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Pathwise Differentiability for SDEs in a Smooth Domain with Reflection

Sebastian Andres*

Abstract

In this paper we study a Skorohod SDE in a smooth domain with normal reflection at the boundary, in particular we prove that the solution is pathwise differentiable with respect to the deterministic starting point. The resulting derivatives evolve according to an ordinary differential equation, when the process is in the interior of the domain, and they are projected to the tangent space, when the process hits the boundary.

Key words: Stochastic differential equation with reflection, normal reflection, local time.

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*University of British Columbia, 121-1984 Mathematics Road Vancouver, B.C. V6T 1Z2, Canada;
Email address: s.andres@math.ubc.ca;
URL: http://www.math.ubc.ca/~s.andres

1 Introduction

This paper contains a pathwise differentiability result for the solution $(X_t(x))_{t\geq 0}$ of a stochastic differential equation (SDE) of the Skorohod type in a smooth bounded domain $G \subset \mathbb{R}^d$, $d \geq 2$, with normal reflection at the boundary. The process $(X_t(x))$ is driven by a d-dimensional standard Brownian motion and by a drift term, whose coefficients are supposed to be continuously differentiable and Lipschitz continuous, i.e. existence and uniqueness of the solution are ensured by the results of Lions and Sznitman in [18].

We prove that for every $t > 0$ the solution $X_t(x)$ is differentiable w.r.t. the deterministic initial value x in every direction $v \in \mathbb{R}^d$ and we give a representation of the derivatives in terms of an ordinary differential equation. As an easy side result, we provide a Bismut-Elworthy formula for the gradient of the transition semigroup.

The resulting derivatives evolve according to a simple linear ordinary differential equation, when the process is away from the boundary, and they have a discontinuity and are projected to the tangent space, when the process hits the boundary. This evolution becomes rather complicated because of the structure of the set of times, when the process is at the boundary, which is known to be a.s. a closed set with zero Lebesgue measure without isolated points. However, this evolution does not give a complete characterization of the derivative process. Therefore, we establish a system of SDE-like equations, whose pathwise unique solution is the derivative process in coordinates w.r.t. a moving frame. This system is similar to the one introduced by Airault in [1] in order to develop probabilistic representations for the solutions of linear PDE systems with mixed Dirichlet-Neumann conditions in a smooth domain in $\mathbb{R}^d$. A further similar system appears in Section V.6 in [13], which deals with the heat equation for diffusion processes on manifolds with boundary conditions. This situation has also been considered in a more recent work by Hsu [12], where similar to [13] the associated matrix-valued Feynman-Kac multiplicative functional is constructed which is determined by the curvature tensor. The multiplicative functional associated with the pathwise derivatives obtained in this paper is very similar and possibly identical to the multiplicative functional in [12]. Nevertheless, the papers [1, 12, 13] deal with PDE systems or the heat equation, respectively, on smooth manifolds with mixed Neumann-Dirichlet boundary conditions, such that the solutions can be interpreted as the derivatives for the transition semigroup of the reflected Brownian motion. In this sense, our result can be considered as a pathwise version of the results in [1, 12, 13] with additional drift term.

In [11] Deuschel and Zambotti proved a pathwise differentiability result w.r.t. the initial data for diffusion processes in the domain $G = [0, \infty)^d$. These results have already been transferred to SDEs in a convex polyhedron with possibly oblique reflection (see [3]). The proof of the main result in [11] is based on the fact that a Brownian path, which is perturbed by adding a Lipschitz path with a sufficiently small Lipschitz constant, attains its minimum at the same time as the original path (see Lemma 1 in [11]). This is due to the fact that a Brownian path leaves its minimum faster than linearly. In [11] this is used in order to provide an exact computation of the reflection term in the difference quotient via Skorohod's lemma.

Our approach is quite similar: Using localization techniques introduced by Anderson and Orey (cf. [2]) we transform the SDE locally into an SDE on a halfspace (cf. Section 2.3 below). Then, in order to compute the local time we need to deal with the pathwise minimum of a continuous martingale in place of the standard Brownian motion. Since the perturbations are now no longer Lipschitz continuous, i.e. Lemma 1 in [11] does not apply, and because of the asymptotics of a Brownian path around its minimum (cf. Lemma 3.7 below) one cannot necessarily expect that an analogous statement to Lemma 1 in [11] holds true in this case. Nevertheless, one can show that the minimum times converge sufficiently fast to obtain differentiability (see Proposition 3.8).

Another crucial ingredient in the proof is the Lipschitz continuity of the solution w.r.t. the initial data. This was proven by Burdzy, Chen and Jones in Lemma 3.8 in [6] for the reflected Brownian motion without drift in planar domains, but the arguments can easily be transferred into our setting (see Proposition 3.2). This will give pathwise convergence of the difference quotients along a subsequence. In order to identify the limit, we shall characterize the limit as the unique solution of the aforementioned SDE-like equation (cf. Section 4 in [1]).

A pathwise differentiability result w.r.t. the initial position of a reflected Brownian motion in smooth domains has also been proven by Burdzy in [4] using excursion theory. The resulting derivative is characterized as a linear map represented by a multiplicative functional for reflected Brownian motion, which has been introduced in Theorem 3.2 of [7]. In constrast to our main results, the SDE considered in [4] does not contain a drift term and the differentiability is shown for the trace process, while we consider the process on the original time-scale. However, we can recover the term, which is mainly characterizing the derivative in [4], describing the influence of curvature of ∂G (cf. Remark 2.7 below).

In a series of papers [20, 21, 22] Pilipenko studies flow properties for SDEs with reflection and obtains Sobolev differentiability in the initial value, see [19] for a review of these results. In general, pathwise differentiability of diffusions processes w.r.t. the initial condition is a classical topic in stochastic analysis, see e.g. Theorem 4.6.5 in [17] for the case without reflection. On the other hand, reflected Brownian motions have been investigated in several articles, where the question of coalescence or noncoalescence of the two-point motion of a Brownian flow is of particular interest. For planar convex domains this has been studied by Cranston and Le Jan in [8] and [9], for some classes of non-smooth domains by Burdzy and Chen in [5], and for two-dimensional smooth domains by Burdzy, Chen and Jones in [6]. In higher dimension the case, where the domain is a sphere, has been considered by Sheu in [24] while the case of a general multi-dimensional smooth domain is still an open problem.

The paper is organized as follows: In Section 2 we give the precise setup and some further preliminaries and we present the main results. Section 3 is devoted to the proof of the main results.

2 Main Results and Preliminaries

2.1 General Notation

Throughout the paper we denote by $\|.\|$ the Euclidian norm, by $\langle .,. \rangle$ the canonical scalar product and by $e = (e^1, \dots, e^d)$ the standard basis in $\mathbb{R}^d$, $d \geq 2$. Let $G \subset \mathbb{R}^d$ be a connected closed bounded domain with C^3-smooth boundary and G_0 its interior and let $n(x)$, $x \in \partial G$, denote the inner normal field. For any $x \in \partial G$, let

$$\pi_x(z) := z - \langle z, n(x) \rangle n(x), \qquad z \in \mathbb{R}^d,$$

denote the orthogonal projection onto the tangent space. The closed ball in $\mathbb{R}^d$ with center x and radius r will be denoted by $B_r(x)$. The transposition of a vector $v \in \mathbb{R}^d$ and of a matrix $A \in \mathbb{R}^{d \times d}$ will be denoted by v^* and A^*, respectively. The set of continuous real-valued functions on G is denoted by $C(G)$. For each $k \in \mathbb{N}$, $C^k(G)$ denotes the set of real-valued functions that are k-times continuously differentiable in G. Furthermore, for $f \in C^1(G)$ we denote by ∇ the gradient of f and in the case where f is $\mathbb{R}^d$-valued by Df the Jacobi matrix. Finally, Δ denotes the Laplace differential operator on $C^2(G)$ and $D_v := \langle v, \nabla \rangle$ the directional derivative operator associated with the direction $v \in \mathbb{R}^d$. The symbols c and c_i, $i \in \mathbb{N}$, will denote constants, whose value may only depend on some quantities specified in a particular context.

2.2 Skorohod SDE

For any starting point $x \in G$, we consider the following stochastic differential equation of the Skorohod type:

$$\begin{aligned} &X_t(x) = x + \int_0^t b(X_r(x))\,dr + w_t + \int_0^t n(X_r(x))\,dl_r(x), \qquad t \geq 0, \\ &X_t(x) \in G, \quad dl_t(x) \geq 0, \quad \int_0^\infty \mathbb{1}_{G_0}(X_t(x))\,dl_t(x) = 0, \qquad t \geq 0, \end{aligned} \tag{2.1}$$

where w is a d-dimensional Brownian motion on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and $l(x)$ denotes the local time of $X(x)$ in ∂G, i.e. it starts at zero, it is non-decreasing and it increases only at those times, when $X(x)$ is at the boundary of G. The components $b^i : G \to \mathbb{R}$ of b are supposed to be in $C^1(G)$, in particular b is Lipschitz continuous. Then, existence and uniqueness of strong solutions of (2.1) are guaranteed by the results in [25] in the case, where G is a convex set, and for arbitrary smooth G by the results in [18]. The local time $l(x)$ is carried by the set

$$C := \{s \geq 0 : X_s(x) \in \partial G\}.$$

We define

$$r(t) := \sup(C \cap [0, t])$$

with the convention $\sup \emptyset := 0$. Then C is known to be a.s. a closed set of zero Lebesgue measure without isolated points. Note that $t \mapsto r(t)$ is constant on each excursion interval of $X(x)$ and is right-continuous. Moreover, for each $t > \inf C$ we have $X_{r(t)}(x) \in \partial G$.

2.3 Localization

In order to prove our main results we shall use the localization technique introduced in [2]. Let $\{U_0, U_1, \ldots\}$ be a countable or finite family of relatively open subsets of G covering G. Every U_m is attached with a coordinate system, i.e. with a mapping $u_m : U_m \to \mathbb{R}^d$, giving each point $x \in U_m$ the coordinates $u_m(x) = (u_m^1(x), \ldots, u_m^d(x))$ such that:

i) $U_0 \subseteq G_0$ and the corresponding coordinates are the original Euclidian coordinates. If $m > 0$ the mapping u_m is one to one and twice continuously differentiable and we have

$$U_m \cap \partial G = \{x \in U_m : u_m^1(x) = 0\}, \qquad U_m \cap G_0 = \{x \in U_m : u_m^1(x) > 0\}.$$

ii) There is a positive constant d_0 such that for every $x \in G$ there exists an index $m(x) \in \mathbb{N}$ such that $B_{d_0}(x) \cap G \subseteq U_{m(x)}$.

iii) For every $m > 0$, $\langle \nabla u^i_m(x), n(x)\rangle = \delta_{1i}$ for all $x \in U_m \cap \partial G$.

iv) For every $m \geq 0$ and $i \in \{1,\dots,d\}$, the functions

$$b^i_m : U_m \to \mathbb{R}\ x \mapsto \langle \nabla u^i_m(x), b(x)\rangle + \tfrac{1}{2}\Delta u^i_m(x),$$

$$\sigma^i_m : U_m \to \mathbb{R}^d\ x \mapsto \nabla u^i_m(x),$$

satisfy

$$\sup_m \sup_{x\in U_m} \left(|b^i_m(x)| + \|\sigma^i_m(x)\|\right) < \infty,$$

and there exists a constant c, not depending on m and i, such that

$$|b^i_m(x) - b^i_m(y)| + \|\sigma^i_m(x) - \sigma^i_m(y)\| \leq c\,\|x - y\|, \qquad \forall x, y \in U_m.$$

Note that these conditions imply $n(x) = \nabla u^1_m(x)$ for all $x \in U_m \cap \partial G$. Since ∂G is supposed to be C^3, the functions b^i_m and σ^i_m are continuously differentiable. We extend the functions b^i_m and σ^i_m to the whole domain G such that they are uniformly bounded and uniformly Lipschitz continuous on G.

We will now define a sequence of stopping times $(\tau_\ell)_\ell$ in such a way that on each interval $[\tau_\ell, \tau_{\ell+1})$ the process $X(x)$ and small perturbations of it are confined to one of the coordinate patches U_m. Fix now an arbitrary $T > 0$ and any $\delta_0 \in (0, d_0)$ and set $\hat{\partial} U_m := \partial U_m \backslash \partial G$ if $U_m \cap \partial G \neq \emptyset$ and $\hat{\partial} U_m := \partial U_m$ if $U_m \subseteq G_0$. Then, we define the sequence of stopping times $(\tau_\ell)_\ell$ by

$$\tau_0 := 0, \qquad \tau_{\ell+1} := \inf\{t > \tau_\ell : \operatorname{dist}(X_t(x), \hat{\partial} U_{m_\ell}) < \delta_0\} \wedge T, \quad \ell \geq 0,$$

where, for every $\ell \geq 0$, $m_\ell := m(X_{\tau_\ell}(x)) \in \mathbb{N}$ such that $B_{d_0}(X_{\tau_\ell}(x)) \subseteq U_{m_\ell}$. Then $\tau_\ell \notin C$ for every ℓ a.s. The dependence of τ_ℓ on x will be suppressed in the notation. Note that by construction $X_t(x) \in U_{m_\ell}$ for all $t \in [\tau_\ell, \tau_{\ell+1})$ and $m_\ell \neq m_{\ell+1}$ for every ℓ since $\delta_0 < d_0$. For abbreviation we set

$$C_\ell := C \cap [\tau_\ell, \tau_{\ell+1}) = \{s \in [\tau_\ell, \tau_{\ell+1}) : X_s(x) \in \partial G\}.$$

Using Itô's formula we get for $t \in [\tau_\ell, \tau_{\ell+1})$:

$$\begin{aligned} u_{m_\ell}(X_t(x)) =& u_{m_\ell}(X_{\tau_\ell}(x)) + \int_{\tau_\ell}^t \left[\langle \nabla u_{m_\ell}(X_r(x)), b(X_r(x))\rangle + \tfrac{1}{2}\Delta u_{m_\ell}(X_r(x))\right] dr \\ &+ \int_{\tau_\ell}^t \nabla u_{m_\ell}(X_r(x))\, dw_r + e^1\left(l_t(x) - l_{\tau_\ell}(x)\right) \\ =& u_{m_\ell}(X_{\tau_\ell}(x)) + \int_{\tau_\ell}^t b_{m_\ell}(X_r(x))\, dr + \int_{\tau_\ell}^t \sigma_{m_\ell}(X_r(x))\, dw_r + e^1\left(l_t(x) - l_{\tau_\ell}(x)\right). \end{aligned} \tag{2.2}$$

For every ℓ we define a continuous semimartingale $(M^{x,\ell}_t)_t$ by

$$M^{x,\ell}_t := \begin{cases} 0 & \text{if } t \in [0, \tau_\ell), \\ \int_{\tau_\ell}^t b^1_{m_\ell}(X_r(x))\, dr + \int_{\tau_\ell}^t \sigma^1_{m_\ell}(X_r(x))\, dw_r & \text{if } t \in [\tau_\ell, \tau_{\ell+1}], \\ M^{x,\ell}_{\tau_\ell+1} & \text{if } t > \tau_{\ell+1}. \end{cases} \tag{2.3}$$

Furthermore, we set $L_t(x) := l_t(x) - l_{\tau_\ell}(x)$ if $t \in [\tau_\ell, \tau_{\ell+1})$, $\ell \geq 0$, so that

$$u^1_{m_\ell}(X_t(x)) = u^1_{m_\ell}(X_{\tau_\ell}(x)) + M^{x,\ell}_t + L_t(x), \qquad t \in [\tau_\ell, \tau_{\ell+1}). \tag{2.4}$$

By the Girsanov Theorem there exists a probability measure $\tilde{\mathbb{P}}_\ell(x)$, which is equivalent to $\mathbb{P}$ and under which $M^{x,\ell}$ is a continuous martingale. The quadratic variation process is given by

$$[M^{x,\ell}]_t = \int_{\tau_\ell}^t \|\sigma^1_{m_\ell}(X_r(x))\|^2 \, dr, \qquad t \in [\tau_\ell, \tau_{\ell+1}),$$

which is strictly increasing in t on $[\tau_\ell, \tau_{\ell+1})$. We set $\rho^\ell_t := \inf\{s : [M^{x,\ell}]_s > t\}$. We can apply the Dambis-Dubins-Schwarz Theorem, in particular its extension in Theorem V.1.7 in [23], since in our case the limit $\lim_{t\to\infty}[M^{x,\ell}]_t = [M^{x,\ell}]_{\tau_{\ell+1}} < \infty$ exists, to conclude that the process

$$B^{x,\ell}_t := M^{x,\ell}_{\rho_t} \quad \text{for } t < [M^{x,\ell}]_{\tau_{\ell+1}}, \qquad B^{x,\ell}_t := M^{x,\ell}_{\tau_{\ell+1}} \quad \text{for } t \geq [M^{x,\ell}]_{\tau_{\ell+1}}, \tag{2.5}$$

is a $\tilde{\mathbb{P}}_\ell(x)$-Brownian motion w.r.t. the time-changed filtration stopped at time $[M^{x,\ell}]_{\tau_{\ell+1}}$ and we have $M^{x,\ell}_t = B^{x,\ell}_{[M^{x,\ell}]_t}$ for all $t \in [\tau_\ell, \tau_{\ell+1})$. In particular, on $[\tau_\ell, \tau_{\ell+1})$ the path of $M^{x,\ell}$ attains a.s. its minimum at a unique time.

Finally we introduce a moving frame. On each coordinate patch U_m of G we define a mapping $x \mapsto O_m(x)$ taking values in the space of orthogonal matrices, which is twice continuously differentiable, such that for $x \in \partial G \cap U_m$ the first row of $O_m(x)$ coincides with $n(x)$. Moreover, there exists a constant c, not depending on m such that

$$\|O_m(x) - O_m(y)\| \leq c\,\|x - y\|, \qquad \forall x, y \in U_m.$$

Again we extend the functions O_m to the whole domain G such that they are uniformly Lipschitz continuous on G.

Now we define the moving frame as the right-continuous process $(O_t)_{t\in[0,T]}$ by $O_t := O_{m_\ell}(X_t(x))$, $t \in [\tau_\ell, \tau_{\ell+1})$, which only jumps at the step times τ_ℓ.

We apply Itô's formula locally on each interval $[\tau_\ell, \tau_{\ell+1})$ to obtain

$$dO_t \cdot O_t^{-1} = \sum_{k=1}^d \alpha_k(X_t(x))\, dw^k_t + \beta(X_t(x))\, dt + \gamma(X_t(x))\, dl_t(x), \tag{2.6}$$

for some coefficient functions α_k, β and γ depending on ℓ.

