conditioned on the fact that one of them at least must be zero is stochastically smaller than the sum of the same number of independent random variables with Poisson distribution, conditioned on the fact that all of them are greater than one. If Z is a Poisson random variable with parameter β, one has:

$$\mathbb{E}(e^{uZ}|Z \geq 1) = \frac{e^{\beta(e^u-1)} - e^{-\beta}}{1-e^{-\beta}}\,.$$

Therefore,

$$\mathbb{E}(e^{u|Ad(\mathbf{Cl}(\mathbf{0}))|_{\mathcal{N}}}|X) \leq \left(\frac{e^{\beta(e^u-1)}}{1-e^{-\beta}}\right)^{7|\mathbf{Cl}(\mathbf{0})|}\,,$$

and thus:

$$\mathbb{P}(|Ad(\mathbf{Cl}(\mathbf{0}))|_{\mathcal{N}} > s|X) \leq e^{-us}\left(\frac{e^{\beta(e^u-1)}}{1-e^{-\beta}}\right)^{7|\mathbf{Cl}(\mathbf{0})|}\,.$$

Let us denote $\phi(t) = t\log t - t + 1$, $K = \frac{1}{(1-e^{-\beta})^{\frac{1}{\beta}}}$ and $K' = \phi^{-1}(2\log K)$. Remark that ϕ is a bijection from $[1, +\infty)$ onto $[0, +\infty)$. Minimizing in u gives:

$$\begin{aligned}\mathbb{P}(|Ad(\mathbf{Cl}(\mathbf{0}))|_{\mathcal{N}} > s|X) &\leq \left[Ke^{-\phi\left(\frac{s}{7\beta|\mathbf{Cl}(\mathbf{0})|}\right)}\right]^{7\beta|\mathbf{Cl}(\mathbf{0})|} \mathbb{1}_{\frac{s}{K'}\geq 7\beta|\mathbf{Cl}(\mathbf{0})|} + \mathbb{1}_{\frac{s}{K'}\leq 7\beta|\mathbf{Cl}(\mathbf{0})|}\,,\\ &\leq e^{-\frac{7}{2}\beta|\mathbf{Cl}(\mathbf{0})|\phi\left(\frac{s}{7\beta|\mathbf{Cl}(\mathbf{0})|}\right)} \mathbb{1}_{\frac{s}{K'}\geq 7\beta|\mathbf{Cl}(\mathbf{0})|} + \mathbb{1}_{\frac{s}{K'}\leq 7\beta|\mathbf{Cl}(\mathbf{0})|}\,.\end{aligned}$$

Remark that:

$$\forall x > 1,\ \phi(x) \geq \min\{x-1, \frac{(x-1)^2}{e^2-1}\}\,.$$

Therefore, if $\frac{s}{7\beta|\mathbf{Cl}(\mathbf{0})|} \geq K' > 1$, denoting: $\bar{K} = \frac{1}{2}\min\{1-\frac{1}{K'}, \frac{(K'-1)^2}{2K'(e^2-1)}\}$,

$$\frac{7}{2}\beta|\mathbf{Cl}(\mathbf{0})|\phi\left(\frac{s}{7\beta|\mathbf{Cl}(\mathbf{0})|}\right) \geq \bar{K}s\ . \tag{5.2}$$

This leads to:

$$\mathbb{P}(|Ad(\mathbf{Cl}(\mathbf{0}))|_{\mathcal{N}} > s|X) \leq e^{-\bar{K}s} +\ \mathbb{1}_{\frac{s}{K'}\leq 7\beta|\mathbf{Cl}(\mathbf{0})|}\ . \tag{5.3}$$

Therefore, denoting $c_9 = \min\{\bar{K}, \frac{c}{7\beta K'}\}$, we get the desired result (notice that β and K' depend on r). □

Lemma 5.2. *Let Λ be a finite subset of $\mathbb{Z}^d$. If f is an increasing function from $\mathbb{N}$ to $[1,+\infty[$,*

$$\mathbb{E}(\Pi_{\mathbf{C}\in\mathbf{Cl}(\Lambda)}f(|\mathbf{C}|)) \leq \mathbb{E}(f(|\mathbf{Cl}(\mathbf{0})|))^{|\Lambda|}\ .$$

