DOI: 10.1214/EJP.v15-746

Electronic Journal of Probability

Vol. 15 (2010), Paper no. 9, pages 241–258.

Journal URL
http://www.math.washington.edu/~ejpecp/

Multidimensional Multifractal Random Measures

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Abstract

We construct and study space homogeneous and isotropic random measures (MMRM) which generalize the so-called MRM measures constructed in [1]. Our measures satisfy an exact scale invariance equation (see equation (1) below) and are therefore natural models in dimension 3 for the dissipation measure in a turbulent flow.

Key words: Random measures, Multifractal processes.

AMS 2000 Subject Classification: Primary 60G57, 28A78, 28A80.

Submitted to EJP on July 26, 2009, final version accepted January 1, 2010.

1 Introduction

The purpose of this paper is to introduce a natural multidimensional generalization (MMRM) of the one dimensional multifractal random measures (MRM) introduced by Bacry and Muzy in [1]. The measures M we introduce are different from zero, homogeneous in space, isotropic and satisfy the following exact scale invariance relation: if T denotes some cutoff parameter we will define below then the following equality in distribution holds for all $\lambda < 1$:

$$(M(\lambda A))_{A\subset B(0,T)} \overset{(law)}{=} \lambda^d e^{\Omega_\lambda}(M(A))_{A\subset B(0,T)}, \tag{1}$$

where $B(0,T)$ is the euclidian ball of radius T in $\mathbb{R}^d$ and Ω_λ is an infinitely divisible random variable independent of $(M(A))_{A\subset B(0,T)}$. In fact, if ν_t denotes the law of $\Omega_{e^{-t}}$ for $t \geq 0$, then it is straightforward to prove that the laws $(\nu_t)_{t\geq 0}$ satisfy the following convolution property:

$$\nu_{t+t'} = \nu_t * \nu_{t'} \tag{2}$$

Therefore, one can find a Levy process $(L_t)_{t\geq 0}$ such that, for each t, ν_t is the law of L_t.

Reciprocaly, consider a cutoff parameter T and a family of laws $(\nu_t)_{t\geq 0}$ satisfying (2), the normalisation condition $\int e^x \nu_t(dx) = 1$ for all t and a technical assumption detailed in section 2. We can construct a measure M different from zero, homogeneous in space, isotropic and which satisfies the exact scale invariance relation (1) for all $\lambda < 1$ with Ω_λ of law $\nu_{\ln(1/\lambda)}$.

Let us note that a multi-dimensional generalization of MRM has already been proposed in the litterature ([5]). However, this generalization is not exactly scale invariant. Let us stress that to our knowledge equation (1) has never been studied mathematically. In dimension 1, MRM are non trivial (i.e. different from 0) homogeneous solutions to (1) for all $\lambda < 1$. In dimension $d \geq 2$, MMRM are non trivial isotropic and homogeneous solutions to (1) for all $\lambda < 1$. If we consider a non negative random variable Y independent from M than the random measure $(YM(A))_{A\subset\mathbb{R}^d}$ is also solution to (1) for all $\lambda < 1$. This leads to the following open problem (unicity):

Open problem 1. *Let $(\nu_t)_{t\geq 0}$ be a fixed family of probability measures which satisfy the convolution property (2), the normalisation condition $\int e^x \nu_t(dx) = 1$ for all t and the non degeneracy condition stated in proposition 2.9.1 (if ψ is the Laplace transform of ν_1, the condition is the existence of $\epsilon > 0$ such that $\psi(1+\epsilon) < d\epsilon$). Consider two non trivial (i.e. different from 0) homogeneous and isotropic random Radon measures M and M' on the Borel σ-algebra $\mathscr{B}(\mathbb{R}^d)$. We suppose that there exists some cutoff parameter T such that M and M' satisfy (1) for all $\lambda < 1$ where the law of Ω_λ is given by $\nu_{\ln(1/\lambda)}$. Can one find (in a possibly extended probability space) a non negative and non trivial (i.e. different from 0) random variable Y (Y') independent of M (M') such that the following equality in law holds:*

$$(YM(A))_{A\subset B(0,T)} \overset{(law)}{=} (Y'M'(A))_{A\subset B(0,T)}$$

The above open problem can be solved in the simple case where the family $(\nu_t)_{t\geq 0}$ is given by $\nu_t = \delta_0$ for all $t \geq 0$ (deterministic scale invariance) and where there is no cutoff parameter. This is the content of the following lemma:

