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Self-interacting diffusions IV: Rate of convergence*

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Abstract

Self-interacting diffusions are processes living on a compact Riemannian manifold defined by a stochastic differential equation with a drift term depending on the past empirical measure μ_t of the process. The asymptotics of μ_t is governed by a deterministic dynamical system and under certain conditions (μ_t) converges almost surely towards a deterministic measure μ^* (see Benaïm, Ledoux, Raimond (2002) and Benaïm, Raimond (2005)). We are interested here in the rate of convergence of μ_t towards μ^*. A central limit theorem is proved. In particular, this shows that greater is the interaction repelling faster is the convergence.

Key words: Self-interacting random processes, reinforced processes.

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1 Introduction

Self-interacting diffusions

Let M be a smooth compact Riemannian manifold and $V : M \times M \to \mathbb{R}$ a sufficiently smooth mapping[1]. For all finite Borel measure μ, let $V\mu : M \to \mathbb{R}$ be the smooth function defined by

$$V\mu(x) = \int_M V(x,y)\mu(dy).$$

Let (e_α) be a finite family of vector fields on M such that $\sum_\alpha e_\alpha(e_\alpha f)(x) = \Delta f(x)$, where Δ is the Laplace operator on M and $e_\alpha(f)$ stands for the Lie derivative of f along e_α. Let (B^α) be a family of independent Brownian motions.

A *self-interacting diffusion* on M associated to V can be defined as the solution to the stochastic differential equation (SDE)

$$dX_t = \sum_\alpha e_\alpha(X_t) \circ dB_t^\alpha - \nabla(V\mu_t)(X_t)dt$$

where $\mu_t = \frac{1}{t}\int_0^t \delta_{X_s} ds$ is the empirical occupation measure of (X_t).

In absence of drift (i.e $V = 0$), (X_t) is just a Brownian motion on M but in general it defines a non Markovian process whose behavior at time t depends on its past trajectory through μ_t. This type of process was introduced in Benaim, Ledoux and Raimond (2002) ([3]) and further analyzed in a series of papers by Benaim and Raimond (2003, 2005, 2007) ([4], [5] and [6]). We refer the reader to these papers for more details and especially to [3] for a detailed construction of the process and its elementary properties. For a general overview of processes with reinforcement we refer the reader to the recent survey paper by Pemantle (2007) ([16]).

Notation and Background

We let $\mathscr{M}(M)$ denote the space of finite Borel measures on M, $\mathscr{P}(M) \subset \mathscr{M}(M)$ the space of probability measures. If I is a metric space (typically, $I = M$, $\mathbb{R}^+ \times M$ or $[0,T] \times M$) we let $C(I)$ denote the space of real valued continuous functions on I equipped with the topology of uniform convergence on compact sets. The normalized Riemann measure on M will be denoted by λ.

Let $\mu \in \mathscr{P}(M)$ and $f : M \to \mathbb{R}$ a nonnegative or $\mu-$integrable Borel function. We write μf for $\int f d\mu$, and $f\mu$ for the measure defined as $f\mu(A) = \int_A f d\mu$. We let $L^2(\mu)$ denote the space of functions for which $\mu|f|^2 < \infty$, equipped with the inner product $\langle f, g\rangle_\mu = \mu(fg)$ and the norm $\|f\|_\mu = \sqrt{\mu f^2}$. We simply write L^2 for $L^2(\lambda)$.

Of fundamental importance in the analysis of the asymptotics of (μ_t) is the mapping $\Pi : \mathscr{M}(M) \to \mathscr{P}(M)$ defined by

$$\Pi(\mu) = \xi(V\mu)\lambda \tag{1}$$

[1] The mapping $V_x : M \to \mathbb{R}$ defined by $V_x(y) = V(x,y)$ is C^2 and its derivatives are continuous in (x,y).

where $\xi : C(M) \to C(M)$ is the function defined by

$$\xi(f)(x) = \frac{e^{-f(x)}}{\int_M e^{-f(y)}\lambda(dy)}. \tag{2}$$

In [3], it is shown that the asymptotics of μ_t can be precisely related to the long term behavior of a certain semiflow on $\mathscr{P}(M)$ induced by the ordinary differential equation (ODE) on $\mathscr{M}(M)$:

$$\dot{\mu} = -\mu + \Pi(\mu). \tag{3}$$

Depending on the nature of V, the dynamics of (3) can either be convergent or nonconvergent leading to similar behaviors for $\{\mu_t\}$ (see [3]). When V is symmetric, (3) happens to be a *quasigradient* and the following convergence result holds.

Theorem 1.1 ([5])**.** *Assume that V is symmetric, i.e. $V(x,y) = V(y,x)$. Then the limit set of $\{\mu_t\}$ (for the topology of weak* convergence) is almost surely a compact connected subset of*

$$\mathsf{Fix}(\Pi) = \{\mu \in \mathscr{P}(M) : \mu = \Pi(\mu)\}.$$

In particular, if $\mathsf{Fix}(\Pi)$ is finite then (μ_t) converges almost surely toward a fixed point of Π. This holds for a generic function V (see [5]). Sufficient conditions ensuring that $\mathsf{Fix}(\Pi)$ has cardinal one are as follows:

Theorem 1.2 ([5], [6])**.** *Assume that V is symmetric and that one of the two following conditions hold*

(i) Up to an additive constant V is a Mercer kernel: For some constant C, $V(x,y) = K(x,y) + C$, and for all $f \in L^2$,

$$\int K(x,y)f(x)f(y)\lambda(dx)\lambda(dy) \geq 0.$$

(ii) For all $x \in M, y \in M, u \in T_xM, v \in T_yM$

$$\mathsf{Ric}_x(u,u) + \mathsf{Ric}_y(v,v) + \mathsf{Hess}_{x,y}V((u,v),(u,v)) \geq K(\|u\|^2 + \|v\|^2)$$

where K is some positive constant. Here Ric_x stands for the Ricci tensor at x and $\mathsf{Hess}_{x,y}$ is the Hessian of V at (x,y).

Then $\mathsf{Fix}(\Pi)$ reduces to a singleton $\{\mu^*\}$ and $\mu_t \to \mu^*$ with probability one.

As observed in [6] the condition (i) in Theorem 1.2 seems well suited to describe *self-repelling diffusions*. On the other hand, it is not clearly related to the geometry of M. Condition (ii) has a more geometrical flavor and is robust to smooth perturbations (of M and V). It can be seen as a Bakry-Emery type condition for self interacting diffusions.

In [5], it is also proved that every stable (for the ODE (3)) fixed point of Π has a positive probability to be a limit point for μ_t; and any unstable fixed point cannot be a limit point for μ_t.

Organisation of the paper

Let $\mu^* \in \mathsf{Fix}(\Pi)$. We will assume that

Hypothesis 1.3. *μ_t converges a.s. towards μ^*.*

In this paper we intend to study the rate of this convergence. Let

$$\Delta_t = e^{t/2}(\mu_{e^t} - \mu^*).$$

It will be shown that, under some conditions to be specified later, for all $g = (g_1, \ldots, g_n) \in C(M)^n$ the process $\left[\Delta_s g_1, \ldots, \Delta_s g_n, V\Delta_s\right]_{s\geq t}$ converges in law, as $t \to \infty$, toward a certain stationary Ornstein-Uhlenbeck process (Z^g, Z) on $\mathbb{R}^n \times C(M)$. This process is defined in Section 2. The main result is stated in section 3 and some examples are developed. It is in particular observed that a strong repelling interaction gives a faster convergence. The section 4 is a proof section.

In the following K (respectively C) denotes a positive constant (respectively a positive random constant). These constants may change from line to line.

2 The Ornstein-Uhlenbeck process (Z^g, Z).

For a more precise definition of Ornstein-Uhlenbeck processes on $C(M)$ and their basic properties, we refer the reader to the appendix (section 5). Throughout all this section we let $\mu \in \mathscr{P}(M)$ and $g = (g_1, ..., g_n) \in C(M)^n$. For $x \in M$ we set $V_x : M \to \mathbb{R}$ defined by $V_x(y) = V(x, y)$.

2.1 The operator G_μ

Let $h \in C(M)$ and let $G_{\mu,h} : \mathbb{R} \times C(M) \to \mathbb{R}$ be the linear operator defined by

$$G_{\mu,h}(u, f) = u/2 + \mathrm{Cov}_\mu(h, f), \tag{4}$$

where Cov_μ is the covariance on $L^2(\mu)$, that is the bilinear form acting on $L^2 \times L^2$ defined by

$$\mathrm{Cov}_\mu(f, h) = \mu(fh) - (\mu f)(\mu h).$$

We define the linear operator $G_\mu : C(M) \to C(M)$ by

$$G_\mu f(x) = G_{\mu,V_x}(f(x), f) = f(x)/2 + \mathrm{Cov}_\mu(V_x, f). \tag{5}$$

It is easily seen that $\|G_\mu f\|_\infty \leq (2\|V\|_\infty + 1/2)\|f\|_\infty$. In particular, G_μ is a bounded operator. Let $\{e^{-tG_\mu}\}$ denote the semigroup acting on $C(M)$ with generator $-G_\mu$. From now on we will assume the following:

Hypothesis 2.1. *There exists $\kappa > 0$ such that $\mu << \lambda$ with $\|\frac{d\mu}{d\lambda}\|_\infty < \infty$, and such that for all $f \in L^2(\lambda)$, $\langle G_\mu f, f\rangle_\lambda \geq \kappa\|f\|_\lambda^2$.*

Let

$$\lambda(-G_\mu) = \lim_{t\to\infty} \frac{\log(\|e^{-tG_\mu}\|)}{t}.$$

This limit exists by subadditivity. Then

Lemma 2.2. *Hypothesis 2.1 implies that* $\lambda(-G_\mu) \le -\kappa < 0$.

Proof : For all $f \in L^2(\lambda)$,

$$\frac{d}{dt}\|e^{-tG_\mu}f\|_\lambda^2 = -2\langle G_\mu e^{-tG_\mu}f, e^{-tG_\mu}f\rangle_\lambda \le -2\kappa\|e^{-tG_\mu}f\|_\lambda.$$

This implies that $\|e^{-tG_\mu}f\|_\lambda \le e^{-\kappa t}\|f\|_\lambda$. Denote by g_t the solution of the differential equation

$$\frac{d}{dt}g_t(x) = \mathrm{Cov}_\mu(V_x, g_t)$$

with $g_0 = f \in C(M)$. Note that $e^{-tG_\mu}f = e^{-t/2}g_t$. It is straightforward to check that (using the fact that $\|\frac{d\mu}{d\lambda}\|_\infty < \infty$) $\frac{d}{dt}\|g_t\|_\lambda \le K\|g_t\|_\lambda$ with K a constant depending only on V and μ. Thus $\sup_{t\in[0,1]}\|g_t\|_\lambda \le K\|f\|_\lambda$. Now, since for all $x \in M$ and $t \in [0,1]$

$$\left|\frac{d}{dt}g_t(x)\right| \le K\|g_t\|_\lambda \le K\|f\|_\lambda,$$

we have $\|g_1\|_\infty \le K\|f\|_\lambda$. This implies that $\|e^{-G_\mu}f\|_\infty \le K\|f\|_\lambda$.
Now for all $t > 1$, and $f \in C(M)$,

$$\begin{aligned}\|e^{-tG_\mu}f\|_\infty &= \|e^{-G_\mu}e^{-(t-1)G_\mu}f\|_\infty \le K\|e^{-(t-1)G_\mu}f\|_\lambda \\ &\le Ke^{-\kappa(t-1)}\|f\|_\lambda \le Ke^{-\kappa t}\|f\|_\infty.\end{aligned}$$

