

Extinction of Fleming-Viot-type particle systems with strong drift*

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Abstract

We consider a Fleming-Viot-type particle system consisting of independently moving particles that are killed on the boundary of a domain. At the time of death of a particle, another particle branches. If there are only two particles and the underlying motion is a Bessel process on $(0, \infty)$, both particles converge to 0 at a finite time if and only if the dimension of the Bessel process is less than 0. If the underlying diffusion is Brownian motion with a drift stronger than (but arbitrarily close to, in a suitable sense) the drift of a Bessel process, all particles converge to 0 at a finite time, for any number of particles.

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1 Introduction

Our paper is motivated by an open problem concerning extinction in a finite time of a branching particle system. We prove two results that are related to the original problem and might shed some light on the still unanswered question.

The following Fleming-Viot-type particle system was studied in [5]. Consider an open bounded set $D \subset \mathbb{R}^d$ and an integer $N \geq 2$. Let $\mathbf{X}_t = (X_t^1, \dots, X_t^N)$ be a process with values in D^N defined as follows. Let $\mathbf{X}_0 = (x^1, \dots, x^N) \in D^N$. Then the processes X_t^1, \dots, X_t^N evolve as independent Brownian motions until the time τ_1 when one of them, say, X^j hits the boundary of D . At this time one of the remaining particles is chosen uniformly, say, X^k , and the process X^j jumps at time τ_1 to $X_{\tau_1}^k$. The processes X_t^1, \dots, X_t^N continue evolving as independent Brownian motions after time τ_1 until the first time $\tau_2 > \tau_1$ when one of them hits the boundary of D . Again at the time τ_2 the particle which approaches the boundary jumps to the current location of a particle chosen uniformly at random from amongst the ones strictly inside D . The subsequent

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evolution of \mathbf{X} proceeds in the same way. We will say that \mathbf{X} constructed above is *driven* by Brownian motion. The main results in this paper are concerned with Fleming-Viot particle systems driven by other processes.

The above recipe defines the process \mathbf{X}_t only for $t < \tau_\infty$, where

$$\tau_\infty = \lim_{k \rightarrow \infty} \tau_k.$$

There is no natural way to define the process \mathbf{X}_t for $t \geq \tau_\infty$, and, therefore, it is of interest to investigate what conditions ensure that $\tau_\infty = \infty$. In Theorem 1.1 of [5] the authors claim that in every domain $D \subset \mathbb{R}^d$ and every $N \geq 2$, we have $\tau_\infty = \infty$, so the Fleming-Viot process is always well-defined. However, the proof of Theorem 1.1 in [5] contains an error which is irreparable in the following sense. That proof is based on only two properties of Brownian motion—the strong Markov property and the fact the the hitting time distribution of a compact set has no atoms (assuming that the starting point lies outside the set). Hence, if some version of that argument were true, it would apply to almost all non-trivial examples of Markov processes with continuous time, and in particular to all diffusions. However, in [3], the authors provided an example of a diffusion X on $D = (0, \infty)$ (a Bessel process with dimension $\nu = -4$), such that $\tau_\infty < \infty$ for the Fleming-Viot process driven by this diffusion with $N = 2$.

It is not known whether Theorem 1.1 of [5] is correct in full generality. It was proved in [3, 10] that the theorem holds in domains which do not have thin channels.

1.1 Main results

We will prove two theorems. The first theorem is concerned with Bessel processes but it is motivated by the original model based on Brownian motion in an open bounded subset of \mathbb{R}^d . Recall that for any real ν , a ν -dimensional Bessel process on $(0, \infty)$ killed at 0 may be defined as a solution to the stochastic differential equation

$$dX_t = dW_t + \frac{\nu - 1}{2X_t} dt, \tag{1.1}$$

where W is the standard Brownian motion. To make a link between Brownian motion in a domain and Bessel processes, we recall that there exists a regularized version ρ of the distance function ([14, Theorem 2, p. 171]). More precisely, there exist $0 < c_1, c_2, c_3, c_4 < \infty$ and a C^∞ function $\rho : D \rightarrow (0, \infty)$ with the following properties,

$$\begin{aligned} c_1 \operatorname{dist}(x, \partial D) &\leq \rho(x) \leq c_2 \operatorname{dist}(x, \partial D), \\ \sup_{x \in D} |\nabla \rho(x)| &\leq c_3, \\ \sup_{x \in D} \left| \rho(x) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_m} \rho(x) \right| &\leq c_4 \quad \text{for } 1 \leq i, m \leq d. \end{aligned}$$

The above estimates and the Itô formula show that if $B = (B^1, \dots, B^d)$ is a d -dimensional Brownian motion and $Z_t = \rho(B_t)$ then

$$dZ_t = \sum_{k=1}^d a_k(Z_t) dB_t^k + \frac{b(Z_t)}{Z_t} dt,$$

where the functions $a_k(\cdot)$ and $b(\cdot)$ are bounded. This shows that the dynamics of Z resembles that of a Bessel process. Note that if $\tau_\infty < \infty$ for the Fleming-Viot process driven by Brownian motion in a domain D then the distances of all particles to ∂D go to 0 as $t \uparrow \tau_\infty$, by Lemma 5.2 of [3]. Hence, it is of some interest to see whether a Fleming-Viot process based on a Bessel process can become extinct in a finite time. We have a complete answer only for $N = 2$, i.e., a two-particle process.

Theorem 1.1. *Let X be a Fleming-Viot process with N particles on $(0, \infty)$ driven by Bessel process of dimension $\nu \in \mathbb{R}$.*

- (i) *If $N = 2$ then $\tau_\infty < \infty$, a.s., if and only if $\nu < 0$.*
- (ii) *If $N\nu \geq 2$ then $\tau_\infty = \infty$, a.s.*

Our second main result is also motivated by some results presented in [5]. Several theorems in [5, 3] are concerned with limits when $N \rightarrow \infty$. To formulate rigorously any of these theorems it would suffice that $\tau_\infty = \infty$, a.s., for all sufficiently large N . In other words, it is not necessary to know whether $\tau_\infty = \infty$ for small values of N . One may wonder whether it is necessarily the case that $\tau_\infty = \infty$ for any Fleming-Viot-type process and sufficiently large N . Our next result shows that once the drift of the diffusion is slightly stronger than the drift of any Bessel process, then $\tau_\infty < \infty$ for the Fleming-Viot process driven by this diffusion and every N .