2.4 Main Results

Theorem 2.1. *For all $t > 0$ and $x \in G$ a.s. the mapping $y \mapsto X_t(y)$ is directional differentiable at x, i.e. the limit $\eta_t := \eta^v_t(x) := D_v X_t(x) = \lim_{\varepsilon\to 0}(X_t(x+\varepsilon v) - X_t(x))/\varepsilon$ exists a.s. for every $v \in \mathbb{R}^d$. Moreover, there exists a right-continuous modification of η such that a.s. for all $t > 0$:*

$$\begin{aligned} \eta_t &= v + \int_0^t Db(X_r(x)) \cdot \eta_r \, dr, && \text{if } t < \inf C, \\ \eta_t &= \pi_{X_{r(t)}(x)}(\eta_{r(t)-}) + \int_{r(t)}^t Db(X_r(x)) \cdot \eta_r \, dr, && \text{if } t \geq \inf C. \end{aligned} \tag{2.7}$$

Remark 2.2. If $x \in \partial G$, $t = 0$ is a.s. an accumulation point of C and we have $r(t) > 0$ a.s. for every $t > 0$. Therefore, in that case $\eta_0 = v$ and $\eta_{0+} = \pi_x(v)$, i.e. there is discontinuity at $t = 0$.

Remark 2.3. The equation (2.7) does not characterize the derivatives, since it does not admit a unique solution. Indeed, if the process (η_t) solves (2.7), then the process $(1 + l_t(x))\eta_t$, $t \geq 0$, also does. A characterizing equation for the derivatives is given Theorem 2.5 below.

Note that this result corresponds to that for the domain $G = [0, \infty)^d$ in Theorem 1 in [11]. The proof of Theorem 2.1 as well as the proofs of Theorem 2.5 and Corollary 2.9 below are postponed to Section 3. As soon as pathwise differentiability is established, we can immediately provide a Bismut-Elworthy formula: Define for all $f \in C_b(G)$ the transition semigroup $P_t f(x) := \mathbb{E}[f(X_t(x))]$, $x \in G$, $t > 0$, associated with X.

Corollary 2.4. *For all $f \in C(G)$, $t > 0$, $x \in G$ and $v \in \mathbb{R}^d$ we have:*

$$D_v P_t f(x) = \frac{1}{t}\, \mathbb{E}\left[f(X_t(x)) \int_0^t \sum_{k=1}^{d} \langle \eta_r^v(x), dw_r \rangle \right], \tag{2.8}$$

and if $f \in C^1(G)$:

$$D_v P_t f(x) = \mathbb{E}\left[\nabla f(X_t(x)) \cdot \eta_t^v(x) \right]. \tag{2.9}$$

Proof. Formula (2.9) is straightforward from the differentiability statement in Theorem 2.1 and the chain rule. For formula (2.8) see the proof of Theorem 2 in [11]. □

From the representation of the derivatives in (2.7) it is obvious that $(\eta_t)_t$ evolves according to a linear differential equation, when the process X is in the interior of G, and that it is projected to the tangent space, when X hits the boundary. Furthermore, if X is at the boundary at some time t_0 and we have $r(t_0-) \neq r(t_0)$, i.e. t_0 is the endpoint of an excursion interval, then also η may have a discontinuity at t_0 and jump as follows:

$$\eta_{t_0} = \pi_{X_{t_0}(x)}(\eta_{t_0-}). \tag{2.10}$$

Consequently, we observe that at each time t_0 as above, η is projected to the tangent space and jumps in direction of $n(X_{t_0}(x))$ or $-n(X_{t_0}(x))$, respectively. Finally, if $X_{t_0}(x) \in \partial G$ and $t \mapsto r(t)$ is continuous in $t = t_0$, there is also a projection of η, but since in this case η_{t_0-} is in the tangent space, the projection has no effect and η is continuous at time t_0.

Set $Y_t := O_t \cdot \eta_t$, $t \geq 0$, where O_t denotes the moving frame introduced in Section 2.3. Let $P = \operatorname{diag}(e^1)$ and $Q = \operatorname{Id} - P$ and

$$Y_t^1 = P \cdot Y_t \qquad \text{and} \qquad Y_t^2 = Q \cdot Y_t$$

to decompose the space $\mathbb{R}^d$ into the direct sum $\operatorname{Im} P \oplus \operatorname{Ker} P$. We define the coefficient functions $c(t)$ and $d(t)$ to be such that

$$\begin{aligned} &\sum_{k=1}^{d} \begin{pmatrix} c_k^1(t) & c_k^2(t) \\ c_k^3(t) & c_k^4(t) \end{pmatrix} dw_t^k + \begin{pmatrix} d^1(t) & d^2(t) \\ d^3(t) & d^4(t) \end{pmatrix} dt \\ =&\sum_{k=1}^{d} \alpha_k(X_t(x))\, dw_t^k + \left[O_t \cdot Db(X_t(x)) \cdot O_t^{-1} + \beta(X_t(x)) \right] dt. \end{aligned}$$

Furthermore, we set $\gamma^2(t) := \gamma(X_t(x)) \cdot Q$.

Theorem 2.5. *There exists a right-continuous modification of η and Y, respectively, such that Y is characterized as the unique solution of*

$$\begin{aligned}
Y_t^1 =& \mathbb{1}_{\{t<\inf C_\ell\}} \left(Y_{\tau_\ell}^1 + \sum_{k=1}^d \int_{\tau_\ell}^t \left(c_k^1(s) Y_s^1 + c_k^2(s) Y_s^2 \right) dw_s^k + \int_{\tau_\ell}^t \left(d^1(s) Y_s^1 + d^2(s) Y_s^2 \right) ds \right) \\
&+ \mathbb{1}_{\{t\geq\inf C_\ell\}} \left(\sum_{k=1}^d \int_{r(t)}^t \left(c_k^1(s) Y_s^1 + c_k^2(s) Y_s^2 \right) dw_s^k + \int_{r(t)}^t \left(d^1(s) Y_s^1 + d^2(s) Y_s^2 \right) ds \right) \\
Y_t^2 =& Y_{\tau_\ell}^2 + \sum_{k=1}^d \int_{\tau_\ell}^t \left(c_k^3(s) Y_s^1 + c_k^4(s) Y_s^2 \right) dw_s^k + \int_{\tau_\ell}^t \left(d^3(s) Y_s^1 + d^4(s) Y_s^2 \right) ds \\
&+ \int_{\tau_\ell}^t \left(\Phi_s^2 + \gamma^2(s) \right) Y_s^2 \, dl_s(x),
\end{aligned}$$

for $t \in [\tau_\ell, \tau_{\ell+1})$, where

$$\Phi_t^2 := Q \cdot O_t \cdot Dn(X_t(x)) \cdot O_t^{-1} \cdot Q, \quad t \in C = \operatorname{supp} dl(x),$$

with the initial condition $Y_{\tau_\ell}^1 = P \cdot O_{\tau_\ell} \cdot O_{\tau_\ell-}^{-1} \cdot Y_{\tau_\ell-}$ and $Y_{\tau_\ell}^2 = Q \cdot O_{\tau_\ell} \cdot O_{\tau_\ell-}^{-1} \cdot Y_{\tau_\ell-}$ for $\ell \geq 1$ as well as $Y_0^1 = P \cdot O_{m_0}(x) \cdot v$ and $Y_0^2 = Q \cdot O_{m_0}(x) \cdot v$ for $\ell = 0$.

Remark 2.6. Here and in the sequel integrals of the form $\int_{r(t)}^t \xi(s)\, dw_s$ are understood as follows: Let $H : C([0,\infty)) \to D([0,\infty))$ be the random map defined by $(Hf)(t) := f(t) - f(r(t))$. Then,

$$\int_{r(t)}^t \xi(s)\, dw_s := (HI)(t)$$

where $I(t)$ is the continuous process $I(t) = \int_0^t \xi(s)\, dw_s$.

The equations in Theorem 2.5 show that on every interval $[\tau_\ell, \tau_{\ell+1})$ the decomposition of Y is a decomposition into a discontinuous and continuous part. The discontinuous part Y^1 becomes zero whenever X hits the boundary, which corresponds to the projection of η to the tangent space described above. On the other hand, Y_2 is continuous which shows that only the component of η in normal direction is affected by the projection.

Remark 2.7. Note that for all $t \in C = \operatorname{supp} dl(x)$,

$$\Phi_t^2 := Q \cdot O_t \cdot Dn(X_t(x)) \cdot O_t^{-1} \cdot Q = -Q \cdot O_t \cdot S(X_t(x)) \cdot O_t^{-1} \cdot Q,$$

where for every $x \in \partial G$, $S(x)$ denotes the symmetric linear endomorphism acting on the tangent space at x, which is known as the shape operator or the Weingarten map, characterized by the relation $S(x)v = -D_v n(x)$ for all v in the tangent space at x. The eigenvalues of $S(x)$ are the principal curvatures of ∂G at x, and in two dimensions its determinant is the Gaussian curvature. Hence, the linear term in the equation for the derivatives in [4] can be recovered in our results. However, because of the presence of stochastic integrals in the characterizing equation in Theorem 2.5 it is unlikely that the result in [4] can be directly deduced from this equation.

Remark 2.8. Define the process $(\mathscr{M}_t)_{t\geq 0}$, taking values in $\mathbb{R}^{d\times d}$, via

$$\eta_t^v(x) = \mathscr{M}_t(x)\cdot v, \qquad v\in\mathbb{R}^d,\ t\geq 0.$$

Then, $\mathscr{M}$ is a multiplicative functional that can possibly be identified with the discontinuous multiplicative functional constructed in [12] (cf. also [1, 13]). Indeed, both functionals satisfy the same Bismut formula, see (2.9) and page 363 in [12]. Nevertheless, the evolution equation for the functional in Theorem 3.4 in [12] is slightly different from the one in Theorem 2.5, since in [12] the geometry of the domain is described in terms of horizontal lifts rather than in terms of a moving frame as in the present paper, which makes a direct identification difficult.

Finally, we give another confirmation of the results, namely they will imply that the Neumann condition holds for X.

Corollary 2.9. *For all $f\in C(G)$ and $t>0$, the transition semigroup $P_t f(x) := \mathbb{E}[f(X_t(x))]$, $x\in G$, satisfies the Neumann condition at ∂G:*

$$x\in\partial G \Longrightarrow D_{n(x)}P_t f(x) = 0.$$

2.5 Example: Processes in the Unit Disc

We end this section by considering the example of the unit disc to illustrate our results. Let the domain $G = B_1(0)$ be the closed unit disc in $\mathbb{R}^2$. Then, for $x\in\partial G$, the inner normal field is given by $n(x) = -x$ and the orthogonal projection onto the tangent space by $\pi_x(z) = z - \langle z, x\rangle x$, $z\in\mathbb{R}^2$. The Skorohod equation can be written as

$$\begin{aligned}
&X_t(x) = x + \int_0^t b(X_r(x))\,dr + w_t - \int_0^t X_r(x)\,dl_r(x), \qquad t\geq 0,\\
&X_t(x)\in G, \quad dl_t(x)\geq 0, \quad \int_0^\infty \mathbb{1}_{\{\|X_t(x)\|<1\}}\,dl_t(x) = 0, \qquad t\geq 0,
\end{aligned}$$

and the system describing the derivatives becomes

$$\begin{aligned}
\eta_t &= v + \int_0^t Db(X_r(x))\cdot\eta_r\,dr, && \text{if } t<\inf C,\\
\eta_t &= \eta_{r(t)-} - \langle \eta_{r(t)-}, X_{r(t)}(x)\rangle X_{r(t)}(x) + \int_{r(t)}^t Db(X_r(x))\cdot\eta_r\,dr, && \text{if } t\geq\inf C.
\end{aligned}$$

In this example a possible choice of the coordinate patches is the following. Let $U_0 := G_0$ be the interior of the disc and u_0 be the identity on U_0. Further, for some small fixed δ we set

$$\begin{aligned}
U_1 &:= \{x = (x^1,x^2)\in G:\ x^1>\delta\}, & U_2 &:= \{x = (x^1,x^2)\in G:\ x^1<-\delta\},\\
U_3 &:= \{x = (x^1,x^2)\in G:\ x^2>\delta\}, & U_4 &:= \{x = (x^1,x^2)\in G:\ x^2<-\delta\},
\end{aligned}$$

and

$$u_m^1(x) := \frac{1}{2}(1-\|x\|^2), \quad m=1,\ldots,4, \qquad u_m^2(x) := \begin{cases}\arctan\frac{x^2}{x^1} & \text{if } m=1,2,\\ \arctan\frac{x^1}{x^2} & \text{if } m=3,4.\end{cases}$$

Finally, in order to define the moving frame, let $O_0 := \mathrm{Id}$ and

$$O_m(x) := \frac{1}{\|x\|}\begin{pmatrix} -x^1 & -x^2 \\ -x^2 & x^1 \end{pmatrix} := O_m^{-1}(x), \quad m = 1, \dots, 4.$$

Hence, $dO_t \cdot O_t^{-1} = 0$ on $[\tau_\ell, \tau_{\ell+1})$ if $m_\ell = 0$, and otherwise we use Itô's formula to obtain

$$\begin{aligned} dO_t \cdot O_t^{-1} &= \frac{X_t^2(x)}{\|X_t(x)\|^3}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} dw_t^1 + \frac{X_t^1(x)}{\|X_t(x)\|^3}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} dw_t^2 \\ &+ \left[\frac{X_t^2(x)}{\|X_t(x)\|^3}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} b^1(X_t(x)) + \frac{X_t^1(x)}{\|X_t(x)\|^3}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} b^2(X_t(x)) + \frac{1}{2\|X_t(x)\|^4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] dt, \end{aligned}$$

from which the coefficient functions α_1, α_2 and β can be defined accordingly. Note that in any case $\gamma = 0$. Furthermore, since $n(x) = -x$ for all $x \in \partial G$, $Dn(x) = -\mathrm{Id}$ and therefore $\Phi_t^2 = -Q$.

For simplicity we restrict ourselves now to the case $b = 0$. Then, the system in Theorem 2.5 can be rewritten as follows: For $t \in [\tau_\ell, \tau_{\ell+1})$, writing $Y_t = (y_t^1, y_t^2)$, we get in the case $m_\ell = 0$ that

$$y_t^1 = 1\!\!1_{\{t < \inf C_\ell\}} y_{\tau_\ell}^1, \qquad y_t^2 = y_{\tau_\ell}^2 - \int_{\tau_\ell}^t y_s^2 \, dl_s(x),$$

and in the case $m_\ell \neq 0$ that

$$\begin{aligned} y_t^1 =& 1\!\!1_{\{t < \inf C_\ell\}} \left(y_{\tau_\ell}^1 + \int_{\tau_\ell}^t \frac{X_s^2(x)}{\|X_s(x)\|^3} y_s^2 \, dw_s^1 - \int_{\tau_\ell}^t \frac{X_s^1(x)}{\|X_s(x)\|^3} y_s^2 \, dw_s^2 + \int_{\tau_\ell}^t \frac{1}{2\|X_s(x)\|^4} y_s^1 \, ds \right) \\ &+ 1\!\!1_{\{t \geq \inf C_\ell\}} \left(\int_{r(t)}^t \frac{X_s^2(x)}{\|X_s(x)\|^3} y_s^2 \, dw_s^1 - \int_{r(t)}^t \frac{X_s^1(x)}{\|X_s(x)\|^3} y_s^2 \, dw_s^2 + \int_{r(t)}^t \frac{1}{2\|X_s(x)\|^3} y_s^1 \, ds \right) \\ y_t^2 =& y_{\tau_\ell}^2 - \int_{\tau_\ell}^t \frac{X_s^2(x)}{\|X_s(x)\|^3} y_s^1 \, dw_s^1 + \int_{\tau_\ell}^t \frac{X_s^1(x)}{\|X_s(x)\|^3} y_s^1 \, dw_s^2 + \int_{\tau_\ell}^t \frac{1}{2\|X_s(x)\|^4} y_s^2 \, ds - \int_{\tau_\ell}^t y_s^2 \, dl_s(x), \end{aligned}$$

with initial value Y_{τ_ℓ} as specified in Theorem 2.5.

3 Proof of the Main Result

3.1 Lipschitz Continuity w.r.t. the Initial Datum

Before adressing the question of differentiability we establish pathwise continuity properties of $x \mapsto (X_t(x))_t$ w.r.t. the sup-norm topology. For this we will need that the mapping $y \mapsto l_t(y)$ is bounded.

Lemma 3.1. *For every $t > 0$ we have $\sup_{x \in G} l_t(x) < \infty$ a.s.*

Proof. See Lemma 3.3 in [4]. □

Proposition 3.2. *Let $T > 0$ be arbitrary and let $(X_t(x))$ and $(X_t(y))$, $t \geq 0$, be two solutions of* (2.1) *for any $x, y \in G$. Then, there exists a positive constant c only depending on T such that*

$$\sup_{t \in [0,T]} \|X_t(x) - X_t(y)\| \leq \|x - y\| \exp(c(T + l_T(x) + l_T(y))) \qquad \textit{for all } x, y \in G.$$

Note that the Lipschitz continuity in the initial condition, which is stated here, becomes effective since the Lipschitz constant can be controlled due to the uniform boundedness of $l_T(x)$ in x established in Lemma 3.1

Proof. The case $x = y$ is clear and it suffices to consider the case $T < \inf\{t : X_t(x) = X_t(y)\}$. We shall proceed similarly to Lemma 3.8 in [6]. Since ∂G is C^2-smooth and G is connected and compact, there exists a positive constant $c_1 < \infty$ such that for all $x \in \partial G$ and all $y \in G$,

$$\langle x-y, n(x)\rangle \leq c_1 \|x-y\|^2. \tag{3.1}$$

Let $T_0 := 0$ and for $k \geq 1$,

$$T_k := \inf\left\{t \geq T_{k-1} : \|X_t(x)-X_t(y)\| \notin \left(\tfrac{1}{2}\|X_{T_{k-1}}(x)-X_{T_{k-1}}(y)\|, 2\|X_{T_{k-1}}(x)-X_{T_{k-1}}(y)\|\right)\right\} \wedge T.$$

Then, by Itô's formula we obtain for any $k \geq 1$ and $t \in (T_{k-1}, T_k]$,

$$\begin{aligned}
&\|X_t(x)-X_t(y)\| - \|X_{T_{k-1}}(x)-X_{T_{k-1}}(y)\| \\
=&\int_{T_{k-1}}^{t} \frac{\langle X_r(x)-X_r(y), b(X_r(x))-b(X_r(y))\rangle}{\|X_r(x)-X_r(y)\|}\,dr \\
&+\int_{T_{k-1}}^{t} \frac{\langle X_r(x)-X_r(y), n(X_r(x))\rangle}{\|X_r(x)-X_r(y)\|}\,dl_r(x) + \int_{T_{k-1}}^{t} \frac{\langle X_r(y)-X_r(x), n(X_r(y))\rangle}{\|X_r(x)-X_r(y)\|}\,dl_r(y) \\
\leq& c_2 \int_{T_{k-1}}^{t} \|X_r(x)-X_r(y)\|\,dr + c_1 \int_{T_{k-1}}^{t} \|X_r(x)-X_r(y)\|\,(dl_r(x)+dl_r(y)) \\
\leq& c_3 \|X_{T_{k-1}}(x)-X_{T_{k-1}}(y)\| \int_{T_{k-1}}^{T_k} (dr + dl_r(x) + dl_r(y)),
\end{aligned}$$

where we have used (3.1) and the Lipschitz continuity of b. Hence, for any $t \in (T_{k-1}, T_k]$,

$$\begin{aligned}
\frac{\|X_t(x)-X_t(y)\|}{\|X_{T_{k-1}}(x)-X_{T_{k-1}}(y)\|} &\leq 1 + c_3\left(T_k - T_{k-1} + l_{T_k}(x) - l_{T_{k-1}}(x) + l_{T_k}(y) - l_{T_{k-1}}(y)\right) \\
&\leq \exp\left(c_3\left(T_k - T_{k-1} + l_{T_k}(x) - l_{T_{k-1}}(x) + l_{T_k}(y) - l_{T_{k-1}}(y)\right)\right),
\end{aligned}$$

and

$$\begin{aligned}
\frac{\|X_t(x)-X_t(y)\|}{\|x-y\|} &= \frac{\|X_t(x)-X_t(y)\|}{\|X_{T_{k-1}}(x)-X_{T_{k-1}}(y)\|} \prod_{j=1}^{k-1} \frac{\|X_{T_j}(x)-X_{T_j}(y)\|}{\|X_{T_{j-1}}(x)-X_{T_{j-1}}(y)\|} \\
&\leq \prod_{j=1}^{k} \exp\left(c_3\left(T_j - T_{j-1} + l_{T_j}(x) - l_{T_{j-1}}(x) + l_{T_j}(y) - l_{T_{j-1}}(y)\right)\right) \\
&\leq \exp\left(c_3\left(T_k + l_{T_k}(x) + l_{T_k}(y)\right)\right) \\
&\leq \exp\left(c_3\left(T + l_T(x) + l_T(y)\right)\right),
\end{aligned}$$

which proves the proposition. □

Remark 3.3. By Proposition 3.2 there exists for every $T>0$ a random $\Delta_T>0$ such that

$$\sup_{t\in[0,T]}\|X_t(x)-X_t(y)\|<\frac{\delta_0}{2},\qquad \forall y\in B_{\Delta_T}(x)\cap G,$$

with δ_0 as in Section 2.3. Then, by the definition of τ_ℓ we have for such y and for every ℓ that $X_t(y)\in U_{m_\ell}$ for all $t\in[\tau_\ell,\tau_{\ell+1})$.