Lemma 1.1. *Let M be a homogeneous random Radon measure on the Borel σ-algebra $\mathscr{B}(\mathbb{R}^d)$ such that there exists $\alpha > 0$ with $\mathbb{E}[M(B(0,1))^\alpha] < \infty$. We suppose that the following equality in distribution holds for all $\lambda < 1$:*

$$(M(\lambda A))_{A\subset\mathbb{R}^d} \overset{(law)}{=} \lambda^d (M(A))_{A\subset\mathbb{R}^d}.$$

Then there exists some (random) constant $C \geq 0$ such that $M = C\mathscr{L}$ a.s. where $\mathscr{L}$ is the Lebesgue measure.

The proof of the above lemma is to be found in section 4.1.

The study of equation (1) is justified on mathematical and physical grounds. Before reviewing in some detail the physical motivation for constructing the MRMM measures and studying equation (1), let us just mention that equation (1) has recently been used in the probabilistic derivation of the KPZ (Knizhnik-Polyakov-Zamolodchikov) equation introduced initially in [12] (see [6], [16]).

1.2 MRMM in dimension 3: a model for the energy dissipation in a turbulent flow

Equations similar to (1) were first proposed in [3] in the context of fully developed turbulence. In [3], the authors conjectured that the following relation should hold between the increments of the longitudinal velocity δv at two scales $l, l' < T$, where T is an integral scale characteristic of the turbulent flow: if $P_l(\delta v)$, $P_{l'}(\delta v)$ denote the probability density functions (p.d.f.) of the longitudinal velocity difference between two points separated by a distance l, l', one has:

$$P_l(\delta v) = \int_{\mathbb{R}} G_{l,l'}(x) P_{l'}(e^{-x}\delta v) e^{-x} dx, \tag{3}$$

where $G_{l,l'}$ is a p.d.f. If one makes the assumption that the $G_{l,l'}$ depend only on the factor $\lambda = l/l'$ (scale invariance), then it is easy to show that the $G_{l,l'}$ are the p.d.f.'s of infinitely divisible laws. Under the assumption of scale invariance, if we take $d = 1$, $A = [0, l']$, $\lambda = l/l'$ and $\Omega_\lambda - d\ln(\lambda)$ of p.d.f. $G_{l,l'}$ in relation (1) (valid in fact simultaneously for all A) then equation (3) is the p.d.f. equivalent to (1).

Following the standard conventions in turbulence, we note ϵ the (random) energy dissipation measure per unit mass and ϵ_l the mean energy dissipation per unit mass in a ball $B(0, l)$:

$$\epsilon_l = \frac{3}{4\pi l^3} \epsilon(B(0, l))$$

We believe the measures we consider in this paper can be used in dimension 3 to model the energy dissipation ϵ in a turbulent flow. Indeed, it is believed that the velocity field of a stationary (in time) turbulent flow at scales smaller than some integral scale T (characteristic of the turbulent flow) is homogeneous in space and isotropic (see [7]). Therefore, the measure ϵ is homogeneous and isotropic (as a function of the velocity field) and according to Kolmogorov's refined similarity hypothesis (see for instance [4], [19] for studies of this hypothesis), one has the following relation in law between the longitudinal velocity increment δv_l of two points separated by a distance l and ϵ_l:

$$\delta v_l \overset{(law)}{=} U(l\epsilon_l)^{1/3},$$

where U is a universal negatively skewed random variable independent of ϵ_l and of law independent of l. Therefore, equations similar to (3) should also hold for the p.d.f. of ϵ_l. Finally, let us note that the statistics of the velocity field and thus also those of ϵ are believed to be universal at scales smaller than T in the sense that they only depend on the average mean energy dissipation per unit mass $< \epsilon >$ defined by (note that the quantity below does not depend on l by homogenity):