This implies that $\|e^{-tG_\mu}\| \le Ke^{-\kappa t}$, which proves the lemma. **QED**

The *adjoint* of G_μ is the operator on $\mathscr{M}(M)$ defined by the relation

$$m(G_\mu f) = (G_\mu^* m)f$$

for all $m \in \mathscr{M}(M)$ and $f \in C(M)$. It is not hard to verify that

$$G_\mu^* m = \frac{1}{2}m + (Vm)\mu - (\mu(Vm))\mu. \tag{6}$$

2.2 The generator A_μ and its inverse Q_μ

Let H^2 be the Sobolev space of real valued functions on M, associated with the norm $\|f\|_H^2 = \|f\|_\lambda^2 + \|\nabla f\|_\lambda^2$. Since $\Pi(\mu)$ and λ are equivalent measures with continuous Radon-Nykodim derivative, $L^2(\Pi(\mu)) = L^2(\lambda)$. We denote by K_μ the projection operator, acting on $L^2(\Pi(\mu))$, defined by

$$K_\mu f = f - \Pi(\mu)f.$$

We denote by A_μ the operator acting on H^2 defined by

$$A_\mu f = \frac{1}{2}\Delta f - \langle \nabla V\mu, \nabla f\rangle.$$

Note that for f and h in H^2 (denoting $\langle\cdot,\cdot\rangle$ the Riemannian inner product on M)

$$\langle A_\mu f, h\rangle_{\Pi(\mu)} = -\frac{1}{2}\int \langle \nabla f, \nabla h\rangle(x)\Pi(\mu)(dx).$$

For all $f \in C(M)$ there exists $\mathsf{Q}_\mu f \in H^2$ such that $\Pi(\mu)(\mathsf{Q}_\mu f) = 0$ and

$$f - \Pi(\mu)f = K_\mu f = -A_\mu \mathsf{Q}_\mu f. \tag{7}$$

It is shown in [3] that $\mathsf{Q}_\mu f$ is C^1 and that there exists a constant K such that for all $f \in C(M)$ and $\mu \in \mathscr{P}(M)$,

$$\|\mathsf{Q}_\mu f\|_\infty + \|\nabla \mathsf{Q}_\mu f\|_\infty \le K\|f\|_\infty. \tag{8}$$

Finally, note that for f and h in L^2,

$$\int \langle \nabla \mathsf{Q}_\mu f, \nabla \mathsf{Q}_\mu h\rangle(x)\Pi(\mu)(dx) = -2\langle A_\mu \mathsf{Q}_\mu f, \mathsf{Q}_\mu h\rangle_{\Pi(\mu)} = 2\langle f, \mathsf{Q}_\mu h\rangle_{\Pi(\mu)}. \tag{9}$$

2.3 The covariance C_μ^g

We let $\widehat{C}_\mu$ denote the bilinear continuous form $\widehat{C}_\mu : C(M) \times C(M) \to \mathbb{R}$ defined by

$$\widehat{C}_\mu(f,h) = 2\langle f, \mathsf{Q}_\mu h\rangle_{\Pi(\mu)}.$$

This form is symmetric (see its expression given by (9)). Note also that for some constant K depending on μ, $|\widehat{C}_\mu(f,h)| \le K\|f\|_\infty \times \|h\|_\infty$.

We let C_μ denote the mapping $C_\mu : M \times M \to \mathbb{R}$ defined by $C_\mu(x,y) = \widehat{C}_\mu(V_x, V_y)$. Let $\tilde{M} = \{1,\dots,n\} \cup M$ and $C_\mu^g : \tilde{M} \times \tilde{M} \to \mathbb{R}$ be the function defined by

$$C_\mu^g(x,y) = \begin{cases} \widehat{C}_\mu(g_x, g_y) & \text{for} \quad x,y \in \{1,\dots,n\}, \\ C_\mu(x,y) & \text{for} \quad x,y \in M, \\ \widehat{C}_\mu(V_x, g_y) & \text{for} \quad x \in M,\ y \in \{1,\dots,n\}. \end{cases}$$

Then C_μ and C_μ^g are covariance functions (as defined in subsection 5.2).

In the following, when $n = 0$, $\tilde{M} = M$ and $C_\mu^g = C_\mu$. When $n \ge 1$, $C(\tilde{M})$ can be identified with $\mathbb{R}^n \times C(M)$.

Lemma 2.3. *There exists a Brownian motion on $\mathbb{R}^n \times C(M)$ with covariance C_μ^g.*

Proof : Since the argument are the same for $n \ge 1$, we just do it for $n = 0$. Let

$$\begin{aligned} d_{C_\mu}(x,y) &:= \sqrt{C_\mu(x,x) - 2C_\mu(x,y) + C_\mu(y,y)} \\ &= \|\nabla \mathsf{Q}_\mu(V_x - V_y)\|_{\Pi(\mu)} \quad \le \quad K\|V_x - V_y\|_\infty \end{aligned}$$

where the last inequality follows from (8). Then $d_{C_\mu}(x,y) \le Kd(x,y)$. Thus d_{C_μ} satisfies (30) and we can apply Theorem 5.4 of the appendix (section 5). **QED**

2.4 The process (Z^g, Z)

Let $G^g_\mu : \mathbb{R}^n \times C(M) \to \mathbb{R}^n \times C(M)$ be the operator defined by

$$G^g_\mu = \begin{pmatrix} I_n/2 & A^g_\mu \\ 0 & G_\mu \end{pmatrix} \tag{10}$$

where I_n is the identity matrix on $\mathbb{R}^n$ and $A^g_\mu : C(M) \to \mathbb{R}^n$ is the linear map defined by $A^g_\mu(f) = \Big(\mathrm{Cov}_\mu(f, g_1), \ldots, \mathrm{Cov}_\mu(f, g_n)\Big)$.

Since G^g_μ is a bounded operator, for any law ν on $\mathbb{R}^n \times C(M)$, there exists $\tilde{Z} = (Z^g, Z)$ an Ornstein-Uhlenbeck process of covariance C^g_μ and drift $-G^g_\mu$, with initial distribution given by ν (using Theorem 5.6). More precisely, $\tilde{Z}$ is the unique solution of

$$\begin{cases} dZ_t &= dW_t - G_\mu Z_t dt \\ dZ^{g_i}_t &= dW^{g_i}_t - \Big(Z^{g_i}_t/2 + \mathrm{Cov}_\mu(Z_t, g_i)\Big)\, dt,\ i = 1, \ldots, n \end{cases} \tag{11}$$

where $\tilde{Z}_0$ is a $\mathbb{R}^n \times C(M)$-valued random variable of law ν and $\tilde{W} = (W^g, W)$ is a $\mathbb{R}^n \times C(M)$-valued Brownian motion of covariance C^g_μ independent of $\tilde{Z}$. In particular, Z is an Ornstein-Uhlenbeck process of covariance C_μ and drift $-G_\mu$. Denote by $\mathsf{P}^{g,\mu}_t$ the semigroup associated to $\tilde{Z}$. Then

Proposition 2.4. *Assume hypothesis 2.1. Then there exists $\pi^{g,\mu}$ the law of a centered Gaussian variable in $\mathbb{R}^n \times C(M)$, with variance $\mathsf{Var}(\pi^{g,\mu})$ where for $(u, m) \in \mathbb{R}^n \times \mathscr{M}(M)$,*

$$\mathsf{Var}(\pi^{g,\mu})(u, m) \;:=\; \mathsf{E}\left((mZ_\infty + \langle u, Z^g_\infty\rangle)^2\right) \;=\; \int_0^\infty \widehat{C}_\mu(f_t, f_t)dt$$

with $f_t = e^{-t/2}\sum_i u_i g_i + V m_t$, and where m_t is defined by

$$m_t f = m_0(e^{-tG_\mu} f) + \sum_{i=1}^n u_i \int_0^t e^{-s/2} Cov_\mu(g_i, e^{-(t-s)G_\mu} f) ds. \tag{12}$$

Moerover,

- *(i) $\pi^{g,\mu}$ is the unique invariant probability measure of P_t.*
- *(ii) For all bounded continuous function φ on $\mathbb{R}^n \times C(M)$ and all $(u, f) \in \mathbb{R}^n \times C(M)$, $\lim_{t\to\infty} \mathsf{P}^{g,\mu}_t \varphi(u, f) = \pi^{g,\mu}\varphi$.*

Proof : This is a consequence of Theorem 5.7. To apply it one can remark that G^g_μ is an operator like the ones given in example 5.11.

The variance $\mathsf{Var}(\pi^{g,\mu})$ is given by $\mathsf{Var}(\pi^{g,\mu})(\nu) = \int_0^\infty \langle \nu, e^{-sG^g_\mu} C^g_\mu e^{s(G^g_\mu)^*} \nu\rangle ds$ for $\nu = (u, m) \in \mathbb{R}^n \times \mathscr{M}(M) = C(\tilde{M})^*$. Thus $\mathsf{Var}(\pi^{g,\mu})(u, m) = \int_0^\infty \widehat{C}_\mu(f_t, f_t) dt$ with $f_t = \sum_i u_t(i) g_i + V m_t$ and where $(u_t, m_t) = e^{-t(G^g_\mu)^*}(u, m)$. Now

$$(G^g_\mu)^* = \begin{pmatrix} I/2 & 0 \\ (A^g_\mu)^* & (G_\mu)^* \end{pmatrix}$$

and $(A^g_\mu)^* u = \sum_i u_i(g_i - \mu g_i)\mu$. Thus $u_t = e^{-t/2}u$ and m_t is the solution with $m_0 = m$ of

$$\frac{dm_t}{dt} = -e^{-t/2}\left(\sum_i u_i(g_i - \mu g_i)\right)\mu - (G_\mu)^* m_t. \tag{13}$$

Note that (13) is equivalent to

$$\frac{d}{dt}(m_t f) = -e^{-t/2}\mathrm{Cov}_\mu\left(\sum_i u_i g_i, f\right) - m_t(G_\mu f)$$

for all $f \in C(M)$, and $m_0 = m$. From which we deduce that

$$m_t = e^{-tG^*_\mu}m_0 - \int_0^t e^{-s/2}e^{-(t-s)G^*_\mu}\left(\sum_i u_i(g_i - \mu g_i)\mu\right) ds$$

which implies the formula for m_t given by (12). **QED**

An Ornstein-Uhlenbeck process of covariance C^g_μ and drift $-G^g_\mu$ will be called *stationary* when its initial distribution is $\pi^{g,\mu}$.

3 A central limit theorem for μ_t

We state here the main results of this article. We assume $\mu^* \in \mathsf{Fix}(\Pi)$ satisfies hypotheses 1.3 and 2.1. Set $\Delta_t = e^{t/2}(\mu_{e^t} - \mu^*)$, $D_t = V\Delta_t$ and $D_{t+\cdot} = (D_{t+s})_{s\geq 0}$. Then

Theorem 3.1. *$D_{t+\cdot}$ converges in law, as $t \to \infty$, towards a stationary Ornstein-Uhlenbeck process of covariance C_{μ^*} and drift $-G_{\mu^*}$.*

For $g \in C(M)^n$, we set $D^g_t = (\Delta_t g, D_t)$ and $D^g_{t+\cdot} = (D^g_{t+s})_{s\geq 0}$. Then

Theorem 3.2. *$D^g_{t+\cdot}$ converges in law towards a stationary Ornstein-Uhlenbeck process of covariance $C^g_{\mu^*}$ and drift $-G^g_{\mu^*}$.*

Define $\widehat{C} : C(M) \times C(M) \to \mathbb{R}$ the symmetric bilinear form defined by

$$\widehat{C}(f,h) = \int_0^\infty \widehat{C}_{\mu^*}(f_t, h_t)dt, \tag{14}$$

with (h_t is defined by the same formula, with h in place of f)

$$f_t(x) = e^{-t/2}f(x) - \int_0^t e^{-s/2}\mathrm{Cov}_{\mu^*}(f, e^{-(t-s)G_{\mu^*}}V_x)ds. \tag{15}$$

Corollary 3.3. *$\Delta_t g$ converges in law towards a centered Gaussian variable Z^g_∞ of covariance*

$$\mathsf{E}[Z^{g_i}_\infty Z^{g_j}_\infty] = \widehat{C}(g_i, g_j).$$

Proof : Follows from theorem 3.2 and the calculus of $\mathsf{Var}(\pi^{g,\mu})(u,0)$. **QED**

3.1 Examples

3.1.1 Diffusions

Suppose $V(x,y) = V(x)$, so that (X_t) is just a standard diffusion on M with invariant measure $\mu^* = \frac{exp(-V)\lambda}{\lambda \exp(-V)}$.