Proof. See Lemma 7 in [14]. □

We may now complete the proof of Lemma 3.5. we fix m and n two integers such that $n \geq m$. First, we get rid of the variables $|Ad(\mathbf{C})|_N$ which are greater than $q(n)$. Remark that, denoting $\Lambda_m^d = [-m,m]^d \cap \mathbb{G}_d$,

$$\max_{\mathbf{A}\in\Phi_m} \sum_{\mathbf{C}\in\mathbf{Cl}(r\mathbf{A})} g(|Ad(\mathbf{C})|_{\mathcal{N}}) \leq \sum_{\mathbf{z}\in\Lambda_m^d} g(|Ad(\mathbf{Cl}(\mathbf{z}))|_{\mathcal{N}})\ .$$

Then,

$$\begin{aligned}\mathbb{P}(\exists\ \mathbf{z}\in\Lambda_m^d \text{ s.t. } g(|Ad(\mathbf{Cl}(\mathbf{z}))|_{\mathcal{N}}) > q(n)) &\leq (2m+1)^d\mathbb{P}_{p_r}(g(|Ad(\mathbf{Cl}(\mathbf{0}))|_{\mathcal{N}}) > \gamma(m))\ ,\\ &\leq (2m+1)^d\mathbb{P}_{p_r}(|Ad(\mathbf{Cl}(\mathbf{0}))|_{\mathcal{N}} > l(q(n)))\ .\end{aligned}$$

Denote by c_9 the constant of Lemma 5.1. For every m:

$$\mathbb{P}(\exists\ z\in\Lambda_m^d \text{ s.t. } g(|Ad(\mathbf{Cl}(\mathbf{z}))|_{\mathcal{N}}) > q(n)) \leq (m+1)^d e^{-c_9 l(q(n))}\ . \tag{5.4}$$

Now, we want to bound from above the following probability:

$$\mathbb{P}(\max_{\mathbf{A}\in\Phi_m} \sum_{\mathbf{Cl}(\mathbf{0})\in\mathbf{Cl}(\mathbf{0})(r\mathbf{A})} g(|Ad(\mathbf{Cl}(\mathbf{0}))|_{\mathcal{N}})\ \mathbb{1}_{\forall\mathbf{z}\in\Lambda_m^d,\ g(|Ad(\mathbf{Cl}(\mathbf{z}))|_{\mathcal{N}})\leq q(n)})\ .$$

First, Lemma 1 in [4] (Lemma 5.3 below) remains true in our setting, without any modification (just remark that the condition $l \leq n$ in their lemma is in fact not needed). Then, we define the following box in $\mathbb{G}_d$, centered at $\mathbf{x}\in\mathbb{G}_d$:

$$\Lambda(\mathbf{x},l) = \{(x_1+k_1,\ldots,x_d+k_d)\in\mathbb{G}_d \text{ s.t. } (k_1,\ldots,k_d)\in[-l,l]^d\}\ .$$

Lemma 5.3. *Let $\mathbf{A}$ be a lattice animal of $\mathbb{G}_d$ containing $\mathbf{0}$, of size $|\mathbf{A}| = m$ and let $1 \leq l$. Then, there exists a sequence $\mathbf{x}_0 = 0, \mathbf{x}_1, \ldots, \mathbf{x}_h \in \mathbb{G}_d$ of $h+1 \leq 1+(2m-2)/l$ points such that*

$$\mathbf{A} \subset \bigcup_{i=0}^{h} \Lambda(l\mathbf{x}_i, 2l),$$

and

$$|\mathbf{x}_{i+1}-\mathbf{x}_i|_\infty \leq 1, \qquad 0 \leq i \leq h-1\ .$$

Continuing to follow [4] , we shall use Lemma 5.3 at different "scales" k, covering a lattice animal by $1+(2m-2)/l_k$ boxes of length $4l_k+1$. We shall choose l_k later. For any animal $\mathbf{A}$ and $0<L,R<\infty$ define:

$$S(L,R;\mathbf{A})=\sum_{\mathbf{C}\in\mathbf{Cl}(r\xi)} g(|Ad(\mathbf{C})|_{\mathcal{N}})\ \mathbb{1}_{L\leq g(|Ad(\mathbf{C})|_{\mathcal{N}})<R}\ .$$

