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Looking for martingales associated to a self-decomposable law

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Abstract

We construct martingales whose 1-dimensional marginals are those of a centered self-decomposable variable multiplied by some power of time t. Many examples involving quadratic functionals of Bessel processes are discussed.

Key words: Convex order, Self-decomposable law, Sato process, Karhunen-Loeve representation, Perturbed Bessel process, Ray-Knight theorem.

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1 Introduction, Motivation

1.1 Notation

We first introduce some notation which will be used throughout our paper.

If A and B are two random variables, $A \overset{\mathrm{d}}{=} B$ means that these variables have the same law.

If $(X_t,\ t \geq 0)$ and $(Y_t,\ t \geq 0)$ are two processes, $(X_t) \overset{(1.\mathrm{d})}{=} (Y_t)$ means that the processes $(X_t,\ t \geq 0)$ and $(Y_t,\ t \geq 0)$ have the same one-dimensional marginals, that is, for any fixed t, $X_t \overset{\mathrm{d}}{=} Y_t$.

If $(X_t,\ t \geq 0)$ and $(Y_t,\ t \geq 0)$ are two processes, $(X_t) \overset{(\mathrm{d})}{=} (Y_t)$ means that the two processes are identical in law.

All random variables and processes which will be considered are assumed to be real valued.

1.2 PCOC's

In a number of applied situations involving randomness, it is a quite difficult problem to single out a certain stochastic process $(Y_t,\ t \geq 0)$, or rather its law, which is coherent with the real-world data.

In some cases, it is already nice to be able to consider that the one-dimensional marginals of (Y_t) are accessible. The random situation being studied may suggest, for instance, that:

(i) there exists a martingale (M_t) such that

$$(Y_t) \overset{(1.\mathrm{d})}{=} (M_t)$$

(this hypothesis may indicate some kind of "equilibrium" with respect to time),

(ii) there exists $H > 0$ such that

$$(Y_t) \overset{(1.\mathrm{d})}{=} (t^H Y_1)$$

(there is a "scaling" property involved in the randomness).

It is a result due to Kellerer [14] that (i) is satisfied for a given process (Y_t) if and only if this process is increasing in the convex order, that is: it is integrable ($\forall t \geq 0$, $\mathbb{E}[|Y_t|] < \infty$), and for every convex function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$,

$$t \geq 0 \longrightarrow \mathbb{E}[\varphi(Y_t)] \in (-\infty, +\infty]$$

is increasing.

In the sequel, we shall use the acronym PCOC for such processes, since, in French, the name of such processes becomes: Processus Croissant pour l'Ordre Convexe.

A martingale (M_t) which has the same one-dimensional marginals as a PCOC is said to be *associated* to this PCOC. Note that several different martingales may be associated to a given PCOC. We shall see several striking occurrences of this in our examples.

We also note that Kellerer's work [14] does not contain a constructive algorithm for a martingale associated to a PCOC.

On the other hand, Roynette [23] has exhibited two large families of PCOC's, denoted by $(F1)$ and $(F2)$: The family $(F1)$ consists of the processes

$$\left(\frac{1}{t}\int_0^t N_s\, \mathrm{d}s,\ t\geq 0\right),$$

and the family $(F2)$ consists of the processes

$$\left(\int_0^t (N_s - N_0)\, \mathrm{d}s,\ t\geq 0\right),$$

where (N_s) above denotes any martingale such that:

$$\forall t\geq 0 \qquad \mathbb{E}[\sup_{s\leq t}|N_s|] < \infty.$$

1.3 Self-decomposability and Sato processes

It is a non-trivial problem to exhibit, for either of these PCOC's, an associated martingale. We have been able to do so concerning some examples in $(F1)$, in the Brownian context, with the help of the Brownian sheet ([9]), and in the more general context of Lévy processes, with the help of Lévy sheets ([10]). Concerning the class $(F2)$, note that, considering a trivial filtration, it follows that (tX), where X is a centered random variable, is a PCOC. Even with this reduction, it is not obvious to find a martingale which is associated to (tX). In order to exhibit examples, we were led to introduce the class (S) of processes (Y_t) satisfying the above condition (ii) and such that Y_1 is a self-decomposable integrable random variable. It is a result due to Sato (see Sato [24, Chapter 3, Sections 15-17]) that, if $(Y_t)\in(S)$, then there exists a process (U_t) which has independent increments, is *H-self-similar* ($\forall c>0$, $(U_{ct}) \overset{(\mathrm{d})}{=} (c^H U_t)$) and satisfies $Y_1 \overset{\mathrm{d}}{=} U_1$. This process (U_t), which is unique in law, will be called the *H-Sato process associated* to Y_1. Clearly, then $(U_t - \mathbb{E}[U_t])$ is a H-self-similar martingale which is associated to the PCOC (V_t) defined by: $V_t = Y_t - \mathbb{E}[Y_t]$. Moreover, $(V_t) \overset{(1.\mathrm{d})}{=} (t^H(Y_1 - \mathbb{E}[Y_1]))$.

We note that the self-decomposability property has also been used in Madan-Yor [18, Theorem 4,Theorem 5] in a very different manner than in this paper, to construct martingales with one-dimensional marginals those of (tX).

1.4 Examples

We look for some interesting processes in the class (S), in a Brownian framework.

Example 1 A most simple example is the process:

$$Y_t := \int_0^t B_s\, \mathrm{d}s, \quad t\geq 0$$

Then,

$$\left(\int_0^t B_s\, \mathrm{d}s\right) \overset{(1.\mathrm{d})}{=} \left(\int_0^t s\, \mathrm{d}B_s\right)$$

and the RHS is a centered (3/2)-Sato process. Moreover the process (Y_t) obviously belongs to the class $(F2)$.

Example 2 The process

$$V_1(t) := \int_0^t (B_s^2 - s)\,\mathrm{d}s, \quad t \geq 0$$

and more generally the process

$$V_N(t) := \int_0^t (R_N^2(s) - Ns)\,\mathrm{d}s, \quad t \geq 0$$

where $(R_N(s))$ is a Bessel process of dimension $N > 0$ starting from 0, belongs to the family $(F2)$ and is 2-self-similar. We show in Section 4 that the centered 2-Sato process:

$$\frac{N^2}{4} \int_0^{\tau_t} 1_{\left(|B_s| \leq \frac{2}{N}\ell_s\right)}\,\mathrm{d}s - \frac{Nt^2}{2}, \quad t \geq 0$$

where (ℓ_s) is the local time in 0 of the Brownian motion B, and

$$\tau_t = \inf\{s;\ \ell_s > t\}$$

is a martingale associated to the PCOC V_N.

Example 3 We extend our discussion of Example 2 by considering, for $N > 0$ and $K > 0$, the process:

$$V_{N,K}(t) := \frac{1}{K^2} \int_0^t s^{2\left(\frac{1}{K}-1\right)} (R_N^2(s) - Ns)\,\mathrm{d}s, \quad t \geq 0$$

Then, in Section 5, a centered $(2/K)$-Sato process (and hence a martingale) associated to the PCOC $V_{N,K}$ may be constructed from the process of first hitting times of a perturbed Bessel process $R_{K,1-\frac{N}{2}}$ as defined and studied first in Le Gall-Yor [16; 17] and then in Doney-Warren-Yor [6]. We remark that, if $0 < K < 2$, then the process

$$V_{N,K}(t^{\frac{K}{2-K}}), \quad t \geq 0$$

belongs to $(F2)$.

Example 4 In Section 6, we generalize again our discussion by considering the process

$$V_N^{(\mu)}(t) := \int_{(0,\infty)} (R_N^2(ts) - Nts)\,\mathrm{d}\mu(s), \quad t \geq 0$$

for μ a nonnegative measure on $(0,\infty)$ such that $\int_{(0,\infty)} s\,\mathrm{d}\mu(s) < \infty$. We show that $V_N^{(\mu)}$ is a PCOC to which we are able to associate two very different martingales. The first one is purely discontinuous and is a centered 1-Sato process, the second one is continuous. The method of proof is based on a Karhunen-Loeve type decomposition (see, for instance, [5] and the references therein, notably Kac-Siegert [13]). For this, we need to develop a precise spectral study of the operator $K^{(\mu)}$ defined on $L^2(\mu)$ by :

$$K^{(\mu)} f(t) = \int_{(0,\infty)} f(s)(t \wedge s)\,\mathrm{d}\mu(s)$$

1.5 Organisation of the paper

We now present more precisely the organisation of our paper:

- in Section 2, we recall some basic results about various representations of self-decomposable variables, and we complete the discussion of Subsection 1.3 above;
- in Section 3, we consider the simple situation, as in Subsection 1.3, where $Y_t = R_N^2(t)$, for R_N a Bessel process of dimension N starting from 0;
- the contents of Sections 4, 5, 6 have already been discussed in the above Subsection 1.4.