Consider the following SDE for a diffusion on $(0, 2]$,

$$X_t = x_0 + W_t - \int_0^t \frac{1}{\beta X_s^{\beta-1}} ds - L_t, \quad t \leq T_0, \tag{1.2}$$

where $x_0 \in (0, 2]$, $\beta > 2$, W is Brownian motion, T_0 is the first hitting time of 0 by X , and L_t is the local time of X at 2, i.e., L_t is a continuous additive functional of X such that

$$\int_0^\infty \mathbf{1}_{\{X_s \neq 2\}} dL_s = 0, \quad \text{a.s.}$$

It is well known that (1.2) has a unique pathwise solution (X, L) (see, e.g., [2], Theorem I.12.1). We will analyze a Fleming-Viot process on $(0, 2]$ driven by the diffusion defined in (1.2). The role of the boundary is played by the point 0, and only this point. In other words, the particles jump only when they reach 0. Let $\mathbb{P}^{\mathbf{x}}$ denote the distribution of the Fleming-Viot particle system starting from $\mathbf{X}_0 = \mathbf{x}$.

Theorem 1.2. *Fix any $\beta > 2$. For every $N \geq 2$, the N -particle Fleming-Viot process on $(0, 2]$ driven by diffusion defined in (1.2) has the property that $\tau_\infty < \infty$, a.s. Moreover,*

$$\mathbb{P}^{\mathbf{x}}(\tau_\infty > t) \leq c_1 e^{-c_2 t}, \quad t \geq 0, \mathbf{x} \in (0, 2]^N, \tag{1.3}$$

where c_1 and c_2 depend only on N and β , and satisfy $0 < c_1, c_2 < \infty$.

Remark 1.3. (i) *If we take $\beta = 2$ in (1.2) then the diffusion is a Bessel process (locally near 0). Hence, we may say that Theorem 1.2 is concerned with a diffusion with a drift “slightly stronger” than the drift of any Bessel process.*

(ii) *The theorem still holds if the constant $1/\beta$ in the drift term in (1.2) is replaced by any other positive constant. We chose $1/\beta$ to simplify some formulas in the proof.*

(iii) *The diffusion (1.2) is reflected at 2 so that we can prove the exponential bound in (1.3). For some Markov processes, the hitting time of a point can be finite almost surely but it may have an infinite expectation; the hitting time of 0 by one-dimensional Brownian motion starting at 1 is a classical example of such situation. The reflection is used in (1.2) to get rid of the effects of excursions of the diffusion far away from the boundary at 0. A different example could be constructed based on a diffusion on $(0, \infty)$ with no reflection but with very strong negative drift far away from 0.*

We end this section with two open problems.

Problem 1.4. *Find necessary and sufficient conditions, in terms of N and ν , for non-extinction in a finite time of an N -particle Fleming-Viot process driven by a ν -dimensional Bessel process.*

Problem 1.5. *Does there exist a Fleming-Viot-type process, not necessarily driven by Brownian motion, such that $\tau_\infty = \infty$, a.s., for the N -particle system, but $\tau_\infty < \infty$ with positive probability for the $(N + 1)$ -particle system, for some $N \geq 2$?*

The rest of the paper contains the proofs of the two main theorems.

2 Proof of Theorem 1.1

2.1 Bessel processes

We start with a review of some facts about Bessel processes and Gamma distributions. Let $Z_t, t \geq 0$, be a square of a Bessel process of dimension $\nu \in \mathbb{R}$ starting at $x \geq 0$, ($Z \sim \text{BESQ}^\nu(x)$, for short), i.e., Z is the unique strong solution to stochastic differential equation

$$dZ_t = \nu dt + 2\sqrt{|Z_t|} dW_t, \quad Z_0 = x,$$

where W is a one-dimensional Brownian motion (see [12, Chapter 11] for the case $\nu \geq 0$ and [8] for the general case).

Squares of Bessel processes have the following scaling property: if $Z_t \sim \text{BESQ}^\nu(x)$ and for some $c > 0$ and all $t \geq 0$ we have $Z'_t = cZ_{c^{-1}t}$, then $Z' \sim \text{BESQ}^\nu(cx)$.

If $Z \sim \text{BESQ}^\nu(x)$ with $x > 0$, and T_0 denotes the first hitting time of 0, then $T_0 = \infty$, a.s., if $\nu \geq 2$, and $T_0 < \infty$, a.s., if $\nu < 2$. Moreover, in the latter case we have

$$T_0 \stackrel{d}{=} \frac{x}{2G}, \tag{2.1}$$

where G is $\Gamma(1 - \frac{\nu}{2})$ -distributed random variable [8, eqn. (15)]. Here and in what follows we say that a random variable is $\Gamma(\alpha)$ -distributed if it has the density

$$f_\alpha(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}, \quad x > 0, \alpha > 0,$$

where

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$$

denotes the standard gamma function. Note that we consider only a one-parameter family of gamma densities, unlike the traditional two-parameter family.

In [12], a Bessel process X of dimension $\nu \geq 0$ starting at $x \geq 0$ ($X \sim \text{Bes}^\nu(x)$), is defined as the square root of $\text{BESQ}^\nu(x^2)$ process Z . If $\nu \geq 0$, then by so called comparison theorems, the paths of Z_t are defined for all $t \geq 0$, so X_t is well defined for all $t \in [0, \infty)$. We define a Bessel process X of dimension $\nu < 0$ starting at $x \geq 0$ as the square root of a $\text{BESQ}^\nu(x^2)$ process Z , i.e., $X_t = \sqrt{Z_t}$ for $t \leq T_0$. For any real ν , these definitions are equivalent to the definition given in (1.1) by the Itô formula.

Processes $\text{Bes}^\nu(x)$ with $\nu \in \mathbb{R}$ scale as follows. If $X \sim \text{Bes}^\nu(x)$ is a Bessel process on $[0, T_0)$, then for all $c > 0$, $cX_{c^{-2}t}$ is a $\text{Bes}^\nu(cx)$ process on $[0, c^2T_0]$. This follows easily from the scaling property of $\text{BESQ}^\nu(x)$ processes.

2.2 Proof of Theorem 1.1 (i)

We start with an alternative construction of the Fleming-Viot process \mathbf{X} . Let $X = (X_t, t \in [0, T_0))$ be a $\text{Bes}^\nu(1)$ process. Let $\mathbf{Y} = (Y_t^1, Y_t^2)$, where Y_t^1 and Y_t^2 are independent copies of X_t and let $\mathbf{Y}_t^i = (Y_t^{i,1}, Y_t^{i,2})$, $i = 1, 2, \dots$, be a sequence of independent copies of \mathbf{Y} . For $i = 1, 2, \dots$ we set

$$\sigma_i = \inf \left\{ t > 0 : Y_t^{i,1} \wedge Y_t^{i,2} = 0 \right\},$$

and

$$\alpha_i = Y_{\sigma_i}^{i,1} \vee Y_{\sigma_i}^{i,2}.$$

It is easily seen that σ_1 may be represented as $\sigma_1 = \min(T'_0, T''_0)$, where T'_0 and T''_0 are independent copies of T_0 , and that $(\sigma_i, i = 1, 2, \dots)$ is a sequence of independent and identically distributed random variables.