Suppose that c_0 and $(t(n,k))_{k\leq\log_2 q(n)}$ are positive real numbers such that:

$$\sum_{k\leq\log_2 q(n)} 2^k t(n,k)\leq c_0 n\ . \tag{5.5}$$

We shall choose these numbers later. Let c' be a positive real number to be fixed later also, and define $a=1+c'c_0$,

$$\begin{aligned}&\mathbb{P}(\exists\ \mathbf{A}\in\Phi_m \text{ with } S(0,q(n);\mathbf{A})>an)\\ \leq\ &\sum_{\substack{k\geq 0\\ 2^k\leq q(n)}}\mathbb{P}(\exists\ \mathbf{A}\in\Phi_m \text{ with } S(2^k,2^{k+1};\mathbf{A})>c't(n,k)2^k)\ .\end{aligned}$$

Now, fix k for the time being, let $\mathbf{A}$ be an animal of size at most m, containing $\mathbf{0}$, let $\mathbf{x}_0=\mathbf{0},\mathbf{x}_1,\ldots,\mathbf{x}_h$ be as in Lemma 5.3 and define $\Lambda_{m,k}:=\bigcup_{i\leq h}\Lambda(l_k\mathbf{x}_i,2l_k)$. Clearly,

$$\begin{aligned}&S(2^k,2^{k+1};\mathbf{A})\\ &\leq 2^{k+1}\left(\text{number of } \mathbf{C}\in\mathbf{Cl}\left(\Lambda_{n,k}\right) \text{ with } g(|Ad(\mathbf{C})|_{\mathcal{N}})\geq 2^k\right)\ .\end{aligned}$$

Lemma 5.4. *There exists $c_{10}\in(0,\infty)$ such that for all $t\geq 3|\Lambda|e^{-c_9 s}$*

$$\mathbb{P}\left(\sum_{\mathbf{C}\in\mathbf{Cl}(\Lambda)}\mathbb{1}_{g(|Ad(\mathbf{C})|_{\mathcal{N}})>2^k}>t\right)\leq e^{-c_{10}t}\ .$$

Proof. Let L be a positive real number. Notice that, conditionally on X, $(|Ad(\mathbf{C})|_{\mathcal{N}})_{\mathbf{C}\in\mathbf{Cl}(\Lambda)}$ are independent. Thus,

$$\mathbb{P}\left(\sum_{\mathbf{C}\in\mathbf{Cl}(\Lambda)}\mathbb{1}_{|Ad(\mathbf{C})|_{\mathcal{N}}>s}>t\right)\leq e^{-Lt}\mathbb{E}\left(\Pi_{\mathbf{C}\in\mathbf{Cl}(\Lambda)}\mathbb{E}(e^{\lambda\ \mathbb{1}_{|Ad(\mathbf{C})|_{\mathcal{N}}>s}}|X)\right)\ . \tag{5.6}$$

But,

$$\begin{aligned}\mathbb{E}(e^{L\ \mathbb{1}_{|Ad(\mathbf{C})|_{\mathcal{N}}>s}}|X)&=e^{\lambda}\mathbb{P}(|Ad(\mathbf{C})|_{\mathcal{N}}>s|X)+(1-\mathbb{P}(|Ad(\mathbf{C})|_{\mathcal{N}}>s|X))\ ,\\ &=1+\mathbb{P}(|Ad(\mathbf{C})|_{\mathcal{N}}>s|X)(e^L-1)\ .\end{aligned}$$

From equation (5.3), we know that:

$$\mathbb{P}(|Ad(\mathbf{C})|_{\mathcal{N}}>s|X)\leq e^{-\bar{K}s}+\ \mathbb{1}_{\frac{s}{K'}\leq 7\beta|\mathbf{C}|}\ .$$

Remark that the right-hand side of this inequality is an increasing function of $|\mathbf{C}|$. Using Lemma 5.2 in equation (5.6), we deduce:

$$\mathbb{P}\left(\sum_{\mathbf{C}\in\mathbf{Cl}(\Lambda)}\mathbb{1}_{|Ad(\mathbf{C})|_{\mathcal{N}}>s}>t\right)\leq e^{-Lt}\left(1+(e^L-1)\left[e^{-\bar{K}s}+\mathbb{P}_{p_r}(|\mathbf{Cl}(\mathbf{0})|>\frac{s}{7\beta K'})\right]\right)^{|\Lambda|}\ .$$