We end this introduction with the following (negative) remark concerning further self-decomposability properties for squared Bessel processes: indeed, it is well-known, and goes back to Shiga-Watanabe [25], that $R_N^2(\bullet)$, considered as a random variable taking values in $C(\mathbb{R}_+, \mathbb{R}_+)$ is infinitely divisible. Furthermore, in the present paper, we show and exploit the self-decomposability of $\int_{(0,\infty)} R_N^2(s)\, d\mu(s)$ for any positive measure μ. It then seems natural to wonder about the self-decomposability of $R_N^2(\bullet)$, but this property is ruled out: the 2-dimensional vectors $(R_N^2(t_1), R_N^2(t_1+t_2))$ are not self-decomposable, as an easy Laplace transform computation implies.

2 Sato processes and PCOC's

2.1 Self-decomposability and Sato processes

We recall, in this subsection, some general facts concerning the notion of self-decomposability. We refer the reader, for background, complements and references, to Sato [24, Chapter 3].

A random variable X is said to be *self-decomposable* if, for each u with $0 < u < 1$, there is the equality in law:

$$X \stackrel{\mathrm{d}}{=} uX + \widehat{X}_u$$

for some variable $\widehat{X}_u$ independent of X.

On the other hand, an *additive process* $(U_t,\ t \geq 0)$ is a stochastically continuous process with càdlàg paths, independent increments, and satisfying $U_0 = 0$.

An additive process (U_t) which is *H-self-similar* for some $H > 0$, meaning that, for each $c > 0$, $(U_{ct}) \stackrel{(\mathrm{d})}{=} (c^H\, U_t)$, will be called a *Sato process* or, more precisely, a *H-Sato process*.

The following theorem, for which we refer to Sato's book [24, Chapter 3, Sections 16-17], gives characterizations of the self-decomposability property.

Theorem 2.1. *Let X be a real valued random variable. Then, X is self-decomposable if and only if one of the following equivalent properties is satisfied:*

1) X is infinitely divisible and its Lévy measure is $\dfrac{h(x)}{|x|}\,\mathrm{d}x$ with h increasing on $(-\infty, 0)$ and decreasing on $(0, +\infty)$.

2) There exists a Lévy process $(C_s,\ s \geq 0)$ such that

$$X \stackrel{\mathrm{d}}{=} \int_0^\infty \mathrm{e}^{-s}\,\mathrm{d}C_s.$$

3) For any (or some) $H > 0$, there exists a H-Sato process $(U_t,\ t \geq 0)$ such that $X \stackrel{\mathrm{d}}{=} U_1$.

In 2) (resp. 3)) the Lévy process (C_s) (resp. the H-Sato process (U_t)) is uniquely determined in law by X, and will be said to be *associated* with X. We note that, if $X \geq 0$, then the function h vanishes on $(-\infty, 0)$, (C_s) is a subordinator and (U_t) is an increasing process.

The relation between (C_s) and (U_t) was made precise by Jeanblanc-Pitman-Yor [11, Theorem 1]:

Theorem 2.2. *If (U_t) is a H-Sato process, then the formulae:*

$$C_s^{(-)} = \int_{\mathrm{e}^{-s}}^1 r^{-H}\,\mathrm{d}U_r \quad \textit{and} \quad C_s^{(+)} = \int_1^{\mathrm{e}^s} r^{-H}\,\mathrm{d}U_r, \quad s \geq 0$$

define two independent and identically distributed Lévy processes from which $(U_t,\ t \geq 0)$ can be recovered by:

$$U_t = \int_{-\log t}^\infty \mathrm{e}^{-sH}\,\mathrm{d}C_s^{(-)} \quad \textit{if} \quad 0 \leq t \leq 1$$

and

$$U_t = U_1 + \int_0^{\log t} \mathrm{e}^{sH}\,\mathrm{d}C_s^{(+)} \quad \textit{if} \quad t \geq 1.$$

In particular, the Lévy process associated with the self-decomposable random variable U_1 is

$$C_s = C_{s/H}^{(-)}, \quad s \geq 0.$$

2.2 Sato processes and PCOC's

We recall (see Subsection 1.2) that a PCOC is an integrable process which is increasing in the convex order. On the other hand, a process $(V_t,\ t \geq 0)$ is said to be a *1-martingale* if there exists, on some filtered probability space, a martingale $(M_t,\ t \geq 0)$ such that $(V_t) \stackrel{(1.\mathrm{d})}{=} (M_t)$. Such a martingale M is said to be *associated* with V. It is a direct consequence of Jensen's inequality that, if V is a 1-martingale, then V is a PCOC. As indicated in Subsection 1.2, the converse holds true (Kellerer [14]).

The following proposition, which is central in the following, summarizes the method sketched in Subsection 1.3.

Proposition 2.3. *Let $H > 0$. Suppose that $Y = (Y_t,\ t \geq 0)$ satisfies:*

(a) Y_1 is an integrable self-decomposable random variable;

(b) $(Y_t) \stackrel{(1.\mathrm{d})}{=} (t^H Y_1)$.

Then the process

$$V_t := Y_t - t^H \,\mathbb{E}[Y_1], \quad t \geq 0$$

is a PCOC, and an associated martingale is

$$M_t := U_t - t^H \,\mathbb{E}[Y_1], \quad t \geq 0$$

where (U_t) denotes the H-Sato process associated with Y_1 according to Theorem 2.1.

3 About the process $(R_N^2(t),\ t \geq 0)$

In the sequel, we denote by $(R_N(t),\ t \geq 0)$ the Bessel process of dimension $N > 0$, starting from 0.

3.1 Self-decomposability of $R_N^2(1)$

As is well-known (see, for instance, Revuz-Yor [22, Chapter XI]) one has

$$\mathbb{E}[\exp(-\lambda R_N^2(1))] = (1 + 2\lambda)^{-N/2}.$$

In other words,

$$R_N^2(1) \stackrel{\mathrm{d}}{=} 2\gamma_{N/2}$$

where, for $a > 0$, γ_a denotes a gamma random variable of index a. Now, the classical Frullani's formula yields:

$$\frac{N}{2}\log(1 + 2\lambda) = \frac{N}{2}\int_0^\infty (1 - \mathrm{e}^{-\lambda t})\,\frac{\mathrm{e}^{-t/2}}{t}\,\mathrm{d}t.$$

Then, $R_N^2(1)$ satisfies the property 1) in Theorem 2.1 with

$$h(x) = \frac{N}{2}\,1_{(0,\infty)}(x)\,\mathrm{e}^{-x/2}$$

and it is therefore self-decomposable.

The process R_N^2 is 1-self-similar and $\mathbb{E}[R_N^2(1)] = N$. By Proposition 2.3, the process

$$V_t^N := R_N^2(t) - t\,N, \quad t \geq 0$$

is a PCOC, and an associated martingale is

$$M_t^N := U_t^N - t\,N, \quad t \geq 0$$

where (U_t^N) denotes the 1-Sato process associated with $R_N^2(1)$ by Theorem 2.1.

We remark that, in this case, the process (V_t^N) itself is a *continuous martingale* and therefore obviously a PCOC. In the following subsections, we give two expressions for the process (U_t^N). As we will see, this process is purely discontinuous with finite variation; consequently, the martingales (V_t^N) and (M_t^N), which have the same one-dimensional marginals, do not have the same law.

3.2 Expression of (U_t^N) from a compound Poisson process

We denote by $(\Pi_s,\ s \geq 0)$ the compound Poisson process with Lévy measure:

$$1_{(0,\infty)}(t)\,\mathrm{e}^{-t}\,\mathrm{d}t.$$

This process allows to compute the distributions of a number of perpetuities

$$\int_0^\infty \mathrm{e}^{-\Lambda_s}\,\mathrm{d}\Pi_s$$

where (Λ_s) is a particular Lévy process, independent of Π; see, e.g., Nilsen-Paulsen [20]. In the case $\Lambda_s = r\,s$, the following result seems to go back at least to Harrison [8].