$$< \epsilon > = < \epsilon_l >$$

where <> denotes an average with respect to the randomness. In particular, the law of $\frac{\epsilon}{<\epsilon>}$ and the p.d.f.'s $G_{l,l'}$ are completely universal, i.e. are the same for all flows; nevertheless, there is still a big debate in the phyics community on the exact form of the p.d.f.'s $G_{l,l'}$. In dimension 3, the measures M we consider are precisely models for $\frac{\epsilon}{<\epsilon>}$.

The rest of this paper is organized as follows: in section 2, we set the notations and give the main results. In section 3, we give a few remarks concerning the important lognormal case. In section 4, we gather the proofs of the main theorems of section 2.

2 Notations and main Results

2.1 Independently scattered infinitely divisible random measure.

The characteristic function of an infinitely divisible random variable X can be written as $\mathbb{E}[e^{iqX}] = e^{\varphi(q)}$, where φ is characterized by the Lévy-Khintchine formula

$$\varphi(q) = imq - \frac{1}{2}\sigma^2 q^2 + \int_{\mathbb{R}^*} (e^{iqx} - 1 - iq\sin(x))\,\nu(dx)$$

and $\nu(dx)$ is the so-called Lévy measure. It satisfies $\int_{\mathbb{R}^*} \min(1, x^2)\,\nu(dx) < +\infty$.

Let G be the unitary group of $\mathbb{R}^d$, that is

$$G = \{M \in M_d(\mathbb{R}); MM^t = \mathrm{I}\}.$$

Since G is a compact separable topological group, we can consider the unique right translation invariant Haar measure H with mass 1 defined on the Borel σ-algebra $\mathscr{B}(G)$. Let S be the half-space

$$S = \{(t, y); t \in \mathbb{R}, y \in \mathbb{R}_+^*\}$$

with which we associate the measure (on the Borel σ-algebra $\mathscr{B}(S)$)

$$\theta(dt, dy) = y^{-2} dt\,dy.$$

Given φ, following [1], we consider an independently scattered infinitely divisible random measure μ associated to $(\varphi, H \otimes \theta)$ and distributed on $G \times S$ (see [14]). More precisely, μ satisfies:

1) For every sequence of disjoint sets $(A_n)_n$ in $\mathscr{B}(G \times S)$, the random variables $(\mu(A_n))_n$ are independent and

$$\mu\Big(\bigcup_n A_n\Big) = \sum_n \mu(A_n) \text{ a.s.,}$$

2) for any measurable set A in $\mathscr{B}(G \times S)$, $\mu(A)$ is an infinitely divisible random variable whose characteristic function is

$$\mathbb{E}(e^{iq\mu(A)}) = e^{\varphi(q) H \otimes \theta(A)}.$$

We stress the fact that μ is not necessarily almost surely a signed measure (undoubtedly, the term random measure is misleading). More precisely, it is not always the case that one can consider a version $\tilde{\mu}$ of μ (i.e. for all A in $\mathscr{B}(G \times S)$, $\tilde{\mu}(A) = \mu(A)$ a.s.) such that almost surely $A \to \tilde{\mu}(A)$ is a signed measure. In other words , it is not always the case that μ (or a version of μ) satisfies the following strong version 1)' of 1):

1)' Almost surely, for every sequence of disjoint sets $(A_n)_n$ in $\mathscr{B}(G \times S)$, the random variables $(\mu(A_n))_n$ are independent and

$$\mu(\bigcup_n A_n) = \sum_n \mu(A_n).$$

Let us additionnally mention that there exists a convex function ψ defined on $\mathbb{R}$ such that for all non empty subset A of $G \times S$:

1. $\psi(q) = +\infty$, if $\mathbb{E}(e^{q\mu(A)}) = +\infty$,
2. $\mathbb{E}(e^{q\mu(A)}) = e^{\psi(q)H\otimes\theta(A)}$ otherwise.