Using inequality (5.1), and recalling that $c_9 = \inf\{\bar{K}, \frac{c}{7\beta K'}\}$,

$$\begin{aligned}\mathbb{P}\left(\sum_{\mathbf{C}\in\mathbf{Cl}(\Lambda)} \mathbb{1}_{|Ad(\mathbf{C})|_{\mathcal{N}}>s} > t\right) &\leq e^{-Lt}\left(1+2(e^{\lambda}-1)e^{-c_9 s}\right)^{|\Lambda|} ,\\ &\leq e^{-Lt}e^{2|\Lambda|(e^{L}-1)e^{-c_9 s}} ,\end{aligned}$$

Now, if $t \geq 2|\Lambda|e^{-c_9 s}$, minimizing over L gives:

$$\mathbb{P}\left(\sum_{\mathbf{C}\in\mathbf{Cl}(\Lambda)} \mathbb{1}_{g(|Ad(\mathbf{C})|_{\mathcal{N}})>2^k} > t\right) \leq e^{t-t\log\frac{t}{2|\Lambda|e^{-c_9 s}}-2|\Lambda|e^{-c_9 s}} = e^{-2|\Lambda|e^{-c_9 s}\phi\left(\frac{t}{2|\Lambda|e^{-c_9 s}}\right)} .$$

Suppose now that $t \geq 3|\Lambda|e^{-c_9 s}$. Using the same argument which led to (5.2), we get that there exists $c_{10} > 0$, depending only on r, such that:

$$\forall\, t \geq 3|\Lambda|e^{-c_9 s},\ \mathbb{P}\left(\sum_{\mathbf{C}\in\mathbf{Cl}(\Lambda)} \mathbb{1}_{g(|Ad(\mathbf{C})|_{\mathcal{N}})>2^k} > t\right) \leq e^{-c_{10}t} .$$

□

Remark that:

$$|\Lambda_{m,k}| \leq \frac{2m}{l_k}(4l_k+1)^2 \leq 50ml_k .$$

Suppose now that

$$t(n,k) \geq 150nl_k e^{-c_9 l(2^k)} . \tag{5.7}$$

The number of choices for $\mathbf{0} = \mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_h$ in Lemma 5.3 is at most 9^h. Therefore,

$$\mathbb{P}(\exists\ \mathbf{A} \in \Phi_m \text{ with } S(2^k, 2^{k+1}; \mathbf{A}) > ct(n,k)2^{k+1}) \leq 9^{\frac{2m}{l_k}} e^{-c_{10}c' t(n,k)} .$$

In view of the last inequality and (5.4), we would be happy if we had

$$t(n,k) \geq \bar{c}l(q(n))$$

and

$$c_{10}c't(n,k) \geq (2\log 9).\frac{2m}{l_k} .$$

Of course, we still have to check conditions (5.5) and (5.7), and we can choose c' as large as we need. A natural way to proceed is first to choose l_k in such a way that the right-hand side in (5.7) is proportional to $\frac{m}{l_k}$, and then to take large enough. Recall that $n \geq m$. Thus, define:

$$l_k = \left\lceil e^{\frac{c_9}{2}l(2^k)} \right\rceil .$$

Choose

$$t(n,k) = \max\{150ml_k e^{-c_9 l(2^k)}, l(q(n))\} .$$

Let c' be such that:

$$c' \geq \frac{4\log 9}{150c_{10}} .$$

This ensures that:

$$c'c_{10}t(n,k) \geq (4\log 9)nl_k e^{-c_9 l(2^k)} \geq (2\log 9)\frac{2n}{l_k} .$$

Therefore,

$$\mathbb{P}(\exists\ \mathbf{A} \in \Phi_m \text{ with } S(2^k, 2^{k+1}; \mathbf{A}) > c' t(n,k) 2^{k+1}) \leq e^{-c_{10} c' t(n,k)/2} \; . \tag{5.8}$$