Lemma 3.4. *For every $t\in[0,T]$ we have that for all $x\in G$ the mapping $y\mapsto l_t(y)$ is continuous at x.*

Proof. Fix $T>0$ and $x\in G$ and set

$$\lambda_t(y):=\int_0^t n(X_r(y))\,dl_r(y),\qquad y\in G,\quad t\in[0,T],$$

which defines for each $y\in G$ a process of bounded variation on $[0,T]$. Then, we get immediately by Proposition 3.2, (2.1) and the Lipschitz property of b that $\lambda(y)$ converges uniformly on $[0,T]$ to $\lambda(x)$ as y tends to x.

Let now $t\in[0,T]$ and ℓ be such that $t\in[\tau_\ell,\tau_{\ell+1})$. Then, for all $y\in B_{\Delta_T}(x)\cap G$ with Δ_T as in Remark 3.3 we have that $X_s(y),X_s(x)\in U_{m_\ell}$ for all $s\in[\tau_\ell,t)$. For such y and s we get

$$\begin{aligned}dl_s(y)-dl_s(x)&=\langle\nabla u^1_{m_\ell}(X_s(y)),n(X_s(y))\rangle\,dl_s(y)-\langle\nabla u^1_{m_\ell}(X_s(x)),n(X_s(x))\rangle\,dl_s(x)\\&=\nabla u^1_{m_\ell}(X_s(y))\,d\lambda_s(y)-\nabla u^1_{m_\ell}(X_s(x))\,d\lambda_s(x)\\&=\sigma^1_{m_\ell}(X_s(y))\,d\lambda_s(y)-\sigma^1_{m_\ell}(X_s(x))\,d\lambda_s(x).\end{aligned}$$

Hence,

$$\begin{aligned}l_t(y)-l_t(x)=&l_{\tau_\ell}(y)-l_{\tau_\ell}(x)+\int_{\tau_\ell}^t\sigma^1_{m_\ell}(X_s(x))\,(d\lambda_s(y)-d\lambda_s(x))\\&+\int_{\tau_\ell}^t\Big(\sigma^1_{m_\ell}(X_s(y))-\sigma^1_{m_\ell}(X_s(x))\Big)\,d\lambda_s(y).\end{aligned}$$

Using Proposition 3.2 and the fact that the functions σ_m are uniformly Lipschitz the last term converges to zero as y tends to x. Recall that $\lambda(y)$ converges uniformly on $[\tau_\ell,t]$ to $\lambda(x)$ as y tends to x. Hence, we have that the associated signed measures $d\lambda(y)$ on $[\tau_\ell,t]$ converge weakly to $d\lambda(x)$ as y tends to x. Since $s\mapsto\sigma^1_{m_\ell}(X_s(x))$ is bounded and continuous on $[\tau_\ell,t]$, the second term converges to zero as y tends to x. We apply the same argument for $l_{\tau_\ell}(y)-l_{\tau_\ell}(x)$ on $[\tau_{\ell-1},\tau_\ell]$ and by iterating this procedure we obtain the claim. □

We fix now an arbitrary $x\in G$, $v\in\mathbb{R}^d$ and $T>0$. Then, we set $x_\varepsilon:=x+\varepsilon\,v$ for all $\varepsilon\in[a_x,b_x]$ with $a_x\le 0$ and $b_x\ge 0$ such that $x_\varepsilon\in G$ for all $\varepsilon\in[a_x,b_x]$. Furthermore, we define for such ε and $t\in[0,T]$,

$$M^{x,\ell}_t(\varepsilon):=\begin{cases}0 & \text{if } t\in[0,\tau_\ell),\\ \int_{\tau_\ell}^t b^1_{m_\ell}(X_r(x_\varepsilon))\,dr+\int_{\tau_\ell}^t\sigma^1_{m_\ell}(X_r(x_\varepsilon))\,dw_r & \text{if } t\in[\tau_\ell,\tau_{\ell+1}],\\ M^{x,\ell}_{\tau_{\ell+1}}(\varepsilon) & \text{if } t>\tau_{\ell+1}.\end{cases}$$

The index x is there to indicate that the stopping times τ_ℓ are the same as in the definition of $M^{x,\ell}$ that are depending on x and not on ε. In particular, $M^{x,\ell}(\varepsilon)$ is a well-defined object, since the coefficient functions b^1_m and σ^1_m have been extended to the whole domain G. Note that $M^{x,\ell}_t = M^{x,\ell}_t(0)$, $t \in [0,T]$. Finally, we set

$$\Delta M^{x,\ell}_t(\varepsilon,\varepsilon') := M^{x,\ell}_t(\varepsilon) - M^{x,\ell}_t(\varepsilon'), \qquad t \in [0,T],\ \varepsilon,\varepsilon' \in [a_x,b_x],$$

so that

$$\Delta M^{x,\ell}_t(\varepsilon,0) = \int_{\tau_\ell}^t \left(b^1_{m_\ell}(X_r(x_\varepsilon)) - b^1_{m_\ell}(X_r(x)) \right) dr + \int_{\tau_\ell}^t \left(\sigma^1_{m_\ell}(X_r(x_\varepsilon)) - \sigma^1_{m_\ell}(X_r(x)) \right) dw_r,$$

for $t \in [\tau_\ell, \tau_{\ell+1})$ and $\varepsilon \in [a_x,b_x]$. In the next lemma we show that $M^{x,\ell}_t(\varepsilon)$ is pathwise jointly continuous in t and ε.

Lemma 3.5. *Let Δ_T be as in Remark 3.3. Then, for a.e. $\omega \in \Omega$ the following holds. For every $\delta_1 \in (0,1)$ and $\delta_2 \in (0,\frac{1}{2})$ there exists a random constant $K = K(\omega,\delta_1,\delta_2,T)$ such that*

$$\left| \Delta M^{x,\ell}_t(\varepsilon,\varepsilon') - \Delta M^{x,\ell}_s(\varepsilon,\varepsilon') \right| \le K\, |\varepsilon - \varepsilon'|^{1-\delta_1}\, |t-s|^{\frac{1}{2}-\delta_2}, \quad \forall s,t \in [0,T], \tag{3.2}$$

for all ε,ε' such that $x_\varepsilon, x_{\varepsilon'} \in B_{\Delta_T}(x) \cap G$. In particular, for every $\delta \in (0,1)$ we have for all such ε

$$\left| M^{x,\ell}_t(\varepsilon) - M^{x,\ell}_t \right| \le K\, |\varepsilon|^{1-\delta}, \qquad \forall t \in [0,T],$$

for some random constant $K = K(\omega,\delta,T)$.

Proof. In a first step we use Kolmogorov's continuity theorem to show the existence of a modification of $(M^{\ell,x}(\varepsilon))_{t,\varepsilon}$ satisfying the above estimate and in a second step we show the claim by a continuity argument.

Step 1: It follows directly from Proposition 3.2, the uniform Lipschitz continuity of b^1_m and σ^1_m and the Burkholder inequality that for every $p > 1$ there exists a positive constant $c_1 = c_1(p,T)$ such that

$$\mathbb{E}\left[\left| M^{x,\ell}_t(\varepsilon) - M^{x,\ell}_t(\varepsilon') \right|^p \right] \le c_1\, |\varepsilon - \varepsilon'|^p, \qquad \forall t \in [0,T],\ \varepsilon,\varepsilon' \in [a_x,b_x].$$

Moreover, the functions b^1_m and σ^1_m are uniformly bounded and again by using Burkholder's inequality we get that for every $p > 1$

$$\mathbb{E}\left[\left| M^{x,\ell}_t(\varepsilon) - M^{x,\ell}_s(\varepsilon) \right|^p \right] \le c_2\, |t-s|^{p/2}, \qquad \forall s,t \in [0,T],\ \varepsilon \in [a_x,b_x]$$

for some constant $c_2 = c_2(p,T)$. Next we will show that for every $p > 1$ there exists a constant $c_3 = c_3(p,T)$ such that

$$\mathbb{E}\left[\left| \Delta M^{x,\ell}_t(\varepsilon,\varepsilon') - \Delta M^{x,\ell}_s(\varepsilon,\varepsilon') \right|^p \right] \le c_3\, |\varepsilon - \varepsilon'|^p\, |t-s|^{p/2}, \qquad \forall s,t \in [0,T],\ \varepsilon,\varepsilon' \in [a_x,b_x].$$

For the rest of the proof the symbol c denotes a constant whose value may change from one occurence to the other one. Let $0 \le s \le t \le T$ and $\varepsilon,\varepsilon' \in [a_x,b_x]$. Recall that both $M^{x,\ell}(\varepsilon)$ and

$M^{x,\ell}(\varepsilon')$ are defined to be constant on $[0,T]\backslash[\tau_\ell,\tau_{\ell+1}]$. Thus, it is enough to consider the case where $[s,t]$ intersects $[\tau_\ell,\tau_{\ell+1}]$. Setting $\hat{s}_i := s\vee\tau_\ell$ and $\hat{t} := t\wedge\tau_{\ell+1}$ we have $|\hat{t}-\hat{s}|\le|t-s|$. By the definition of $M^{x,\ell}(\varepsilon)$ and $M^{x,\ell}(\varepsilon')$ we have

$$\mathbb{E}\left[\left|\Delta M_t^{x,\ell}(\varepsilon,\varepsilon')-\Delta M_s^{x,\ell}(\varepsilon,\varepsilon')\right|^p\right] \le c\,\mathbb{E}\left[\left|\int_{\hat{s}}^{\hat{t}}(b^1_{m_\ell}(X_r(x_\varepsilon))-b^1_{m_\ell}(X_r(x_{\varepsilon'})))\,dr\right|^p\right] + c\,\mathbb{E}\left[\left|\int_{\hat{s}}^{\hat{t}}(\sigma^1_{m_\ell}(X_r(x_\varepsilon))-\sigma^1_{m_\ell}(X_r(x_{\varepsilon'})))\,dw_r\right|^p\right].$$

By the uniform Lipschitz continuity of b_m and Proposition 3.2 the first term can be estimated by

$$c\,|t-s|^p\,\mathbb{E}\left[\sup_{r\in[\hat{s},\hat{t}]}\|X_r(x_\varepsilon)-X_r(x_{\varepsilon'})\|^p\right]\le c\,|\varepsilon-\varepsilon'|^p\,|t-s|^p.$$

For the second term we get the following estimate by Burkholder's inequality, the uniform Lipschitz continuity of σ_m and again by Proposition 3.2:

$$\begin{aligned}
&c\,\mathbb{E}\left[\sup_{r\in[\hat{s},\hat{t}]}\left|\int_{\hat{s}}^{r}\left(\sigma^1_{m_\ell}(X_r(x_\varepsilon))-\sigma^1_{m_\ell}(X_r(x_{\varepsilon'}))\right)\,dw_r\right|^p\right]\\
\le&c\,\mathbb{E}\left[\left(\int_{\hat{s}}^{\hat{t}}\|X_r(x_\varepsilon)-X_r(x_{\varepsilon'})\|^2dr\right)^{p/2}\right]\\
\le&c\,|t-s|^{p/2}\,\mathbb{E}\left[\sup_{r\in[\hat{s},\hat{t}]}\|X_r(x_\varepsilon)-X_r(x_{\varepsilon'})\|^p\right]\\
\le&c\,|\varepsilon-\varepsilon'|^p\,|t-s|^{p/2}
\end{aligned}$$

and we obtain the desired estimate. We apply now Kolmogorov's continuity theorem, in particular the version for double parameter random fields in Theorem 1.4.4 in [17], which implies that for any given $\delta_1\in(0,1)$ and $\delta_2\in(0,\frac{1}{2})$ there exists a modification of the random field $(M^{x,\ell}(\varepsilon))_{t,\varepsilon}$ satisfying (3.2) for some random constant $K=K(\omega,\delta_1,\delta_2,T)$.

Step 2: The existence of a modification shown in Step 1 immediately implies that a.s. (3.2) holds for all $s,t\in[0,T]\cap\mathbb{Q}$ and $\varepsilon,\varepsilon'\in[a_x,b_x]\cap\mathbb{Q}$. The claim follows if $\Delta M_t^{x,\ell}(\varepsilon,\varepsilon')-\Delta M_s^{x,\ell}(\varepsilon,\varepsilon')$ is pathwise continuous in s and t as well as in ε and ε'. It is enough to show the continuity of $M_t^{x,\ell}(\varepsilon)$ in t and ε. The continuity in t is obvious and for every ε such that $x_\varepsilon\in B_{\Delta_T}(x)\cap G$ we get by an application of Itô's formula as in (2.2)

$$M_t^{x,\ell}(\varepsilon)=u^1_{m_\ell}(X_t(x_\varepsilon))-u^1_{m_\ell}(X_{\tau_\ell}(x_\varepsilon))-L_t(x_\varepsilon),$$

where the right hand side is continuous in ε by Proposition 3.2 and Lemma 3.4. □

3.2 Convergence of Minimum Times

In this section we investigate the behaviour of the local time, when the starting point x of $X(x)$ has been perturbed by a small ε. To that purpose we shall transform the process locally into a process on the halfspace as indicated in Section 2.3, which allows us to use Skorohod's Lemma to compute the local time in terms of the time when the continuous martingale $M^{x,\ell}$ attains its minimum. As a result we shall obtain that for ε tending to zero the minimum time of $M^{x,\ell}$ converges almost surely faster than polynomially to the minimum time of $M^{x,\ell}$.

We fix from now on an arbitrary $T > 0$. In the following let (A_n) be the family of connected components of $[0,T]\backslash C$. Then, A_n is open and recall that for every ℓ the mapping $t \mapsto [M^{x,\ell}]_t$ is continuous and increasing on $[\tau_\ell, \tau_{\ell+1})$. Thus, for every n we may choose a random $q_n \in A_n$ a follows: Let ℓ be such that $\inf A_n \in [\tau_\ell, \tau_{\ell+1})$, then we choose $q_n \in [\tau_\ell, \tau_{\ell+1}) \cap A_n$ such that $[M^{x,\ell}]_{q_n} \in \mathbb{Q}$.

In order to compute the local time $l(x)$, recall that on every interval $[\tau_\ell, \tau_{\ell+1})$, $\ell \geq 0$, $l(x)$ is carried by the set of times t, when $u^1_{m_\ell}(X_t(x)) = 0$. Therefore, we can apply Skorohod's Lemma (see e.g. Lemma VI.2.1 in [23]) to equation (2.4) to obtain

$$L_t(x) = \left[-u^1_{m_\ell}(X_{\tau_\ell}(x)) - \inf_{\tau_\ell \leq s \leq t} M^{x,\ell}_s \right]^+, \qquad t \in [\tau_\ell, \tau_{\ell+1}).$$

Fix any $q_n > \inf C$ and ℓ such that $q_n \in [\tau_\ell, \tau_{\ell+1})$. Since $u^1_{m_\ell}(X_{r(q_n)}(x)) = 0$ and $t \mapsto L_t(x)$ is non-decreasing, we have for all $\tau_\ell \leq s \leq r(q_n)$:

$$\begin{aligned} M^{x,\ell}_{r(q_n)} &= -u^1_{m_\ell}(X_{\tau_\ell}(x)) - L_{r(q_n)}(x) \leq -u^1_{m_\ell}(X_{\tau_\ell}(x)) - L_s(x) \\ &= -u^1_{m_\ell}(X_s(x)) + M^{x,\ell}_s \leq M^{x,\ell}_s. \end{aligned}$$

Moreover, $L(x)$ is constant on $[r(q_n), t]$ for all $t \in A_n \cap [\tau_\ell, \tau_{\ell+1})$, so that

$$L_t(x) = L_{r(q_n)}(x) = \left[-u^1_{m_\ell}(X_{\tau_\ell}(x)) - M^{x,\ell}_{r(q_n)} \right]^+, \qquad t \in A_n \cap [\tau_\ell, \tau_{\ell+1}). \tag{3.3}$$

Note that $M^{x,\ell}_{r(q_n)} \leq M^{x,\ell}_s$ for all $s \in [\tau_\ell, q_n]$. Further, with probability one we have that $r(q_n)$ is the unique time in $[\tau_\ell, q_n]$, when $M^{x,\ell}$ attains its minimum. Analogously we compute the local time of the process with perturbed starting point. For fixed $v \in \mathbb{R}^d$ we set $x_\varepsilon := x + \varepsilon v$, $\varepsilon \in \mathbb{R}$, where $|\varepsilon|$ is always supposed to be sufficiently small, such that x_ε lies in G. Furthermore, there exists a random $\Delta_n > 0$ such that for all $\varepsilon \in (-\Delta_n, \Delta_n)$ we have $X_t(x_\varepsilon) \in U_{m_\ell}$ for all $t \in [\tau_\ell, q_n]$ (cf. Remark 3.3). As above we obtain for such ε:

$$L_{q_n}(x_\varepsilon) = L_{r_\varepsilon(q_n)}(x_\varepsilon) = \left[-u^1_{m_\ell}(X_{\tau_\ell}(x_\varepsilon)) - M^{x,\ell}_{r_\varepsilon(q_n)}(\varepsilon) \right]^+, \tag{3.4}$$

where $r_\varepsilon(q_n)$ is defined similarly as $r(q_n)$. In particular, $M^{x,\ell}_{r_\varepsilon(q_n)}(\varepsilon) \leq M^{x,\ell}_s(\varepsilon)$ for all $s \in [\tau_\ell, q_n]$.