We construct a two-particle Fleming-Viot type process $\mathbf{X}_t = (X_t^1, X_t^2)$ as follows. First let $\tau_1 = \sigma_1$ and set $\mathbf{X}_t = \mathbf{Y}_t^1$ for $t \in [0, \tau_1)$. At τ_1 one of the particles hits the boundary of $D = (0, \infty)$, and it jumps to $\xi_1 = \alpha_1$. To continue the process we use the scaling property of \mathbf{Y}_t : let $\tau_2 = \tau_1 + \xi_1^2 \sigma_2$ and set $\mathbf{X}_t = \xi_1 \mathbf{Y}_{\xi_1^{-2}(t-\tau_1)}^2$ for $t \in [\tau_1, \tau_2)$. At τ_2 , one of the particles hits the boundary and jumps, this time to $\xi_2 = \alpha_2 \xi_1$. We continue the process in the same way by setting

$$\xi_j = \prod_{i=1}^j \alpha_i,$$

$$\tau_n = \sum_{j=1}^n \xi_{j-1}^2 \sigma_j,$$

and

$$\mathbf{X}_t = \xi_n \mathbf{Y}_{\xi_n^{-2}(t-\tau_n)}^n, \quad \text{for } t \in [\tau_n, \tau_{n+1}).$$

It is easy to see that the construction of \mathbf{X} given above is equivalent to that given in the Introduction, except that the driving process is a ν -dimensional Bessel process. The process \mathbf{X}_t is well defined up until τ_∞ , and we will show now that $\tau_\infty < \infty$ almost surely if and only if $\nu < 0$.

Note that $\mathbf{X}_0 = (1, 1)$ for the process constructed above. However, it is easy to see that for any two starting points $\mathbf{X}_0 = (x_0^1, x_0^2)$ and $\mathbf{X}_0 = (z_0^1, z_0^2)$ with $x_0^1, x_0^2, z_0^1, z_0^2 > 0$, the distributions of \mathbf{X}_{τ_1} are mutually absolutely continuous. This implies that the argument given below proves the theorem for any initial value of \mathbf{X} .

The case $\nu \geq 2$ is very simple: then $\sigma_1 = \infty$, a.s., so $\tau_\infty = \infty$, a.s. So for the rest of this section we assume that $\nu < 2$.

To check whether $\tau_\infty < \infty$ or $\tau_\infty = \infty$, we will apply the following theorem. Let $\log^+ x = \max(\log x, 0)$.

Theorem 2.1. ([6]; see also [4] or [9]) *Let $\{(A_n, B_n), n \geq 1\}$ be a sequence of independent and identically distributed random variables such that $A_n, B_n \in \mathbb{R}$ and*

$$\mathbb{E}(\log^+ |A_1|) < \infty, \quad \mathbb{E}(\log^+ |B_1|) < \infty.$$

Then the infinite random series

$$\sum_{n=1}^{\infty} \left(\prod_{j=1}^{n-1} A_j \right) B_n$$

converges a.s. to a finite limit if and only if

$$\mathbb{E} \log |A_1| < 0.$$

We will apply Theorem 2.1 with $A_n = \alpha_n^2$ and $B_n = \sigma_n$. Thus, in order to prove Theorem 1.1 (i), it suffices to show that

- (a) $\mathbb{E} \log \sigma_1 < \infty$ for $\nu < 2$;

(b) $\mathbb{E} \log(\alpha_1^2) < 0$ for $\nu < 0$ and $\mathbb{E} \log(\alpha_1^2) \geq 0$ for $\nu \geq 0$.

Proof of (a). Note that, in view of (2.1),

$$\mathbb{E} \log \sigma_1 \leq \mathbb{E} \log T_0 = \mathbb{E} \log \frac{1}{2G} = -\log 2 - \mathbb{E} \log G,$$

where $G \sim \Gamma(1 - \frac{\nu}{2})$. But for $G \sim \Gamma(\alpha)$ with $\alpha = 1 - \frac{\nu}{2} > 0$, we have

$$\begin{aligned} \mathbb{E} \log G &= \int_0^\infty \log x f_\alpha(x) dx \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} \log x e^{-x} dx \\ &= \frac{1}{\Gamma(\alpha)} \frac{d}{d\alpha} \Gamma(\alpha) = \psi(\alpha) < \infty, \end{aligned} \tag{2.2}$$

where ψ is the well known digamma function ([1, Section 6.3]) defined as

$$\psi(x) = \frac{d}{dx} \log \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}. \quad \square$$

Proof of (b). By Theorem 8 part (ii) of [11] we get that the density of α_1^2 is given by

$$\begin{aligned} h_\nu(y) &= \frac{(y+2)^{\nu-3}}{\Gamma(1-\frac{\nu}{2})} \sum_{n=0}^\infty \frac{\Gamma(3-\nu+2n)}{n! \Gamma(2-\frac{\nu}{2}+n)} \left(\frac{y}{(y+2)^2}\right)^n \\ &= \frac{(y+2)^{\nu-3}}{\Gamma(1-\frac{\nu}{2})} g\left(\frac{y}{(y+2)^2}\right), \end{aligned}$$

where

$$g(z) = \sum_{n=0}^\infty c_n z^n$$

with

$$c_n = \frac{\Gamma(2n+3-\nu)}{n! \Gamma(n+2-\frac{\nu}{2})}, \quad n = 0, 1, 2, \dots$$