Let $f \in C(M)$. Since $e^{-tG_{\mu^*}}1 = e^{-t/2}1$, f_t defined by (15) is equal to $e^{-t/2}f$. Thus

$$\widehat{C}(f,g) = 2\mu^*(f\,\mathsf{Q}_{\mu^*}g). \tag{16}$$

Corollary 3.3 says that

Theorem 3.4. *For all $g \in C(M)^n$, Δ_t^g converges in law toward a centered Gaussian variable $(Z_\infty^{g_1},\dots,Z_\infty^{g_n})$, with covariance given by*

$$\mathsf{E}(Z_\infty^{g_i}Z_\infty^{g_j}) = 2\mu^*(g_i\mathsf{Q}_{\mu^*}g_j).$$

Remark 3.5. *This central limit theorem for Brownian motions on compact manifolds has already been considered by Baxter and Brosamler in [1] and [2]; and by Bhattacharya in [7] for ergodic diffusions.*

3.1.2 The case $\mu^* = \lambda$ and V symmetric.

Suppose here that $\mu^* = \lambda$ and that V is symmetric. We assume (without loss of generality since $\Pi(\lambda) = \lambda$ implies that $V\lambda$ is a constant function) that $V\lambda = 0$.

Since V is compact and symmetric, there exists an orthonormal basis $(e_\alpha)_{\alpha\geq 0}$ in $L^2(\lambda)$ and a sequence of reals $(\lambda_\alpha)_{\alpha\geq 0}$ such that e_0 is a constant function and

$$V = \sum_{\alpha\geq 1}\lambda_\alpha e_\alpha \otimes e_\alpha.$$

Assume that for all α, $1/2+\lambda_\alpha > 0$. Then hypothesis 2.1 is satisfied, and the convergence of μ_t towards λ holds with positive probability (see [6]).

Let $f \in C(M)$ and f_t defined by (15), denoting $f^\alpha = \langle f, e_\alpha\rangle_\lambda$ and $f_t^\alpha = \langle f_t, e_\alpha\rangle_\lambda$, we have $f_t^0 = e^{-t/2}f^0$ and for $\alpha \geq 1$,

$$\begin{aligned} f_t^\alpha &= e^{-t/2}f^\alpha - \lambda_\alpha e^{-(1/2+\lambda_\alpha)t}\left(\frac{e^{\lambda_\alpha t}-1}{\lambda_\alpha}\right)f^\alpha \\ &= e^{-(1/2+\lambda_\alpha)t}f^\alpha. \end{aligned}$$

Using the fact that $\widehat{C}_\lambda(f,g) = 2\lambda(f\,\mathsf{Q}_\lambda g)$, this implies that

$$\widehat{C}(f,g) = 2\sum_{\alpha\geq 1}\sum_{\beta\geq 1}\frac{1}{1+\lambda_\alpha+\lambda_\beta}\langle f, e_\alpha\rangle_\lambda\langle g, e_\beta\rangle_\lambda\lambda(e_\alpha\mathsf{Q}_\lambda e_\beta).$$

This, with corollary 3.3, proves

Theorem 3.6. *Assume hypothesis 1.3 and that $1/2+\lambda_\alpha > 0$ for all α. Then for all $g \in C(M)^n$, Δ_t^g converges in law toward a centered Gaussian variable $(Z_\infty^{g_1},\dots,Z_\infty^{g_n})$, with covariance given by $\mathsf{E}(Z_\infty^{g_i}Z_\infty^{g_j})=\widehat{C}(g_i,g_j)$.*

In particular,

$$\mathsf{E}(Z_\infty^{e_\alpha}Z_\infty^{e_\beta})=\frac{2}{1+\lambda_\alpha+\lambda_\beta}\lambda(e_\alpha \mathsf{Q}_\lambda e_\beta).$$

When all λ_α are positive, which corresponds to what is named a self-repelling interaction in [6], the rate of convergence of μ_t towards λ is bigger than when there is no interaction, and the bigger is the interaction (that is larger λ_α's) faster is the convergence.

4 Proof of the main results

We assume hypothesis 1.3 and μ^* satisfies hypothesis 2.1. For convenience, we choose for the constant κ in hypothesis 2.1 a constant less than 1/2. In all this section, we fix $g=(g_1,...,g_n)\in C(M)^n$.

4.1 A lemma satisfied by Q_μ

We denote by $\mathscr{X}(M)$ the space of continuous vector fields on M, and equip the spaces $\mathscr{P}(M)$ and $\mathscr{X}(M)$ respectively with the weak convergence topology and with the uniform convergence topology.

Lemma 4.1. *For all $f\in C(M)$, the mapping $\mu\mapsto\nabla\mathsf{Q}_\mu f$ is a continuous mapping from $\mathscr{P}(M)$ in $\mathscr{X}(M)$.*

Proof : Let μ and ν be in $\mathscr{M}(M)$, and $f\in C(M)$. Set $h=\mathsf{Q}_\mu f$. Then $f=-A_\mu h+\Pi(\mu)f$ and

$$\begin{aligned}\|\nabla\mathsf{Q}_\mu f-\nabla\mathsf{Q}_\nu f\|_\infty &= \|-\nabla\mathsf{Q}_\mu A_\mu h+\nabla\mathsf{Q}_\nu A_\mu h\|_\infty\\ &= \|\nabla h+\nabla\mathsf{Q}_\nu A_\mu h\|_\infty\\ &\le \|\nabla(h+\mathsf{Q}_\nu A_\nu h)\|_\infty+\|\nabla\mathsf{Q}_\nu(A_\mu-A_\nu)h\|_\infty.\end{aligned}$$

Since $\nabla(h+\mathsf{Q}_\nu A_\nu h)=0$ and $(A_\mu-A_\nu)h=\langle\nabla V_{\mu-\nu},\nabla h\rangle$, we get

$$\|\nabla\mathsf{Q}_\mu f-\nabla\mathsf{Q}_\nu f\|_\infty\le K\|\langle\nabla V_{\mu-\nu},\nabla h\rangle\|_\infty. \tag{17}$$

Using the fact that $(x,y)\mapsto\nabla V_x(y)$ is uniformly continuous, the right hand term of (17) converges towards 0, when $d(\mu,\nu)$ converges towards 0, d being a distance compatible with the weak convergence. **QED**

4.2 The process Δ

Set $h_t=V\mu_t$ and $h^*=V\mu^*$. Recall $\Delta_t=e^{t/2}(\mu_{e^t}-\mu^*)$ and $D_t(x)=V\Delta_t(x)=\Delta_t V_x$. Then (D_t) is a continuous process taking its values in $C(M)$ and $D_t=e^{t/2}(h_{e^t}-h^*)$.

To simplify the notation, we set $K_s = K_{\mu_s}$, $Q_s = Q_{\mu_s}$ and $A_s = A_{\mu_s}$. Let $(M_t^f)_{t\geq 1}$ be the martingale defined by $M_t^f = \sum_\alpha \int_1^t e_\alpha(Q_s f)(X_s) dB_s^\alpha$. The quadratic covariation of M^f and M^h (with f and h in $C(M)$) is given by

$$\langle M^f, M^h\rangle_t = \int_1^t \langle \nabla Q_s f, \nabla Q_s h\rangle(X_s) ds.$$

Then for all $t \geq 1$ (with $\dot{Q}_t = \frac{d}{dt} Q_t$) ,

$$Q_t f(X_t) - Q_1 f(X_1) = M_t^f + \int_1^t \dot{Q}_s f(X_s) ds - \int_1^t K_s f(X_s) ds.$$

Thus

$$\begin{aligned}
\mu_t f &= \frac{1}{t}\int_1^t K_s f(X_s) ds + \frac{1}{t}\int_1^t \Pi(\mu_s) f\, ds + \frac{1}{t}\int_0^1 f(X_s) ds \\
&= -\frac{1}{t}\left(Q_t f(X_t) - Q_1 f(X_1) - \int_1^t \dot{Q}_s f(X_s) ds \right) \\
&\quad + \frac{M_t^f}{t} + \frac{1}{t}\int_1^t \langle \xi(h_s), f\rangle_\lambda ds + \frac{1}{t}\int_0^1 f(X_s) ds.
\end{aligned}$$

For $f \in C(M)$ (using the fact that $\mu^* f = \langle \xi(h^*), f\rangle_\lambda$), $\Delta_t f = \sum_{i=1}^5 \Delta_t^i f$ with

$$\begin{aligned}
\Delta_t^1 f &= e^{-t/2}\left(-Q_{e^t} f(X_{e^t}) + Q_1 f(X_1) + \int_1^{e^t} \dot{Q}_s f(X_s) ds \right) \\
\Delta_t^2 f &= e^{-t/2} M_{e^t}^f \\
\Delta_t^3 f &= e^{-t/2}\int_1^{e^t} \langle \xi(h_s) - \xi(h^*) - D\xi(h^*)(h_s - h^*), f\rangle_\lambda ds \\
\Delta_t^4 f &= e^{-t/2}\int_1^{e^t} \langle D\xi(h^*)(h_s - h^*), f\rangle_\lambda ds \\
\Delta_t^5 f &= e^{-t/2}\left(\int_0^1 f(X_s) ds - \mu^* f \right).
\end{aligned}$$

Then $D_t = \sum_{i=1}^5 D_t^i$, where $D_t^i = V\Delta_t^i$. Finally, note that

$$\langle D\xi(h^*)(h - h^*), f\rangle_\lambda = -\mathrm{Cov}_{\mu^*}(h - h^*, f). \tag{18}$$

4.3 First estimates

We recall the following estimate from [3]: There exists a constant K such that for all $f \in C(M)$ and $t > 0$,

$$\|\dot{Q}_t f\|_\infty \leq \frac{K}{t}\|f\|_\infty.$$

This estimate, combined with (8), implies that for f and h in $C(M)$,

$$\langle M^f - M^h\rangle_t \le K\|f-h\|_\infty \times t$$

and that

Lemma 4.2. *There exists a constant K depending on $\|V\|_\infty$ such that for all $t \ge 1$, and all $f \in C(M)$*

$$\|\Delta^1_t f\|_\infty + \|\Delta^5_t f\|_\infty \le K \times (1+t)e^{-t/2}\|f\|_\infty, \tag{19}$$

which implies that $((\Delta^1+\Delta^5)_{t+s})_{s\ge 0}$ and $((D^1+D^5)_{t+s})_{s\ge 0}$ both converge towards 0 (respectively in $\mathscr{M}(M)$ and in $C(\mathbb{R}^+ \times M)$).