Lemma 3.6. *For all n we have $r_\varepsilon(q_n) \to r(q_n)$ a.s. for $\varepsilon \to 0$.*

Proof. Consider some q_n and let ℓ be such that $q_n \in [\tau_\ell, \tau_{\ell+1})$. We fix now a typical ω such that $r(q_n)$ is the unique time in $[\tau_\ell, q_n]$, when $M^{x,\ell}$ attains its minimum and such that Lemma 3.5 holds. For every sequence $(\varepsilon_k)_k$ converging to zero we can extract a subsequence of $(r_{\varepsilon_k}(q_n))$, still denoted by $(r_{\varepsilon_k}(q_n))$, converging to some $\hat{r}(q_n)$. By construction we have

$$M^{x,\ell}_{r_{\varepsilon_k}(q_n)}(\varepsilon_k) \leq M^{x,\ell}_{r(q_n)}(\varepsilon_k)$$

for every k. Note that on one hand the right hand side converges to $M^{x,\ell}_{r(q_n)}$ as $k \to \infty$ by Lemma 3.5. On the other hand the left hand side converges to $M^{x,\ell}_{\hat{r}(q_n)}$, since

$$\left| M^{x,\ell}_{r_{\varepsilon_k}(q_n)}(\varepsilon_k) - M^{x,\ell}_{\hat{r}(q_n)} \right| \le \left| M^{x,\ell}_{r_{\varepsilon_k}(q_n)}(\varepsilon_k) - M^{x,\ell}_{r_{\varepsilon_k}(q_n)} \right| + \left| M^{x,\ell}_{r_{\varepsilon_k}(q_n)} - M^{x,\ell}_{\hat{r}(q_n)} \right|,$$

where the first term tends to zero for $k \to \infty$ by Lemma 3.5 and the second term by the continuity of $M^{x,\ell}$. Thus, $M^{x,\ell}_{\hat{r}(q_n)} \le M^{x,\ell}_{r(q_n)}$. Since $r(q_n)$ is unique time in $[\tau_\ell, q_n]$, when $M^{x,\ell}$ attains its minimum, this implies $\hat{r}(q_n) = r(q_n)$. □

Lemma 3.7. *Let $(W_t)_{t\ge 0}$ be a Brownian motion on $(\Omega, \mathscr{F}, \mathbb{P})$. For all $T > 0$, let $\vartheta : \Omega \to [0, T]$ be the random variable such that a.s.*

$$W_\vartheta < W_s \qquad \forall s \in [0, T] \backslash \{\vartheta\}.$$

Then,

$$\liminf_{s\to\vartheta} \frac{W_s - W_\vartheta}{\sqrt{|s-\vartheta|\, h(|s-\vartheta|)}} \ge 1 \quad \textit{a.s.}, \tag{3.5}$$

for every function h on $[0, \infty)$ satisfying $0 < h(t) \downarrow 0$ as $t \downarrow 0$ and $\int_0^{r_0} h(t)\frac{dt}{t} < \infty$ for some $r_0 > 0$.

Proof. It suffices to consider the case $T = 1$. We recall the following path decomposition of a Brownian motion, proven in [10]. Denoting by $(M, \hat{M})$ two independent copies of the standard Brownian meander (see [23]), we set for all $r \in (0, 1)$,

$$V_r(t) := \begin{cases} -\sqrt{r}M(1) + \sqrt{r}M(\frac{r-t}{r}), & t \in [0, r] \\ -\sqrt{r}M(1) + \sqrt{1-r}\hat{M}(\frac{t-r}{1-r}), & t \in (r, 1] \end{cases}$$

Let now $(\tau, M, \hat{M})$ be an independent triple, such that τ has the arcsine law. Then, $V_\tau \overset{d}{=} W$. This formula has the following meaning: τ is the unique time in $[0, 1]$, when the path attains minimum $-\sqrt{\tau}M(1)$. The path starts in zero at time $t = 0$ and runs backward the path of M on $[0, \tau]$ and then it runs the path of $\hat{M}$. Moreover, it was proved in [14] that the law of the Brownian meander is absolutely continuous w.r.t. the law of the three-dimensional Bessel process $(R_t)_{t\ge 0}$ on the time interval $[0, 1]$ starting in zero. We recall that a.s.

$$\liminf_{t\to 0} \frac{R_t}{\sqrt{t}\, h(t)} \ge 1$$

for every function h satisfying the conditions in the statement (see [15], p. 164). Since the same asymptotics hold for the Brownian meander at zero, the claim follows. □

In the next proposition we will apply Lemma 3.7 to the Brownian motions $B^{x,\ell}$ defined in Section 2.3. More precisely, Lemma 3.7 gives that a.s. for every ℓ and every nonnegative $q \in \mathbb{Q}$ the following holds: If $q \le [M^{x,\ell}]_{\tau_{\ell+1}}$, denoting by ϑ^ℓ_q the unique time when $B^{x,\ell}$ attains its minimum over $[0, q]$, $B^{x,\ell}$ satisfies the asymptotic behaviour stated in (3.5) at ϑ^ℓ_q.

Proposition 3.8. *Let $\delta > 0$ be arbitrary. Then, for all n we have*

$$\frac{|r_\varepsilon(q_n) - r(q_n)|^\delta}{\varepsilon} \longrightarrow 0 \quad a.s. \ as \ \varepsilon \to 0.$$

Proof. First we fix $0 < \delta' < 1$. By construction we have for every q_n and ℓ such that $q_n \in [\tau_\ell, \tau_{\ell+1})$ and for ε small enough,

$$M^{x,\ell}_{r_\varepsilon(q_n)}(\varepsilon) \le M^{x,\ell}_{r(q_n)}(\varepsilon).$$

Since $M^{x,\ell}_t(\varepsilon) = M^{x,\ell}_t + \Delta M^{x,\ell}_t(\varepsilon, 0)$ for every $t \in [\tau_\ell, q_n]$ with $\Delta M^{x,\ell}_t(\varepsilon, 0)$ as in Lemma 3.5, this can be rewritten as

$$M^{x,\ell}_{r_\varepsilon(q_n)} - M^{x,\ell}_{r(q_n)} \le \Delta M^{x,\ell}_{r(q_n)}(\varepsilon, 0) - \Delta M^{x,\ell}_{r_\varepsilon(q_n)}(\varepsilon, 0),$$

which implies

$$\frac{M^{x,\ell}_{r_\varepsilon(q_n)} - M^{x,\ell}_{r(q_n)}}{|r_\varepsilon(q_n) - r(q_n)|^{(1-\delta')/2}} \mathbb{1}_{\{r_\varepsilon(q_n) \neq r(q_n)\}} \le \frac{\left|\Delta M^{x,\ell}_{r(q_n)}(\varepsilon, 0) - \Delta M^{x,\ell}_{r_\varepsilon(q_n)}(\varepsilon, 0)\right|}{|r_\varepsilon(q_n) - r(q_n)|^{(1-\delta')/2}} \mathbb{1}_{\{r_\varepsilon(q_n) \neq r(q_n)\}}. \tag{3.6}$$

Recall that $M^{x,\ell}_\cdot = B^{x,\ell}_{[M^{x,\ell}]_\cdot}$, where $B^{x,\ell}$ is a $\tilde{\mathbb{P}}_\ell(x)$-Brownian motion (see (2.5)) and $B^{x,\ell}$ attains its minimum over $\left[0, [M^{x,\ell}]_{q_n}\right]$ at time $\vartheta^\ell_{[M^{x,\ell}]_{q_n}} = [M^{x,\ell}]_{r(q_n)}$. Note that q_n has been chosen such that $[M^{x,\ell}]_{q_n} \in \mathbb{Q}$. Hence, applying Lemma 3.7 with $h(t) = t^{\delta'/2}$ it follows that a.s.

$$\begin{aligned} M^{x,\ell}_{r_\varepsilon(q_n)} - M^{x,\ell}_{r(q_n)} = B^{x,\ell}_{[M^{x,\ell}]_{r_\varepsilon(q_n)}} - B^{x,\ell}_{[M^{x,\ell}]_{r(q_n)}} &\ge \tfrac{1}{2} \left| [M^{x,\ell}]_{r_\varepsilon(q_n)} - [M^{x,\ell}]_{r(q_n)} \right|^{(1+\delta')/2} \\ &= \tfrac{1}{2} \left| \int_{r(q_n)}^{r_\varepsilon(q_n)} \|\sigma^1_{m_\ell}(X_r(x))\|^2 \, dr \right|^{(1+\delta')/2} \end{aligned}$$

for all $\varepsilon \in (-\Delta_n, \Delta_n)$ for some positive Δ_n. Since

$$\|\sigma^1_{m_\ell}(X_{r(q_n)}(x))\|^2 = \|\nabla u^1_{m_l}(X_{r(q_n)}(x))\|^2 = \|n(X_{r(q_n)}(x))\|^2 = 1,$$

we have by Lemma 3.6, possibly after choosing a smaller Δ_n, that $\|\sigma^1_{m_\ell}(X_r(x))\|^2$ is bounded away from zero uniformly in r between $r(q_n)$ and $r_\varepsilon(q_n)$. Thus,

$$M^{x,\ell}_{r_\varepsilon(q_n)} - M^{x,\ell}_{r(q_n)} \ge c_1 \left| r_\varepsilon(q_n) - r(q_n) \right|^{(1+\delta')/2}$$

and we derive from (3.6) that a.s.

$$c_1 \left| r_\varepsilon(q_n) - r(q_n) \right|^{\delta'} \le \sup_{\substack{s,t \in [0,T] \\ s \neq t}} \frac{\left| \Delta M^{x,\ell}_t(\varepsilon, 0) - \Delta M^{x,\ell}_s(\varepsilon, 0) \right|}{|t-s|^{(1-\delta')/2}} \le K \, |\varepsilon|^{1-\delta'}$$

for some random constant $K = K(\omega, \delta', T)$, where we have used Lemma 3.5. Hence, for every $\delta > 0$ we have $|\varepsilon|^{-1} |r_\varepsilon(q_n) - r(q_n)|^\delta \le K_\delta$ for some random constant K_δ. In particular, since

$$\frac{|r_\varepsilon(q_n) - r(q_n)|^\delta}{\varepsilon} \le K_{\delta/2} \, |r_\varepsilon(q_n) - r(q_n)|^{\delta/2},$$

the claim follows by Lemma 3.6. $\square$

Corollary 3.9. *For any n and let ℓ be such that $q_n \in [\tau_\ell, \tau_{\ell+1})$. Then,*

i) $\frac{1}{\varepsilon}\left|M^{x,\ell}_{r(q_n)}(\varepsilon) - M^{x,\ell}_{r_\varepsilon(q_n)}(\varepsilon)\right| \longrightarrow 0$ *a.s. as* $\varepsilon \to 0$,

ii) $\frac{1}{\varepsilon}\left|l_{r(q_n)}(x_\varepsilon) - l_{r_\varepsilon(q_n)}(x_\varepsilon)\right| \longrightarrow 0$ *a.s. as* $\varepsilon \to 0$.

Proof. For arbitrary $\delta \in (0, \frac{1}{2})$ we have

$$\frac{1}{\varepsilon}\left|M^{x,\ell}_{r(q_n)}(\varepsilon) - M^{x,\ell}_{r_\varepsilon(q_n)}(\varepsilon)\right| \leq \mathbb{1}_{\{r_\varepsilon(q_n)\neq r(q_n)\}} \frac{|r_\varepsilon(q_n) - r(q_n)|^{\frac{1}{2}-\delta}}{\varepsilon} \frac{\left|M^{x,\ell}_{r(q_n)}(\varepsilon) - M^{x,\ell}_{r_\varepsilon(q_n)}(\varepsilon)\right|}{|r_\varepsilon(q_n) - r(q_n)|^{\frac{1}{2}-\delta}}$$
$$\leq \frac{|r_\varepsilon(q_n) - r(q_n)|^{\frac{1}{2}-\delta}}{\varepsilon} \sup_{\substack{s,t\in[0,T]\\ s\neq t}} \frac{\left|M^{x,\ell}_t(\varepsilon) - M^{x,\ell}_s(\varepsilon)\right|}{|t-s|^{\frac{1}{2}-\delta}}.$$

Since

$$\sup_{s\neq t} \frac{\left|M^{x,\ell}_t(\varepsilon) - M^{x,\ell}_s(\varepsilon)\right|}{|t-s|^{\frac{1}{2}-\delta}} \leq \sup_{s\neq t} \frac{\left|\Delta M^{x,\ell}_t(\varepsilon,0) - \Delta M^{x,\ell}_s(\varepsilon,0)\right|}{|t-s|^{\frac{1}{2}-\delta}} + \sup_{s\neq t} \frac{\left|M^{x,\ell}_t - M^{x,\ell}_s\right|}{|t-s|^{\frac{1}{2}-\delta}},$$

is a.s bounded by a random constant due to Lemma 3.5 and due to the fact that $M^{x,\ell}$ is Hölder continuous of order $\frac{1}{2} - \delta$, we obtain i) from Proposition 3.8.
ii) follows from i). Indeed, by Proposition 3.2 and Lemma 3.6 we have for ε sufficiently small that $l_{r(q_n)}(x_\varepsilon) = l_{r_\varepsilon(q_n)}(x_\varepsilon) = 0$ if $q_n < \inf C$ and $l_{r(q_n)}(x_\varepsilon), l_{r_\varepsilon(q_n)}(x_\varepsilon) > 0$ if $q_n > \inf C$. In the first case ii) is trivial and the latter case we have by (3.4)

$$\begin{aligned}\left|l_{r(q_n)}(x_\varepsilon) - l_{r_\varepsilon(q_n)}(x_\varepsilon)\right| &= \left|L_{r(q_n)}(x_\varepsilon) - L_{r_\varepsilon(q_n)}(x_\varepsilon)\right| = M^{x,\ell}_{r_\varepsilon(r(q_n))}(\varepsilon) - M^{x,\ell}_{r_\varepsilon(q_n)}(\varepsilon) \\ &\leq M^{x,\ell}_{r(q_n)}(\varepsilon) - M^{x,\ell}_{r_\varepsilon(q_n)}(\varepsilon),\end{aligned} \tag{3.7}$$

where we have used the fact that $M^{x,\ell}(\varepsilon)$ attains its minimum over $[\tau_\ell, q_n]$ at time $r_\varepsilon(q_n)$ and its minimum over $[\tau_\ell, r(q_n)]$ at time $r_\varepsilon(r(q_n))$, respectively. □

3.3 Proof of the Differentiability

The main idea in order to prove the differentiability result is based on a pathwise argument similar to Step 5 in the proof of Theorem 1 in [11]. Denote by $\eta_t(\varepsilon) := \frac{1}{\varepsilon}(X_t(x_\varepsilon) - X_t(x))$ the difference quotient, $x_\varepsilon = x + \varepsilon v$ for any fixed vector $v \in \mathbb{R}^d$ and let $T > 0$ be fixed. Then, the first step is to prove the following

Proposition 3.10. *There exists a set $\Omega_0 \subseteq \Omega$ with $\mathbb{P}[\Omega_0] = 1$ such that for every $\omega \in \Omega_0$ the following holds. Let $(\varepsilon_\nu)_\nu = (\varepsilon_\nu(\omega))_\nu$ be any random sequence such that $\lim_{\nu\to\infty} \varepsilon_\nu(\omega) = 0$ for every $\omega \in \Omega_0$. Then, there exists a subsequence $(\varepsilon_{\nu_l})_l$ with $(\nu_l)_l = (\nu_l(\omega))_l$ such that for all $t \in [0, T]\backslash C(\omega)$ the limit of $\eta_t(\varepsilon)$ along the subsequence $(\varepsilon_{\nu_l})_l$, i.e.*

$$\lim_{l\to\infty} \eta_t(\omega, \varepsilon_{\nu_l(\omega)}(\omega)) =: \hat{\eta}_t(\omega, (\varepsilon_n)) =: \hat{\eta}_t(\omega),$$

exists and is measurable. Furthermore, for all $\omega \in \Omega_0$, $(\hat{\eta}_t)_{t\in[0,T]\setminus C}$ satisfies

$$\begin{aligned}\hat{\eta}_t &= v + \int_0^t Db(X_r(x)) \cdot \hat{\eta}_r^k \, dr, && t \in [0, \inf C),\\ \hat{\eta}_t &= \hat{\eta}_{r(t)} + \int_{r(t)}^t Db(X_r(x)) \cdot \hat{\eta}_r^k \, dr, && t \in [\inf C, T]\setminus C.\end{aligned} \tag{3.8}$$

We stress here that the typical ω is fixed at the beginning, in particular *before* considering any sequences or subsequences. At the *beginning* of the proof of Proposition 3.10 we will choose the set Ω_0 with full measure. No sequence or subsequence is involved in this choice. The statement of the Proposition will then follow by completely *pathwise* arguments, which are *purely deterministic* and do not require any other choice of events.

In the next step we extend $\hat{\eta}(\omega)$, $\omega \in \Omega_0$, to a right-continuous trajectory on $[0, T]$. Then, we prove that in coordinates of the moving frame introduced in Section 2.2 $\hat{\eta}$ solves the evolution equation appearing in Theorem 2.5, which is shown to admit a pathwise unique solution.

Finally, we outline how the almost sure directional differentiability can be deduced from this. First note that $\eta_T(\varepsilon)$ converges a.s. if and only if for every component $\eta_T^i(\varepsilon)$, $i = 1, \ldots, d$,

$$\liminf_{\varepsilon\to 0} \eta_T^i(\varepsilon) = \limsup_{\varepsilon\to 0} \eta_T^i(\varepsilon) \quad \text{a.s.} \tag{3.9}$$

Let now $(\varepsilon_\nu^{i,-})_\nu$ and $(\varepsilon_\nu^{i,+})_\nu$ be two random sequences, along which $\eta_T^i(\varepsilon)$ converges to its limes inferior and its limes superior, respectively. We apply Proposition 3.10 to these two sequences and get two limiting objects $\hat{\eta}^-$ and $\hat{\eta}^+$, being two trajectories in $\mathbb{R}^d$ whose i-th components at time T, $\hat{\eta}_T^{i,-}$ and $\hat{\eta}_T^{i,+}$, coincide with the limes inferior and limes inferior of $\eta_T^i(\varepsilon)$, of course. From the fact that both $\hat{\eta}^-$ and $\hat{\eta}^+$ solve an equation having a pathwise unique solution we conclude that $\mathbb{P}\left[\hat{\eta}_t^- = \hat{\eta}_t^+, \, \forall t \in [0, T]\right] = 1$, which implies $\hat{\eta}_T^{i,-} = \hat{\eta}_T^{i,+}$ a.s. and we obtain (3.9).