Proposition 3.1. *The Lévy process (C_s^N) associated with the self-decomposable random variable $R_N^2(1)$ in the sense of Theorem 2.1 is*

$$C_s^N = 2\,\Pi_{Ns/2}, \quad s \geq 0.$$

Proof

a) First, recall that for a subordinator $(\tau_s,\ s \geq 0)$ and $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ Borel, there is the formula:

$$\mathbb{E}\left[\exp\left(-\int_0^\infty f(s)\,\mathrm{d}\tau_s\right)\right] = \exp\left(-\int_0^\infty \Phi(f(s))\,\mathrm{d}s\right),$$

where Φ is the Lévy exponent of $(\tau_s,\ s \geq 0)$. Consequently, a slight amplification of this formula is:

$$\mathbb{E}\left[\exp\left(-\mu\int_0^\infty f(s)\,\mathrm{d}\tau_{As}\right)\right] = \exp\left(-\int_0^\infty \Phi(\mu f(u/A))\,\mathrm{d}u\right)$$

for every $\mu, A > 0$.

b) We set $C_s^N = 2\,\Pi_{Ns/2}$. Then, as a consequence of the previous formula with $\mu = 2, f(s) = \lambda\,\mathrm{e}^{-s}, A = N/2$, we get:

$$\mathbb{E}\left[\exp\left(-\lambda\int_0^\infty \mathrm{e}^{-s}\,\mathrm{d}C_s^N\right)\right] = \exp\left(-\int_0^\infty \Phi(2\,\lambda\,\mathrm{e}^{-\frac{2}{N}u})\,\mathrm{d}u\right)$$

with, for $x > 0$,

$$\Phi(x) = \int_0^\infty (1-\mathrm{e}^{-tx})\,\mathrm{e}^{-t}\,\mathrm{d}t = \frac{x}{1+x}.$$

c) We obtain by change of variable:

$$\begin{aligned}
\exp\left(-\int_0^\infty \Phi(2\,\lambda\,\mathrm{e}^{-\frac{2}{N}u})\,\mathrm{d}u\right) &= \exp\left(-\int_0^\infty \frac{2\,\lambda\,\mathrm{e}^{-\frac{2}{N}u}}{1+2\,\lambda\,\mathrm{e}^{-\frac{2}{N}u}}\,\mathrm{d}u\right) \\
&= \exp\left(-\frac{N}{2}\int_0^{2\lambda} \frac{1}{1+x}\,\mathrm{d}x\right) \\
&= (1+2\,\lambda)^{-N/2}.
\end{aligned}$$

Consequently,

$$\mathbb{E}\left[\exp\left(-\lambda\int_0^\infty \mathrm{e}^{-s}\,\mathrm{d}C_s^N\right)\right]=(1+2\lambda)^{-N/2},$$

which proves the result.

□

By application of Theorem 2.2 we get:

Corollary 3.1.1. *Let $\Pi^{(+)}$ and $\Pi^{(-)}$ two independent copies of the Lévy process Π. Then*

$$U_t^N = 2\int_{-\frac{N}{2}\log t}^{\infty} \mathrm{e}^{-2s/N}\,\mathrm{d}\Pi_s^{(-)} \quad \textit{if} \quad 0\le t\le 1$$

and

$$U_t^N = U_1^N + 2\int_0^{\frac{N}{2}\log t} \mathrm{e}^{2s/N}\,\mathrm{d}\Pi_s^{(+)} \quad \textit{if} \quad t\ge 1.$$

3.3 Expression of (U_t^N) from the local time of a perturbed Bessel process

There is by now a wide literature on perturbed Bessel processes, a notion originally introduced by Le Gall-Yor [16; 17], and then studied by Chaumont-Doney [3], Doney-Warren-Yor [6]. We also refer the interested reader to Doney-Zhang [7].

We first introduce the perturbed Bessel process $(R_{1,\alpha}(t),\ t\ge 0)$ starting from 0, for $\alpha<1$, as the nonnegative continuous strong solution of the equation

$$R_{1,\alpha}(t)=B_t+\frac{1}{2}L_t(R_{1,\alpha})+\alpha\, M_t(R_{1,\alpha}) \tag{1}$$

where $L_t(R_{1,\alpha})$ is the semi-martingale local time of $R_{1,\alpha}$ in 0 at time t, and

$$M_t(R_{1,\alpha})=\sup_{0\le s\le t}R_{1,\alpha}(s),$$

(B_t) denoting a standard linear Brownian motion starting from 0. (The strong solution property has been established in Chaumont-Doney [3].)
It is clear that the process $R_{1,0}$ is nothing else but the Bessel process R_1 (reflected Brownian motion).
We also denote by $T_t(R_{1,\alpha})$ the hitting time:

$$T_t(R_{1,\alpha})=\inf\{s;\ R_{1,\alpha}(s)>t\}.$$

We set $L_{T_t}(R_{1,\alpha})$ for $L_{T_t(R_{1,\alpha})}(R_{1,\alpha})$.
Finally, in the sequel, we set

$$\alpha_N=1-\frac{N}{2}.$$

Proposition 3.2. *For any $\alpha<1$, the process $(L_{T_t}(R_{1,\alpha}),\ t\ge 0)$ is a 1-Sato process, and we have*

$$(U_t^N)\overset{(\mathrm{d})}{=}(L_{T_t}(R_{1,\alpha_N})).$$

Proof

By the uniqueness in law of the solution to the equation (1), the process $R_{1,\alpha}$ is (1/2)-self-similar. As a consequence, the process $(L_{T_t}(R_{1,\alpha}),\ t \geq 0)$ is 1-self-similar.

On the other hand, the pair $(R_{1,\alpha}, M(R_{1,\alpha}))$ is strong Markov (see Doney-Warren-Yor [6, p. 239]). As

$$R_{1,\alpha}(u) = M_u(R_{1,\alpha}) = t \quad \text{ if } \quad u = T_t(R_{1,\alpha}),$$

the fact that $(L_{T_t}(R_{1,\alpha}),\ t \geq 0)$ is an additive process follows from the strong Markov property.

Finally, we need to prove:

$$R_N^2(1) \stackrel{\mathrm{d}}{=} L_{T_1}(R_{1,\alpha_N}).$$

For the remainder of the proof, we denote R_{1,α_N} by R, and $L_t(R_{1,\alpha_N})$, $T_t(R_{1,\alpha_N})$, $M_t(R_{1,\alpha_N}) \cdots$ are simply denoted respectively by $L_t,\ T_t,\ M_t\ \cdots$ As a particular case of the "balayage formula" (Yor [26], Revuz-Yor [22, VI.4]) we deduce from equation (1), that:

$$\begin{aligned} \exp(-\lambda L_t) R_t &= \int_0^t \exp(-\lambda L_s)\, \mathrm{d}R_s \\ = \int_0^t \exp(-\lambda L_s)\, \mathrm{d}B_s &+ \frac{1-\exp(-\lambda L_t)}{2\lambda} + \alpha_N \int_0^t \exp(-\lambda L_s)\, \mathrm{d}M_s. \end{aligned}$$

Hence,

$$\begin{aligned} \exp(-\lambda L_t)(1+2\lambda R_t) &= 1 + 2\lambda \int_0^t \exp(-\lambda L_s)\, \mathrm{d}B_s \\ &\quad + 2\lambda\, \alpha_N \int_0^t \exp(-\lambda L_s)\, \mathrm{d}M_s. \end{aligned}$$

From this formula, we learn that the martingale

$$\left(\int_0^{u \wedge T_t} \exp(-\lambda L_s)\, \mathrm{d}B_s,\ u \geq 0 \right)$$

is bounded; hence, by applying the optional stopping theorem, we get:

$$\begin{aligned} \mathbb{E}[\exp(-\lambda L_{T_t})](1+2\lambda t) &= 1 + 2\lambda\, \alpha_N \mathbb{E}\left[\int_0^{T_t} \exp(-\lambda L_s)\, \mathrm{d}M_s \right] \\ &= 1 + 2\lambda\, \alpha_N \int_0^t \mathbb{E}[\exp(-\lambda L_{T_u})]\, \mathrm{d}u, \end{aligned}$$

by time-changing. Setting

$$\varphi_\lambda(t) = \mathbb{E}[\exp(-\lambda L_{T_t})],$$

we obtain:

$$\varphi_\lambda(t) = \frac{1}{1+2\lambda t} + \frac{2\lambda\, \alpha_N}{1+2\lambda t} \int_0^t \varphi_\lambda(u)\, \mathrm{d}u.$$

Consequently

$$\varphi_\lambda(t) = (1 + 2\lambda t)^{-N/2}.$$

Therefore,

$$\mathbb{E}[\exp(-\lambda L_{T_1})] = (1 + 2\lambda)^{-N/2} = \mathbb{E}[\exp(-\lambda R_N^2(1)],$$

which proves the desired result.