Condition (5.7) is trivially verified from the definition of $t(n,k)$. Now, let us check condition (5.5). Now, assume that f is $\frac{4}{c_9}$-nice. Then, using the definition of l_k,

$$\sum_{k \leq \log_2 q(n)} 2^{k+1} 150 n l_k e^{-c_9 l(2^k)} \leq 300 n \sum_k e^{\log(2^k) - \frac{c_9}{2} l(2^k)} \; .$$

By our assumption,

$$\limsup_{k \to \infty} \frac{e^{\log(2^k) - \frac{c_9}{2} l(2^k)}}{(2^{\frac{1}{3}})^k} \leq 1 \; ,$$

and therefore,

$$\Sigma := \sum_k e^{\log(2^k) - \frac{c_9}{2} l(2^k)} < \infty \; .$$

On the other hand,

$$\sum_{k \leq \log_2 q(n)} 2^{k+1} l(q(n)) \leq 4 q(n) l(q(n)) \leq 4n \; ,$$

by definition of $q(n)$. Therefore, condition (5.5) is checked with

$$c_0 = 300\Sigma + 4 \; .$$

Remark also that $g(x) \geq x$, and hence $l(y) \leq y$. Therefore:

$$n \leq (q(n)+1) l(q(n)+1) \leq (q(n)+1)^2 \; ,$$

$$q(n) \geq m^{1/2} - 1 \; ,$$

and thus, if

$$\liminf_{y \to \infty} \frac{l(y)}{\log y} \geq \frac{(4d+1)}{c_9} \tag{5.9}$$

then

$$d \log(m+1) - \frac{c_9}{2} l(q(n)) \xrightarrow[n \to \infty]{} -\infty \; ,$$

and therefore, there exist a constant c_{11} such that:

$$\forall n \geq m,\ (m+1)^d e^{-c_9 l(q(n))} \leq c_{11} e^{-\frac{c_9}{2} l(q(n))} \; .$$

Since $q(n) \leq n$, we get in the same way that there is a constant c_{12} such that:

$$\sum_{k \leq \log_2 q(n)} e^{-c_{10} c t(n,k)/2} \leq \log_2 q(n) e^{-c_{10} c l(q(n))/2} \leq c_{12} e^{-c_{10} c l(q(n))/4} \; ,$$

provided that (5.9) holds. Therefore, we define $c_5 = \sup\{\frac{6}{c}, \frac{4d+1}{c_9}\}$, and we suppose that

$$\liminf_{y \to \infty} \frac{l(y)}{\log y} \geq c_5 \; . \tag{5.10}$$

Then, for $c_6 = a = 1 + c c_0$, there exists a positive constant c_7 such that:

$$\begin{aligned} \mathbb{P}(\sup_{\mathbf{A} \in \Phi_m} \sum_{\mathbf{C} \in \mathbf{Cl}(r\xi)} g(|Ad(\mathbf{Cl}(\mathbf{0}))|_{\mathcal{N}}) > c_6 n) \quad &\leq \quad (m+1)^2 e^{-\bar{c} l(q(n))} \\ &+ \quad \sum_{k \leq \log_2 q(n)} e^{-c_{10} c t(n,k)/2} \; , \\ &\leq \quad e^{-c_7 l(q(n))} \; , \end{aligned}$$

where the first inequality follows from (5.4) and (5.8). This concludes the proof of Lemma 3.5. □

5.2 Two simple geometric lemmas

Lemma 5.5. *Let $\mathcal{N}$ be a locally finite set of points in $\mathbb{R}^d$ and $\mathcal{D}$ the Delaunay triangulation based on $\mathcal{N}$. Let u and v be two distinct points in $\mathcal{N}$. Then, there is a path on $\mathcal{D}$ going from v to u and totally included in the (closed) ball of center u and radius $|u-v|$.*

Proof. First we show that there is a neighbour w of v in $\mathcal{D}$ inside the (closed) ball of diameter $[u,v]$. To see this, let us define by $(B_\alpha)_{\alpha\in[0,1]}$ the collection of euclidean balls such that B_α has diameter $[v,x_\alpha]$, where $x_\alpha = v+\alpha(u-v)$. Notice that $B_\alpha \subset B_{\alpha'}$ as soon as $\alpha \le \alpha'$. Define:

$$\alpha_0 := \min\{\alpha \in [0,1] \text{ s.t. } \exists w' \in \mathcal{N}\setminus\{v\}\cap B_\alpha\} \ .$$