We also have

Lemma 4.3. *There exists a constant K such that for all $t \ge 0$ and all $f \in C(M)$,*

$$\begin{aligned}
\mathsf{E}[(\Delta^2_t f)^2] &\le K\|f\|^2_\infty,\\
|\Delta^3_t f| &\le K\|f\|_\lambda \times e^{-t/2}\int_0^t \|D_s\|^2_\lambda ds,\\
|\Delta^4_t f| &\le K\|f\|_\lambda \times e^{-t/2}\int_0^t e^{s/2}\|D_s\|_\lambda ds.
\end{aligned}$$

Proof : The first estimate follows from

$$\mathsf{E}[(\Delta^2_t f)^2] = e^{-t}\mathsf{E}[(M^f_{e^t})^2] = e^{-t}\mathsf{E}[\langle M^f\rangle_{e^t}] \le K\|f\|^2_\infty.$$

The second estimate follows from the fact that

$$\|\xi(h)-\xi(h^*)-D\xi(h^*)(h-h^*)\|_\lambda = O(\|h-h^*\|^2_\lambda).$$

The last estimate follows easily after having remarked that

$$|\langle D\xi(h^*)(h_s-h^*), f\rangle| = |\mathrm{Cov}_{\mu^*}(h_s-h^*, f)| \le K\|f\|_\lambda \times \|h_s-h^*\|_\lambda.$$

This proves this lemma. **QED**

4.4 The processes Δ' and D'

Set $\Delta' = \Delta^2+\Delta^3+\Delta^4$ and $D' = D^2+D^3+D^4$. For $f \in C(M)$, set

$$\epsilon^f_t = e^{t/2}\langle \xi(h_{e^t}) - \xi(h^*) - D\xi(h^*)(h_{e^t}-h^*), f\rangle_\lambda.$$

Then

$$d\Delta'_t f = -\frac{\Delta'_t f}{2}dt + dN^f_t + \epsilon^f_t dt + \langle D\xi(h^*)(D_t), f\rangle_\lambda dt$$

where for all $f \in C(M)$, N^f is a martingale. Moreover, for f and h in $C(M)$,

$$\langle N^f, N^h\rangle_t = \int_0^t \langle \nabla \mathsf{Q}_{e^s} f(X_{e^s}), \nabla \mathsf{Q}_{e^s} h(X_{e^s})\rangle ds.$$

Then, for all x,

$$dD'_t(x) = -\frac{D'_t(x)}{2}dt + dM_t(x) + \epsilon_t(x)dt + \langle D\xi(h^*)(D_t), V_x\rangle_\lambda dt$$

where M is the martingale in $C(M)$ defined by $M(x) = N^{V_x}$ and $\epsilon_t(x) = \epsilon_t^{V_x}$. We also have

$$G_{\mu^*}(D')_t(x) = \frac{D'_t(x)}{2} - \langle D\xi(h^*)(D'_t), V_x\rangle_\lambda.$$

Denoting $L_{\mu^*} = L_{-G_{\mu^*}}$ (defined by equation (32) in the appendix (section 5)),

$$dL_{\mu^*}(D')_t(x) \quad = \quad dD'_t(x) + G_{\mu^*}(D')_t(x)dt$$

and we have

$$L_{\mu^*}(D')_t(x) = M_t(x) + \int_0^t \epsilon'_s(x)ds$$

with $\epsilon'_s(x) = {\epsilon'}_s V_x$ where for all $f \in C(M)$,

$$\epsilon'_s f = \epsilon_s^f + \langle D\xi(h^*)((D^1 + D^5)_s), f\rangle_\lambda.$$

Using lemma 5.5,

$$D'_t = L_{\mu^*}^{-1}(M)_t + \int_0^t e^{-(t-s)G_{\mu*}}\epsilon'_s ds. \tag{20}$$

Denote $\Delta_t g = (\Delta_t g_1, \dots, \Delta_t g_n)$, $\Delta'_t g = (\Delta'_t g_1, \dots, \Delta'_t g_n)$, $N^g = (N^{g_1}, \dots, N^{g_n})$ and $\epsilon'_t g = (\epsilon'_t g_1, \dots, \epsilon'_t g_n)$. Then, denoting $L^g_{\mu^*} = L_{-G^g_{\mu^*}}$ (with $G^g_{\mu^*}$ defined by (10)) we have

$$L^g_{\mu^*}(\Delta' g, D')_t = (N^g_t, M_t) + \int_0^t ({\epsilon'}_s g, \epsilon'_s)ds$$

so that (using lemma 5.5 and integrating by parts)

$$(\Delta'_t g, D'_t) = (L^g_{\mu^*})^{-1}(N^g, M)_t + \int_0^t e^{-(t-s)G^g_{\mu^*}}({\epsilon'}_s g, \epsilon'_s)ds. \tag{21}$$

Moreover

$$(L^g_{\mu^*})^{-1}(N^g, M)_t = \left(\widehat{N}^{g_1}_t, \dots, \widehat{N}^{g_n}_t, L^{-1}_{\mu^*}(M)_t\right),$$

where

$$\widehat{N}^{g_i}_t = N^{g_i}_t - \int_0^t \left(\frac{N^{g_i}_s}{2} + \widehat{C}_{\mu^*}(L^{-1}_{\mu^*}(M)_s, g_i)\right)ds.$$

4.5 Estimation of ϵ'_t

4.5.1 Estimation of $\|L_{\mu^*}^{-1}(M)_t\|_\lambda$

Lemma 4.4. *(i) For all $\alpha \geq 2$, there exists a constant K_α such that for all $t \geq 0$,*

$$\mathsf{E}[\|L_{\mu^*}^{-1}(M)_t\|_\lambda^\alpha]^{1/\alpha} \leq K_\alpha.$$

(ii) a.s. there exists C with $\mathsf{E}[C] < \infty$ such that for all $t \geq 0$,

$$\|L_{\mu^*}^{-1}(M)_t\|_\lambda \leq C(1+t).$$

Proof : We have

$$dL_{\mu^*}^{-1}(M)_t = dM_t - G_{\mu^*}L_{\mu^*}^{-1}(M)_t dt.$$

Let N be the martingale defined by

$$N_t = \int_0^t \left\langle \frac{L_{\mu^*}^{-1}(M)_s}{\|L_{\mu^*}^{-1}(M)_s\|_\lambda}, dM_s \right\rangle_\lambda .$$

We have $\langle N\rangle_t \leq Kt$ for some constant K. Then

$$\begin{aligned} d\|L_{\mu^*}^{-1}(M)_t\|_\lambda^2 &= 2\|L_{\mu^*}^{-1}(M)_t\|_\lambda dN_t - 2\langle L_{\mu^*}^{-1}(M)_t, G_{\mu^*}L_{\mu^*}^{-1}(M)_t\rangle_\lambda dt \\ &+ d\left(\int \langle M(x)\rangle_t \lambda(dx)\right). \end{aligned}$$

Note that there exists a constant K such that

$$\frac{d}{dt}\left(\int \langle M(x)\rangle_t \lambda(dx)\right) \leq K$$

and that (see hypothesis 2.1)

$$\langle L_{\mu^*}^{-1}(M)_t, G_{\mu^*}L_{\mu^*}^{-1}(M)_t\rangle_\lambda \geq \kappa\|L_{\mu^*}^{-1}(M)_t\|_\lambda^2.$$

This implies that

$$\frac{d}{dt}\mathsf{E}[\|L_{\mu^*}^{-1}(M)_t\|_\lambda^2] \leq -2\kappa\mathsf{E}[\|L_{\mu^*}^{-1}(M)_t\|_\lambda^2] + K$$

which implies (i) for $\alpha = 2$. For $\alpha > 2$, we find that

$$\begin{aligned} \frac{d}{dt}\mathsf{E}[\|L_{\mu^*}^{-1}(M)_t\|_\lambda^\alpha] &\leq -\alpha\kappa\mathsf{E}[\|L_{\mu^*}^{-1}(M)_t\|_\lambda^\alpha] + K\mathsf{E}[\|L_{\mu^*}^{-1}(M)_t\|_\lambda^{\alpha-2}] \\ &\leq -\alpha\kappa\mathsf{E}[\|L_{\mu^*}^{-1}(M)_t\|_\lambda^\alpha] + K\mathsf{E}[\|L_{\mu^*}^{-1}(M)_t\|_\lambda^\alpha]^{\frac{\alpha-2}{\alpha}} \end{aligned}$$

which implies that $\mathsf{E}[\|L_{\mu^*}^{-1}(M)_t\|_\lambda^\alpha]$ is bounded.

We now prove (ii). Fix $\alpha > 1$. Then there exists a constant K such that

$$\frac{\|L_{\mu^*}^{-1}(M)_t\|_\lambda^2}{(1+t)^\alpha} \leq \|L_{\mu^*}^{-1}(M)_0\|_\lambda^2 + 2\int_0^t \frac{\|L_{\mu^*}^{-1}(M)_s\|_\lambda}{(1+s)^\alpha} dN_s + K.$$

Then Bürkholder-Davies-Gundy inequality (BDG inequality in the following) inequality implies that

$$\mathsf{E}\left[\sup_{t\geq 0}\frac{\|L_{\mu^*}^{-1}(M)_t\|_\lambda^2}{(1+t)^\alpha}\right]\leq K+2\sup_{t\geq 0}\left(\int_0^t\frac{Kds}{(1+s)^{2\alpha}}\right)^{1/2}$$

which is finite. This implies the lemma by taking $\alpha=2$. **QED**

4.5.2 Estimation of $\|D_t\|_\lambda$

Note that for all $f\in C(M)$, $|\epsilon_t^f|\leq Ke^{-t/2}\|D_t\|_\lambda^2\times\|f\|_\infty$. Thus

$$|\epsilon'_t f|\leq Ke^{-t/2}(1+t+\|D_t\|_\lambda^2)\times\|f\|_\infty.$$

This implies (using lemma 2.2 and the fact that $0<\kappa<1/2$)

Lemma 4.5. *There exists K such that*

$$\left\|\int_0^t e^{-(t-s)G_{\mu^*}}\epsilon'_s ds\right\|_\infty\leq Ke^{-\kappa t}\left(1+\int_0^t e^{-(1/2-\kappa)s}\|D_s\|_\lambda^2 ds\right).\tag{22}$$

This lemma with lemma 4.4-(ii) implies the following

Lemma 4.6. *a.s. there exists C with* $\mathsf{E}[C]<\infty$ *such that*

$$\|D_t\|_\lambda\leq C\times\left[1+t+\int_0^t e^{-s/2}\|D_s\|_\lambda^2 ds\right].\tag{23}$$

Proof : First note that

$$\|D_t\|_\lambda\leq\|D'_t\|_\lambda+K(1+t)e^{-t/2}.$$

Using the expression of D'_t given by (20), we get

$$\begin{aligned}\|D'_t\|_\lambda &\leq \|L_{\mu^*}^{-1}(M)_t\|_\lambda+\left\|\int_0^t e^{-(t-s)G_{\mu^*}}\epsilon'_s ds\right\|_\infty\\ &\leq C(1+t)+Ke^{-\kappa t}\left(1+\int_0^t e^{-(1/2-\kappa)s}\|D_s\|_\lambda^2 ds\right)\end{aligned}$$

(with $\mathsf{E}[C]<\infty$) which implies the lemma. **QED**

Lemma 4.7. *Let x and* ϵ *be real functions, and* α *a real constant. Assume that for all* $t\geq 0$*, we have* $x_t\leq\alpha+\int_0^t\epsilon_s x_s ds$*. Then* $x_t\leq\alpha\exp\left(\int_0^t\epsilon_s ds\right)$.