3.3.1 The Limit along a Subsequence

Proof of Proposition 3.10. Let $T > 0$ still be fixed. We choose $\Omega_0 \subseteq \Omega$ with full measure such that for all $\omega \in \Omega_0$ Lemma 3.1 holds and Corollary 3.9 holds for all n. Let now $\omega \in \Omega_0$ be fixed. Let $t \in [0, T]\setminus C$ and n be such that $t \in A_n$. Using Proposition 3.2 there exists $\Delta_n = \Delta_n(\omega) > 0$ such that $l_{q_n}(x) = l_{q_n}(x_\varepsilon) = 0$ if $q_n < \inf C$ and both of them are strictly positive if $q_n > \inf C$ for all $|\varepsilon| \in (0, \Delta_n)$. Then,

$$\begin{aligned}\eta_t(\varepsilon) =& \eta_{r(q_n)}(\varepsilon) + \frac{1}{\varepsilon}\int_{r(q_n)}^t \left(b(X_r(x_\varepsilon)) - b(X_r(x))\right) dr + \frac{1}{\varepsilon}\int_0^{r_\varepsilon(q_n)} n(X_r(x_\varepsilon))\, dl_r(x_\varepsilon)\\ &- \frac{1}{\varepsilon}\int_0^{r(q_n)} n(X_r(x_\varepsilon))\, dl_r(x_\varepsilon)\\ =& \eta_{r(q_n)}(\varepsilon) + \sum_{k=1}^d \int_{r(q_n)}^t \left[\int_0^1 \frac{\partial b}{\partial x^k}(X_r^{\alpha,\varepsilon})\, d\alpha\right] \eta_r^k(\varepsilon)\, dr + R_{q_n}(x_\varepsilon),\end{aligned} \tag{3.10}$$

where $X_r^{\alpha,\varepsilon} := \alpha X_r(x_\varepsilon) + (1-\alpha)X_r(x)$, $\alpha \in [0,1]$, and

$$R_{q_n}(x_\varepsilon) := \frac{1}{\varepsilon}\int_{r(q_n)}^{r_\varepsilon(q_n)} n(X_r(x_\varepsilon))\, dl_r(x_\varepsilon). \tag{3.11}$$

Note that if $q_n < \inf C$ we have $r(q_n) = 0$, $\eta_{r(q_n)}(\varepsilon) = \nu$ and $R_{q_n}(x_\varepsilon) = 0$. In any case,

$$\|R_{q_n}(x_\varepsilon)\| \leq \frac{1}{\varepsilon}\left|\int_{r(q_n)}^{r_\varepsilon(q_n)} \|n(X_r(x_\varepsilon))\|\, dl_s(x_\varepsilon)\right| = \left|\frac{l_{r_\varepsilon(q_n)}(x_\varepsilon) - l_{r(q_n)}(x_\varepsilon)}{\varepsilon}\right| \longrightarrow 0 \quad \text{as } \varepsilon \to 0, \tag{3.12}$$

by Corollary 3.9. Recall that $\|\eta_t(\varepsilon)\| \leq \exp(c_1(T + l_T(x) + l_T(x_\varepsilon))) \leq c$ for all $t \in [0,T]$ and $\varepsilon \neq 0$ by Proposition 3.2 and Lemma 3.1. Let $(\varepsilon_\nu)_\nu = (\varepsilon_\nu(\omega))_\nu$ be any random sequence converging to zero. By a diagonal procedure, we can extract a subsequence $(\varepsilon_{\nu_l})_l$ with $(\nu_l)_l = (\nu_l(\omega))_l$ such that $\eta_{r(q_n)}(\varepsilon_{\nu_l})$ has a limit $\hat{\eta}_{r(q_n)} \in \mathbb{R}^d$ as $l \to \infty$ for all $n \in \mathbb{N}$.

Let now $\hat{\eta} : [0,T]\backslash C \to \mathbb{R}^d$ be the unique solution of

$$\hat{\eta}_t := \hat{\eta}_{r(q_n)} + \int_{r(q_n)}^{t} Db(X_r(x)) \cdot \hat{\eta}_r\, dr, \qquad t \in A_n.$$

By (3.10) and Proposition 3.2, we get for $|\varepsilon| \in (0, \Delta_n)$ and $t \in A_n$,

$$\begin{aligned}\|\eta_t(\varepsilon) - \hat{\eta}_t\| \leq & \|\eta_{r(q_n)}(\varepsilon) - \hat{\eta}_{r(q_n)}\| + \|R_{q_n}(x_\varepsilon)\| \\ & + \sup_{r\in A_n}\left\|Db(X_r^{\alpha,\varepsilon}) - Db(X_r(x))\right\| \exp(c_1(T + l_T(x) + l_T(x_\varepsilon))) \\ & + c_2 \int_0^t \|\eta_r(\varepsilon) - \hat{\eta}_r\|\, dr.\end{aligned}$$

Since $\eta_{r(q_n)}(\varepsilon_{\nu_l}) \to \hat{\eta}_{r(q_n)}$, $\|R_{q_n}(x_\varepsilon)\| \to 0$, $X_r^{\alpha,\varepsilon_{\nu_l}} \to X_r(x)$ uniformly in $r \in [0,t]$ and since the derivatives of b are continuous, we obtain by Gronwall's Lemma that $\eta_t(\varepsilon_{\nu_l})$ converges to $\hat{\eta}_t$ uniformly in $t \in A_n$ for every n. Thus, since C has zero Lebesgue measure, $\eta_t(\varepsilon_{\nu_l})$ converges to $\hat{\eta}_t$ for all $t \in [0,T]\backslash C$ as $l \to \infty$ and by the dominated convergence theorem we get that $(\hat{\eta}_t(\omega))_{t\in[0,T]\backslash C}$ satisfies (3.8). Finally, the measurability of $\hat{\eta}$ is immediate from its construction. □

From now on we will denote by $\hat{\eta}$ the limiting object constructed in Proposition 3.10 from any arbitrary but fixed random sequence (ε_n). The next lemma shows that $\hat{\eta}_{r(q_n)}$ is in the tangent at $X_{r(q_n)}(x)$ for every q_n.

Lemma 3.11. *For every $q_n > \inf C$,*

i) $\langle \eta_{r(q_n)}(\varepsilon), n(X_{r(q_n)}(x))\rangle \to 0$ a.s. and in L^p, $p > 1$, as $\varepsilon \to 0$,

ii) $\langle \eta_{r_\varepsilon(q_n)}(\varepsilon), n(X_{r_\varepsilon(q_n)}(x_\varepsilon))\rangle \to 0$ a.s. and in L^p, $p > 1$, as $\varepsilon \to 0$.

Proof. By dominated convergence it suffices to prove convergence almost surely. Let ℓ be such that $q_n \in [\tau_\ell, \tau_{\ell+1})$. Then, clearly $X_{r(q_n)}(x) \in U_{m_\ell} \cap \partial G$. Recall that $n(X_{r(q_n)}(x)) = \nabla u^1_{m_\ell}(X_{r(q_n)}(x))$, and by Taylor's formula we get

$$\langle \eta_{r(q_n)}(\varepsilon), n(X_{r(q_n)}(x))\rangle = \frac{1}{\varepsilon}\left(u^1_{m_\ell}(X_{r(q_n)}(x_\varepsilon)) - u^1_{m_\ell}(X_{r(q_n)}(x))\right) + O(\varepsilon).$$

Note that the term of second order in the Taylor expansion is in $O(\varepsilon)$ by Proposition 3.2. Recall that $u^1_{m_\ell}(X_{r(q_n)}(x)) = 0$, and combining formula (2.4) and (3.4), we get

$$u^1_{m_\ell}(X_{r(q_n)}(x_\varepsilon)) = u^1_{m_\ell}(X_{\tau_\ell}(x_\varepsilon)) + M^{x,\ell}_{r(q_n)}(\varepsilon) + L_{r(q_n)}(x_\varepsilon) = M^{x,\ell}_{r(q_n)}(\varepsilon) - M^{x,\ell}_{r_\varepsilon(r(q_n))}(\varepsilon)$$

for all $\varepsilon \in (-\Delta_n, \Delta_n)$ for some positive Δ_n. Arguing similarly as in (3.7) we obtain from Corollary 3.9 i) that

$$\left| \frac{u^1_{m_\ell}(X_{r(q_n)}(x_\varepsilon)) - u^1_{m_\ell}(X_{r(q_n)}(x))}{\varepsilon} \right| \leq 2 \left| \frac{M^{x,\ell}_{r(q_n)}(\varepsilon) - M^{x,\ell}_{r_\varepsilon(q_n)}(\varepsilon)}{\varepsilon} \right| \longrightarrow 0 \qquad \text{as } \varepsilon \to 0,$$

and i) follows. The proof of ii) is rather analogous. For an appropriate $\Delta_n > 0$ we have $r_\varepsilon(q_n) \in [\tau_\ell, \tau_{\ell+1})$ and $l_{r_\varepsilon(q_n)}(x) > 0$ for all $|\varepsilon| \in (0, \Delta_n)$. Then, for such ε we get again by using Taylor's formula and the fact that $u^1_{m_\ell}(X_{r_\varepsilon(q_n)}(x_\varepsilon)) = 0$,

$$\begin{aligned}
\langle \eta_{r_\varepsilon(q_n)}(\varepsilon), n(X_{r_\varepsilon(q_n)}(x_\varepsilon)) \rangle =& \langle \eta_{r_\varepsilon(q_n)}(\varepsilon), \nabla u^1_{m_\ell}(X_{r_\varepsilon(q_n)}(x_\varepsilon)) \rangle \\
=& -\frac{1}{\varepsilon} \left(u^1_{m_\ell}(X_{r_\varepsilon(q_n)}(x_\varepsilon)) - u^1_{m_\ell}(X_{r_\varepsilon(q_n)}(x)) \right) + O(\varepsilon) \\
=& \frac{1}{\varepsilon} \left(M^{x,\ell}_{r_\varepsilon(q_n)} - M^{x,\ell}_{r(r_\varepsilon(q_n))} \right) + O(\varepsilon).
\end{aligned}$$

Since $M^{x,\ell}$ attains its minimum over $[\tau_\ell, q_n]$ at time $r(q_n)$ and its minimum over $[\tau_\ell, r_\varepsilon(q_n)]$ at time $r(r_\varepsilon(q_n))$, respectively, we finally get

$$\begin{aligned}
|\langle \eta_{r_\varepsilon(q_n)}(\varepsilon), n(X_{r_\varepsilon(q_n)}(x_\varepsilon)) \rangle| &\leq \frac{1}{|\varepsilon|} \left(M^{x,\ell}_{r_\varepsilon(q_n)} - M^{x,\ell}_{r(q_n)} + M^{x,\ell}_{r(r_\varepsilon(q_n))} - M^{x,\ell}_{r(q_n)} \right) + O(\varepsilon) \\
&\leq \frac{2}{|\varepsilon|} \left(M^{x,\ell}_{r_\varepsilon(q_n)} - M^{x,\ell}_{r(q_n)} \right) + O(\varepsilon),
\end{aligned}$$

which tends to zero again by Corollary 3.9 i). □

So far $\hat{\eta}_t$ is not defined for every $t \in C$, only for the left endpoints $r(q_n)$ of the excursion intervals. We will now extend the trajectories of $\hat{\eta}$ to the set C. To that aim, note that since for every $m \geq 0$ the coordinate mapping u_m is a diffeomorphism, the set $\{\nabla u^i_m(x),\, i = 2, \ldots, d\}$ is linear independent for all $x \in U_m$ and by construction it is also a basis of the tangent space at x if $x \in \partial G \cap U_m$. Let $\{\bar{n}^m_2(x), \ldots, \bar{n}^m_d(x)\}$ be the Gram-Schmidt orthonormalization of $\{\nabla u^i_m(x),\, i = 2, \ldots, d\}$ for every $x \in U_m$ and for every m. Then, $\bar{n}^m(x) := \{n(x), \bar{n}^m_2(x), \ldots, \bar{n}^m_d(x)\}$ is an ONB of $\mathbb{R}^d$ for all $x \in U_m \cap \partial G$. We define now $\hat{\eta}_t$ for $t \in C \cap [\tau_\ell, \tau_{\ell+1})$ in the coordinates w.r.t. the basis $n^m_\ell(X_t(x))$ on $U_{m_\ell} \cap \partial G$. For that purpose it is sufficient to define $\langle \hat{\eta}_t, \nabla u^i_{m_\ell}(X_t(x)) \rangle$ for $i \in \{1, \ldots, d\}$. We set

$$\eta^*_t := \begin{cases} \hat{\eta}_t & \text{if } t \in [0, T] \backslash C, \\ 0 & \text{if } t \in [0, T] \cap C \end{cases}$$

and for $i = 1, \ldots, d$ we define the process $(I^*_i(t))_{t \in [0,T]}$ on $[0, T]$ by

$$I^*_i(t) := \nabla u^i_{m_\ell}(X_{\tau_\ell}(x)) \cdot \hat{\eta}_{\tau_\ell} + \int_{\tau_\ell}^t \nabla b^i_{m_\ell}(X_r(x)) \cdot \eta^*_r \, dr + \sum_{j=1}^d \int_{\tau_\ell}^t \nabla \sigma^{ij}_{m_\ell}(X_r(x)) \cdot \eta^*_r \, dw^j_r, \quad \text{if } t \in [\tau_\ell, \tau_{\ell+1}).$$

Now we define for $t \in C \cap [\tau_\ell, \tau_{\ell+1})$

$$\langle \hat{\eta}_t, \nabla u^1_{m_\ell}(X_t(x))\rangle = \langle \hat{\eta}_t, n(X_t(x))\rangle := 0 \quad \text{and} \quad \langle \hat{\eta}_t, \nabla u^i_{m_\ell}(X_t(x))\rangle := I^*_i(t)$$

for $i = 2, \ldots, d$. Having extended $\hat{\eta}$ to a trajectory on $[0, T]$ we can define $\hat{I}_i(t)$, $t \in [0, T]$, similarly to $I^*_i(t)$ locally on each interval $[\tau_\ell, \tau_{\ell+1})$. Note that I^*_i and $\hat{I}_i$ are continuous on each interval $[\tau_\ell, \tau_{\ell+1})$, in particular they are right-continuous on $[0, T]$, and since C has zero Lebesgue measure, $I^*_i(t) = \hat{I}_i(t)$ a.s. for every t. Thus, by a continuity argument

$$\mathbb{P}\left[I^*_i = \hat{I}_i \text{ on } [0, T]\right] = 1. \tag{3.13}$$

Lemma 3.12. *For every $\ell \geq 0$, $i = 1, \ldots, d$ and $p \geq 2$ we have*

$$\mathbb{E}\left[\int_{\tau_\ell}^{\tau_{\ell+1}} \left|\int_0^1 \nabla b^i_{m_\ell}(X^{\alpha,\varepsilon_{\nu_k}}_r)\, d\alpha \cdot \eta_r(\varepsilon_{\nu_k}) - \nabla b^i_{m_\ell}(X_r(x)) \cdot \hat{\eta}_r\right|^p dr\right] \to 0 \qquad \text{as } k \to \infty,$$

and

$$\mathbb{E}\left[\sup_{s\in[\tau_\ell,\tau_{\ell+1})}\left|\int_{\tau_\ell}^{s} \left(\int_0^1 \nabla \sigma^{ij}_{m_\ell}(X^{\alpha,\varepsilon_{\nu_k}}_r)\, d\alpha \cdot \eta_r(\varepsilon_{\nu_k}) - \nabla \sigma^{ij}_{m_\ell}(X_r(x)) \cdot \hat{\eta}_r\right) dw^j_r\right|^p\right] \to 0 \qquad \text{as } k \to \infty,$$

for every $j = 1, \ldots, d$, where as before $X^{\alpha,\varepsilon}_r := \alpha X_r(x_\varepsilon) + (1-\alpha)X_r(x)$, $\alpha \in [0, 1]$.

Proof. By Proposition 3.2 and Lemma 3.1 the first term can be estimated as follows

$$\begin{aligned}
&\mathbb{E}\left[\int_{\tau_\ell}^{\tau_{\ell+1}} \left|\int_0^1 \nabla b^i_{m_\ell}(X^{\alpha,\varepsilon}_r)\, d\alpha \cdot \eta_r(\varepsilon) - \nabla b^i_{m_\ell}(X_r(x)) \cdot \hat{\eta}_r\right|^p dr\right] \\
\leq &c_1\, \mathbb{E}\left[\int_{\tau_\ell}^{\tau_{\ell+1}} \left|\int_0^1 \nabla b^i_{m_\ell}(X^{\alpha,\varepsilon}_r)\, d\alpha - \nabla b^i_{m_\ell}(X_r(x))\right|^p dr\right] + c_2 \mathbb{E}\left[\int_{\tau_\ell}^{\tau_{\ell+1}} \|\eta_r(\varepsilon) - \hat{\eta}_r\|^p dr\right].
\end{aligned}$$

For the second term we get similarly, using Burkholder's inequality,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[\tau_\ell,\tau_{\ell+1})}\left|\int_{\tau_\ell}^{s} \left(\int_0^1 \nabla \sigma^{ij}_{m_\ell}(X^{\alpha,\varepsilon}_r)\, d\alpha \cdot \eta_r(\varepsilon) - \nabla \sigma^{ij}_{m_\ell}(X_r(x)) \cdot \hat{\eta}_r\right) dw^j_r\right|^p\right] \\
&\leq c_3\, \mathbb{E}\left[\int_{\tau_\ell}^{\tau_{\ell+1}} \left|\int_0^1 \nabla \sigma^{ij}_{m_\ell}(X^{\alpha,\varepsilon}_r)\, d\alpha \cdot \eta_r(\varepsilon) - \nabla \sigma^{ij}_{m_\ell}(X_r(x)) \cdot \hat{\eta}_r\right|^p dr\right] \\
&\leq c_4\, \mathbb{E}\left[\int_{\tau_\ell}^{\tau_{\ell+1}} \left|\int_0^1 \nabla \sigma^{ij}_{m_\ell}(X^{\alpha,\varepsilon}_r)\, d\alpha - \nabla \sigma^{ij}_{m_\ell}(X_r(x))\right|^p dr\right] + c_5\, \mathbb{E}\left[\int_{\tau_\ell}^{\tau_{\ell+1}} \|\eta_r(\varepsilon) - \hat{\eta}_r\|^p dr\right].
\end{aligned}$$

Hence both terms converges to zero along (ε_{ν_k}) by dominated convergence, since $X^{\alpha,\varepsilon_{\nu_k}}_r \to X_r(x)$ uniformly in $r \in [0, T]$, $\nabla b^i_{m_\ell}$ and $\nabla \sigma^{ij}_{m_\ell}$ are continuous and $\eta_r(\varepsilon_{\nu_k})$ converges to $\hat{\eta}_r$ uniformly in $r \in A_n$ for every n. □

For every $i = 1 \ldots, d$ let $(F_i(t))_{t\in[0,T]}$ be the process defined by $F_i(t) := \langle \hat{\eta}_t, \nabla u^i_{m_\ell}(X_t(x))\rangle$ if $t \in [\tau_\ell, \tau_{\ell+1})$.

Lemma 3.13. *For almost every ω the following holds.*

i) For all $t \in [0,T]$

$$F_1(t) = \mathscr{I}(t) := \begin{cases} \hat{I}_1(t) & \text{if } t < \inf C_\ell, \\ \hat{I}_1(t) - \hat{I}_1(r(t)) & \text{if } t \geq \inf C_\ell, \end{cases}$$

with ℓ such that $t \in [\tau_\ell, \tau_{\ell+1})$.

ii) For every $i \in \{2, \dots, d\}$ we have for all $t \in [0,T]$

$$F_i(t) = \hat{I}_i(t) = \nabla u^i_{m_\ell}(X_{\tau_\ell}(x)) \cdot \hat{\eta}_{\tau_\ell} + \int_{\tau_\ell}^t \nabla b^i_{m_\ell}(X_r(x)) \cdot \hat{\eta}_r \, dr + \sum_{j=1}^d \int_{\tau_\ell}^t \nabla \sigma^{ij}_{m_\ell}(X_r(x)) \cdot \hat{\eta}_r \, dw^j_r, \tag{3.14}$$

with ℓ such that $t \in [\tau_\ell, \tau_{\ell+1})$.