□

4 About the process $\left(\int_0^t R_N^2(s)\,\mathrm{d}s,\ t \geq 0\right)$

4.1 A class of Sato processes

Let $(\ell_t,\ t \geq 0)$ be the local time in 0 of a linear Brownian motion $(B_t,\ t \geq 0)$ starting from 0. We denote, as usual, by $(\tau_t,\ t \geq 0)$ the inverse of this local time:

$$\tau_t = \inf\{s \geq 0;\ \ell_s > t\}.$$

Proposition 4.1. *Let $f(x,u)$ be a Borel function on $\mathbb{R}_+ \times \mathbb{R}_+$ such that*

$$\forall t > 0 \qquad \iint_{\mathbb{R}_+ \times [0,t]} |f(x,u)|\,\mathrm{d}x\,\mathrm{d}u < \infty. \tag{2}$$

Then the process $A^{(f)}$ defined by:

$$A_t^{(f)} = \int_0^{\tau_t} f(|B_s|, \ell_s)\,\mathrm{d}s, \quad t \geq 0$$

is an integrable additive process. Furthermore,

$$\mathbb{E}[A_t^{(f)}] = 2 \iint_{\mathbb{R}_+ \times [0,t]} f(x,u)\,\mathrm{d}x\,\mathrm{d}u.$$

Proof

Assume first that f is nonnegative. Then,

$$A_t^{(f)} = \sum_{0 \leq u \leq t} \int_{\tau_{u-}}^{\tau_u} f(|B_s|, u)\,\mathrm{d}s.$$

By the theory of excursions (Revuz-Yor [22, Chapter XII, Proposition 1.10]) we have

$$\mathbb{E}[A_t^{(f)}] = \int_0^t \mathrm{d}u \int n(\mathrm{d}\varepsilon) \int_0^{V(\varepsilon)} \mathrm{d}s\, f(|\varepsilon_s|, u)$$

where n denotes the Itô measure of Brownian excursions and $V(\varepsilon)$ denotes the life time of the excursion ε. The entrance law under n is given by:

$$n(\varepsilon_s \in \mathrm{d}x;\ s < V(\varepsilon)) = (2\pi s^3)^{-1/2}\,|x|\,\exp(-x^2/(2s))\,\mathrm{d}x.$$

Therefore

$$\mathbb{E}[A_t^{(f)}] = 2\int_0^t \mathrm{d}u \int_0^\infty \mathrm{d}x\, f(x,u).$$

The additivity of the process $A^{(f)}$ follows easily from the fact that, for any $t \geq 0$, $(B_{\tau_t+s},\ s \geq 0)$ is a Brownian motion starting from 0, which is independent of $\mathscr{B}_{\tau_t}$ (where $(\mathscr{B}_u)$ is the natural filtration of B).

□

Corollary 4.1.1. *We assume that f is a Borel function on $\mathbb{R}_+ \times \mathbb{R}_+$ satisfying (2) and which is m-homogeneous for $m > -2$, meaning that*

$$\forall a > 0,\ \forall (x,u) \in \mathbb{R}_+ \times \mathbb{R}_+, \quad f(ax, au) = a^m f(x,u).$$

Then the process $A^{(f)}$ is a $(m+2)$-Sato process.

Proof

This is a direct consequence of the scaling property of Brownian motion.

□

4.2 A particular case

Let $N > 0$. We denote by $A^{(N)}$ the process $A^{(f)}$ with

$$f(x,u) = \frac{N^2}{4}\,1_{(x \leq \frac{2}{N}u)}.$$

By Proposition 4.1, $(A_t^{(N)})$ is an integrable process and

$$\mathbb{E}[A_t^{(N)}] = \frac{N\,t^2}{2}.$$

We now consider the process Y_N defined by

$$Y_N(t) = \int_0^t R_N^2(s)\,\mathrm{d}s, \quad t \geq 0.$$

Theorem 4.2. *The process $A^{(N)}$ is a 2-Sato process and*

$$(Y_N(t)) \overset{(1.d)}{=} (A_t^{(N)}).$$

Proof

It is a direct consequence of Corollary 4.1.1 that $A^{(N)}$ is a 2-Sato process.

By Mansuy-Yor [19, Theorem 3.4, p.38], the following extension of the Ray-Knight theorem holds: For any $u > 0$,

$$(L_{\tau_u}^{a-(2u/N)},\ 0 \le a \le (2u/N)) \stackrel{(\mathrm{d})}{=} (R_N^2(a),\ 0 \le a \le (2u/N))$$

where L_t^x denotes the local time of the semi-martingale $(|B_s| - \frac{2}{N}\,\ell_s,\ s \ge 0)$ in x at time t.
We remark that

$$s \in [0, \tau_t] \Longrightarrow |B_s| - \frac{2}{N}\,\ell_s \ge -\frac{2t}{N}.$$

Therefore, the occupation times formula entails:

$$A_t^{(N)} = \frac{N^2}{4} \int_{-2t/N}^{0} L_{\tau_t}^x \,\mathrm{d}x = \frac{N^2}{4} \int_0^{2t/N} L_{\tau_t}^{x-(2t/N)} \,\mathrm{d}x.$$

Thus, by the above mentioned extension of the Ray-Knight theorem,

$$(A_t^{(N)}) \stackrel{(1.\mathrm{d})}{=} \left(\frac{N^2}{4} \int_0^{2t/N} R_N^2(s)\,\mathrm{d}s \right).$$

The scaling property of R_N also yields the identity in law:

$$(A_t^{(N)}) \stackrel{(1.\mathrm{d})}{=} \left(\int_0^{t} R_N^2(s)\,\mathrm{d}s \right),$$

and the result follows from the definition of Y_N.

□

We may now apply Proposition 2.3 to get:

Corollary 4.2.1. *The process V_N defined by:*

$$V_N(t) = Y_N(t) - \frac{N t^2}{2}, \quad t \ge 0$$

is a PCOC and an associated martingale is M_N defined by:

$$M_N(t) = A_t^{(N)} - \frac{N t^2}{2}, \quad t \ge 0.$$

Moreover, M_N is a centered 2-Sato process.

4.3 Representation of $A^{(N)}$ as a process of hitting times

Theorem 4.3. *The process $A^{(N)}$ is identical in law to the process*

$$T_t(R_{1,\alpha_N}), \quad t \ge 0$$

where R_{1,α_N} denotes the perturbed Bessel process defined in Subsection 3.3 and

$$T_t(R_{1,\alpha_N}) = \inf\{s;\ R_{1,\alpha_N}(s) > t\}.$$

The proof can be found in Le Gall-Yor [17]. Nevertheless, for the convenience of the reader, we give again the proof below. A more general result, based on Doney-Warren-Yor [6], shall also be stated in the next section.

Proof

In this proof, we adopt the following notation: (B_t) still denotes a standard linear Brownian motion starting from 0, $S_t = \sup_{0\le s\le t} B_s$ and $\sigma_t = \inf\{s;\ B_s > t\}$. Moreover, for $a < 1$ and $t \ge 0$, we set:

$$H_t^a = (B_t - a\,S_t), \quad X_t^a = \int_0^t 1_{(B_s > a\,S_s)}\,\mathrm{d}s \quad \text{and} \quad Z_t^a = \inf\{s;\ X_s^a > t\}.$$

Lemma 4.3.1. *Let $a < 1$. Then*

$$M_t(H^a) := \sup_{0\le s\le t} H_s^a = (1-a)\,S_t.$$

Consequently, $-a\,S_t = \alpha\,M_t(H^a)$, with $\alpha = -a/(1-a)$.

Proof

Since $a < 1$, we have, for $0 \le s \le t$,

$$(B_s - a\,S_s) \le (1-a)\,S_s \le (1-a)\,S_t.$$

Moreover, there exists $s_t \in [0,t]$ such that $B_{s_t} = S_t$ and therefore $S_{s_t} = S_t$. Hence, $B_{s_t} - a\,S_{s_t} = (1-a)\,S_t$.