Proof : Similarly to the proof of Gronwall's lemma, we set $y_t=\int_0^t\epsilon_s x_s ds$ and take $\lambda_t=y_t\exp\left(-\int_0^t\epsilon_s ds\right)$. Then $\dot\lambda_t\leq\alpha\epsilon_t\exp\left(-\int_0^t\epsilon_s ds\right)$ and

$$y_t\quad\leq\quad\alpha\int_0^t\epsilon_s\exp\left(\int_s^t\epsilon_u du\right)ds\quad\leq\quad\alpha\exp\left(\int_0^t\epsilon_u du\right)-\alpha.$$

This implies the lemma. **QED**

Lemma 4.8. *a.s., there exists C such that for all t, $\|D_t\|_\lambda \leq C(1+t)$.*

Proof : Lemmas 4.6 and 4.7 imply that

$$\|D_t\|_\lambda \leq C(1+t) \times \exp\left(C\int_0^t e^{-s/2}\|D_s\|_\lambda ds\right).$$

Since hypothesis 1.3 implies that $\lim_{s\to\infty} e^{-s/2}\|D_s\|_\lambda = 0$, then a.s. for all $\epsilon > 0$, there exists C_ϵ such that $\|D_t\|_\lambda \leq C_\epsilon e^{\epsilon t}$. Taking $\epsilon < 1/4$, we get

$$\int_0^\infty e^{-s/2}\|D_s\|_\lambda^2 ds \leq C_\epsilon.$$

This proves the lemma. **QED**

4.5.3 Estimation of ϵ'_t

Lemma 4.9. *a.s. there exists C such that for all $f \in C(M)$,*

$$|\epsilon'{}_t f| \quad\leq\quad C(1+t)^2 e^{-t/2}\|f\|_\infty$$

Proof : We have $|\epsilon'{}_t f| \leq |\epsilon_t^f| + K(1+t)e^{-t/2}\|f\|_\infty$ and

$$|\epsilon_t^f| \quad\leq\quad K\|f\|_\lambda \times e^{-t/2}\|D_t\|_\lambda^2 \quad\leq\quad C\|f\|_\infty \times (1+t)^2 e^{-t/2}$$

by lemma 4.8. **QED**

4.6 Estimation of $\|D_t - L_{\mu^*}^{-1}(M)_t\|_\infty$

Lemma 4.10. *(i) $\|D_t - L_{\mu^*}^{-1}(M)_t\|_\infty \leq Ce^{-\kappa t}$.*

(ii) $\|(\Delta_t g, D_t) - (L_{\mu^*}^g)^{-1}(N^g, M)_t\|_\infty \leq C(1+\|g\|_\infty)e^{-\kappa t}$.

Proof : Note that (i) is implied by (ii). We prove (ii). We have $\|(\Delta_t g, D_t) - (\Delta'_t g, D'_t)\|_\infty \leq K(1+\|g\|_\infty)(1+t)e^{-\kappa t}$. So to prove this lemma, using (21), it suffices to show that

$$\left\|\int_0^t e^{-(t-s)G_{\mu^*}^g}(\epsilon'{}_s g, \epsilon'_s)ds\right\|_\infty \leq K(1+\|g\|_\infty)e^{-\kappa t}. \tag{24}$$

Using hypothesis 2.1 and the definition of $G_{\mu^*}^g$, we have that for all positive t, $\|e^{-tG_{\mu^*}^g}\|_\infty \leq Ke^{-\kappa t}$. This implies $\|e^{-(t-s)G_{\mu^*}^g}(\epsilon'{}_s g, \epsilon'_s)\|_\infty \leq Ke^{-\kappa(t-s)}\|\epsilon'_s\|_\infty \times (1+\|g\|_\infty)$. Thus the term (24) is dominated by

$$K(1+\|g\|_\infty)\int_0^t e^{-\kappa(t-s)}\|\epsilon'_s\|_\infty ds,$$

from which we prove (24) like in the previous lemma. **QED**

4.7 Tightness results

We refer the reader to section 5.1 in the appendix (section 5), where tightness criteria for families of $C(M)$-valued random variables are given. They will be used in this section.

4.7.1 Tightness of $(L_{\mu^*}^{-1}(M)_t)_{t\geq 0}$

In this section we prove the following lemma which in particular implies the tightness of $(D_t)_{t\geq 0}$ and of $(D'_t)_{t\geq 0}$.

Lemma 4.11. *$(L_{\mu^*}^{-1}(M)_t)_{t\geq 0}$ is tight.*

Proof : We have the relation (that defines $L_{\mu^*}^{-1}(M)$)

$$dL_{\mu^*}^{-1}(M)_t(x) = -G_{\mu^*}L_{\mu^*}^{-1}(M)_t(x)dt + dM_t(x).$$

Thus, using the expression of G_{μ^*}

$$dL_{\mu^*}^{-1}(M)_t(x) = -\frac{1}{2}L_{\mu^*}^{-1}(M)_t(x)dt + A_t(x)dt + dM_t(x),$$

with

$$A_t(x) = \widehat{C}_{\mu^*}(V_x, L_{\mu^*}^{-1}(M)_t).$$

Since μ^* is absolutely continuous with respect to λ, we have that (with $\mathrm{Lip}(A_t)$ the Lipschitz constant of A_t, see (36)).

$$\|A_t\|_\infty + \mathrm{Lip}(A_t) \leq K\|L_{\mu^*}^{-1}(M)_t\|_\lambda.$$

Therefore (using lemma 4.4 (i) for $\alpha = 2$), $\sup_t \mathsf{E}[\|A_t\|_\infty^2] < \infty$.

To prove this tightness result, we first prove that for all x, $(L_{\mu^*}^{-1}(M)_t(x))_t$ is tight. Setting $Z_t^x = L_{\mu^*}^{-1}(M)_t(x)$ we have

$$\begin{aligned}
\frac{d}{dt}\mathsf{E}[(Z_t^x)^2] &\leq -\mathsf{E}[(Z_t^x)^2] + 2\mathsf{E}[|Z_t^x| \times |A_t(x)|] + \frac{d}{dt}\mathsf{E}[\langle M(x)\rangle_t] \\
&\leq -\mathsf{E}[(Z_t^x)^2] + K\mathsf{E}[(Z_t^x))^2]^{1/2} + K
\end{aligned}$$

which implies that $(L_{\mu^*}^{-1}(M)_t(x))_t$ is bounded in $L^2(\mathsf{P})$ and thus tight.

We now estimate $\mathsf{E}[|Z_t^x - Z_t^y|^\alpha]^{1/\alpha}$ for α greater than 2 and the dimension of M. Setting $Z_t^{x,y} =$

$Z_t^x - Z_t^y$, we have (using lemma 4.4 (i) for the last inequality)

$$\begin{aligned}
\frac{d}{dt}\mathsf{E}[(Z_t^{x,y})^\alpha] &\leq -\frac{\alpha}{2}\mathsf{E}[(Z_t^{x,y})^\alpha] + \alpha\mathsf{E}[(Z_t^{x,y})^{\alpha-1}|A_t(x)-A_t(y)|] \\
&\quad +\frac{\alpha(\alpha-1)}{2}\mathsf{E}\left[(Z_t^{x,y})^{\alpha-2}\frac{d}{dt}\langle M(x)-M(y)\rangle_t\right] \\
&\leq -\frac{\alpha}{2}\mathsf{E}[(Z_t^{x,y})^\alpha] + Kd(x,y)\mathsf{E}[(Z_t^{x,y})^{\alpha-1}\|L^{-1}(M)_t\|_\lambda] \\
&\quad +Kd(x,y)^2\mathsf{E}[(Z_t^{x,y})^{\alpha-2}] \\
&\leq -\frac{\alpha}{2}\mathsf{E}[(Z_t^{x,y})^\alpha] + Kd(x,y)\mathsf{E}[(Z_t^{x,y})^\alpha]^{\frac{\alpha-1}{\alpha}}\mathsf{E}[\|L^{-1}(M)_t\|_\lambda^\alpha]^{1/\alpha} \\
&\quad +Kd(x,y)^2\mathsf{E}[(Z_t^{x,y})^\alpha]^{\frac{\alpha-2}{\alpha}} \\
&\leq -\frac{\alpha}{2}\mathsf{E}[(Z_t^{x,y})^\alpha] + Kd(x,y)\mathsf{E}[(Z_t^{x,y})^\alpha]^{\frac{\alpha-1}{\alpha}} \\
&\quad +Kd(x,y)^2\mathsf{E}[(Z_t^{x,y})^\alpha]^{\frac{\alpha-2}{\alpha}}.
\end{aligned}$$

Thus, if $x_t = \mathsf{E}[(Z_t^{x,y})^\alpha]/d(x,y)^\alpha$,

$$\frac{dx_t}{dt} \leq -\frac{\alpha}{2}x_t + Kx_t^{\frac{\alpha-1}{\alpha}} + Kx_t^{\frac{\alpha-2}{\alpha}}.$$

It is now an exercise to show that $x_t \leq K$ and so that $\mathsf{E}[(Z_t^{x,y})^\alpha]^{1/\alpha} \leq Kd(x,y)$. Using proposition 5.2, this completes the proof for the tightness of $(L_{\mu^*}^{-1}(M)_t)_t$. **QED**

Remark 4.12. *Kolmogorov's theorem (see theorem 1.4.1 and its proof in Kunita (1990)), with the estimates given in the proof of this lemma, implies that*

$$\sup_t \mathsf{E}[\|L_{\mu^*}^{-1}(M)_t\|_\infty] < \infty.$$

4.7.2 Tightness of $((L_{\mu^*}^g)^{-1}(N^g, M)_t)_{t\geq 0}$

Let $\widehat{\Delta}g$ be defined by the relation

$$(\widehat{\Delta}g, L_{\mu^*}^{-1}(M)) = (L_{\mu^*}^g)^{-1}(N^g, M).$$

Set $A_t g = (A_t g_1, \dots, A_t g_n)$ with $A_t g_i = \widehat{C}_{\mu^*}(g_i, L_{\mu^*}^{-1}(M)_t)$. Then

$$d\widehat{\Delta}_t g = dN_t^g - \frac{\widehat{\Delta}_t g}{2}dt + A_t g dt.$$

Thus,

$$\widehat{\Delta}_t g = e^{-t/2}\int_0^t e^{s/2}dN_s^g + e^{-t/2}\int_0^t e^{s/2}A_s g ds.$$

Using this expression it is easy to prove that $(\widehat{\Delta}_t g)_{t\geq 0}$ is bounded in $L^2(\mathsf{P})$. This implies, using also lemma 4.11

Lemma 4.13. $((L_{\mu^*}^g)^{-1}(N^g, M)_t)_{t\geq 0}$ *is tight.*

4.8 Convergence in law of $(N^g, M)_{t+\cdot} - (N^g, M)_t$

In this section, we denote by E_t the conditional expectation with respect to $\mathscr{F}_{e^t}$. We also set $\mathsf{Q} = \mathsf{Q}_{\mu^*}$ and $C = \widehat{C}_{\mu^*}$.

4.8.1 Preliminary lemmas.

For $f \in C(M)$ and $t \geq 0$, set $N_s^{f,t} = N_{t+s}^f - N_t^f$.