Let q_c be defined as $q_c = \sup\{q \geq 0; \psi(q) < +\infty\}$. For any $q \in [0, q_c[$, $\psi(q) < +\infty$ and $\psi(q) = \varphi(-iq)$.

2.2 Multidimensional Multifractal Random Measures (MMRM).

We further assume that the independently scattered infinitely divisible random measure μ associated to $(\varphi, H \otimes \theta)$ satisfies:

$$\psi(2) < +\infty,$$

and $\psi(1) = 0$. The condition on $\psi(1)$ is just a normalisation condition. The condition $\psi(2) < +\infty$ is technical and can probably be relaxed to the condition used in [1]: there exists $\epsilon > 0$ such that $\psi(1+\epsilon) < +\infty$. However, in the multidimensional setting, the situation is more complicated because there is no strict decorrelation property similar to the one dimensional setting: in dimension $d \geq 2$, there does not exist some distance R such that $M(A)$ and $M(B)$ are independent for two Lebesgue measurable sets $A, B \subset \mathbb{R}^d$ separated by a distance of at least R (see also lemma 3.2 below). In [1], the proof of the non-triviality of the MRM deeply relies on a decomposition of the unit interval into independent blocks, which is only possible because such an independent scale R exists in dimension 1. Nevertheless, the condition $\psi(2) < +\infty$ is enough general to cover the cases considered in turbulence: see [3], [17], [18].

Definition 2.3. Filtration $\mathscr{F}_l$. *Let Ω be the probability space on which μ is defined. $\mathscr{F}_l$ is defined as the σ-algebra generated by $\{\mu(A \times B); A \subset G, B \subset S, \mathrm{dist}(B, \mathbb{R}^2 \setminus S) \geq l\}$.*

Given T, let us now define the function $f : \mathbb{R}_+ \to \mathbb{R}$ by

$$f(l) = \begin{cases} l, & \text{if } l \leq T \\ T & \text{if } l \geq T \end{cases}$$

The cone-like subset $A_l(t)$ of S is defined by

$$A_l(t) = \{(s, y) \in S; y \geq l, -f(y)/2 \leq s - t \leq f(y)/2\}.$$

For forthcoming computations, we stress that for s, t real we have:

$$\theta(A_l(s) \cap A_l(t)) = g_l(|t - s|)$$

where $g_l : \mathbb{R}_+ \to \mathbb{R}$ is given by (with the notation $x^+ = \max(x, 0)$):

$$g_l(x) = \begin{cases} \ln(T/l) + 1 - \frac{x}{l}, & \text{if } x \le l \\ \ln^+(T/x) & \text{if } x \ge l \end{cases}$$

For any $x \in \mathbb{R}^d$ and $m \in G$, we denote by x_1^m the first coordinate of the vector mx. The cone product $C_l(x)$ is then defined as

$$C_l(x) = \{(m, t, y) \in G \times S; (t, y) \in A_l(x_1^m)\}.$$

Definition 2.4. $\omega_\mathbf{l}(\mathbf{x})$ process. *The process $\omega_l(x)$ is defined as $\omega_l(x) = \mu(C_l(x))$.*

Remark 2.5. *We make the additional assumption*

$$\int_{[-1,1]} |x| \, \nu(dx) < +\infty,$$

which ensures the process $(t, l) \in \mathbb{R} \times]0, +\infty[\mapsto \omega_l(t)$ is locally bounded. The process $l \mapsto M_l(A)$ is thus a continuous martingale for each measurable subset A. It is not clear how to drop that condition and make the whole machinery work all the same. According to the authors, that condition has been omitted in [1].

Definition 2.6. $\mathbf{M_l(dx)}$ measure. *For any $l > 0$, we define the measure $M_l(dx) = e^{\omega_l(x)} \, dx$, that is*

$$M_l(A) = \int_A e^{\omega_l(x)} \, dx$$

for any Lebesgue measurable subset $A \subset \mathbb{R}^d$.