This is indeed a minimum because $\mathcal{N}$ is locally finite. Notice also that the set is non-empty since it contains $\alpha = 1$. Thus the interior of B_{α_0} does not intersect $\mathcal{N}$, but ∂B_{α_0} contains (at least) two points of $\mathcal{N}$ (including v). This implies that there is a point on the sphere ∂B_{α_0} which is a neighbour of v. So we have proved that there is a neighbour w_1 of v inside the ball of diameter $[u,v]$. Then, as long as the neighbour obtained is different from u, we may iterate this construction to get a sequence of neighbours $w_0 = v, w_1, w_2, \ldots$ such that w_{i+1} belongs to the ball of diameter $[u,w_i]$. All these balls are included in the ball of center u and radius $|u-v|$. Since $\mathcal{N}$ is locally finite, this construction has to stop at some k, when the condition that w_k is distinct from u is no longer satisfied. Then the desired path is constructed. □

For the next lemma, recall that a locally finite set of points $\mathcal{N}$ in $\mathbb{R}^d$ is said to be "in generic position" if every subset of points of cardinal $d+1$ can be circumscribed a unique d-dimensional sphere and if no $d+2$ points in $\mathcal{N}$ are co-spherical, i.e. lie on a common sphere in $\mathbb{R}^d$. It is well known that a Poisson random set with intensity comparable to the Lebesgue measure is in generic position with probability 1.

Lemma 5.6. *Let $\mathcal{N}$ be a locally finite set of points in $\mathbb{R}^d$ in generic position and define, for any edge e of the Delaunay graph $\mathcal{D}(\mathcal{N})$:*

$$\Gamma(\mathcal{N},e) = \{x \in \mathbb{R}^d \text{ s.t. } e \not\subset \mathcal{D}(\mathcal{N}\cup\{x\}\} \ .$$

Then,

$$\Gamma(\mathcal{N},e) = \bigcap_{\Delta\ni e} \overset{o}{B}(\Delta) \ ,$$

where the intersection is taken over all Delaunay cells Δ which contain e.

Proof. Let us define, for any pair of vertices $\{u,v\}$ in $\mathcal{N}$, the following convex $(d-1)$-dimensional polytope:

$$P(\{u,v\}) := \{y \in \mathbb{R}^d \text{ s.t. } d(y,u) = d(y,v) \le d(y,w)\ \forall w \in \mathcal{N}\} \ ,$$

where d is the euclidean distance. $P(\{u,v\})$ is non-empty if and only if C_u and C_v share a $(d-1)$-dimensional face i.e when $\{u,v\}$ is an edge of $\mathcal{D}(\mathcal{N})$, and in this case, $P(\{u,v\})$ is precisely this common face. Now, let $e=\{u,v\}$ be an edge of $\mathcal{D}(\mathcal{N})$, and let x be a point of $\mathbb{R}^d$. Then, $e \not\subset \mathcal{D}(\mathcal{N}\cup\{x\})$ if and only if all the points of $P(e)$ are (strictly) closer to x than to u (or v, but this is the same since $d(y,u)=d(y,v)$ for y in $P(e)$). But this is a convex condition: this merely means that $P(e)$ is included in the open half space containing x and with boundary the median hyperplane of $[x,u]$. Thus $e \not\subset \mathcal{D}(\mathcal{N}\cup\{x\})$ if and only if all the extreme points of the convex $(d-1)$-dimensional polytope $P(e)$ are closer to x than to u. But these extreme points are those points $y \in P(e)$ for which there are $d-1$ different points w in $\mathcal{N}\setminus\{u,v\}$ such that $d(y,u)=d(y,v)=d(y,w)$ (recall that $\mathcal{N}$ is in generic position). These are the centers of the circumballs of the Delaunay cells. Thus $e \not\subset \mathcal{D}(\mathcal{N}\cup\{x\})$ if and only x belongs to $\bigcap_{\Delta\ni e} \overset{o}{B}(\Delta)$. □

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