In particular, the trajectories of $\hat{\eta}$ are right-continuous on $[0,T]$.

Proof. i) First note that since $\hat{I}_1$ is continuous on each interval $[\tau_\ell, \tau_{\ell+1})$ and $t \mapsto r(t)$ is right-continuous, the paths of $\mathscr{I}$ are also right-continuous on $[0,T]$. Let $t \in [0,T]$ be fixed, ℓ be such that $t \in [\tau_\ell, \tau_{\ell+1})$ and $\Delta_T > 0$ be as in Remark 3.3. Then, $t \notin C$ a.s. Further, if $t < \inf C_\ell$ and $|\varepsilon| < \Delta_T$ then $l_t(x) = l_{\tau_\ell}(x)$ and $l_t(x_\varepsilon) = l_{\tau_\ell}(x_\varepsilon)$. So we have by Taylor's formula and (2.2) that

$$\begin{aligned} F_1(t) =& \langle \hat{\eta}_t, \nabla u^1_{m_\ell}(X_t(x)) \rangle = \lim_{k\to\infty} \frac{1}{\varepsilon_{\nu_k}} \left(u^1_{m_\ell}(X_t(x_{\varepsilon_{\nu_k}})) - u^1_{m_\ell}(X_t(x)) \right) 1\!\!1_{\{0<|\varepsilon_{\nu_k}|<\Delta_T\}} \\ =& \lim_{k\to\infty} \frac{1}{\varepsilon_{\nu_k}} \Bigg(u^1_{m_\ell}(X_{\tau_\ell}(x_{\varepsilon_{\nu_k}})) - u^1_{m_\ell}(X_{\tau_\ell}(x)) + \int_{\tau_\ell}^t \left(b^1_{m_\ell}(X_r(x_{\varepsilon_{\nu_k}})) - b^1_{m_\ell}(X_r(x)) \right) dr \\ & + \sum_{j=1}^d \int_{\tau_\ell}^t \left(\sigma^{1j}_{m_\ell}(X_r(x_{\varepsilon_{\nu_k}})) - \sigma^{1j}_{m_\ell}(X_r(x)) \right) dw^j_r \Bigg) \\ =& \lim_{k\to\infty} \Bigg(\nabla u^1_{m_\ell}(X_{\tau_\ell}(x)) \cdot \eta_{\tau_\ell}(\varepsilon_{\nu_k}) + \int_{\tau_\ell}^t \int_0^1 \nabla b^1_{m_\ell}(X^{\alpha,\varepsilon_{\nu_k}}_r) \cdot \eta_r(\varepsilon_{\nu_k}) \, d\alpha \, dr \\ & + \sum_{j=1}^d \int_{\tau_\ell}^t \int_0^1 \nabla \sigma^{1j}_{m_\ell}(X^{\alpha,\varepsilon_{\nu_k}}_r) \cdot \eta_r(\varepsilon_{\nu_k}) \, d\alpha \, dw^j_r \Bigg), \end{aligned}$$

where as before $X^{\alpha,\varepsilon}_r := \alpha X_r(x_\varepsilon) + (1-\alpha) X_r(x)$, $\alpha \in [0,1]$. On the other hand, if $t > \inf C_\ell$ we get

$$\begin{aligned} F_1(t) =& \langle \hat{\eta}_t, \nabla u^1_{m_\ell}(X_t(x)) \rangle \\ =& \lim_{k\to\infty} \Bigg(\nabla u^1_{m_\ell}(X_{r(t)}(x)) \cdot \eta_{r(t)}(\varepsilon_{\nu_k}) + \int_{r(t)}^t \int_0^1 \nabla b^1_{m_\ell}(X^{\alpha,\varepsilon_{\nu_k}}_r) \cdot \eta_r(\varepsilon_{\nu_k}) \, d\alpha \, dr \\ & + \sum_{j=1}^d \int_{r(t)}^t \int_0^1 \nabla \sigma^{1j}_{m_\ell}(X^{\alpha,\varepsilon_{\nu_k}}_r) \cdot \eta_r(\varepsilon_{\nu_k}) \, d\alpha \, dw^j_r + \frac{1}{\varepsilon_{\nu_k}} \left(l_{r_{\varepsilon_{\nu_k}}(t)}(x_{\varepsilon_{\nu_k}}) - l_{r(t)}(x_{\varepsilon_{\nu_k}}) \right) \Bigg), \end{aligned}$$

where the first term and the last term converge to zero by Corollary 3.9ii) and Lemma 3.11, respectively. The remaining terms converge in L^2 to the corresponding terms in the definition of $\hat{I}_1$ by Lemma 3.12. Hence, we obtain that $F_1(t) = \mathscr{I}(t)$ a.s. for every t. Since the trajectories of $\hat{\eta}$ are continuous on every excursion interval A_n, the paths of F_1 are right-continuous on every A_n and have only possibly jumps at the step times τ_ℓ. Hence,

$$\mathbb{P}\left[F_1(t) = \mathscr{I} \text{ on } [0,T]\backslash C\right] = \mathbb{P}\left[\langle \hat{\eta}_\cdot, \nabla u^i_{m_\ell}(X_\cdot(x))\rangle = \mathscr{I} \text{ on } A_n \text{ for all n}\right] = 1.$$

Finally, since by definition $F_1 = \langle \hat{\eta}_\cdot, \nabla u^i_{m_\ell}(X_\cdot(x))\rangle = 0 = \mathscr{I}$ on $C \cap [\tau_\ell, \tau_{\ell+1})$ for every ℓ, we obtain

$$\mathbb{P}\left[F_1 = \mathscr{I} \text{ on } [0,T]\right] = 1,$$

which gives i).

ii) We proceed similarly to i). Let $i \in \{2,\dots,d\}$ and $t \in [0,T]$ be fixed and ℓ be such that $t \in [\tau_\ell, \tau_{\ell+1})$. Then, $t \notin C$ a.s. and we have by Taylor's formula and (2.2)

$$\begin{aligned} F_i(t) =& \langle \hat{\eta}_t, \nabla u^i_{m_\ell}(X_t(x))\rangle = \lim_{k\to\infty}\Bigg(\nabla u^i_{m_\ell}(X_{\tau_\ell}(x))\cdot \eta_{\tau_\ell}(\varepsilon_{\nu_k}) + \int_{\tau_\ell}^t \int_0^1 \nabla b^i_{m_\ell}(X_r^{\alpha,\varepsilon_{\nu_k}})\cdot \eta_r(\varepsilon_{\nu_k})\,d\alpha\,dr \\ &+ \sum_{j=1}^d \int_{\tau_\ell}^t \int_0^1 \nabla \sigma^{ij}_{m_\ell}(X_r^{\alpha,\varepsilon_{\nu_k}})\cdot \eta_r(\varepsilon_{\nu_k})\,d\alpha\,dw^j_r\Bigg), \end{aligned}$$

The sequence on the right hand side converges in L^2 to the right hand side of (3.14) by Lemma 3.12. Hence, we obtain that $F_i(t) = \hat{I}_i(t) = I^*_i(t)$ a.s. for every t. Since the trajectories of $\hat{\eta}$ are continuous on every excursion interval A_n, the path of F_i are right-continuous on every A_n and we get

$$\mathbb{P}\left[F_i = \hat{I}_i = I^*_i \text{ on } [0,T]\backslash C\right] = \mathbb{P}\left[F_i = \hat{I}_i = I^*_i \text{ on } A_n \text{ for all n}\right] = 1.$$

Finally, since by definition $F_i = \langle \hat{\eta}_\cdot, \nabla u^i_{m_\ell}(X_\cdot(x))\rangle = I^*_i$ on $C \cap [\tau_\ell, \tau_{\ell+1})$, we use (3.13) to obtain

$$\mathbb{P}\left[F_i = \hat{I}_i = I^*_i \text{ on } [0,T]\right] = 1,$$

and we obtain ii).

The right-continuity of the trajectories of $\hat{\eta}$ is now immediate from i) and ii). Indeed, writing $\hat{\eta}_t$ in the basis $n^m_\ell(X_t(x))$, we get that on one hand the basis vectors are continuous in t on $[\tau_\ell, \tau_{\ell+1})$ and on the other hand the coordinates are right-continuous in t by i) and ii). □

The extension of $\hat{\eta}$ on C and Lemma 3.11 imply that $\langle \hat{\eta}_t, \nabla u^1_{m_\ell}(X_t(x))\rangle = \langle \hat{\eta}_t, n(X_t(x))\rangle = 0$ for all $t \in [\tau_\ell, \tau_{\ell+1}) \cap C$, i.e. when the process $X(x)$ is at the boundary $\hat{\eta}$ is at the tangent space, while the projection of $\hat{\eta}$ is a continuous process as indicated by (3.14).

Let now for all $x \in U_m$, $m \geq 0$ and $\eta \in \mathbb{R}^d$

$$\tilde{\Pi}^m_x(\eta) := \sum_{k=2}^d \langle \eta, \bar{n}^m_k(x)\rangle\, \bar{n}^m_k(x), \tag{3.15}$$

so that obviously

$$\tilde{\Pi}^m_x(\eta) = \pi_x(\eta), \qquad \forall x \in \partial G \cap U_m,\ \forall \eta \in \mathbb{R}^d.$$

For later use we prove now uniform convergence of $\tilde{\Pi}^{m_\ell}_{X_t(x_\varepsilon)}(\eta_t(\varepsilon))$ to $\tilde{\Pi}^{m_\ell}_{X_t(x)}(\hat{\eta}_t)$ along the chosen subsequence. The proof is based on the fact that there are no local time terms appearing in equation (2.2) for $u^i_{m_\ell}$, $i = 2,\dots,d$. In particular, note that $\tilde{\Pi}^{m_\ell}_{X_t(x)}(\hat{\eta}_t)$ is not the same as $Q \cdot O_t \cdot \hat{\eta}_t$. Later we will identify that process with Y^2_t appearing in Theorem 2.5, which does depend on the local time. Both processes do only coincide for $t \in [\tau_\ell, \tau_{\ell+1}) \cap C$.

Lemma 3.14. *Let $\Delta_T > 0$ be as in Remark 3.3 such that, for every $\ell \geq 0$, $\tilde{\Pi}^{m_\ell}_{X_s(x_\varepsilon)}(\eta_s(\varepsilon))$ is well defined for all $s \in [\tau_\ell, \tau_{\ell+1})$ and all $0 < |\varepsilon| < \Delta_T$. Then,*

$$\sup_{s\in[\tau_\ell,\tau_{\ell+1})} \left| \tilde{\Pi}^{m_\ell}_{X_s(x_{\varepsilon_{\nu_k}})}(\eta_s(\varepsilon_{\nu_k})) - \tilde{\Pi}^{m_\ell}_{X_s(x)}(\hat{\eta}_s) \right| 1\!\!1_{\{0<|\varepsilon_{\nu_k}|<\Delta_T\}} \to 0 \quad \text{in } L^p,\ p \geq 2 \text{ as } k \to \infty.$$

Proof. Since every function $\bar{n}^m_k$ is continuous on U_m, it suffices by Proposition 3.2 to show that

$$\sup_{s\in[\tau_\ell,\tau_{\ell+1})} \left| \tilde{\Pi}^{m_\ell}_{X_s(x)}(\eta_s(\varepsilon_{\nu_k})) - \tilde{\Pi}^{m_\ell}_{X_s(x)}(\hat{\eta}_s) \right| 1\!\!1_{\{0<|\varepsilon_{\nu_k}|<\Delta_T\}} \to 0 \quad \text{in } L^p,\ p \geq 2, \text{ as } k \to \infty,$$

and for this it is enough to prove that for every $i \in \{2,\dots,d\}$,

$$\sup_{s\in[\tau_\ell,\tau_{\ell+1})} \left| \langle \eta_s(\varepsilon_{\nu_k}), \nabla u^i_{m_\ell}(X_s(x)) \rangle - \langle \hat{\eta}_s, \nabla u^i_{m_\ell}(X_s(x)) \rangle \right| 1\!\!1_{\{0<|\varepsilon_{\nu_k}|<\Delta_T\}} \to 0 \quad \text{in } L^p \text{ as } k \to \infty.$$

For $|\varepsilon| < \Delta_T$ we use as before Taylor's formula and (2.2) to obtain

$$\begin{aligned} \langle \eta_s(\varepsilon), \nabla u^i_{m_\ell}(X_s(x)) \rangle =& \frac{1}{\varepsilon}\left(u^i_{m_\ell}(X_s(x_\varepsilon)) - u^i_{m_\ell}(X_s(x)) \right) + O(\varepsilon) \\ =& \nabla u^i_{m_\ell}(X_{\tau_\ell}(x)) \cdot \eta_{\tau_\ell}(\varepsilon) + \int_{\tau_\ell}^s \int_0^1 \nabla b^i_{m_\ell}(X^{\alpha,\varepsilon}_r) \cdot \eta_r(\varepsilon)\, d\alpha\, dr \\ &+ \sum_{j=1}^d \int_{\tau_\ell}^s \int_0^1 \nabla \sigma^{ij}_{m_\ell}(X^{\alpha,\varepsilon}_r) \cdot \eta_r(\varepsilon)\, d\alpha\, dw^j_r + O(\varepsilon), \end{aligned} \tag{3.16}$$

where again $X^{\alpha,\varepsilon}_r := \alpha X_r(x_\varepsilon) + (1-\alpha) X_r(x)$, $\alpha \in [0,1]$. Comparing (3.14) and (3.16) leads to

$$\begin{aligned} &\mathbb{E}\left[\sup_{s\in[\tau_\ell,\tau_{\ell+1})} \left| \langle \eta_s(\varepsilon), \nabla u^i_{m_\ell}(X_s(x)) \rangle - \langle \hat{\eta}_s, \nabla u^i_{m_\ell}(X_s(x)) \rangle \right|^p 1\!\!1_{\{0<|\varepsilon|<\Delta_T\}} \right] \\ \leq& c_1\, \mathbb{E}\left[\|\nabla u^i_{m_\ell}(X_{\tau_\ell}(x))\|^p \, \|\eta_{\tau_\ell}(\varepsilon) - \hat{\eta}_{\tau_\ell}\|^p \right] \\ &+ c_1\, \mathbb{E}\left[\int_{\tau_\ell}^{\tau_{\ell+1}} \left| \int_0^1 \nabla b^i_{m_\ell}(X^{\alpha,\varepsilon}_r)\, d\alpha \cdot \eta_r(\varepsilon) - \nabla b^i_{m_\ell}(X_r(x)) \cdot \hat{\eta}_r \right|^p dr \right] \\ &+ c_1 \sum_{j=1}^d \mathbb{E}\left[\sup_{s\in[\tau_\ell,\tau_{\ell+1})} \left| \int_{\tau_\ell}^s \left(\int_0^1 \nabla \sigma^{ij}_{m_\ell}(X^{\alpha,\varepsilon}_r)\, d\alpha \cdot \eta_r(\varepsilon) - \nabla \sigma^{ij}_{m_\ell}(X_r(x)) \cdot \hat{\eta}_r \right) dw^j_r \right|^p \right] + O(\varepsilon). \end{aligned}$$

The claim follows now from Lemma 3.12 and the fact that $\eta_{\tau_\ell}(\varepsilon_{\nu_k}) \to \hat{\eta}_{\tau_\ell}$. □

3.3.2 A Characterizing Equation for the Derivatives

The next step to prove the differentiability result is to identify the derivative. To that aim we shall establish a system of SDE-like equations, which admits a unique solution and which is solved by $\hat{Y}_t := O_t \cdot \hat{\eta}_t$, $t \in [0,T]$, O_t denoting the moving frame defined in Section 2.3. In other words, we shall show that $\hat{Y}$ is the unique solution of the system in Theorem 2.5.

We shall proceed similarly to Section 4 in [1] (see also Section V.6 in [13]), namely we shall derive an equation for $Y_t(\varepsilon) := O_t \cdot \eta_t(\varepsilon)$, $t \in [0,T]$, which converges in L^2 to the equation in Theorem 2.5. However, it is a general principle in the theory of stochastic differential equations that if pathwise uniqueness holds, then any 'reasonable' approximation converges at least in probability to the solution (see [16]).