□

Lemma 4.3.2. *Let $a < 1$ and recall $\alpha = -a/(1-a)$. We set:*

$$D_t^a = H_{Z_t^a}^a,\ t \ge 0.$$

Then the processes D^a and $R_{1,\alpha}$ are identical in law.

Proof

a) Since Z_t^a is a time of increase of the process

$$X_s^a = \int_0^s 1_{(H_u^a > 0)}\,\mathrm{d}u,\ s \ge 0,$$

we get: $D_t^a = H_{Z_t^a}^a \ge 0$. Moreover, since the process $(H^a)^+$ is obviously Z^a-continuous, the process D^a is continuous.

b) By Tanaka's formula,

$$(H_t^a)^+ = \int_0^t 1_{(H_s^a > 0)}\,\mathrm{d}(B_s - a\,S_s) + \frac{1}{2}\,L_t(H^a)$$

where $L_t(H^a)$ denotes the local time of the semi-martingale H^a in 0 at time t. If $s > 0$ belongs to the support of dS_s, then $B_s = S_s$ and, since $a < 1$, $B_s - a S_s > 0$. Therefore,

$$(H_t^a)^+ = \int_0^t 1_{(B_s - a S_s > 0)} \, dB_s - a S_t + \frac{1}{2} L_t(H^a).$$

By Lemma 4.3.1, $-a S_t = \alpha M_t(H^a)$. Consequently,

$$D_t^a = \int_0^{Z_t^a} 1_{(B_s - a S_s > 0)} \, dB_s + \frac{1}{2} L_{Z_t^a}(H^a) + \alpha M_{Z_t^a}(H^a).$$

By the Dubins-Schwarz theorem, the process

$$\int_0^{Z_t^a} 1_{(B_s - a S_s > 0)} \, dB_s, \quad t \geq 0$$

is a Brownian motion.

On the other hand, it is easy to see that

$$L_{Z_t^a}(H^a) = L_t(D^a) \quad \text{and} \quad M_{Z_t^a}(H^a) = M_t(D^a).$$

Therefore, the process D^a is a continuous and nonnegative solution to equation (1).

□

To conclude the proof of Theorem 4.3, we observe that by Lévy's equivalence theorem ([22, Theorem VI.2.3]), the process $A^{(N)}$ is identical in law to the process

$$\frac{N^2}{4} \int_0^{\sigma_t} 1_{(B_s > (1 - \frac{2}{N}) S_s)} \, ds, \quad t \geq 0.$$

By the scaling property of B, the above process has the same law as

$$\int_0^{\sigma_{Nt/2}} 1_{(B_s > (1 - \frac{2}{N}) S_s)} \, ds = X_{\sigma_{Nt/2}}^{1 - \frac{2}{N}}, \quad t \geq 0.$$

Now,

$$X_{\sigma_{Nt/2}}^{1 - \frac{2}{N}} = \inf\{X_u^{1 - \frac{2}{N}};\, S_u > \frac{Nt}{2}\}$$

and, by Lemma 4.3.1,

$$S_u = \frac{N}{2} M_u(H^{1 - \frac{2}{N}}).$$

Thus,

$$\begin{aligned} X_{\sigma_{Nt/2}}^{1 - \frac{2}{N}} &= \inf\{X_u^{1 - \frac{2}{N}};\, M_u(H^{1 - \frac{2}{N}}) > t\} \\ &= \inf\{v;\, M_v(D^{1 - \frac{2}{N}}) > t\} = \inf\{v;\, D_v^{1 - \frac{2}{N}} > t\}. \end{aligned}$$

The result then follows from Lemma 4.3.2.

□

Corollary 4.3.1. *The process*

$$T_t(R_{1,\alpha_N}), \quad t \geq 0$$

is a 2-Sato process and

$$\left(\int_0^t R_N^2(s)\,\mathrm{d}s\right) \stackrel{(1.d)}{=} \left(T_t(R_{1,\alpha_N})\right).$$

5 About the process $\left(\frac{1}{K^2}\int_0^t s^{\frac{2(1-K)}{K}} R_N^2(s)\,\mathrm{d}s,\ t \geq 0\right)$

In this section we extend Corollary 4.3.1. We fix two positive real numbers N and K. We first recall some important results on general perturbed Bessel processes $R_{K,\alpha}$ with $\alpha < 1$.

5.1 Perturbed Bessel processes

We follow, in this subsection, Doney-Warren-Yor [6]. We first recall the definition of the process $R_{K,\alpha}$ with $K > 0$ and $\alpha < 1$.
The case $K = 1$ was already introduced in Subsection 3.3. For $K > 1$, $R_{K,\alpha}$ is defined as a continuous nonnegative solution to

$$R_t = B_t + \frac{K-1}{2}\int_0^t \frac{1}{R_s}\,\mathrm{d}s + \alpha\, M_t(R), \tag{3}$$

and, for $0 < K < 1$, $R_{K,\alpha}$ is defined as the square root of a continuous nonnegative solution to

$$X_t = 2\int_0^t \sqrt{X_s}\,\mathrm{d}B_s + K\,t + \alpha\, M_t(X). \tag{4}$$

We note that, for any $K > 0$, $(R_{K,0}(t)) \stackrel{(d)}{=} (R_K(t))$. As in the case $K = 1$, for any $K > 0$, the pair $(R_{K,\alpha}, M(R_{K,\alpha}))$ is strong Markov.
We denote, as before,

$$T_t(R_{K,\alpha}) = \inf\{s;\ R_{K,\alpha}(s) > t\}.$$

The following theorem, due to Doney-Warren-Yor [6, Theorem 5.2, p. 246] is an extension of the Ciesielski-Taylor theorem and of the Ray-Knight theorem.

Theorem 5.1. *1)*

$$\int_0^\infty 1_{(R_{K+2,\alpha}(s)\leq 1)}\,\mathrm{d}s \stackrel{\mathrm{d}}{=} T_1(R_{K,\alpha})$$

2)

$$(L_\infty^a(R_{K+2,\alpha}),\ a \geq 0) \stackrel{(d)}{=} (\frac{a^{1-K}}{K} R_{2(1-\alpha)}(a^K),\ a \geq 0)$$

5.2 Identification of the Sato process associated to $Y_{N,K}$

We denote, for $N > 0$ and $K > 0$, by $Y_{N,K}$ the process:

$$Y_{N,K}(t) = \frac{1}{K^2} \int_0^t s^{\frac{2(1-K)}{K}} R_N^2(s)\,\mathrm{d}s, \quad t \geq 0.$$

We also recall the notation:

$$\alpha_N = 1 - \frac{N}{2}.$$

Theorem 5.2. *The process*

$$T_{t^{1/K}}(R_{K,\alpha_N}), \quad t \geq 0$$

is a $(2/K)$*-Sato process and*

$$(Y_{N,K}(t)) \overset{(1.\mathrm{d})}{=} (T_{t^{1/K}}(R_{K,\alpha_N})).$$

Proof

In the following proof, we denote R_{K,α_N} simply by R, and we set T_t and M_t for, respectively, $T_t(R)$ and $M_t(R)$.

The first part of the statement follows from the $(1/2)$-self-similarity of R and from the strong Markovianity of (R, M), taking into account that, for any $t \geq 0$,

$$R_{T_t} = M_{T_t} = t.$$

By occupation times formula, we deduce from 1) in Theorem 5.1,

$$\int_0^1 L_\infty^x(R_{K+2,\alpha_N})\,\mathrm{d}x \overset{\mathrm{d}}{=} T_1.$$

Using then 2) in Theorem 5.1, we obtain:

$$\int_0^1 L_\infty^x(R_{K+2,\alpha_N})\,\mathrm{d}x \overset{\mathrm{d}}{=} \int_0^1 \frac{x^{1-K}}{K} R_N^2(x^K)\,\mathrm{d}x.$$

By change of variable, the last integral is equal to $Y_{N,K}(1)$, and hence,

$$Y_{N,K}(1) \overset{\mathrm{d}}{=} T_1.$$

The final result now follows by self-similarity.