By the duplication formula for the gamma function ([1, eqn. 6.1.18]), i.e.,

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right),$$

we have

$$\begin{aligned} c_n &= \frac{2^{2n+2-\nu}}{\sqrt{\pi}} \cdot \frac{\Gamma(n + \frac{3-\nu}{2})}{n!} \\ &= \frac{2^{2n+2-\nu}}{\sqrt{\pi}} \binom{n + \frac{1-\nu}{2}}{n} \Gamma\left(\frac{3-\nu}{2}\right), \end{aligned}$$

where for $x > -1$ and $k \in \mathbb{N}$,

$$\binom{x}{k} = \frac{\Gamma(x+1)}{k! \Gamma(x-k+1)}$$

is a generalized binomial coefficient. Therefore,

$$\begin{aligned} g(z) &= \frac{2^{2-\nu}}{\sqrt{\pi}} \Gamma\left(\frac{3-\nu}{2}\right) \sum_{n=0}^\infty \binom{n + \frac{1-\nu}{2}}{n} (4z)^n \\ &= \frac{2^{2-\nu}}{\sqrt{\pi}} \Gamma\left(\frac{3-\nu}{2}\right) \left(\frac{1}{1-4z}\right)^{\frac{3-\nu}{2}}, \end{aligned}$$

as for $a \in \mathbb{R}$

$$\sum_{n=0}^{\infty} \binom{n+a}{n} z^n = (1-z)^{-a-1}.$$

Now

$$g\left(\frac{y}{(y+2)^2}\right) = \frac{2^{2-\nu}}{\sqrt{\pi}} \Gamma\left(\frac{3-\nu}{2}\right) \frac{(y+2)^{3-\nu}}{(y^2+4)^{\frac{3-\nu}{2}}}$$

and therefore for $y \geq 0$

$$h_\nu(y) = \frac{2^{2-\nu}}{\sqrt{\pi}} \frac{\Gamma\left(\frac{3-\nu}{2}\right)}{\Gamma\left(1-\frac{\nu}{2}\right)} \frac{1}{(y^2+4)^{\frac{3-\nu}{2}}}.$$

So, to prove (b) we need to study the sign of the integral

$$I(\nu) = \int_0^\infty h_\nu(y) \log y \, dy.$$

Recall the Student's t -distribution with $a > 0$ degrees of freedom ([1, section 26.7]). The density for this distribution is given by

$$f(x; a) = \frac{\Gamma\left(\frac{a+1}{2}\right)}{\sqrt{\pi a} \Gamma\left(\frac{a}{2}\right)} \left(1 + \frac{x^2}{a}\right)^{-\frac{a+1}{2}}, \quad -\infty < x < \infty.$$

Changing the variable $y = \frac{2x}{\sqrt{2-\nu}}$ in $I(\nu)$ we get

$$\begin{aligned} I(\nu) &= \int_0^\infty f(x; 2-\nu) \log \frac{2x}{\sqrt{2-\nu}} \, dx \\ &= \frac{1}{2} \mathbb{E} \log \frac{2|X|}{\sqrt{2-\nu}}, \end{aligned}$$

where X is a random variable with t -distribution with $(2-\nu)$ -degrees of freedom.

It is well known ([1, section 26.7]) that

$$X \stackrel{d}{=} \frac{Z\sqrt{2-\nu}}{\sqrt{V}},$$

where Z has standard normal distribution and V has chi-squared distribution with $(2-\nu)$ degrees of freedom, and Z and V are independent. Therefore

$$\begin{aligned} I(\nu) &= \frac{1}{2} \mathbb{E} \log \frac{2|Z|}{\sqrt{V}} \\ &= \frac{1}{2} \log 2 + \frac{1}{4} \left(\mathbb{E} \log \frac{Z^2}{2} - \mathbb{E} \log \frac{V}{2} \right). \end{aligned}$$

Note that $\frac{Z^2}{2}$ has $\Gamma\left(\frac{1}{2}\right)$ distribution and $\frac{V}{2}$ has $\Gamma\left(\frac{2-\nu}{2}\right)$ distribution. Therefore, by (2.2),

$$I(\nu) = \frac{1}{2} \log 2 + \frac{1}{4} \left(\psi\left(\frac{1}{2}\right) - \psi\left(\frac{2-\nu}{2}\right) \right).$$

The function ψ is strictly increasing with $\psi\left(\frac{1}{2}\right) = -2 \log 2 - \gamma$ and $\psi(1) = -\gamma$ where γ is the Euler constant ([1, eqns. 6.3.2, 6.3.3]). Using these facts we see that

$$I(\nu) = \frac{1}{4} \left(\psi(1) - \psi\left(\frac{2-\nu}{2}\right) \right),$$

and therefore $I(\nu) < 0$ iff $\frac{2-\nu}{2} > 1$ iff $\nu < 0$. This completes the proof of (b) and of Theorem 1.1 (i). \square

2.3 Proof of Theorem 1.1 (ii)

Suppose that $\mathbf{X} = (X^1, \dots, X^N)$ is a Fleming-Viot process driven by ν -dimensional Bessel process, driven by a ν -dimensional Bessel process, $\mathbf{X}_0 = (x^1, \dots, x^N)$, $x^j > 0$ for all $1 \leq j \leq N$, and $N\nu \geq 2$. Let $Z_t = (X_t^1)^2 + \dots + (X_t^N)^2$ and $z_0 = (x^1)^2 + \dots + (x^N)^2 > 0$. According to [13, Thm. 2.1], the process $\{Z_t, t \in [0, \tau_1]\}$ is an $(N\nu)$ -dimensional square of Bessel process, i.e., it has distribution $\text{BESQ}^{N\nu}(z_0)$. More generally, $\{Z_t, t \in [\tau_k, \tau_{k+1}]\}$ has distribution $\int \text{BESQ}^{N\nu}(z) \mathbb{P}(Z_{\tau_k} \in dz)$ for $k \geq 0$, where, by convention, $\tau_0 = 0$. Let $Y_t = Z_t^{1/2}$, $B_0 = 0$ and define B inductively on intervals $(\tau_k, \tau_{k+1}]$ by

$$B_t = B_{\tau_k} + \int_{\tau_k}^t dY_s - \int_{\tau_k}^t \frac{N\nu - 1}{2Y_s} ds.$$

Then, by the Itô formula, B is a Brownian motion and

$$dZ_t = 2\sqrt{Z_t} dB_t + N\nu dt,$$

for $t \in (\tau_k, \tau_{k+1})$, $k \geq 0$. Let \widehat{Z}^t be defined by $\widehat{Z}_0 = z_0$ and

$$\widehat{Z}_t = \int_0^t 2\sqrt{\widehat{Z}_s} dB_s + N\nu t, \quad t \geq 0.$$