Let the rows of O_t be denoted by $n_t^k = n^k(X_t(x))$, $k = 1, \dots d$. Then, we obtain by the chain rule that for every t

$$\frac{1}{\varepsilon}\left[b(X_t(x_\varepsilon)) - b(X_t(x))\right] = \sum_{k=1}^{d} \int_0^1 D_{n_t^k} b(X_t^{\alpha,\varepsilon}) \cdot \langle \eta_t(\varepsilon), n_t^k \rangle \, d\alpha = \int_0^1 Db(X_t^{\alpha,\varepsilon})\, d\alpha \cdot O_t^{-1} \cdot Y_t(\varepsilon),$$

where again $X_t^{\alpha,\varepsilon} := \alpha X_t(x_\varepsilon) + (1-\alpha) X_t(x)$, $\alpha \in [0,1]$. By applying Itô's integration by parts formula on each interval $[\tau_\ell, \tau_{\ell+1})$ we get

$$\begin{aligned}
dY_t(\varepsilon) =& O_t \cdot d\eta_t(\varepsilon) + dO_t \cdot \eta_t(\varepsilon) \\
=& O_t \cdot \frac{1}{\varepsilon}\left[b(X_t(x_\varepsilon)) - b(X_t(x))\right] dt + O_t \cdot \frac{1}{\varepsilon}\left[n(X_t(x_\varepsilon)) dl_t(x_\varepsilon) - n(X_t(x)) dl_t(x)\right] \\
& + dO_t \cdot O_t^{-1} \cdot Y_t(\varepsilon) \\
=& \left[O_t \cdot \int_0^1 Db(X_t^{\alpha,\varepsilon})\, d\alpha \cdot O_t^{-1} + \beta(X_t(x)) \right] \cdot Y_t(\varepsilon)\, dt \\
& + \sum_{k=1}^{d} \alpha_k(X_t(x)) \cdot Y_t(\varepsilon)\, dw_t^k + \gamma(X_t(x)) \cdot Y_t(\varepsilon)\, dl_t(x) \\
& + O_t \cdot \frac{1}{\varepsilon}\left[n(X_t(x_\varepsilon)) dl_t(x_\varepsilon) - n(X_t(x)) dl_t(x)\right],
\end{aligned}$$

with coefficient functions α_k and β and γ as in (2.6). Let $P = \operatorname{diag}(e^1)$ and $Q = \operatorname{Id} - P$ and set

$$Y_t^{1,\varepsilon} = P \cdot Y_t(\varepsilon) \qquad \text{and} \qquad Y_t^{2,\varepsilon} = Q \cdot Y_t(\varepsilon)$$

to decompose the space $\mathbb{R}^d$ into the direct sum $\operatorname{Im} P \oplus \operatorname{Ker} P$. We define the coefficients c and d_ε to be such that

$$\begin{aligned}
& \sum_{k=1}^{d} \begin{pmatrix} c_k^1(t) & c_k^2(t) \\ c_k^3(t) & c_k^4(t) \end{pmatrix} dw_t^k + \begin{pmatrix} d_\varepsilon^1(t) & d_\varepsilon^2(t) \\ d_\varepsilon^3(t) & d_\varepsilon^4(t) \end{pmatrix} dt \\
=& \sum_{k=1}^{d} \alpha_k(X_t(x))\, dw_t^k + \left[O_t \cdot \int_0^1 Db(X_t^{\alpha,\varepsilon})\, d\alpha \cdot O_t^{-1} + \beta(X_t(x)) \right] dt.
\end{aligned}$$

As before let $t \in [0,T]\backslash C$ and let n and ℓ be such that $t \in A_n \cap [\tau_\ell, \tau_{\ell+1})$. Then we choose $\Delta_T > 0$ as in Remark 3.3 and such that a.s. $l_{q_n}(x) = l_{q_n}(x_\varepsilon) = 0$ if $q_n < \inf C$ and both of them are strictly positive if $q_n > \inf C$ for all $0 < |\varepsilon| < \Delta_T$. For such ε we get

$$Y_t^{1,\varepsilon} = Y_{\tau_\ell}^{1,\varepsilon} + \sum_{k=1}^{d} \int_{\tau_\ell}^{t} \left(c_k^1(s)\, Y_s^{1,\varepsilon} + c_k^2(s)\, Y_s^{2,\varepsilon}\right) dw_s^k + \int_{\tau_\ell}^{t} \left(d_\varepsilon^1(s)\, Y_s^{1,\varepsilon} + d_\varepsilon^2(s)\, Y_s^{2,\varepsilon}\right) ds, \tag{3.17}$$

if $t < \inf C_\ell$ and

$$Y_t^{1,\varepsilon} = Y_{r(t)}^{1,\varepsilon} + \sum_{k=1}^{d} \int_{r(t)}^{t} \left(c_k^1(s)\, Y_s^{1,\varepsilon} + c_k^2(s)\, Y_s^{2,\varepsilon}\right) dw_s^k + \int_{r(t)}^{t} \left(d_\varepsilon^1(s)\, Y_s^{1,\varepsilon} + d_\varepsilon^2(s)\, Y_s^{2,\varepsilon}\right) ds + R_t(\varepsilon), \tag{3.18}$$

if $t \geq \inf C_\ell$, where $C_\ell := C \cap [\tau_\ell, \tau_{\ell+1})$ and $R_t(\varepsilon) := P \cdot O_t \cdot R_{q_n}(x_\varepsilon)$ with $R_{q_n}(x_\varepsilon)$ as in (3.11). Moreover, since $\eta_t(\varepsilon)$ is continuous in t, the initial value is given by

$$Y_{\tau_\ell}^{1,\varepsilon} = Y_{\tau_\ell -}^{1,\varepsilon} + P \cdot \left(O_{\tau_\ell} - O_{\tau_\ell -}\right) \cdot \eta_{\tau_\ell -}(\varepsilon) = P \cdot O_{\tau_\ell} \cdot O_{\tau_\ell -}^{-1} \cdot Y_{\tau_\ell -}(\varepsilon)$$

for $\ell \geq 1$ and $Y_0^{1,\varepsilon} = P \cdot O_{m_0}(x) \cdot v$.

Next we compute the corresponding equation for $Y^{2,\varepsilon}$. For $s \in [\tau_\ell, t]$ let the rows of $O_{m_\ell}(X_s(x_\varepsilon))$ be denoted by $n^k(X_s(x_\varepsilon))$, $k = 1, \dots d$. In particular, for $k \in \{2, \dots, d\}$ we have $n^k(X_s(x)) \cdot n(X_s(x))\, dl_s(x) = 0$ and $n^k(X_s(x_\varepsilon)) \cdot n(X_s(x_\varepsilon))\, dl_s(x_\varepsilon) = 0$. For such k we use again Taylor's formula to obtain

$$\begin{aligned}
&n^k(X_s(x)) \cdot \tfrac{1}{\varepsilon} \left[n(X_s(x_\varepsilon))\, dl_s(x_\varepsilon) - n(X_s(x))\, dl_s(x)\right] \\
=&\tfrac{1}{\varepsilon} n^k(X_s(x)) \cdot n(X_s(x_\varepsilon))\, dl_s(x_\varepsilon) \\
=&\tfrac{1}{\varepsilon} \left[(n^k(X_s(x)) - n^k(X_s(x_\varepsilon))) \cdot n(X_s(x_\varepsilon))\, dl_s(x_\varepsilon)\right] \\
=&- \eta_s(\varepsilon)^* \cdot Dn^k(X_s(x_\varepsilon))^* \cdot n(X_s(x_\varepsilon))\, dl_s(x_\varepsilon) + O(\varepsilon) \\
=&\eta_s(\varepsilon)^* \cdot (Dn(X_s(x_\varepsilon)))^* \cdot n^k(X_s(x_\varepsilon))^*\, dl_s(x_\varepsilon) + O(\varepsilon) \\
=&n^k(X_s(x_\varepsilon)) \cdot Dn(X_s(x_\varepsilon)) \cdot \eta_s(\varepsilon)\, dl_s(x_\varepsilon) + O(\varepsilon) \\
=&n^k(X_s(x_\varepsilon)) \cdot Dn(X_s(x_\varepsilon)) \cdot O_s^{-1} \cdot Y_s(\varepsilon)\, dl_s(x_\varepsilon) + O(\varepsilon).
\end{aligned}$$

Hence,

$$\begin{aligned}
Q \cdot O_s \cdot \tfrac{1}{\varepsilon} \left[n(X_s(x_\varepsilon)) dl_s(x_\varepsilon) - n(X_s(x)) dl_s(x)\right] &= \Phi_\varepsilon(s) \cdot Y_s(\varepsilon)\, dl_s(x_\varepsilon) + O(\varepsilon) \\
&= \left[\Phi_\varepsilon^1(s) \cdot Y_s^{1,\varepsilon} + \Phi_\varepsilon^2(s) \cdot Y_s^{2,\varepsilon}\right] + O(\varepsilon),
\end{aligned}$$

where

$$\Phi_\varepsilon(s) := Q \cdot O_{m_\ell}(X_s(x_\varepsilon)) \cdot Dn(X_s(x_\varepsilon)) \cdot O_s^{-1} \quad \text{and} \quad \Phi_\varepsilon^1(s) := \Phi_\varepsilon(s) \cdot P, \quad \Phi_\varepsilon^2(s) := \Phi_\varepsilon(s) \cdot Q.$$

Finally, we obtain the following equation for $Y^{2,\varepsilon}$:

$$\begin{aligned}
Y_t^{2,\varepsilon} =& Y_{\tau_\ell}^{2,\varepsilon} + \sum_{k=1}^{d} \int_{\tau_\ell}^{t} \left(c_k^3(s)\, Y_s^{1,\varepsilon} + c_k^4(s)\, Y_s^{2,\varepsilon}\right) dw_s^k + \int_{\tau_\ell}^{t} \left(d_\varepsilon^3(s)\, Y_s^{1,\varepsilon} + d_\varepsilon^4(s)\, Y_s^{2,\varepsilon}\right) ds \\
&+ \int_{\tau_\ell}^{t} \left(\Phi_\varepsilon^1(s) Y_s^{1,\varepsilon} + \Phi_\varepsilon^2(s)\, Y_s^{2,\varepsilon}\right) dl_s(x_\varepsilon) + \int_{\tau_\ell}^{t} \left(\gamma^1(s)\, Y_s^{1,\varepsilon} + \gamma^2(s)\, Y_s^{2,\varepsilon}\right) dl_s(x) + O(\varepsilon),
\end{aligned} \tag{3.19}$$

with $\gamma^1(s) := \gamma(X_s(x)) \cdot P$ and $\gamma^2(s) := \gamma(X_s(x)) \cdot Q$. The initial value is given by

$$Y^{2,\varepsilon}_{\tau_\ell} = Y^{2,\varepsilon}_{\tau_\ell -} + Q \cdot \left(O_{\tau_\ell} - O_{\tau_\ell -} \right) \cdot \eta_{\tau_\ell -}(\varepsilon) = Q \cdot O_{\tau_\ell} \cdot O^{-1}_{\tau_\ell -} \cdot Y_{\tau_\ell -}(\varepsilon)$$

for $\ell \geq 1$ and $Y^{2,\varepsilon}_0 = Q \cdot O_{m_0}(x) \cdot v$.

Setting $\hat{Y}_t = O_t \cdot \hat{\eta}_t$ and $\hat{Y}^1_t = P \cdot \hat{Y}_t$, $\hat{Y}^2_t = Q \cdot \hat{Y}_t$, $t \in [0, T]$, the next step is to prove the following

Proposition 3.15. $(\hat{Y}_t)_{t \in [0,T]}$ *solves the equation in Theorem 2.5.*

Obviously, from the second equation in Theorem 2.5 it follows that $\hat{Y}^2_t$ is a continuous semimartingale in t on every interval $[\tau_\ell, \tau_{\ell+1})$. Hence, the mapping $t \mapsto \pi_{X_{r(q_n)}(x)}(\hat{\eta}_t)$ is continuous at time $t = r(q_n)$ for every n. To complete the proof of Theorem 2.1 we need to show Proposition 3.15 and that the system in Theorem 2.5 admits a unique solution. First, we prove two preparing lemmas.

Lemma 3.16. *For every* $t \in [0, T]$ *and* ℓ *such that* $t \in [\tau_\ell, \tau_{\ell+1})$,

i) $\int_{\tau_\ell}^t \Phi^1_\varepsilon(s) Y^{1,\varepsilon}_s \, dl_s(x_\varepsilon) \to 0$ *in* L^2 *as* $\varepsilon \to 0$,

ii) $\int_{\tau_\ell}^t \gamma^1(s) Y^{1,\varepsilon}_s \, dl_s(x) \to 0$ *in* L^2 *as* $\varepsilon \to 0$.

Proof. Since Φ^1_ε is uniformly bounded, we get

$$\left\| \int_{\tau_\ell}^t \Phi^1_\varepsilon(s) Y^{1,\varepsilon}_s \, dl_s(x_\varepsilon) \right\| \leq c_1 \int_{\tau_\ell}^t \left| \langle \eta_s(\varepsilon), n^1(X_s(x)) \rangle \right| dl_s(x_\varepsilon)$$

$$\leq c_1 \int_0^t \left| \langle \eta_s(\varepsilon), n(X_s(x_\varepsilon)) \rangle \right| dl_s(x_\varepsilon) + c_1 \int_{\tau_\ell}^t \left| \langle \eta_s(\varepsilon), n^1(X_s(x_\varepsilon)) - n^1(X_s(x)) \rangle \right| dl_s(x_\varepsilon), \tag{3.20}$$

where the second term tends to zero by Proposition 3.2. Let now $\sigma^\varepsilon_s := \inf\{r : l_r(x_\varepsilon) \geq s\}$ be the left-continuous inverse of $l(x_\varepsilon)$. For any fixed $s > 0$ we have a.s. $X_s(x) \notin \partial G$ and by Proposition 3.2 $X_s(x_\varepsilon) \notin \partial G$ if $|\varepsilon| < \Delta_s$ for some positive random Δ_s. Hence, for such ε, σ^ε_s is a.s. the left endpoint of an excursion interval of $X(x_\varepsilon)$. In particular, $\sigma^\varepsilon_s = r_\varepsilon(q_n)$ a.s. for some q_n depending on s. Then, by Lemma 3.11ii) we get

$$\mathbb{E}\left[\left| \langle \eta_{\sigma^\varepsilon_s}(\varepsilon), n(X_{\sigma^\varepsilon_s}(x_\varepsilon)) \rangle \right|^2 \mathbb{1}_{\{|\varepsilon| < \Delta_s\}} \right] = \mathbb{E}\left[\left| \langle \eta_{r_\varepsilon(q_n)}(\varepsilon), n(X_{r_\varepsilon(q_n)}(x_\varepsilon)) \rangle \right|^2 \mathbb{1}_{\{|\varepsilon| < \Delta_s\}} \right] \to 0,$$

as $\varepsilon \to 0$. Thus, $\mathbb{E}\left[\left| \langle \eta_{\sigma^\varepsilon_s}(\varepsilon), n(X_{\sigma^\varepsilon_s}(x_\varepsilon)) \rangle \right|^2 \right]$ tends to zero for every s, so we can apply the dominated convergence theorem to obtain that for every positive M

$$\mathbb{E}\left[\int_0^M \left| \langle \eta_{\sigma^\varepsilon_s}(\varepsilon), n(X_{\sigma^\varepsilon_s}(x_\varepsilon)) \rangle \right|^2 ds \right] = \int_0^M \mathbb{E}\left[\left| \langle \eta_{\sigma^\varepsilon_s}(\varepsilon), n(X_{\sigma^\varepsilon_s}(x_\varepsilon)) \rangle \right|^2 \right] ds \to 0, \quad \text{as } \varepsilon \to 0. \tag{3.21}$$

We show now that also the first term in (3.20) tends to zero. On one hand, we use the change of variables formula for Stieltjes integrals (see e.g. Proposition 4.9 in Chapter 0 in [23]) to obtain for an arbitrary $M > 0$

$$\int_0^t \left|\langle \eta_s(\varepsilon), n(X_s(x_\varepsilon))\rangle\right| dl_s(x_\varepsilon)\, \mathbb{1}_{\{l_t(x_\varepsilon)\le M\}} = \int_0^{l_t(x_\varepsilon)} \left|\langle \eta_{\sigma_s^\varepsilon}(\varepsilon), n(X_{\sigma_s^\varepsilon}(x_\varepsilon))\rangle\right| ds\, \mathbb{1}_{\{l_t(x_\varepsilon)\le M\}}$$
$$\le \int_0^M \left|\langle \eta_{\sigma_s^\varepsilon}(\varepsilon), n(X_{\sigma_s^\varepsilon}(x_\varepsilon))\rangle\right| ds,$$

which converges to zero in L^2 by (3.21). On the other hand, using the Cauchy-Schwarz inequality and Proposition 3.2 we get

$$\mathbb{E}\left[\left(\int_0^t \left|\langle \eta_s(\varepsilon), n(X_s(x_\varepsilon))\rangle\right| dl_s(x_\varepsilon)\right)^2 \mathbb{1}_{\{l_t(x_\varepsilon)>M\}}\right] \le \mathbb{E}\left[\exp(c_2(t+l_t(x)+l_t(x_\varepsilon)))\, l_t(x_\varepsilon)^4\right]^{1/2}$$
$$\times\, \mathbb{P}[l_t(x_\varepsilon)>M]^{1/2}.$$

Hence,

$$\limsup_{\varepsilon\to 0} \mathbb{E}\left[\left(\int_0^t \left|\langle \eta_s(\varepsilon), n(X_s(x_\varepsilon))\rangle\right| dl_s(x_\varepsilon)\right)^2\right] \le \mathbb{E}[\exp(c_4(l_t(x)+t))l_t(x)^4]^{1/2}\, \mathbb{P}[l_t(x)>M]^{1/2}.$$

Finally, we let M tend to infinity and obtain that i) holds. ii) follows by an analogous, simpler proceeding. Note that the finiteness of the exponential moments of the local time, which is needed in the final step, can be deduced for instance from (2.4). Indeed, the stochastic integral does have finite exponential moments and the remaining terms are uniformly bounded. □

So far $\Phi(t)$ and $\Phi_\varepsilon(t)$ are only defined on the support of $l(x)$ and $l(x_\varepsilon)$, respectively. For the next proof we extend them to the whole interval $[0,T]$ by setting

$$\Phi_\varepsilon(t) := Q\cdot O_{m_\ell}(X_t(x_\varepsilon))\cdot Dn^1(X_t(x_\varepsilon))\cdot O_t^{-1}, \quad \Phi(t) := Q\cdot O_t\cdot Dn^1(X_t(x))\cdot O_t^{-1},$$

if $t\in[\tau_\ell,\tau_{\ell+1})$ and we define $\Phi_\varepsilon^1(t)$, $\Phi_\varepsilon^2(t)$ as well as $\Phi^1(t)$ and $\Phi^2(t)$ as before.

Lemma 3.17. *For every $t\in[0,T]$ and ℓ such that $t\in[\tau_\ell,\tau_{\ell+1})$,*

i) $\int_{\tau_\ell}^t \Phi^2_{\varepsilon_{\nu_k}}(s)\, Y_s^{2,\varepsilon_{\nu_k}}\, dl_s(x_{\varepsilon_{\nu_k}}) \to \int_{\tau_\ell}^t \Phi^2(s)\, \hat{Y}_s^2\, dl_s(x)$ *in* L^2 *as* $k\to\infty$,

ii) $\int_{\tau_\ell}^t \gamma^2(s)\, Y_s^{2,\varepsilon_{\nu_k}}\, dl_s(x) \to \int_{\tau_\ell}^t \gamma^2(s)\, \hat{Y}_s^2\, dl_s(x)$ *in* L^2 *as* $k\to\infty$.