□

Corollary 5.2.1. *The process*

$$V_{N,K}(t) := Y_{N,K}(t) - \frac{N}{2K} t^{2/K}, \quad t \geq 0$$

is a PCOC, and an associated martingale is

$$M_{N,K}(t) := T_{t^{1/K}}(R_{K,\alpha_N}) - \frac{N}{2K} t^{2/K}, \quad t \geq 0,$$

which is a centered $(2/K)$*-Sato process.*

Finally, we have proven, in particular, that for any $\rho > -2$ and any $N > 0$, the random variable

$$\int_0^1 s^\rho R_N^2(s)\,\mathrm{d}s$$

is self-decomposable. This result will be generalized and made precise in the next section, using completely different arguments.

6 About the random variables $\int R_N^2(s)\,\mathrm{d}\mu(s)$

In this section, we consider a fixed measure μ on $\mathbb{R}_+^* = (0, \infty)$ such that

$$\int_{\mathbb{R}_+^*} s\,\mathrm{d}\mu(s) < \infty.$$

6.1 Spectral study of an operator

We associate with μ an operator $K^{(\mu)}$ on $E = L^2(\mu)$ defined by

$$\forall f \in E \quad K^{(\mu)} f(t) = \int_{\mathbb{R}_+^*} f(s)(t \wedge s)\,\mathrm{d}\mu(s)$$

where $\wedge$ denotes the infimum.

Lemma 6.1. *The operator $K^{(\mu)}$ is a nonnegative symmetric Hilbert-Schmidt operator.*

Proof

As a consequence of the obvious inequality:

$$(t \wedge s)^2 \leq t\,s,$$

we get

$$\int\int_{(\mathbb{R}_+^*)^2} (t \wedge s)^2\,\mathrm{d}\mu(t)\,\mathrm{d}\mu(s) \leq \left(\int_{\mathbb{R}_+^*} s\,\mathrm{d}\mu(s)\right)^2,$$

and therefore $K^{(\mu)}$ is a Hilbert-Schmidt operator.
On the other hand, denoting by $(\bullet, \bullet)_E$ the scalar product in E, we have:

$$(K^{(\mu)} f, g)_E = \mathbb{E}\left[\int f(t) B_t\,\mathrm{d}\mu(t) \int g(s) B_s\,\mathrm{d}\mu(s)\right]$$

where B is a standard Brownian motion starting from 0. This entails that $K^{(\mu)}$ is nonnegative symmetric.

□

Lemma 6.2. *Let $\lambda \in \mathbb{R}$. Then λ is an eigenvalue of $K^{(\mu)}$ if and only if $\lambda > 0$ and there exists $f \in L^2(\mu)$, $f \neq 0$, such that:*

i)

$$\lambda f'' + f \cdot \mu = 0 \quad \text{in the distribution sense on } \mathbb{R}_+^* \tag{5}$$

ii) f admits a representative which is absolutely continuous on $\mathbb{R}_+$, f' admits a representative which is right-continuous on $\mathbb{R}_+^*$;
(In the sequel, f and f' respectively always denote such representatives.)

iii)

$$f(0) = 0 \quad \text{and} \quad \lim_{t\to\infty} f'(t) = 0.$$

Proof

Let $f \in L^2(\mu)$ and $g = K^{(\mu)} f$. We have, for μ-a.e. $t > 0$,

$$g(t) = \int_0^t \mathrm{d}u \int_{(u,\infty)} f(s)\, \mathrm{d}\mu(s). \tag{6}$$

Thus g admits a representative (still denoted by g) which is absolutely continuous on $\mathbb{R}_+$ and $g(0) = 0$. Moreover, g' admits a representative which is right-continuous on $\mathbb{R}_+^*$ and is given by:

$$g'(t) = \int_{(t,\infty)} f(s)\, \mathrm{d}\mu(s). \tag{7}$$

In particular

$$|g'(t)| \le t^{-1/2} \left[\int_{(t,\infty)} f^2(s)\, \mathrm{d}\mu(s) \int_{(t,\infty)} u\, \mathrm{d}\mu(u) \right]^{1/2}. \tag{8}$$

Hence:

$$\lim_{t\to\infty} g'(t) = 0.$$

Besides, (7) entails:

$$g'' + f \cdot \mu = 0 \quad \text{in the distribution sense on } \mathbb{R}_+^*.$$

Consequently, 0 is not an eigenvalue of $K^{(\mu)}$ and the “only if” part is proven.

Conversely, let $f \in L^2(\mu)$, $f \neq 0$, and $\lambda > 0$ such that properties i),ii),iii) hold. Then

$$\lambda f'(t) = \int_{(t,\infty)} f(s)\, \mathrm{d}\mu(s).$$

Hence

$$\lambda f(t) = \int_0^t \mathrm{d}u \int_{(u,\infty)} f(s)\, \mathrm{d}\mu(s) = K^{(\mu)} f(t),$$

which proves the “if” part.

□

We note that, since 0 is not an eigenvalue of $K^{(\mu)}$, $K^{(\mu)}$ is actually a *positive* symmetric operator. On the other hand, by the previous proof, the functions $f \in L^2(\mu)$, $f \neq 0$, satisfying properties i),ii),iii) in the statement of Lemma 6.2, are the eigenfunctions of the operator $K^{(\mu)}$ corresponding to the eigenvalue $\lambda > 0$.

Lemma 6.3. *Let f be an eigenfunction of $K^{(\mu)}$. Then,*

$$|f(t)| = o(t^{1/2}) \quad \textit{and} \quad |f'(t)| = o(t^{-1/2})$$

when t tends to ∞.

Proof

This is a direct consequence of (8).

□

Lemma 6.4. *Let f_1 and f_2 be eigenfunctions of $K^{(\mu)}$ with respect to the same eigenvalue. Then,*

$$\forall t > 0 \quad f_1'(t) f_2(t) - f_1(t) f_2'(t) = 0.$$

Proof

By (5),

$$(f_1' f_2 - f_1 f_2')' = 0 \quad \text{in the sense of distributions on } \mathbb{R}_+^*.$$

By right-continuity, there exists $C \in \mathbb{R}$ such that

$$\forall t > 0 \quad f_1'(t) f_2(t) - f_1(t) f_2'(t) = C.$$

Letting t tend to ∞, we deduce from Lemma 6.3 that $C = 0$.

□

Lemma 6.5. *Let f be a solution of (5) with $\lambda > 0$, and let $a > 0$. We assume as previously that f (resp. f') denotes the representative which is absolutely continuous (resp. right-continuous) on $\mathbb{R}_+^*$. If $f(a) = f'(a) = 0$, then, for any $t \geq a$, $f(t) = 0$.*

Proof

This lemma is quite classical if the measure μ admits a continuous density with respect to the Lebesgue measure (see, for instance, [4]). The proof may be easily adapted to this more general case.

□

We are now able to state the main result of this section.

Theorem 6.6. *The operator $K^{(\mu)}$ is a positive symmetric compact operator whose all eigenvalues are simple, i.e. the dimension of each eigenspace is 1.*

Proof

It only remains to prove that the eigenvalues are simple. For this purpose, let $\lambda > 0$ be an eigenvalue and let f_1 and f_2 be eigenfunctions with respect to this eigenvalue. Let $a > 0$ with $\mu(\{a\}) = 0$. By Lemma 6.4,

$$f_1'(a) f_2(a) - f_1(a) f_2'(a) = 0.$$

Hence, there exist c_1 and c_2 with $c_1^2 + c_2^2 > 0$ such that, setting $f = c_1 f_1 + c_2 f_2$, we have

$$f(a) = f'(a) = 0.$$

By Lemma 6.5, $f(t) = 0$ for any $t \geq a$. But, since $\mu(\{a\}) = 0$, f' is also left-continuous at a. Then, we may reason on $(0, a]$ as on $[a, \infty)$ and therefore we also have $f(t) = 0$ for $0 < t \leq a$. Finally,

$$c_1 f_1 + c_2 f_2 = 0,$$

which proves the result.

$\square$

In the following, we denote by $\lambda_1 > \lambda_2 > \cdots$ the decreasing (possibly finite) sequence of the eigenvalues of $K^{(\mu)}$. Of course, this sequence depends on μ, which we omit in the notation. The following corollary plays an essential role in the sequel.

Corollary 6.6.1. *There exists a Hilbert basis $(f_n)_{n\geq 1}$ in $L^2(\mu)$ such that*

$$\forall n \geq 1 \quad K^{(\mu)} f_n = \lambda_n f_n.$$

Since $K^{(\mu)}$ is Hilbert-Schmidt,

$$\sum_{n\geq 1} \lambda_n^2 < \infty.$$

It will be shown in Subsection 6.3 (see Theorem 6.7) that actually

$$\sum_{n\geq 1} \lambda_n < \infty,$$

i.e. $K^{(\mu)}$ is trace-class.