By definition, \widehat{Z} is an $(N\nu)$ -dimensional squared Bessel process on $[0, \infty)$. We assumed that $N\nu \geq 2$ and $z_0 > 0$ so we have $\widehat{Z}_t > 0$ for all $t \geq 0$, a.s. Since \widehat{Z} is continuous, for every integer $j > 0$ there exists a random variable a_j such that $\widehat{Z}_t > a_j > 0$ for all $t \in [0, j]$, a.s. Note that $\widehat{Z}_t = Z_t$ for $t \in [0, \tau_1)$ and $\widehat{Z}_{\tau_1} < Z_{\tau_1}$ because Z has a positive jump at time τ_1 . Strong existence and uniqueness for SDE's with smooth coefficients implies that $\widehat{Z}_t \leq Z_t$ for all $t \in [\tau_1, \tau_2)$, because if the trajectories of \widehat{Z} and Z ever meet then they have to be identical after that time up to τ_2 . Once again, $\widehat{Z}_{\tau_2} < Z_{\tau_2}$ because Z has a positive jump at time τ_2 . By induction, $\widehat{Z}_t \leq Z_t$ for all $t \in [\tau_k, \tau_{k+1})$, $k \geq 0$, a.s. Hence, $Z_t > a_j > 0$ for all $t \in [0, j]$ and $j > 0$, a.s. This implies that $\tau_\infty = \infty$, a.s., by an argument similar to that in Lemma 5.2 of [3].

3 Proof of Theorem 1.2

3.1 Preliminaries

We will give new meanings to some symbols used in the previous section. Constants denoted by c with subscripts will be tacitly assumed to be strictly positive and finite; in addition, they may be assumed to satisfy some other conditions.

(i) Let W be one-dimensional Brownian motion and let b be a Lipschitz function defined on an interval in \mathbb{R} , i.e., $|b(x_1) - b(x_2)| \leq L|x_1 - x_2|$ for some $L < \infty$ and all x_1 and x_2 in the domain of b . Consider a diffusion $X_t, t \in [s, u]$, satisfying the following stochastic differential equation,

$$dX_t = dW_t + b(X_t) dt, \quad X_s = a. \tag{3.1}$$

Let y_t be the solution to the ordinary differential equation

$$\frac{d}{dt} y_t = b(y_t), \quad y_s = a.$$

We will later write $y' = b(y)$ instead of $\frac{d}{dt} y_t = b(y_t)$.

The following inequality appears in Ch. 3, Sect. 1 of the book by Freidlin and Wentzell [7]. For every $\delta > 0$,

$$\mathbb{P} \left(\sup_{s \leq t \leq u} |X_t - y_t| > \delta \right) \leq \mathbb{P} \left(\sup_{s \leq t \leq u} |W_t| > \delta e^{-L(u-s)} \right),$$

where L is a Lipschitz constant of b . It follows that

$$\begin{aligned} \mathbb{P}\left(\sup_{s \leq t \leq u} |X_t - y_t| > \delta\right) &\leq \mathbb{P}\left(\sup_{s \leq t \leq u} W_t > \delta e^{-L(u-s)}\right) + \mathbb{P}\left(\inf_{s \leq t \leq u} W_t < -\delta e^{-L(u-s)}\right) \\ &= 2\mathbb{P}\left(\sup_{s \leq t \leq u} W_t > \delta e^{-L(u-s)}\right) \\ &= 4\mathbb{P}\left(W_u - W_s > \delta e^{-L(u-s)}\right) \\ &\leq c_0 \exp\left(-\frac{\delta^2}{2(u-s)} e^{-2L(u-s)}\right), \end{aligned} \tag{3.2}$$

where c_0 is an absolute constant.

(ii) Recall that $\beta > 2$ and consider the function

$$b(x) = -\frac{1}{\beta x^{\beta-1}}, \quad x > 0. \tag{3.3}$$

We need the assumption that $\beta > 2$ for the main part of the argument but many calculations given below hold for a larger family of β 's. It is easy to check that

$$y_{s,a}(t) := (a^\beta + s - t)^{1/\beta}, \quad s \leq t \leq s + a^\beta, \tag{3.4}$$

is the solution to the ordinary differential equation

$$y' = b(y) \tag{3.5}$$

with the initial condition $y_{s,a}(s) = a$, where $s \in \mathbb{R}$, $a > 0$. Note that the function $y_{s,a}(t)$ approaches 0 vertically at $t = s + a^\beta$.

(iii) Fix any $\gamma \in (0, 1)$ and let L be the Lipschitz constant of b on the interval $[a(\gamma/2)^{1/\beta}/2, 2a]$. Then

$$L = b'(a(\gamma/2)^{1/\beta}/2), \quad b'(x) = \frac{\beta - 1}{\beta x^\beta},$$

and, therefore,

$$L = \frac{\beta - 1}{\beta(a(\gamma/2)^{1/\beta}/2)^\beta} = \frac{\beta - 1}{\beta \gamma 2^{1-\beta} a^\beta}. \tag{3.6}$$

Let X be the solution to (3.1) with b defined in (3.3). Assume that $\delta > 0$ is so small that

$$a(\gamma/2)^{1/\beta}/2 \leq y_{0,a}((1 - \gamma/2)a^\beta) - \delta < y_{0,a}(0) + \delta \leq 2a. \tag{3.7}$$

It follows that if $\sup_{0 \leq t \leq (1-\gamma/2)a^\beta} |X_t - y_{0,a}(t)| \leq \delta$ then $X_t \in [a(\gamma/2)^{1/\beta}/2, 2a]$ for $0 \leq t \leq (1 - \gamma/2)a^\beta$. Hence, we can apply (3.2) with L given by (3.6) to obtain the following estimate

$$\mathbb{P}\left(\sup_{0 \leq t \leq (1-\gamma/2)a^\beta} |X_t - y_{0,a}(t)| > \delta \mid X_0 = a\right) \leq c_0 \exp\left(-c_1 \frac{\delta^2}{a^\beta}\right), \tag{3.8}$$

where X_t satisfies (3.1) and c_1 depends on β and γ , but it does not depend on δ and a .