Lemma 4.14. *For all f and h in $C(M)$, $\lim_{t\to\infty} \langle N^{f,t}, N^{h,t}\rangle_s = s \times C(f,h)$.*

Proof : For $z \in M$ and $u > 0$, set

$$\begin{cases} G(z) &= \langle \nabla \mathsf{Q} f, \nabla \mathsf{Q} h\rangle(z) - C(f,h); \\ G_u(z) &= \langle \nabla \mathsf{Q}_u f, \nabla \mathsf{Q}_u h\rangle(z) - C(f,h). \end{cases}$$

We have

$$\begin{aligned} \langle N^{f,t}, N^{h,t}\rangle_s - s \times C(f,h) &= \int_{e^t}^{e^{t+s}} G_u(X_u)\frac{du}{u} \\ &= \int_{e^t}^{e^{t+s}} (G_u - G)(X_u)\frac{du}{u} + \int_{e^t}^{e^{t+s}} G(X_u)\frac{du}{u}. \end{aligned}$$

Integrating by parts, we get that

$$\int_{e^t}^{e^{t+s}} G(X_u)\frac{du}{u} = (\mu_{e^{t+s}} G - \mu_{e^t} G) + \int_0^s (\mu_{e^{t+u}} G) du.$$

Since $\mu^* G = 0$, this converges towards 0 on the event $\{\mu_t \to \mu^*\}$. The term $\int_{e^t}^{e^{t+s}} (G_u - G)(X_u)\frac{du}{u}$ converges towards 0 because $(\mu, z) \mapsto \nabla \mathsf{Q}_\mu f(z)$ is continuous. This proves the lemma. **QED**

Let $f_1, \dots, f_n$ be in $C(M)$. Let (t_k) be an increasing sequence converging to ∞ such that the conditional law of $M^{n,k} = (N^{f_1,t_k}, \dots, N^{f_n,t_k})$ given $\mathscr{F}_{e^{t_k}}$ converges in law towards a $\mathbb{R}^n$-valued process $W^n = (W_1, \dots, W_n)$.

Lemma 4.15. *W^n is a centered Gaussian process such that for all i and j,*

$$\mathsf{E}[W_i^n(s) W_j^n(t)] = (s \wedge t) C(f_i, f_j).$$

Proof : We first prove that W^n is a martingale. For all k, $M^{n,k}$ is a martingale. For all $u \leq v$, BDG inequality implies that $(M^{n,k}(v) - M^{n,k}(u))_k$ is bounded in L^2.

Let $l \geq 1$, $\varphi \in C(\mathbb{R}^l)$, $0 \leq s_1 \leq \cdots \leq s_l \leq u$ and $(i_1, \dots, i_l) \in \{1, \dots, n\}^l$. Then for all k and $i \in \{1, \dots, n\}$, the martingale property implies that

$$\mathsf{E}_{t_k}[(M_i^{n,k}(v) - M_i^{n,k}(u)) Z_k] = 0$$

where Z_k is of the form

$$Z_k = \varphi(M^{n,k}_{i_1}(s_1), \dots, M^{n,k}_{i_l}(s_l)). \tag{25}$$

Using the convergence of the conditional law of $M^{n,k}$ given $\mathscr{F}_{e^{t_k}}$ towards the law of W^n and since $(M_i^{n,k}(v) - M_i^{n,k}(u))_k$ is uniformly integrable (because it is bounded in L^2), we prove that $\mathsf{E}[(W_i^n(v) - W_i^n(u))Z] = 0$ where Z is of the form

$$Z = \varphi(W^n_{i_1}(s_1), \dots, W^n_{i_l}(s_l)). \tag{26}$$

This implies that W^n is a martingale.

We now prove that for $(i, j) \in \{1, \dots, n\}$ (with $C = C_{\mu^*}$),

$$\langle W_i^n, W_j^n \rangle_s = s \times C(f_i, f_j).$$

By definition of $\langle M_i^{n,k}, M_j^{n,k} \rangle$ (in the following $\langle \cdot, \cdot \rangle_u^v = \langle \cdot, \cdot \rangle_v - \langle \cdot, \cdot \rangle_u$)

$$\mathsf{E}_{t_k}\left[\left((M_i^{n,k}(v) - M_i^{n,k}(u))(M_j^{n,k}(v) - M_j^{n,k}(u)) - \langle M_i^{n,k}, M_j^{n,k} \rangle_u^v\right) Z_k\right] = 0 \tag{27}$$

where Z_k is of the form (25). Using the convergence in law and the fact that $(M^{n,k}(v) - M^{n,k}(u))_k^2$ is bounded in L^2 (still using BDG inequality), we prove that as $k \to \infty$,

$$\mathsf{E}_{t_k}[(M_i^{n,k}(v) - M_i^{n,k}(u))(M_j^{n,k}(v) - M_j^{n,k}(u))Z_k]$$

converges towards $\mathsf{E}[(W_i^n(v) - W_i^n(u))(W_j^n(v) - W_j^n(u))Z]$ with Z of the form (26). Now,

$$\begin{aligned} &\mathsf{E}_{t_k}[\langle M_i^{n,k}, M_j^{n,k} \rangle_v Z_k] - v \times \mathsf{E}[Z] \times C(x_i, x_j) \\ &\quad = \quad \mathsf{E}_{t_k}[(\langle M_i^{n,k}, M_j^{n,k} \rangle_v - v \times C(f_i, f_j))Z_k] \quad + \quad v \times (\mathsf{E}_{t_k}[Z_k] - \mathsf{E}[Z]) \times C(f_i, f_j) \end{aligned}$$

The convergence in L^2 of $\langle M_i^{n,k}, M_j^{n,k} \rangle_v$ towards $v \times C(f_i, f_j)$ shows that the first term converges towards 0. The convergence of the conditional law of $M^{n,k}$ with respect to $\mathscr{F}_{e^{t_k}}$ towards W^n shows that the second term converges towards 0. Thus

$$\mathsf{E}\left[\left((W_i^n(v) - W_i^n(u))(W_j^n(v) - W_j^n(u)) - (v - u)C(f_i, f_j)\right) Z\right] = 0.$$

This shows that $\langle W_i^n, W_j^n \rangle_s = s \times C(f_i, f_j)$. We conclude using Lévy's theorem. **QED**

4.8.2 Convergence in law of $M_{t+\cdot} - M_t$

In this section, we denote by $\mathscr{L}_t$ the conditional law of $M_{t+\cdot} - M_t$ knowing $\mathscr{F}_{e^t}$. Then $\mathscr{L}_t$ is a probability measure on $C(\mathbb{R}^+ \times M)$.

Proposition 4.16. *When $t \to \infty$, $\mathscr{L}_t$ converges weakly towards the law of a $C(M)$-valued Brownian motion of covariance C_{μ^*}.*

Proof : In the following, we will denote $M_{t+\cdot} - M_t$ by M^t. We first prove that

Lemma 4.17. *$\{\mathscr{L}_t : t \geq 0\}$ is tight.*

Proof : For all $x \in M$, t and u in $\mathbb{R}^+$,

$$\mathsf{E}_t[(M_u^t(x))^2] = \mathsf{E}_t\left[\int_t^{t+u} d\langle M(x)\rangle_s\right] \le Ku.$$

This implies that for all $u \in \mathbb{R}^+$ and $x \in M$, $(M_u^t(x))_{t\ge 0}$ is tight.

Let $\alpha > 0$. We fix $T > 0$. Then for (u,x) and (v,y) in $[0,T]\times M$, using BDG inequality,

$$\begin{aligned}\mathsf{E}_t[|M_u^t(x) - M_v^t(y)|^\alpha]^{\frac{1}{\alpha}} &\le \mathsf{E}_t[|M_u^t(x) - M_u^t(y)|^\alpha]^{\frac{1}{\alpha}} + \mathsf{E}_t[|M_u^t(y) - M_v^t(y)|^\alpha]^{\frac{1}{\alpha}} \\ &\le K_\alpha \times (\sqrt{T} d(x,y) + \sqrt{|v-u|})\end{aligned}$$

where K_α is a positive constant depending only on α, $\|V\|_\infty$ and $\mathrm{Lip}(V)$ the Lipschitz constant of V. We now let D_T be the distance on $[0,T]\times M$ defined by

$$D_T((u,x),(v,y)) = K_\alpha \times (\sqrt{T} d(x,y) + \sqrt{|v-u|}).$$

The covering number $N([0,T]\times M, D_T, \epsilon)$ is of order $\epsilon^{-d-1/2}$ as $\epsilon \to 0$. Taking $\alpha > d + 1/2$, we conclude using proposition 5.2. **QED**

Let (t_k) be an increasing sequence converging to ∞ and N a $C(M)$-valued random process (or a $C(\mathbb{R}^+ \times M)$ random variable) such that $\mathscr{L}_{t_k}$ converges in law towards N.

Lemma 4.18. *N is a $C(M)$-valued Brownian motion of covariance C_{μ^*}.*

Proof : Let W be a $C(M)$-valued Brownian motion of covariance C_{μ^*}. Using lemma 4.15, we prove that for all $(x_1,\dots,x_n)\in M^n$, $(N(x_1),\dots,N(x_n))$ has the same distribution as $(W(x_1),\dots,X(x_n))$. This implies the lemma. **QED**

Since $\{\mathscr{L}_t\}$ is tight, this lemma implies that $\mathscr{L}_t$ converges weakly towards the law of a $C(M)$-valued Brownian motion of covariance C_{μ^*}. **QED**

4.8.3 Convergence in law of $(N^g, M)_{t+\cdot} - (N^g, M)_t$

Let $\mathscr{L}_t^g$ denote the conditional law of $(N^g,M)_{t+\cdot} - (N^g,M)_t$ knowing $\mathscr{F}_{e^t}$. Then $\mathscr{L}_t^g$ is a probability measure on $C(\mathbb{R}^+ \times M \cup \{1,\dots,n\})$. Let $(N^{g,t}, M^t)$ denote the process $(N^g,M)_{t+\cdot} - (N^g,M)_t$.

Let $(W_t^f)_{(t,f)\in\mathbb{R}^+\times C(M)}$ be a $\mathscr{X}(M)$-valued Brownian motion of covariance $\widehat{C}_{\mu^*}$. Denoting $W_t(x) = W_t^{V_x}$, then $W = (W_t(x))_{(t,x)\in\mathbb{R}^+\times M}$ is a $C(M)$-valued Brownian motion of covariance C_{μ^*}. Let W^g denote $(W^{g_1},\dots,W^{g_n})$, and let (W^g,W) denote the process $(W_t^g,(W_t(x))_{x\in M})_{t\ge 0}$.

Proposition 4.19. *As t goes to ∞, $\mathscr{L}_t^g$ converges weakly towards the law of (W^g,W).*

Proof : We first prove that $\{\mathscr{L}_t^g : t \ge 0\}$ is tight. This is a straightforward consequence of the tightness of $\{\mathscr{L}_t\}$ and of the fact that for all $\alpha > 0$, there exists K_α such that for all nonnegative u and v, $\mathsf{E}_t[|N_u^{g,t} - N_v^{g,t}|^\alpha]^{\frac{1}{\alpha}} \le K_\alpha\sqrt{|v-u|}$.