6.2 Examples

In this subsection, we consider two particular types of measures μ.

6.2.1 $\mu = \sum_{j=1}^{n} a_j \delta_{t_j}$

Let $a_1, \cdots, a_n$ positive real numbers and $0 < t_1 < \cdots < t_n$. We denote by δ_t the Dirac measure at t and we consider, in this paragraph,

$$\mu = \sum_{j=1}^{n} a_j \delta_{t_j}.$$

By the previous study, the sequence of eigenvalues of $K^{(\mu)}$ is finite if and only if the space $L^2(\mu)$ is finite dimensional, that is if μ is of the above form. In this case, the eigenvalues of $K^{(\mu)}$ are the eigenvalues of the matrix $(m_{i,j})_{1\le i,j\le n}$ with

$$m_{i,j} = \sqrt{a_i\, a_j}\, t_{i\wedge j}.$$

In particular, by the previous study, such a matrix has n distinct eigenvalues, which are > 0.

6.2.2 $\mu = C\, t^\rho\, 1_{(0,1]}(t)\,\mathrm{d}t$

In this paragraph, we consider

$$\mu = C\, t^\rho\, 1_{(0,1]}(t)\,\mathrm{d}t$$

with $C > 0$ and $\rho > -2$. By Lemma 6.2, the eigenfunctions f of $K^{(\mu)}$ associated with $\lambda > 0$ are characterized by:

$$\begin{aligned} \lambda f''(x) + C\, x^\rho\, f(x) = 0 \quad \text{on} \quad (0,1), \\ f(0) = 0, \quad f'(1) = 0. \end{aligned} \tag{9}$$

We set $\sigma = (\rho+2)^{-1}$ and $\nu = \sigma - 1$. For $a > -1$, we recall the definition of the Bessel function J_a:

$$J_a(x) = \sum_{k=0}^{\infty} \frac{(-1)^k\,(x/2)^{a+2k}}{k!\,\Gamma(a+k+1)}.$$

Then, the only function f satisfying (9) and $f(0) = 0$ is, up to a multiplicative constant,

$$f(x) = x^{1/2}\, J_\sigma\left(2\sigma\sqrt{\frac{C}{\lambda}}\, x^{1/2\sigma}\right).$$

We deduce from the equality, which is valid for $a > 1$,

$$a\, J_a(x) + x\, J_a'(x) = x\, J_{a-1}(x)$$

that $f'(1) = 0$ if and only if

$$J_\nu\left(2\sigma\sqrt{\frac{C}{\lambda}}\right) = 0.$$

Denote by $(j_{\nu,k},\ k \ge 1)$ the sequence of the positive zeros of J_ν. Then the sequence $(\lambda_k,\ k \ge 1)$ of eigenvalues of $K^{(\mu)}$ is given by:

$$2\sigma\sqrt{\frac{C}{\lambda_k}} = j_{\nu,k}, \quad k \ge 1$$

or, since $\sigma = \nu + 1$,

$$\lambda_k = 4\, C\, (\nu+1)^2\, j_{\nu,k}^{-2}, \quad k \ge 1.$$

Particular case Suppose $\rho = 0$. Then $\nu = -1/2$ and

$$J_\nu(x) = J_{-1/2}(x) = \left(\frac{2}{\pi\, x}\right)^{1/2} \cos(x).$$

Hence,

$$\lambda_k = 4\, C\, \pi^{-2}\, (2k-1)^{-2}, \quad k \ge 1.$$

6.3 Representation of $\int B_s^2 \, \mathrm{d}\mu(s)$

We again consider the general setting defined in Subsection 6.1, the notation of which we keep. In this subsection, we study the random variable

$$Y_1^{(\mu)} := \int B_s^2 \, \mathrm{d}\mu(s).$$

The use of the operator $K^{(\mu)}$ and of its spectral decomposition in the type of study we develop below, is called the Karhunen-Loeve decomposition method. It has a long history which goes back at least to Kac-Siegert [12; 13]. We also refer to the recent paper [5] and to the references therein.

Theorem 6.7. *The eigenvalues* $(\lambda_k,\ k \geq 1)$ *of the operator* $K^{(\mu)}$ *satisfy*

$$\sum_{k\geq 1} \lambda_k = \int_{\mathbb{R}_+^*} t \, \mathrm{d}\mu(t) \, (< \infty, \text{ by hypothesis}).$$

Moreover, there exists a sequence $(\Gamma_n,\ n \geq 1)$ *of independent normal variables such that:*

$$Y_1^{(\mu)} \stackrel{\mathrm{d}}{=} \sum_{n\geq 1} \lambda_n \Gamma_n^2.$$

Proof

We deduce from Corollary 6.6.1, by the Bessel-Parseval equality, that:

$$Y_1^{(\mu)} = \sum_{n\geq 1} \left(\int B_s f_n(s) \, \mathrm{d}\mu(s) \right)^2 \quad \text{a.s.}$$

Taking expectations, we get

$$\int_{\mathbb{R}_+^*} t \, \mathrm{d}\mu(t) = \sum_{n\geq 1} (K^{(\mu)} f_n, f_n)_E = \sum_{n\geq 1} \lambda_n.$$

We set, for $n \geq 1$,

$$\Gamma_n = \frac{1}{\sqrt{\lambda_n}} \int B_s f_n(s) \, \mathrm{d}\mu(s).$$

Then $(\Gamma_n,\ n \geq 1)$ is a Gaussian sequence and

$$\mathbb{E}[\Gamma_n \Gamma_m] = \frac{1}{\sqrt{\lambda_n \lambda_m}} (K^{(\mu)} f_n, f_m)_E = \delta_{n,m}$$

where $\delta_{n,m}$ denotes Kronecker's symbol. Hence, the result follows.

□

Corollary 6.7.1. *The Laplace transform of $Y_1^{(\mu)}$ is*

$$F_1^{(\mu)}(t) = \prod_{n\geq 1}(1+2\,t\,\lambda_n)^{-1/2}.$$

Proof

This is a direct consequence of the previous theorem, taking into account that, if Γ is a normal variable, then

$$\Gamma^2 \overset{\mathrm{d}}{=} 2\,\gamma_{1/2}.$$

$\square$

6.4 Representation of $\int R_N^2(s)\,\mathrm{d}\mu(s)$

We now consider the random variable

$$Y_N^{(\mu)} := \int R_N^2(s)\,\mathrm{d}\mu(s).$$

A number of explicit computations of the Laplace transforms of these variables are found in Pitman-Yor ([21, Section 2]).

Theorem 6.8. *There exists a sequence $(\Theta_{N,n},\ n \geq 1)$ of independent variables with, for any $n \geq 1$,*

$$\Theta_{N,n} \overset{\mathrm{d}}{=} R_N^2(1) \overset{\mathrm{d}}{=} 2\,\gamma_{N/2}$$

such that

$$Y_N^{(\mu)} \overset{\mathrm{d}}{=} \sum_{n\geq 1} \lambda_n\,\Theta_{N,n}. \tag{10}$$

Moreover, the Laplace transform of $Y_N^{(\mu)}$ is

$$F_N^{(\mu)}(t) = \prod_{n\geq 1}(1+2\,t\,\lambda_n)^{-N/2}. \tag{11}$$

Proof

It is clear, for instance from Revuz-Yor [22, Chapter XI, Theorem 1.7], that

$$F_N^{(\mu)}(t) = [F_1^{(\mu)}(t)]^N.$$

Therefore, by Corollary 6.7.1, formula (11) holds. Formula (10) then follows directly by the injectivity of the Laplace transform.

$\square$

Corollary 6.8.1. *The random variable $Y_N^{(\mu)}$ is self-decomposable. The function h, which is decreasing on $(0,\infty)$ and associated with $Y_N^{(\mu)}$ in Theorem 2.1, is*

$$h(x) = \frac{N}{2}\sum_{n\geq 1} \exp\left(-\frac{1}{2\lambda_n}\,x\right).$$

Proof

We saw in Subsection 3.1 that $R_N^2(1)$ satisfies the property 1) in Theorem 2.1 with

$$h(x) = \frac{N}{2}\,1_{(0,\infty)}(x)\,\mathrm{e}^{-x/2}$$

and it is therefore self-decomposable. Using then the representation (10) of $Y_N^{(\mu)}$, we obtain the desired result.