(iv) Suppose that $a > 0$ and $u = (1 - \gamma)a^\beta$. Then $y_{0,a}(u) = a\gamma^{1/\beta}$. For $\varepsilon \in (0, 1)$, let

$$\bar{\delta} = \bar{\delta}(\varepsilon) = a \left[1 - (1 - \varepsilon\gamma)^{1/\beta}\right]. \tag{3.9}$$

We fix $\varepsilon \in (0, 1)$ so small that for all $a > 0$ the inequality (3.7) is satisfied with $\bar{\delta}$ in place of δ and, moreover,

$$(1 - \gamma/2)(a - \bar{\delta})^\beta > (1 - \gamma)a^\beta \tag{3.10}$$

and

$$\gamma^{1/\beta} + 1 - (1 - \varepsilon\gamma)^{1/\beta} < 1. \tag{3.11}$$

If $\delta \in [-\bar{\delta}, \bar{\delta}]$ then

$$y_{0,a+\delta}(t) = \left((a + \delta)^\beta - t \right)^{1/\beta}, \quad 0 \leq t \leq (a + \delta)^\beta.$$

It is straightforward to check that

$$y_{0,a+\delta}(u) \geq a\gamma^{1/\beta}(1 - \varepsilon)^{1/\beta} > 0.$$

We will estimate the difference $y_{0,a+\delta}(u) - y_{0,a}(u)$ as a function of δ . Let

$$f(\delta) = y_{0,a+\delta}(u) = \left((a + \delta)^\beta - (1 - \gamma)a^\beta \right)^{1/\beta}.$$

Then

$$y_{0,a+\delta}(u) - y_{0,a}(u) = f(\delta) - f(0) = \delta f'(\hat{\delta}),$$

where $\hat{\delta}$ is between 0 and δ . But

$$f'(\delta) = (a + \delta)^{\beta-1} \left((a + \delta)^\beta - (1 - \gamma)a^\beta \right)^{1/\beta-1}$$

and

$$f''(\delta) = -a^\beta(\beta - 1)(1 - \gamma)(a + \delta)^{\beta-2} \left((a + \delta)^\beta - (1 - \gamma)a^\beta \right)^{1/\beta-2}.$$

It follows from (3.9) and the fact that $\varepsilon \in (0, 1)$ that $(a + \delta)^\beta - (1 - \gamma)a^\beta > 0$. Thus $f'' < 0$ and f' is strictly decreasing. Therefore, for $\delta \in [-\bar{\delta}, \bar{\delta}]$ we have

$$f'(\delta) \leq f'(-\bar{\delta}) = \left(\frac{1 - \varepsilon\gamma}{\gamma(1 - \varepsilon)} \right)^{1-1/\beta} =: M > 1. \tag{3.12}$$

Hence $|y_{0,a+\delta}(u) - y_{0,a}(u)| \leq M|\delta|$ for $\delta \in [-\bar{\delta}, \bar{\delta}]$. Suppose that $\delta' \in (0, \bar{\delta}]$ and $\delta = \frac{\delta'}{M+1}$. If $a_0 \in [a - \delta, a + \delta]$, then

$$y_{0,a}(u) - \delta' \leq y_{0,a_0}(u) \leq y_{0,a}(u) + \delta'. \tag{3.13}$$

The function $b(x)$ is strictly increasing on $(0, \infty)$. This easily implies that if $0 < a_1 < a_2$ and $0 \leq s < t \leq a_1^\beta$ then

$$y_{0,a_2}(s) - y_{0,a_1}(s) < y_{0,a_2}(t) - y_{0,a_1}(t). \tag{3.14}$$

Hence, inequality (3.13) holds in fact for all $t \in [0, u]$ in place of u and, moreover, $y_{0,a-\bar{\delta}}(t) \leq y_{0,a}(t) - \delta'$. Another consequence of (3.13) and (3.14) is that if $s \in [0, u]$ and $a_0 \in [y_{0,a}(s) - \delta, y_{0,a}(s) + \delta]$ then for $t \in [s, u]$,

$$y_{0,a-\bar{\delta}}(t) \leq y_{0,a}(t) - \delta' \leq y_{s,a_0}(t) \leq y_{0,a}(t) + \delta'. \tag{3.15}$$

The following generalization of (3.10) also follows from (3.14). If $s \in [0, u]$ and $a_0 \geq y_{0,a}(s) - \delta$ then

$$s + (1 - \gamma/2)a_0^\beta > (1 - \gamma)a^\beta. \tag{3.16}$$

3.2 Proof of Theorem 1.2

Suppose that $\mathbf{X}_t = (X_t^1, \dots, X_t^N)$ is a Fleming-Viot process on $(0, 2]$ driven by the diffusion defined in (1.2), with an arbitrary $2 \leq N < \infty$. Recall that the role of the boundary is played by the point 0, and only this point. In other words, the particles jump only when they approach 0.

Step 1. Let $\mathbf{x} = (x^1, \dots, x^N)$, $[N] = \{1, \dots, N\}$, and let j_1 be the smallest integer in $[N]$ with

$$x^{j_1} = \max_{1 \leq j \leq N} x^j.$$

Consider process \mathbf{X} starting from $\mathbf{X}_0 = \mathbf{x}$ and let $I_1 = \{j_1\}$, $J_1 = [N] \setminus I_1$ and $S_0 = 0$. Let $u = (1 - \gamma)(x^{j_1})^\beta$ and

$$S_1 = u \wedge \inf \left\{ t \geq 0 : \exists_{j \in J_1} X_t^j = X_t^{j_1} \right\}.$$

Note that two processes X^i and X^j can meet either when their paths intersect at a time when both processes are continuous or when one of the processes jumps onto the other. Let j_2 be the smallest index in J_1 such that the equality in the definition of S_1 holds with $j = j_2$. Let $I_2 = \{j_1, j_2\}$ and $J_2 = [N] \setminus I_2$.

Next we proceed by induction. Assume that, for some $n < N$, the sets I_1, \dots, I_n , J_1, \dots, J_n , and stopping times $S_1 < S_2 < \dots < S_{n-1}$ are defined. Then we let

$$S_n = u \wedge \inf \left\{ t \geq S_{n-1} : \exists_{i \in I_n} \exists_{j \in J_n} X_t^i = X_t^j \right\},$$

$I_{n+1} = I_n \cup \{j_{n+1}\}$ and $J_{n+1} = [N] \setminus I_{n+1}$, where j_{n+1} is the smallest index in J_n such that the equality in the definition of S_n holds with $j = j_{n+1}$.

The set I_n has n elements which are indices of particles which are “descendants” of the particle X^{j_1} that was the highest at time 0. By convention, we let $I_n = I_N$ and $S_n = u$ for $n \geq N$.