Let (t_k) be an increasing sequence converging to ∞ and $(\tilde{N}^g, \tilde{M})$ a $\mathbb{R}^n \times C(M)$-valued random process (or a $C(\mathbb{R}^+ \times M \cup \{1,\dots,n\})$ random variable) such that $\mathscr{L}_{t_k}^g$ converges in law towards $(\tilde{N}^g,\tilde{M})$. Then lemmas 4.14 and 4.15 imply that $(\tilde{N}^g,\tilde{M})$ has the same law as (W^g,W). Since $\{\mathscr{L}_t^g\}$ is tight, $\mathscr{L}_t^g$ convergences towards the law of (W^g,W). **QED**

4.9 Convergence in law of D

4.9.1 Convergence in law of $(D_{t+s} - e^{-sG_{\mu^*}} D_t)_{s\geq 0}$

We have

$$D'_{t+s} - e^{-sG_{\mu^*}} D'_t = L_{\mu^*}^{-1}(M^t)_s + \int_0^s e^{-(s-u)G_{\mu^*}} \epsilon'_{t+u} du.$$

Since (using lemma 4.9) $\left\|\int_0^s e^{-(s-u)G_{\mu^*}} \epsilon'_{t+u} du\right\|_\infty \leq K e^{-\kappa t}$ and $\|D_t - D'_t\|_\infty \leq K(1+t)e^{-t/2}$, this proves that $(D_{t+s} - e^{-sG_{\mu^*}} D_t - L_{\mu^*}^{-1}(M_{t+\cdot} - M_t)_s)_{s\geq 0}$ converges towards 0. Since $L_{\mu^*}^{-1}$ is continuous, this proves that the law of $L_{\mu^*}^{-1}(M_{t+\cdot} - M_t)$ converges weakly towards $L_{\mu^*}^{-1}(W)$. Since $L_{\mu^*}^{-1}(W)$ is an Ornstein-Uhlenbeck process of covariance C_{μ^*} and drift $-G_{\mu^*}$ started from 0, we have

Theorem 4.20. *The conditional law of $(D_{t+s} - e^{-sG_{\mu^*}} D_t)_{s\geq 0}$ given $\mathscr{F}_{e^t}$ converges weakly towards an Ornstein-Uhlenbeck process of covariance C_{μ^*} and drift $-G_{\mu^*}$ started from* 0.

4.9.2 Convergence in law of $D_{t+\cdot}$

We can now prove theorem 3.1. We here denote by P_t the semigroup of an Ornstein-Uhlenbeck process of covariance C_{μ^*} and drift $-G_{\mu^*}$, and we denote by π its invariant probability measure.

Since $(D_t)_{t\geq 0}$ is tight, there exists $\nu \in \mathscr{P}(C(M))$ and an increasing sequence t_n converging towards ∞ such that D_{t_n} converges in law towards ν. Then $D_{t_n+\cdot}$ converges in law towards $(L_{\mu^*}^{-1}(W)_s + e^{-sG_{\mu^*}} Z_0)$, with Z_0 independent of W and distributed like ν. This proves that $D_{t_n+\cdot}$ converges in law towards an Ornstein-Uhlenbeck process of covariance C_{μ^*} and drift $-G_{\mu^*}$.

We now fix $t > 0$. Let s_n be a subsequence of t_n such that $D_{s_n-t+\cdot}$ converges in law. Then D_{s_n-t} converges towards a law we denote by ν_t and $D_{s_n-t+\cdot}$ converges in law towards an Ornstein-Uhlenbeck process of covariance C_{μ^*} and drift $-G_{\mu^*}$. Since $D_{s_n} = D_{s_n-t+t}$, D_{s_n} converges in law towards $\nu_t \mathsf{P}_t$. On the other hand D_{s_n} converges in law towards ν. Thus $\nu_t \mathsf{P}_t = \nu$.

Let φ be a Lipschitz bounded function on $C(M)$. Then

$$\begin{aligned} |\nu_t \mathsf{P}_t \varphi - \pi\varphi| &= \left| \int (\mathsf{P}_t\varphi(f) - \pi\varphi)\nu_t(df) \right| \\ &\leq \int |\mathsf{P}_t\varphi(f) - \mathsf{P}_t\varphi(0)|\nu_t(df) + |\mathsf{P}_t\varphi(0) - \pi\varphi| \qquad (28) \end{aligned}$$

where the second term converges towards 0 (using proposition 2.4 (ii) or theorem 5.7 (ii)) and the first term is dominated by (using lemma 5.8) $Ke^{-\kappa t} \int \|f\|_\infty \nu_t(df)$.

It is easy to check that

$$\begin{aligned} \int \|f\|_\infty \nu_t(df) &= \lim_{k\to\infty} \int (\|f\|_\infty \wedge k)\nu_t(df) \\ &= \lim_{k\to\infty} \lim_{n\to\infty} \mathsf{E}[\|D_{s_n-t}\|_\infty \wedge k] \quad \leq \quad \sup_t \mathsf{E}[\|D_t\|_\infty]. \end{aligned}$$

Since

$$\|D_t\|_\infty \leq \|D_t^1 + D_t^5\|_\infty + \|L_{\mu^*}^{-1}(M)_t\|_\infty + \left\| \int_0^t e^{(t-s)G_{\mu^*}} \epsilon'_s ds \right\|_\infty,$$

using the estimates (19), the proof of lemma 4.10 and remark 4.12, we get that

$$\sup_{t\geq 0} \mathsf{E}[\|D_t\|_\infty] < \infty.$$

Taking the limit in (28), we prove $\nu\varphi = \pi\varphi$ for all Lipschitz bounded function φ on $C(M)$. This implies $\nu = \pi$, which proves the theorem. **QED**

4.9.3 Convergence in law of D^g

Set $D_t'^g = (\Delta'_t g, D'_t)$. Since $\|D_t^g - D_t'^g\|_\infty \leq K(1+t)e^{-t/2}$, instead of studying D^g, we can only study $D_t'^g$. Then

$$D'^g_{t+s} - e^{-sG^g_{\mu^*}} D'^g_t = (L^g_{\mu^*})^{-1}(N^{g,t}, M^t)_s + \int_0^s e^{-(s-u)G^g_{\mu^*}} (\epsilon'_{t+u} g, \epsilon'_{t+u}) du.$$

The norm of the second term of the right hand side (using the proof of lemma 4.10) is dominated by

$$K(1+\|g\|_\infty)\int_0^s e^{-\kappa(s-u)} \|\epsilon'_{t+u}\|_\infty du \leq K\int_0^s e^{-\kappa(s-u)}(1+t+u)^2 e^{-(t+u)/2} du$$

whcih is less than $Ke^{-\kappa t}$. Like in section 4.9.1, since $(L^g_{\mu^*})^{-1}(W^g, W)$ is an Ornstein-Uhlenbeck process of covariance $C^g_{\mu^*}$ and drift $-G^g_{\mu^*}$ started from 0,

Theorem 4.21. *The conditional law of $((\Delta^g, D)_{t+s} - e^{-sG^g_{\mu^*}}(\Delta^g, D)_t)_{s\geq 0}$ given $\mathscr{F}_{e^t}$ converges weakly towards an Ornstein-Uhlenbeck process of covariance $C^g_{\mu^*}$ and drift $-G^g_{\mu^*}$ started from 0.*

From this theorem, like in section 4.9.2, we prove theorem 3.2. **QED**

5 Appendix : Ornstein-Uhlenbeck processes on $C(M)$

5.1 Tighness in $\mathscr{P}(C(M))$

Let (M, d) be a compact metric space. Denote by $\mathscr{P}(C(M))$ the space of Borel probability measures on $C(M)$. Since $C(M)$ is separable and complete, Prohorov theorem (see [8]) asserts that $\mathscr{X} \subset \mathscr{P}(C(M))$ is tight if and only if it is relatively compact.

The next proposition gives a useful criterium for a class of random variables to be tight. It follows directly from [15] (Corollary 11.7 p. 307 and the remark following Theorem 11.2). A function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a Young function if it is convex, increasing and $\psi(0) = 0$. If Z is a real valued random variable, we let

$$\|Z\|_\psi = \inf\{c > 0 : \mathsf{E}\big(\psi(|Z|/c)\big) \leq 1\}.$$

For $\epsilon > 0$, we denote by $N(M, d; \epsilon)$ the covering number of E by balls of radius less than ϵ (i.e. the minimal number of balls of radius less than ϵ that cover E), and by D the diameter of M.

Proposition 5.1. *Let $(F_t)_{t\in I}$ be a family of $C(M)$-valued random variables and ψ a Young function. Assume that*

(i) There exists $x \in E$ such that $(F_t(x))_{t\in I}$ is tight;

(ii) $\|F_t(x) - F_t(y)\|_\psi \le Kd(x,y)$;

(iii) $\int_0^D \psi^{-1}(N(M,d;\epsilon))d\epsilon < \infty$.

Then $(F_t)_{t\ge 0}$ is tight.

Proposition 5.2. *Suppose M is a compact finite dimensional manifold of dimension r, d is the Riemannian distance, and*

$$[\mathsf{E}|F_t(x) - F_t(y)|^\alpha]^{1/\alpha} \le Kd(x,y)$$

for some $\alpha > r$. Then conditions (ii) and (iii) of Proposition 5.1 hold true.

Proof : One has $N(E,d;\epsilon)$ of order ϵ^{-r}; and for $\psi(x) = x^\alpha$, $\|\cdot\|_\psi$ is the L^α norm. Hence the result. **QED**

5.2 Brownian motions on $C(M)$.

Let $C : M \times M \to \mathbb{R}$ be a covariance function, that is a continuous symmetric function such that $\sum_{ij} a_i a_j C(x_i, x_j) \ge 0$ for every finite sequence (a_i, x_i) with $a_i \in \mathbb{R}$ and $x_i \in M$.

A *Brownian motion on $C(M)$ with covariance C* is a continuous $C(M)$-valued stochastic process $W = \{W_t\}_{t\ge 0}$ such that $W_0 = 0$ and for every finite subset $S \subset \mathbb{R}^+ \times \tilde{M}$, $\{W_t(x)\}_{(t,x)\in S}$ is a centered Gaussian random vector with

$$\mathsf{E}[W_s(x)W_t(y)] = (s \wedge t)C(x,y).$$

For d' a pseudo-distance on M and for $\epsilon > 0$, let

$$\omega(\epsilon) = \sup\{\eta > 0 : d(x,y) \le \eta \Rightarrow d'(x,y) \le \epsilon\}. \tag{29}$$

Then $N(M,d;\omega_C(\epsilon)) \ge N(M,d';\epsilon)$. We will consider the following hypothsis that d' may or may not satisfy:

$$\int_0^1 \log(N(M,d;\omega(\epsilon))\, d\epsilon < \infty. \tag{30}$$

Let d_C be the pseudo-distance on M defined by

$$d_C(x,y) = \sqrt{C(x,x) - 2C(x,y) + C(y,y)}.$$

When $d' = d_C$, the function ω defined by (29) will be denoted by ω_C.

Remark 5.3. *Assume that M is a compact finite dimensional manifold and that $d_C(x,y) \le Kd(x,y)^\alpha$ for some $\alpha > 0$. Then $\omega_C(\epsilon) \le (\frac{\epsilon}{K})^{1/\alpha}$ and $N(M,d;\eta) = O(\eta^{-dim(M)})$; so that d_C satisfies* (30).

Theorem 5.4. *Assume d_C satisfies* (30). *Then there exists a Brownian motion on $C(M)$ with covariance C.*

Proof : By Mercer Theorem (see e.g [11]) there exists a countable family of function $\Psi_i \in C(M)$, $i \in \mathbb{N}$, such that $C(x, y) = \sum_i \Psi_i(x)\Psi_i(y)$, and the convergence is uniform. Let $B^i, i \in \mathbb{N}$, be a family of independent standard Brownian motions. Set $W_t^n(x) = \sum_{i \le n} B_t^i \Psi_i(x)$, $n \ge 0$. Then, for each $(t, x) \in \mathbb{R}^+ \times M$, the sequence $(W_t^n(x))_{n \ge 1}$ is a martingale. It is furthermore bounded in L^2 since

$$\mathsf{E}[(W_t^n(x))^2] = t \sum_{i \le n} \Psi_i(x)^2 \le tC(x, x).$$

Hence by Doob's convergence theorem one may define $W_t(x) = \sum_{i \ge 0} B_t^i \Psi_i(x)$. Let now $S \subset \mathbb{R}^+ \times M$ be a countable and dense set. It is easily checked that the family $(W_t(x))_{(t,x) \in S}$ is a centered Gaussian family with covariance given by

$$\mathsf{E}[W_s(x)W_t(y)] = (s \wedge t)C(x, y),$$

In particular, for $t \ge s$

$$\begin{aligned} \mathsf{E}[(W_s(x) - W_t(y))^2] &= sC(x, x) - 2sC(x, y) + tC(y, y) \\ &\le K(t - s) + s d_C(x, y)^2 \end{aligned}$$

This later bound combined with classical results on Gaussian processes (see e.g Theorem 11.17 in [15]) implies that $(t, x) \mapsto W_t(x)$ admits a version uniformly continuous over $S_T = \{(t, x) \in S : t \le T\}$. By density it can be extended to a continuous (in (t, x)) process $W = (W_t(x))_{\{(t,x) \in \mathbb{R}^+ \times M\}}$. **QED**

5.3 Ornstein-Ulhenbeck processes

Let $A : C(M) \to C(M)$ be a bounded operator and C a covariance satisfying hypothesis 30. Let W be $C(M)$-valued Brownian motion with covariance C.