Proof. Again we only prove i). Recall the definition of $\tilde{\Pi}_x^m$ in (3.15). By construction we have for $s\in[\tau_\ell,\tau_{\ell+1})$,

$$\Phi^2(s)\,\hat{Y}_s^2\, dl_s(x) = \Phi^2(s)\cdot Q\cdot O_s\cdot \hat{\eta}_s\, dl_s(x) = \Phi^2(s)\cdot Q\cdot O_s\cdot \pi_{X_s(x)}(\hat{\eta}_s)\, dl_s(x)$$
$$= \Phi^2(s)\cdot Q\cdot O_s\cdot \tilde{\Pi}^{m_\ell}_{X_s(x)}(\hat{\eta}_s)\, dl_s(x).$$

Analogously, setting $O_s^\varepsilon := O_{m_\ell}(X_s(x_\varepsilon))$, we have for sufficiently small ε

$$\begin{aligned}\Phi_\varepsilon^2(s)\, Y_s^{2,\varepsilon}\, dl_s(x_\varepsilon) &= \left\{\Phi_\varepsilon^2(s)\cdot Q\cdot O_s^\varepsilon\cdot \eta_s(\varepsilon) + \Phi_\varepsilon^2(s)\cdot Q\cdot \left[O_s - O_s^\varepsilon\right]\cdot \eta_s(\varepsilon)\right\} dl_s(x_\varepsilon)\\ &= \left\{\Phi_\varepsilon^2(s)\cdot Q\cdot O_s^\varepsilon\cdot \tilde{\Pi}^{m_\ell}_{X_s(x_\varepsilon)}(\eta_s(\varepsilon)) + \Phi_\varepsilon^2(s)\cdot Q\cdot \left[O_s - O_s^\varepsilon\right]\cdot \eta_s(\varepsilon)\right\} dl_s(x_\varepsilon).\end{aligned}$$

Hence,

$$\begin{aligned}&\int_{\tau_\ell}^t \Phi_\varepsilon^2(s)\, Y_s^{2,\varepsilon}\, dl_s(x_\varepsilon) - \Phi^2(s)\, Y_s^2\, dl_s(x)\\ =&\int_{\tau_\ell}^t \left[\Phi_\varepsilon^2(s)\cdot Q\cdot O_s^\varepsilon\cdot \tilde{\Pi}^{m_\ell}_{X_s(x_\varepsilon)}(\eta_s(\varepsilon)) - \Phi^2(s)\cdot Q\cdot O_s\cdot \tilde{\Pi}^{m_\ell}_{X_s(x)}(\hat{\eta}_s)\right] dl_s(x_\varepsilon)\\ &+\int_{\tau_\ell}^t \Phi_\varepsilon^2(s)\cdot Q\cdot \left[O_s - O_s^\varepsilon\right]\cdot \eta_s(\varepsilon)\, dl_s(x_\varepsilon) + \int_{\tau_\ell}^t \Phi^2(s)\cdot Q\cdot O_s\cdot \tilde{\Pi}^{m_\ell}_{X_s(x)}(\hat{\eta}_s)(dl_s(x_\varepsilon) - dl_s(x)).\end{aligned}$$

The first term converges to zero in L^2 along ε_{ν_k} for $k\to\infty$ by Lemma 3.14 and Proposition 3.2. The second term converges also to zero in L^2 again by Proposition 3.2. Finally, the third term tends to zero by the weak convergence of $l(x_\varepsilon)$ to $l(x)$ on $[\tau_\ell, t]$ and i) is proven. Note that $\tilde{\Pi}^{m_\ell}_{X_s(x)}(\hat{\eta}_s)$ is continuous in s on $[\tau_\ell, t]$. □

Proof of Proposition 3.15. Let $t\in[0,T]$ be fixed. Since $t\notin C$ a.s. we have by Proposition 3.10 that $Y_t^{1,\varepsilon}$ and $Y_t^{2,\varepsilon}$ converge a.s. to $\hat{Y}_t^1$ and $\hat{Y}_t^2$, respectively, along the chosen subsequence ε_{ν_l}, and by the dominated convergence theorem we have also convergence in L^2 . Furthermore, the right hand sides in (3.17), (3.18) and (3.19) converge along ε_{ν_l} in L^2 to the corresponding terms in the equation describing Y^1 and Y^2. Indeed, Lemma 3.11 gives convergence of $Y^{1,\varepsilon}_{r(t)}$ to zero and $R_t(\varepsilon)$ tends to zero arguing as in (3.12) and in Corollary 3.9. The convergence of the terms involving the local times follows from Lemma 3.16 and Lemma 3.17. The convergence of the remaining integral terms is clear. Hence, a.s. $\hat{Y}_t$ satisfies the system in Theorem 2.5. Since $t\in[0,T]$ is arbitrary, by the right-continuity of the paths we finally get that with probability one this holds for all $t\in[0,T]$. □

It remains to show uniqueness, which is carried out in the next proposition.

Proposition 3.18. *The system in Theorem 2.5 admits a pathwise unique solution on $[0,T]$, i.e. if (Y_t) and $(\tilde{Y}_t)$ are two solutions then $Y_t=\tilde{Y}_t$ for all $t\in[0,T]$ a.s.*

Proof. Let $(U_t)_{t\in[0,T]}$ be a right-continuous process defined on every interval $[\tau_\ell,\tau_{\ell+1})$ as the unique solution of the matrix-valued equation

$$U_t = \mathrm{Id} - \int_{\tau_\ell}^t U_s\cdot(\Phi^2(s)+\gamma^2(s))\, dl_s(x),\quad t\in[\tau_\ell,\tau_{\ell+1}).$$

Then, introducing the stopping times $T_{\ell,N} := \inf\{s\ge\tau_\ell : \max(\|U_s^{-1}\|,\|U_s\|)\ge N\}\wedge\tau_{\ell+1}$, $N\in\mathbb{N}$, we have $T_{\ell,N}\uparrow\tau_{\ell+1}$ a.s. as N tends to infinity. Using integration by parts we get

$$\begin{aligned}d(U_t\cdot Y_t^2) =& \sum_{k=1}^d U_t\left(c_k^3(t)\, Y_t^1 + c_k^4(t)\, Y_t^2\right) dw_t^k + U_t\left(d^3(t)\, Y_t^1 + d^4(t)\, Y_t^2\right) dt\\ &+ U_t\,(\Phi^2(t)+\gamma^2(t))\, Y_t^2\, dl_t(x) - U_t\,(\Phi^2(t)+\gamma^2(t))\, Y_t^2\, dl_t(x).\end{aligned}$$

The last two terms cancel, so we can rewrite the system in Theorem 2.5 as

$$
\begin{aligned}
Y_t^1 =& 1\!\!1_{\{t<\inf C_\ell\}} \left(Y_{\tau_\ell}^1 + \sum_{k=1}^d \int_{\tau_\ell}^t \left(c_k^1(s)\, Y_s^1 + c_k^2(s)\, Y_s^2 \right) dw_s^k + \int_{\tau_\ell}^t \left(d^1(s)\, Y_s^1 + d^2(s)\, Y_s^2 \right) ds \right) \\
&+ 1\!\!1_{\{t\geq\inf C_\ell\}} \left(\sum_{k=1}^d \int_{r(t)}^t \left(c_k^1(s)\, Y_s^1 + c_k^2(s)\, Y_s^2 \right) dw_s^k + \int_{r(t)}^t \left(d^1(s)\, Y_s^1 + d^2(s)\, Y_s^2 \right) ds \right) \\
Y_t^2 =& U_t^{-1} \cdot Y_{\tau_\ell}^2 + U_t^{-1} \cdot \sum_{k=1}^d \int_{\tau_\ell}^t U_s \cdot \left(c_k^3(s)\, Y_s^1 + c_k^4(s)\, Y_s^2 \right) dw_s^k \\
&+ U_t^{-1} \cdot \int_{\tau_\ell}^t U_s \cdot \left(d^3(s)\, Y_s^1 + d^4(s)\, Y_s^2 \right) ds,
\end{aligned}
$$

for $t \in [\tau_\ell, \tau_{\ell+1})$ with the initial condition $Y_{\tau_\ell}^1 = P \cdot O_{\tau_\ell} \cdot O_{\tau_\ell -}^{-1} \cdot Y_{\tau_\ell -}$ and $Y_{\tau_\ell}^2 = Q \cdot O_{\tau_\ell} \cdot O_{\tau_\ell -}^{-1} \cdot Y_{\tau_\ell -}$ for $\ell \geq 1$ as well as $Y_0^1 = P \cdot O_{m_0}(x) \cdot v$ and $Y_0^2 = Q \cdot O_{m_0}(x) \cdot v$ for $\ell = 0$.

We shall prove existence and pathwise uniqueness of the solution on every interval $[\tau_\ell, \tau_{\ell+1})$ by induction over ℓ. For every interval we shall first show existence and uniqueness of Y on $[\tau_\ell, T_{\ell,N})$ for every N, from which we shall derive existence and uniqueness of Y on $[\tau_\ell, \tau_{\ell+1})$ by taking the limit $N \to \infty$.

We proceed as in Lemma 4.3 in [1]. Let H be the totality of $\mathbb{R}^d$-valued adapted processes (φ_t), $t \in [0, T]$, whose paths are a.s. càdlàg and which satisfy $\sup_{t\in[0,T]} \mathbb{E}[\|\varphi_t\|^2] < \infty$. On H we introduce the norm

$$
\|\varphi\|_H = \sup_{t\in[0,T]} \mathbb{E}\left[\|\varphi_t\|^2 \right]^{1/2}.
$$

Fix now an arbitrary ℓ. Then, the initial condition Y_{τ_ℓ} is either uniquely specified in terms of $Y_{\tau_\ell -}$ by the induction assumption if $\ell > 0$ or it is given by Y_0 if $\ell = 0$. For any $\varphi \in H$ we define the process $I(\varphi)$ by

$$
\begin{aligned}
I(\varphi)_t^1 =& 1\!\!1_{\{t<\inf C_\ell\}} \left(Y_{\tau_\ell}^1 + \sum_{k=1}^d \int_{\tau_\ell}^t \left(c_k^1(s)\, \varphi_s^1 + c_k^2(s)\, \varphi_s^2 \right) dw_s^k + \int_{\tau_\ell}^t \left(d^1(s)\, \varphi_s^1 + d^2(s)\, \varphi_s^2 \right) ds \right) \\
&+ 1\!\!1_{\{t\geq\inf C_\ell\}} \left(\sum_{k=1}^d \int_{r(t)}^t \left(c_k^1(s)\, \varphi_s^1 + c_k^2(s)\, \varphi_s^2 \right) dw_s^k + \int_{r(t)}^t \left(d^1(s)\, \varphi_s^1 + d^2(s)\, \varphi_s^2 \right) ds \right) \\
I(\varphi)_t^2 =& U_t^{-1} \cdot Y_{\tau_\ell}^2 + U_t^{-1} \cdot \sum_{k=1}^d \int_{\tau_\ell}^t U_s \cdot \left(c_k^3(s)\, \varphi_s^1 + c_k^4(s)\, \varphi_s^2 \right) dw_s^k \\
&+ U_t^{-1} \cdot \int_{\tau_\ell}^t U_s \cdot \left(d^3(s) \cdot \varphi_s^1 + d^4(s)\, \varphi_s^2 \right) ds,
\end{aligned}
$$

if $t \in [\tau_\ell, T_{\ell,N})$ and $I(\varphi)_t = 0$ if $t \in [0, T] \backslash [\tau_\ell, T_{\ell,N})$. By definition of $T_{\ell,N}$, we have for fixed N that all the coefficient functions are uniformly bounded in $t \in [\tau_\ell, T_{\ell,N}]$. Hence, one can easily verify that

$$
\mathbb{E}\left[\|I(\varphi)_t\|^2 \right] \leq c_1 \left\{ 1 + \int_0^t \mathbb{E}\left[\|\varphi_r\|^2 \right] dr \right\} \leq c_2 \left\{ 1 + \sup_{s\in[0,T]} \mathbb{E}\left[\|\varphi_s\|^2 \right] \right\},
$$

where the constants only depend on N and T and not on φ and t. This proves that $I(\varphi) \in H$ for every $\varphi \in H$. Similarly, one can show that for any $\varphi, \psi \in H$,

$$\mathbb{E}\left[\|I(\varphi)_t - I(\psi)_t\|^2\right] \le c_3 \int_0^t \mathbb{E}\left[\|\varphi_s - \psi_s\|^2\right] ds \le c_4 \sup_{s\in[0,T]} \mathbb{E}[\|\varphi_s - \psi_s\|^2],$$

and hence,

$$\|I(\varphi) - I(\psi)\|_H \le c\|\varphi - \psi\|_H.$$

For every N, existence and uniqueness of Y on $[\tau_\ell, T_{\ell,N})$ follow now by standard arguments via Picard-iteration, the details are omitted. Since $T_{\ell,N} \uparrow \tau_{\ell+1}$ a.s. as $N \to \infty$ we get existence of Y on $[\tau_\ell, \tau_{\ell+1})$. Moreover, for any two solutions (Y_t) and $(\tilde{Y}_t)$ we have for every ℓ and every N

$$\mathbb{E}\left[\|\tilde{Y}_t - Y_t\|^2 \mathbb{1}_{\{t\in[\tau_\ell, T_{\ell,N})\}}\right] \le \sup_{s\in[0,T]} \mathbb{E}\left[\|\tilde{Y}_s - Y_s\|^2 \mathbb{1}_{\{s\in[\tau_\ell, T_{\ell,N})\}}\right] = 0.$$

By taking the limit $N \to \infty$ we obtain $\mathbb{E}\left[\|\tilde{Y}_t - Y_t\|^2 \mathbb{1}_{\{t\in[\tau_\ell,\tau_{\ell+1})\}}\right] = 0$ for every ℓ, so that

$$\mathbb{E}\left[\|\tilde{Y}_t - Y_t\|^2\right] = \sum_\ell \mathbb{E}\left[\|\tilde{Y}_t - Y_t\|^2 \mathbb{1}_{\{t\in[\tau_\ell,\tau_{\ell+1})\}}\right] = 0,$$

and therefore $\tilde{Y}_t = Y_t$ a.s. for all $t \in [0, T]$. The claim follows by the right-continuity of the trajectories. □

Let Ω_0 be the subset of Ω constructed in Proposition 3.10. Combining Proposition 3.10, Proposition 3.15 and Proposition 3.18 gives immediately

Corollary 3.19. *Fix $\omega \in \Omega_0$. Let $(\varepsilon_\nu^{(1)})_\nu$, $(\varepsilon_\nu^{(2)})_\nu$ two random sequences converging to zero. There exist subsequences $(\varepsilon_{\nu_l}^{(i)})_l$ such that $Y_t(\varepsilon_{\nu_l}^{(i)})$ has a limit $\hat{Y}_t^{(i)}$ as $l \to \infty$ for all $t \in [0,T]\backslash C$. Moreover, the paths of $\hat{Y}^{(i)}$ can be extended to right-continuous trajectories on $[0,T]$, so that $\hat{Y}_t^{(i)}$ is a solution to the equation in Theorem 2.5. Therefore,*

$$\mathbb{P}\left[\hat{Y}_t^{(1)} = \hat{Y}_t^{(2)}, \forall t \in [0,T]\right] = 1.$$

Proof of Theorem 2.1 and Theorem 2.5. It remains to show that $\eta_T(\varepsilon)$ converges a.s. as ε tends to zero or equivalently that $Y_T(\varepsilon) = O_T \cdot \eta_T(\varepsilon)$ converges a.s.

With a slight abuse of notation we denote by $Y_T^i(\varepsilon)$, $i \in \{1,\dots,d\}$, the components of $Y_T(\varepsilon) \in \mathbb{R}^d$. Then, it is enough to show that for every $i \in \{1,\dots,d\}$

$$Y_T^{i,-} := \liminf_{\varepsilon\to 0} Y_T^i(\varepsilon) = \limsup_{\varepsilon\to 0} Y_T^i(\varepsilon) =: Y_T^{i,+} \qquad \text{a.s.} \tag{3.22}$$

Fix an arbitrary $i \in \{1,\dots,d\}$. Let now $(\varepsilon_\nu^+)_\nu$ be a random sequence converging to zero such that $\lim_{\nu\to\infty} Y_T^i(\varepsilon_\nu^+) = Y_T^{i,+}$. By Proposition 3.10 there exists a subsequence $\varepsilon_{\nu_l}^+$ such that $Y_T(\varepsilon_{\nu_l}^+)$ has a limit Y_T^+ as $l \to \infty$ for every $\omega \in \Omega_0$. In particular, the i-th component of Y_T^+ is equal to $Y_T^{i,+}$.

Analogously, we choose a random sequence $(\varepsilon_\nu^-)_\nu$ converging to zero such that $\lim_{\nu\to\infty} Y_T^i(\varepsilon_\nu^-) = Y_T^{i,-}$. Then, there exists a subsequence $\varepsilon_{\nu_l}^-$ such that $Y_T(\varepsilon_{\nu_l}^-)$ converges to Y_T^- as $l \to \infty$ and $Y_T^{i,-}$ is the i-th component of Y_T^-.

Corollary 3.19 implies $Y_T^+ = Y_T^-$ a.s., in particular $Y_T^{i,+} = Y_T^{i,-}$ a.s., and since i is arbitrary, we obtain that (3.22) holds for every i. Hence, $\eta_T(\varepsilon)$ converges a.s.

Finally, since also $T > 0$ is arbitrary, Theorem 2.1 and Theorem 2.5 follow. □

3.4 The Neumann Condition

In this final section we prove the Neumann condition stated in Corollary 2.9. Let $x \in \partial G$. By a density argument it is sufficient to consider bounded functions f, which are continuously differentiable and have bounded derivatives. Then, for each $t > 0$ we obtain by dominated convergence and the chain rule:

$$D_{n(x)} P_t f(x) = \mathbb{E}\left[\nabla f(X_t(x))\, D_{n(x)} X_t(x)\right].$$

Thus, it suffices to show $D_{n(x)} X_t(x) = 0$ for all $t \in [0, T]$ for some arbitrary fixed T, which is equivalent to $Y_t = 0$ for all $t \in [0, T]$, where $Y_t = O_t \cdot \eta_t^v$ with $v = n(x)$. Again we shall prove this separately on every interval $[\tau_\ell, \tau_{\ell+1})$ by an induction argument over ℓ. We shall use the same notation as in Proposition 3.18. For $t \in [0, \tau_1)$ Y_t satisfies

$$\begin{aligned} Y_t^1 =& \sum_{k=1}^{d} \int_{r(t)}^{t} \left(c_k^1(s)\, Y_s^1 + c_k^2(s)\, Y_s^2\right) dw_s^k + \int_{r(t)}^{t} \left(d^1(s)\, Y_s^1 + d^2(s)\, Y_s^2\right) ds \\ Y_t^2 =& U_t^{-1} \cdot \sum_{k=1}^{d} \int_0^t U_s \cdot \left(c_k^3(s)\, Y_s^1 + c_k^4(s)\, Y_s^2\right) dw_s^k + U_t^{-1} \cdot \int_0^t U_s \cdot \left(d^3(s)\, Y_s^1 + d^4(s)\, Y_s^2\right) ds. \end{aligned}$$

Note that $\inf C = 0$ and $Y_0^2 = Q \cdot O_{m_0}(x) \cdot n(x) = 0$. Setting $Y_t^N := Y_{t \wedge T_{0,N}}$ and $Y_t^{1,N} = P \cdot Y_t^N$, $Y_t^{2,N} = Q \cdot Y_t^N$ as well as $M_t := \sum_{k=1}^d \int_0^t \left(c_k^1(s)\, Y_s^1 + c_k^2(s)\, Y_s^2\right) dw_s^k$ for $t \in [0, T]$, we obtain by Doob's inequality for every N that

$$\begin{aligned} \mathbb{E}\left[\sup_{t \in [0,T]} \left\|Y_t^N\right\|^2\right] \le& \mathbb{E}\left[\sup_{t \in [0,T_{0,N}]} \left\|Y_t^{1,N}\right\|^2\right] + \mathbb{E}\left[\sup_{t \in [0,T_{0,N}]} \left\|Y_t^{2,N}\right\|^2\right] \\ \le& \mathbb{E}\left[\sup_{t \in [0,T_{0,N}]} \left\|M_t - M_{r(t)}\right\|^2\right] + c_1 \int_0^T \mathbb{E}\left[\sup_{r \in [0,s]} \left\|Y_r^N\right\|^2\right] ds \\ \le& 2\, \mathbb{E}\left[\sup_{t \in [0,T_{0,N}]} \left\|M_t\right\|^2\right] + c_1 \int_0^T \mathbb{E}\left[\sup_{r \in [0,s]} \left\|Y_r^N\right\|^2\right] ds \\ \le& c_2 \int_0^T \mathbb{E}\left[\sup_{r \in [0,s]} \left\|Y_r^N\right\|^2\right] ds, \end{aligned}$$

which implies by Gronwall's Lemma that $Y_t^N = 0$ for all $t \in [0, T]$ a.s. We let N tend to infinity to obtain that $Y_t = 0$ for all $t \in [0, \tau_1)$ a.s. Similarly one obtains $Y_t = 0$ on $[\tau_\ell, \tau_{\ell+1})$ for an arbitrary ℓ, note that $Y_{\tau_\ell} = 0$ by the induction assumption.

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