Step 2. Write $a = x^{j_1}$ and $u = (1 - \gamma)a^\beta$. Then $\mathbf{x} \in (0, a]^N$. Recall $\bar{\delta}$ and M defined in (3.9) and (3.12), and for $1 \leq n \leq N$ define

$$\hat{\delta}_n = \frac{\bar{\delta}}{(M + 1)^{N-n}}. \tag{3.17}$$

Note that $\hat{\delta}_n = (M + 1)\hat{\delta}_{n-1}$. Consider events

$$F_n = \bigcup_{j \in I_n} \left\{ \sup_{S_{n-1} \leq t < S_n} |X_t^j - y_{0,a}(t)| > \hat{\delta}_n \right\}.$$

Note that for every t , $\max_{j \in I_n} X_t^j \geq \max_{j \in J_n} X_t^j$. Hence,

$$\bigcup_{1 \leq j \leq N} \left\{ \sup_{0 \leq t < u} X_t^j - y_{0,a}(t) > \bar{\delta} \right\} \subset \bigcup_{1 \leq n \leq N} F_n,$$

and, therefore,

$$\begin{aligned}
 & \mathbb{P}^{\mathbf{x}} \left(\bigcup_{1 \leq j \leq N} \left\{ \sup_{0 \leq t < u} X_t^j - y_{0,a}(t) > \bar{\delta} \right\} \right) \\
 & \leq \mathbb{P}^{\mathbf{x}} \left(\bigcup_{1 \leq n \leq N} F_n \right) \\
 & = \mathbb{P}^{\mathbf{x}} \left(\bigcup_{1 \leq n \leq N} F_n \cap F_1^c \cap \dots \cap F_{n-1}^c \right) \\
 & \leq \sum_{1 \leq n \leq N} \mathbb{P}^{\mathbf{x}} (F_n \cap F_1^c \cap \dots \cap F_{n-1}^c) \\
 & \leq \sum_{1 \leq n \leq N} \mathbb{P}^{\mathbf{x}} (F_n \mid F_1^c, \dots, F_{n-1}^c) \\
 & \leq \sum_{1 \leq n \leq N} \sum_{j \in I_n} \mathbb{P}^{\mathbf{x}} \left(\sup_{S_{n-1} \leq t < S_n} |X_t^j - y_{0,a}(t)| > \hat{\delta}_n \mid F_1^c, \dots, F_{n-1}^c \right),
 \end{aligned} \tag{3.18}$$

where we adopted the convention $\mathbb{P}^{\mathbf{x}}(F_1 \mid F_0^c) = \mathbb{P}^{\mathbf{x}}(F_1)$.

Suppose that F_{n-1}^c holds and $j \in I_n$. Then $|X_{S_{n-1}}^j - y_{0,a}(S_{n-1})| \leq \hat{\delta}_{n-1}$. Let y_t^j , $t \geq S_{n-1}$, be a solution to $y' = b(y)$ with $y_{S_{n-1}}^j = X_{S_{n-1}}^j$. By (3.15),

$$\mathbb{P}^{\mathbf{x}} \left(\sup_{S_{n-1} \leq t < S_n} |X_t^j - y_{0,a}(t)| > \hat{\delta}_n \mid F_{n-1}^c \right) \leq \mathbb{P}^{\mathbf{x}} \left(\sup_{S_{n-1} \leq t < S_n} |X_t^j - y_t^j| > \hat{\delta}_n \mid F_{n-1}^c \right).$$

It follows from (3.16) that we can apply (3.8) (with an appropriate shift of the time scale) to X^j , assuming that F_{n-1}^c holds, on the interval $[S_{n-1}, S_n] \subset [0, u] = [0, (1 - \gamma)a^\beta]$. We obtain

$$\begin{aligned}
 & \mathbb{P}^{\mathbf{x}} \left(\sup_{S_{n-1} \leq t < S_n} |X_t^j - y_t^j| > \hat{\delta}_n \mid F_{n-1}^c \right) \leq c_0 \exp \left(-c_2 \frac{\hat{\delta}_n^2}{(y_{0,a}(S_{n-1}) + \hat{\delta}_{n-1})^\beta} \right) \\
 & \leq c_0 \exp \left(-c_2 \frac{\bar{\delta}^2 (M+1)^{-2N}}{(a + \bar{\delta})^\beta} \right) = c_0 \exp \left(-c_3 \frac{\bar{\delta}^2}{(a + \bar{\delta})^\beta} \right).
 \end{aligned}$$

We combine this estimate with (3.18) to see that

$$\begin{aligned}
 & \mathbb{P}^{\mathbf{x}} \left(\bigcup_{1 \leq j \leq N} \left\{ \sup_{0 \leq t < u} X_t^j - y_{0,a}(t) > \bar{\delta} \right\} \right) \\
 & \leq \sum_{1 \leq n \leq N} \sum_{j \in I_n} \mathbb{P}^{\mathbf{x}} \left(\sup_{S_{n-1} \leq t < S_n} |X_t^j - y_{0,a}(t)| > \hat{\delta}_n \mid F_1^c, \dots, F_{n-1}^c \right) \\
 & \leq c_0 N^2 \exp \left(-c_3 \frac{\bar{\delta}^2}{(a + \bar{\delta})^\beta} \right).
 \end{aligned} \tag{3.19}$$

Step 3. We will prove that there exist $v < \infty$ and $r \in (0, 2)$ such that if $\mathbf{x} \in (0, r]^N$ then

$$\mathbb{P}^{\mathbf{x}}(\tau_\infty > v) \leq \frac{1}{2}. \tag{3.20}$$

Consider an $r \in (0, 2)$ and for $\mathbf{x} = (x^1, \dots, x^N)$, let $A_0 = \max_j x^j$, $U_0 = 0$, and for

$k = 0, 1, 2, \dots$, let

$$U_{k+1} = U_k + (1 - \gamma)A_k^\beta,$$

$$A_{k+1} = \max_{1 \leq j \leq N} X_{U_{k+1}}^j.$$

Let Y_t^k denote the solution to ODE (3.5) with the initial condition $Y_{U_k}^k = A_k$. Recall ε from (3.10) and let $\Delta_k = A_k [1 - (1 - \varepsilon\gamma)^{1/\beta}]$. For $k = 0, 1, 2, \dots$ define events

$$\Gamma_k = \left[\max_{1 \leq j \leq N} \sup_{U_k \leq t < U_{k+1}} X_t^j > Y_t^k + \Delta_k \right].$$