An *Ornstein-Ulhenbeck process* with drift A, covariance C and initial condition $F_0 = f \in C(M)$ is defined to be a continuous $C(M)$-valued stochastic process such that

$$F_t - f = \int_0^t AF_s ds + W_t. \tag{31}$$

We let $(e^{tA})_{t \in \mathbb{R}}$ denote the linear flow induced by A. For each t, e^{tA} is a bounded operator on $C(M)$. Let $L_A : C(\mathbb{R}^+ \times M) \to C(\mathbb{R}^+ \times M)$ be defined by

$$L_A(f)_t = f_t - f_0 - \int_0^t Af_s ds, \qquad t \ge 0. \tag{32}$$

Lemma 5.5. *The restriction of L_A to $C_0(\mathbb{R}^+ \times M) = \{f \in C(\mathbb{R}^+ \times M) : f_0 = 0\}$ is bijective with inverse $(L_A)^{-1}$ defined by*

$$L_A^{-1}(g)_t = g_t + \int_0^t e^{(t-s)A} A g_s ds. \tag{33}$$

Proof : Observe that $L_A(f) = 0$ implies that $f_t = e^{tA} f_0$. Hence L_A restricted to $C_0(\mathbb{R}^+ \times M)$ is injective. Let $g \in C_0(\mathbb{R}^+ \times M)$ and let f_t be given by the right hand side of (33). Then

$$h_t = L_A(f)_t - g_t = \int_0^t e^{(t-s)A} A g_s ds - \int_0^t A f_s ds.$$

It is easily seen that h is differentiable and that $\frac{d}{dt} h_t = 0$. This proves that $h_t = h_0 = 0$. **QED**

This lemma implies for all $f \in C(M)$, $g \in C_0(\mathbb{R}^+ \times M)$ the solution to $L_A(f) = g$, with $f_0 = f$ is given by $f_t = e^{tA} f + L_A^{-1}(g)_t$. This implies

Theorem 5.6. *Let A be a bounded operator acting on $C(M)$. Let C be a covariance function satisfying hypothesis 30. Then there exists a unique solution to (31), given by*

$$F_t = e^{tA} f + L_A^{-1}(W)_t.$$

Note that $L_A^{-1}(W)_t$ is Gaussian and its variance $\mathsf{Var}_{F_t}(\mu) := \mathsf{E}[\langle \mu, F_t \rangle^2]$ (with $\mu \in \mathscr{M}(M)$) is given by

$$\mathsf{Var}_{F_t}(\mu) = \int_0^t \langle \mu, e^{sA} C e^{sA^*} \mu \rangle ds. \tag{34}$$

where $C : \mathscr{M}(M) \to C(M)$ is the operator defined by $C\mu(x) = \int_M C(x,y)\mu(dy)$. *** We refer to [10] for the calculation of Var_{F_t}. Note that the results given in Theorem 5.6 are not included in [10].

5.3.1 Asymptotic Behaviour

Let $\lambda(A) = \lim_{t\to\infty} \frac{\log(\|e^{tA}\|}{t}$. Denote by P_t the semigroup associated to an Ornstein-Uhlenbeck process of covariance C and drift A. Then for all bounded measurable $\varphi : C(M) \to \mathbb{R}$ and $f \in C(M)$,

$$\mathsf{P}_t \varphi(f) = \mathsf{E}[\varphi(F_t)], \tag{35}$$

where F_t is the solution to (31), with $F_0 = f$.

Theorem 5.7. *Assume that $\lambda(A) < 0$. Then there exists a centered Gaussian variable in $C(M)$, with variance V given by*

$$\mathsf{V}(\mu) = \int_0^\infty \langle \mu, e^{sA} C e^{sA*} \mu \rangle ds.$$

Let π denote the law of this Gaussian variable. Let d_{V} be the pseudo-distance defined by $d_{\mathsf{V}}(x,y) = \sqrt{\mathsf{V}(\delta_x - \delta_y)}$. Assume furthermore that d_C and d_{V} satisfy (30). *Then*

(i) π is the unique invariant probability measure of P_t.

(ii) For all bounded continuous function φ on $C(M)$ and all $f \in C(M)$,

$$\lim_{t\to\infty} \mathsf{P}_t \varphi(f) = \pi\varphi.$$

Proof : The fact that $\lambda(A) < 0$ implies that $\lim_{t\to\infty} \mathsf{Var}_{F_t}(\mu) = \mathsf{V}(\mu) < \infty$. Let ν_t denote the law of F_t, where F_t is the solution to (31), with $F_0 = f$. Since F_t is Gaussian, every limit point of $\{\nu_t\}$ (for the weak* topology) is the law of a $C(M)$-valued Gaussian variable with variance V. The proof then reduces to show that (ν_t) is relatively compact or equivalently that $\{F_t\}$ is tight. We use Proposition 5.1. The first condition is clearly satisfied. Let $\psi(x) = e^{x^2} - 1$. It is easily verified that for any real valued Gaussian random variable Z with variance σ^2, $\|Z\|_\Psi = \sigma\sqrt{8/3}$. Hence $\|F_t(x) - F_t(y)\|_\psi \leq 2d_{\mathsf{V}}(x,y)$ so that condition (ii) holds with d_{V}. Denoting ω (defined by (29)) by ω_{V} when $d' = d_{\mathsf{V}}$, $N(M,d;\omega_{\mathsf{V}}(\epsilon)) \geq N(M,d_{\mathsf{V}};\epsilon)$ and since $\psi^{-1}(u) = \sqrt{\log(u-1)}$ condition (iii) is verified. **QED**

Even thought we don't have the speed of convergence in (ii), we have

Lemma 5.8. *Assume that $\lambda(A) < 0$. For all bounded Lipschitz continuous $\varphi : C(M) \to \mathbb{R}$, all f and g in $C(M)$,*

$$|\mathsf{P}_t\varphi(f) - \mathsf{P}_t\varphi(g)| \leq Ke^{\lambda(A)t}\|f - g\|_\infty.$$

Proof : We have $\mathsf{P}_t\varphi(f) = \mathsf{E}[\varphi(L_A^{-1}(W)_t + e^{tA}f)]$. So, using the fact that φ is Lipschitz,

$$|\mathsf{P}_t\varphi(f) - \mathsf{P}_t\varphi(g)| \ \leq \ K\|e^{tA}(f-g)\|_\infty \ \leq \ Ke^{\lambda(A)t}\|f-g\|_\infty. \quad \textbf{QED}$$

To conclude this section we give a set of simple sufficient conditions ensuring that d_{V} satisfies (30). For $f \in C(M)$ we let

$$\mathrm{Lip}(f) = \sup_{x\neq y} \frac{|f(x) - f(y)|}{d(x,y)} \in \mathbb{R}^+ \cup \{\infty\}. \tag{36}$$

A map f is said to be Lipschitz provided $Lip(f) < \infty$.

Proposition 5.9. *Assume that*

(i) $N(d,M;\epsilon) = O(\epsilon^{-r})$ *for some* $r > 0$*;*

(ii) C *is Lipschitz;*

(iii) *There exists* $K > 0$ *such that* $Lip(Af) \leq K(Lip(f) + \|f\|_\infty)$*;*

(iv) $\lambda(A) < 0$.

Then d_C *and* d_{V} *satisfy* (30).

Note that (i) holds when M is a finite dimensional manifold. We first prove

Lemma 5.10. *Under hypotheses (iii) and (iv) of proposition 5.9, there exist constants K and α such that*

$$Lip(e^{tA}f) \leq e^{\alpha t}(Lip(f) + K\|f\|_\infty).$$

Proof : For all x, y

$$\begin{aligned} |e^{tA}f(x) - e^{tA}f(y)| &= \left| \int_0^t [Ae^{sA}f(x) - Ae^{sA}f(y)]ds + f(x) - f(y) \right| \\ &\leq K\left(\int_0^t \left[Lip(e^{sA}f) + \|e^{sA}f\|_\infty \right] ds + Lip(f) \right) d(x,y). \end{aligned}$$

Since $\lambda(A) = -\lambda < 0$, there exists $K' > 0$ such that $\|e^{sA}\| \leq K' e^{-s\lambda}$. Thus

$$Lip(e^{tA} f) \leq K \int_0^t Lip(e^{sA} f) ds + \frac{KK'}{\lambda} \|f\|_\infty + Lip(f)$$

and the result follows from Gronwall's lemma. **QED**

Proof of proposition 5.9 : Set $\mu = \delta_x - \delta_y$ and $f_s = Ce^{sA^*}\mu$ so that

$$\langle \mu, e^{sA} C e^{sA^*} \mu \rangle = e^{sA} f_s(x) - e^{sA} f_s(y).$$

It follows from (ii) and (iv) that $Lip(f_s) + \|f_s\|_\infty \leq Ke^{-s\lambda}$. Therefore, by the preceding lemma, $Lip(e^{sA} f_s) \leq Ke^{\alpha s}$ and we have

$$\begin{aligned} d_{\mathsf{V}}(x,y)^2 &\leq d(x,y) \int_0^T Lip(e^{sA} f_s) ds + \int_T^\infty \left| e^{sA} f(x) - e^{sA} f(y) \right| ds \\ &\leq d(x,y) \int_0^T Ke^{\alpha s} ds + 2 \int_T^\infty \|e^{sA} f_s\|_\infty ds \\ &\leq K \left(d(x,y) e^{\alpha T} + \int_T^\infty e^{-s\lambda} ds \right) \\ &\leq K(d(x,y) e^{\alpha T} + e^{-\lambda T}). \end{aligned}$$

Let $\gamma = \frac{\alpha}{\lambda}$, $\epsilon > 0$, and $T = -\ln(\epsilon)/\lambda$. Then $d_{\mathsf{V}}^2(x,y) \leq K(\epsilon^{-\gamma} d(x,y) + \epsilon)$. Therefore $d(x,y) \leq \epsilon^{\gamma+1} \Rightarrow d_{\mathsf{V}}^2(x,y) \leq K\epsilon$, so that $N(d, M; \omega_{\mathsf{V}}(\epsilon)) = O\left(\epsilon^{-2r(\gamma+1)}\right)$ and d_{V} satisfies (30). **QED**

Example 5.11. Let

$$Af(x) = \int f(y) k_0(x,y) \mu(dy) + \sum_{i=1}^n a_i(x) f(b_i(x))$$

where μ is a bounded measure on M, $k_0(x,y)$ is bounded and uniformly Lipschitz in x, $a_i : M \to \mathbb{R}$ and $b_i : M \to M$ are Lipschitz. Then hypothesis (iii) of proposition 5.9 is verified.

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