□

As a consequence, following Bondesson [1], we see that $Y_N^{(\mu)}$ is a generalized gamma convolution (GGC) whose Thorin measure is the discrete measure:

$$\frac{N}{2}\sum_{n\geq 1}\delta_{1/2\lambda_n}.$$

Particular case We consider here, as in Section 5, the particular case:

$$\mu = \frac{1}{K^2}\,t^{\frac{2(1-K)}{K}}\,1_{(0,1]}(t)\,\mathrm{d}t.$$

Then, $Y_N^{(\mu)}$ is the random variable $Y_{N,K}(1)$ studied in Section 5. As a consequence of Paragraph 6.2.2 with

$$C = \frac{1}{K^2} \quad \text{and} \quad \rho = \frac{2}{K} - 2,$$

we have

$$\lambda_k = j_{\nu,k}^{-2}, \quad k \geq 1$$

with $\nu = \frac{K}{2} - 1$. Moreover, by Theorem 5.2,

$$Y_2^{(\mu)} \overset{\mathrm{d}}{=} T_1(R_K).$$

It is known (see Kent [15] and, for instance, Borodin-Salminen [2, formula 2.0.1, p. 387]) that

$$\mathbb{E}[\exp(-t\,T_1(R_K))] = \frac{2^{-\nu}}{\Gamma(\nu+1)}\,\frac{(\sqrt{2t})^{\nu}}{I_\nu(\sqrt{2t})}$$

where I_ν denotes the modified Bessel function:

$$I_\nu(x) = \sum_{k=0}^{\infty}\frac{(x/2)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)}.$$

We set:

$$\widehat{I}_\nu(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k}}{k!\,\Gamma(\nu+k+1)}.$$

Therefore, by formula (11) in the case $N=2$, we recover the following representation:

$$\widehat{I}_\nu(x) = \frac{1}{\Gamma(\nu+1)} \prod_{k\geq 1} \left(1+\frac{x^2}{j_{\nu,k}^2}\right).$$

In particular ($\nu = -1/2$),

$$\cosh(x) = \prod_{k\geq 1} \left(1+\frac{4\,x^2}{\pi^2\,(2k-1)^2}\right).$$

Likewise we obtain, for $\nu = 1/2$,

$$\frac{\sinh(x)}{x} = \prod_{k\geq 1} \left(1+\frac{x^2}{\pi^2\,k^2}\right).$$

6.5 Sato process associated to $Y_N^{(\mu)}$

Theorem 6.9. *Let (U_t^N) be the 1-Sato process associated to $R_N^2(1)$ (cf. Section 3). Then, the 1-Sato process associated to $Y_N^{(\mu)}$ is $(U_t^{(N,\mu)})$ defined by:*

$$U_t^{(N,\mu)} = \sum_{n\geq 1} \lambda_n\, U_t^{N,n}, \quad t \geq 0$$

where $((U_t^{N,n}),\ n\geq 1)$ denotes a sequence of independent processes such that, for $n \geq 1$,

$$(U_t^{N,n}) \overset{(\mathrm{d})}{=} (U_t^N).$$

Proof

This is a direct consequence of Theorem 6.8.

□

Corollary 6.9.1. *The process*

$$V_t^{(N,\mu)} := \int_{\mathbb{R}_+^*} (R_N^2(t\,s) - N\,t\,s)\,\mathrm{d}\mu(s), \quad t \geq 0$$

is a PCOC and an associated martingale is

$$M_t^{(N,\mu)} := U_t^{(N,\mu)} - N\,t \int_{\mathbb{R}_+^*} s\,\mathrm{d}\mu(s), \quad t \geq 0.$$

The above martingale $(M_t^{(N,\mu)})$ is purely discontinuous. We also may associate to the PCOC $(V_t^{(N,\mu)})$ a continuous martingale, as we now state.

Theorem 6.10. *A continuous martingale associated to the PCOC $(V_t^{(N,\mu)})$ is*

$$\sum_{n\geq 1} \lambda_n \left((R_N^{(n)})^2(t) - Nt\right), \quad t \geq 0$$

where $((R_N^{(n)}(t)),\ n \geq 1)$ denotes a sequence of independent processes such that, for $n \geq 1$,

$$(R_N^{(n)}(t)) \overset{(\mathrm{d})}{=} (R_N(t)).$$

Proof

This is again a direct consequence of Theorem 6.8.

□

We can also explicit the relation between $U^{(N,\mu)}$ and $U^{(N',\mu)}$. Let $C^{(N,\mu)}$ (resp. $C^{(N',\mu)}$) be the Lévy process associated with $Y_N^{(\mu)}$ (resp. $Y_{N'}^{(\mu)}$). We see, by Laplace transform, that

$$(C_s^{(N',\mu)},\ s \geq 0) \overset{(\mathrm{d})}{=} (C_{N's/N}^{(N,\mu)},\ s \geq 0). \tag{12}$$

Then, using the relations between the processes U and C given in Theorem 2.2, we obtain:

Proposition 6.11. *We have:*

$$(U_t^{(N',\mu)},\ t \geq 0) \overset{(\mathrm{d})}{=} \left(\int_0^{t^{N'/N}} s^{\frac{N-N'}{N'}}\, \mathrm{d}U_s^{(N,\mu)},\ t \geq 0\right).$$

Proof

By Theorem 2.2,

$$(U_t^{(N',\mu)},\ 0 \leq t \leq 1) \overset{(\mathrm{d})}{=} \left(\int_{-\log t}^{\infty} \mathrm{e}^{-s}\, \mathrm{d}C_s^{(N',\mu)},\ 0 \leq t \leq 1\right).$$

Therefore, we obtain by (12),

$$(U_t^{(N',\mu)},\ 0 \leq t \leq 1) \overset{(\mathrm{d})}{=} \left(\int_{-\log t}^{\infty} \mathrm{e}^{-s}\, \mathrm{d}C_{N's/N}^{(N,\mu)},\ 0 \leq t \leq 1\right)$$

or, after the change of variable: $N's/N = u$,

$$(U_t^{(N',\mu)},\ 0 \leq t \leq 1) \overset{(\mathrm{d})}{=} \left(\int_{-\log t^{N'/N}}^{\infty} \mathrm{e}^{-Nu/N'}\, \mathrm{d}C_u^{(N,\mu)},\ 0 \leq t \leq 1\right).$$

By Theorem 2.2,

$$C_u^{(N,\mu)} = \int_{\mathrm{e}^{-u}}^{1} \frac{1}{r}\, \mathrm{d}U_r^{(N,\mu)} = \int_0^u \mathrm{e}^{v}\, \mathrm{d}U_{\mathrm{e}^{-v}}^{(N,\mu)}.$$

Therefore,

$$(U_t^{(N',\mu)},\ 0 \le t \le 1) \stackrel{(\mathrm{d})}{=} \left(\int_{-\log t^{N'/N}}^{\infty} \mathrm{e}^{\frac{N'-N}{N'}u}\, \mathrm{d}U_{\mathrm{e}^{-u}}^{(N,\mu)},\ 0 \le t \le 1 \right)$$

and, after the change of variable: $\mathrm{e}^{-u} = s$,

$$(U_t^{(N',\mu)},\ 0 \le t \le 1) \stackrel{(\mathrm{d})}{=} \left(\int_0^{t^{N'/N}} s^{\frac{N-N'}{N'}}\, \mathrm{d}U_s^{(N,\mu)},\ 0 \le t \le 1 \right).$$

By a similar computation for $t \ge 1$, we finally obtain the desired result.

□

Corollary 6.11.1. *For $N > 0$ and $K > 0$, we set, with the notation of Section 5,*

$$T_t^{N,K} = T_t(R_{K,\alpha_N}), \quad t \ge 0.$$

Then, for $N > 0$, $N' > 0$ and $K > 0$, for any $t \ge 0$,

$$T_t^{N',K} \stackrel{\mathrm{d}}{=} \int_0^{t^{N'/N}} s^{2\,\frac{N-N'}{N'}}\, \mathrm{d}T_s^{N,K}.$$

Proof

By Theorem 5.2, $(T_{t^{1/2}}^{N,K})$ is the 1-Sato process associated with $Y_N^{(\mu)}$ defined from

$$\mu = \frac{1}{K^2}\, t^{\frac{2(1-K)}{K}}\, 1_{(0,1]}(t)\, \mathrm{d}t.$$

□

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