Note that $Y_{U_{k+1}}^k = \gamma^{1/\beta} A_k$. Suppose that $\bigcap_{k=0}^\infty \Gamma_k^c$ holds. Then

$$A_{k+1} \leq \gamma^{1/\beta} A_k + A_k [1 - (1 - \varepsilon\gamma)^{1/\beta}] = c_4 A_k,$$

for all k , where $c_4 = \gamma^{1/\beta} + 1 - (1 - \varepsilon\gamma)^{1/\beta} < 1$, by (3.11). Hence, $A_k \leq c_4^k A_0$ and, therefore, $\sum_k A_k^\beta < \infty$. If we let $v = (1 - \gamma) \sum_{k=1}^\infty r^\beta c_4^{k\beta} < \infty$ then $\lim_{k \rightarrow \infty} U_k \leq (1 - \gamma) \sum_{k=1}^\infty A_0^\beta c_4^{k\beta} \leq v$ and $\limsup_{t \uparrow v} \max_{1 \leq j \leq N} X_t^j = 0$. This implies easily that $\tau_\infty \leq v$. Thus, to prove (3.20), it will suffice to show that there exists $r \in (0, 2)$ such that if $\mathbf{x} \in (0, r]^N$, then

$$\mathbb{P}^\mathbf{x} \left(\bigcup_{k=0}^\infty \Gamma_k \right) < \frac{1}{2}. \tag{3.21}$$

But

$$\mathbb{P}^\mathbf{x} \left(\bigcup_{k=0}^\infty \Gamma_k \right) \leq \mathbb{P}^\mathbf{x}(\Gamma_0) + \sum_{k=1}^\infty \mathbb{P}^\mathbf{x}(\Gamma_k \mid \Gamma_0^c, \dots, \Gamma_{k-1}^c). \tag{3.22}$$

By (3.19) and the strong Markov property applied at U_k ,

$$\begin{aligned} \mathbb{P}^\mathbf{x}(\Gamma_k \mid \Gamma_0^c, \dots, \Gamma_{k-1}^c) &\leq c_0 N^2 \exp \left(-c_3 \frac{\Delta_k^2}{(A_k + \Delta_k)^\beta} \right) \\ &= c_0 N^2 \exp \left(-c_3 A_k^{2-\beta} \frac{(1 - (1 - \varepsilon\gamma)^{1/\beta})^2}{(2 - (1 - \varepsilon\gamma)^{1/\beta})^\beta} \right) \\ &\leq c_0 N^2 \exp \left(-c_5 A_0^{2-\beta} c_4^{k(2-\beta)} \right), \end{aligned}$$

where $c_4 < 1$. So by (3.22), if $\max_j x^j \leq r$, then

$$\mathbb{P}^\mathbf{x} \left(\bigcup_{k=0}^\infty \Gamma_k \right) \leq c_0 N^2 \sum_{k=0}^\infty \exp \left(-c_5 r^{2-\beta} c_4^{k(2-\beta)} \right),$$

which is convergent. Since $2 - \beta < 0$, we can choose $r > 0$ so small that the above sum is less than $1/2$, proving (3.21).

Step 4. Let $r \in (0, 2)$ and v be as in Step 3. Partition the set $(0, 2]^N$ into two sets $A = (0, r]^N$ and A^c . First we will show that the time when process \mathbf{X} enters the set A has a distribution with an exponentially decreasing tail.

So assume that $\mathbf{X}_0 \in A^c$ and let

$$I_1 = \left\{ j \in [N] : X_0^j \in (r, 2] \right\}, \quad I_2 = [N] \setminus I_1.$$

Let τ_1^j be the the first hitting time of 0 by the process X^j and let

$$\eta = \begin{cases} 0 & \text{if } I_2 = \emptyset, \\ \max_{j \in I_2} \left\{ \tau_1^j \right\}, & \text{otherwise.} \end{cases}$$

Consider

$$p_1(\mathbf{x}) = \mathbb{P}^{\mathbf{x}} \left\{ \forall_{j \in I_1} \forall_{0 \leq t \leq 1/2} X_t^j \in \left[\frac{r}{2}, 2 \right]; \eta < 1/2; \forall_{i \in I_2} \forall_{\tau_1^i < t \leq 1/2} X_t^i \in \left[\frac{r}{4}, 2 \right] \right\}.$$

We will argue that for $\mathbf{x} \in A^c$ we have

$$p_1(\mathbf{x}) \geq p_1 > 0. \tag{3.23}$$

Indeed, with probability at least $q_1 > 0$ any particle from I_1 stays in the interval $[r/2, 2]$ up to time $t = 1/2$. With probability at least $q_2 > 0$ any particle from I_2 hits 0 before time $t = 1/2$; with probability at least $1/N$ it jumps onto a particle in I_1 ; and then with probability at least $q_3 > 0$ it stays in the interval $[r/4, 2]$ up to time $t = 1/2$. Therefore (3.23) holds with $p_1 = (q_1 q_2 q_3 / N)^N$. Obviously q_1, q_2, q_3 and p_1 depend on r .

Next, if we define

$$p_2(\mathbf{x}) = \mathbb{P}^{\mathbf{x}} \left\{ \forall_{j \in [N]} \forall_{0 < t < 1/2} X_t^j \in \left[\frac{r}{8}, 2 \right]; X_{1/2}^j \in \left[\frac{r}{8}, r \right] \right\},$$

and $B = [\frac{r}{4}, 2]^N$, then an argument similar to that proving (3.23) shows that for $\mathbf{x} \in B$ we have $p_2(\mathbf{x}) \geq p_2 > 0$, where p_2 depends on r . Therefore, by the Markov property at time $t = 1/2$, for $\mathbf{x} \in A^c$ we have

$$\mathbb{P}^{\mathbf{x}}(\mathbf{X}_1 \in A) \geq p := p_1 p_2 > 0. \tag{3.24}$$

Now let

$$T = \inf \{ t \geq 0 : \mathbf{X}_t \in A \}.$$

By (3.24), for all $\mathbf{x} \in (0, 2]^N$,

$$\mathbb{P}^{\mathbf{x}}(T \leq 1) \geq p > 0.$$

Applying the Markov property at $t = 1, 2, \dots$ we obtain

$$\mathbb{P}^{\mathbf{x}}(T \geq k) \leq (1 - p)^k.$$

Choose k so large that $(1 - p)^k < \frac{1}{2}$. Recall that r and v are as in Step 3. Let θ denote the usual Markovian shift operator. Then for any $\mathbf{x} \in (0, 2]^N$,

$$\mathbb{P}^{\mathbf{x}}(\tau_\infty \geq k + v) \leq \mathbb{P}^{\mathbf{x}}(T \geq k) + \mathbb{P}^{\mathbf{x}}(\tau_\infty \circ \theta_T \geq v) \leq (1 - p)^k + \frac{1}{2} := q < 1.$$

Therefore, applying the Markov property at times $k + v, 2(k + v), 3(k + v), \dots$, we obtain,

$$\mathbb{P}^{\mathbf{x}}(\tau_\infty \geq n(k + v)) \leq q^n, \quad n = 1, 2, \dots,$$

which proves (1.3). This implies that $\tau_\infty < \infty$, a.s.

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