Theorem 2.7. Multidimensional Multifractal Random Measure (MMRM).
Let φ and T be given as above and let ω_l and M_l be defined as above. With probability one, there exists a limit measure (in the sense of weak convergence of measures)

$$M(dx) = \lim_{l \to 0^+} M_l(dx).$$

This limit is called the Multidimensional Multifractal Random Measure. The scaling exponent of M is defined by

$$\forall q \ge 0, \quad \zeta(q) = dq - \psi(q).$$

Moreover:

i) $\forall x \in \mathbb{R}^d$, $M(\{x\}) = 0$

ii) for any bounded subset K of $\mathbb{R}^d$, $M(K) < +\infty$ and $E[M(K)] \le |K|$.

Proposition 2.8. Homogenity and isotropy

1. *The measure M is homogeneous in space, i.e. the law of $(M(A))_{A \subset \mathbb{R}^d}$ coincides with the law of $(M(x + A))_{A \subset \mathbb{R}^d}$ for each $x \in \mathbb{R}^d$.*

2. *The measure M is isotropic, i.e. the law of $(M(A))_{A \subset \mathbb{R}^d}$ coincide with the law of $(M(mA))_{A \subset \mathbb{R}^d}$ for each $m \in G$.*

Proposition 2.9. Main properties of the MRM.

1. *The measure M is different from 0 if and only if there exists $\epsilon > 0$ such that $\zeta(1+\epsilon) > d$; in that case, $\mathbb{E}(M(A)) = |A|$.*

2. *Let $q > 1$ and consider the unique $n \in \mathbb{N}$ such that $n < q \leq n+1$. If $\zeta(q) > d$ and $\psi(n+1) < \infty$, then $\mathbb{E}[M(A)^q] < +\infty$.*

3. *For any fixed $\lambda \in]0,1]$ and $l \leq T$, the two processes $(\omega_{\lambda l}(\lambda A))_{A\subset B(0,T)}$ and $(\Omega_\lambda + \omega_l(A))_{A\subset B(0,T)}$ have the same law, where Ω_λ is an infinitely divisible random variable independent from the process $(\omega_l(A))_{A\subset B(0,T)}$ and its law is characterized by $\mathbb{E}[e^{iq\Omega_\lambda}] = \lambda^{-\varphi(q)}$.*

4. *For any $\lambda \in]0,1]$, the law of $(M(\lambda A))_{A\subset B(0,T)}$ is equal to the law of $(W_\lambda M(A))_{A\subset B(0,T)}$, where $W_\lambda = \lambda^d e^{\Omega_\lambda}$ and Ω_λ is an infinitely divisible random variable (independent of $(M(A))_{A\subset B(0,T)}$) and its characteristic function is*
$$\mathbb{E}[e^{iq\Omega_\lambda}] = \lambda^{-\varphi(q)}.$$

5. *If $\zeta(q) \neq -\infty$ and $0 < t < T$ then*
$$\mathbb{E}\big[M(B(0,t))^q\big] = (t/T)^{\zeta(q)}\mathbb{E}\big[M(B(0,T))^q\big].$$

3 The limit lognormal case

In the gaussian case, we have $\psi(q) = \gamma^2 q^2/2$ and the condition $\zeta(1+\epsilon) > d$ corresponds to $\gamma^2 < 2d$. The approximating measures M_l is thus defined as:

$$M_l(A) = \int_A e^{\gamma X_l(x) - \frac{\gamma^2 \mathbb{E}[X_l(x)^2]}{2}}\, dx$$

where X_l is a centered gaussian field (equal to $(\omega_l - \mathbb{E}[\omega_l])/\gamma$) with correlations given by:

$$\begin{aligned}\mathbb{E}[X_l(x)X_l(y)] &= H \otimes \theta(C_l(x) \cap C_l(y))\\ &= \int_G g_l(|x_1^m - y_1^m|) H(dm).\end{aligned}$$

The limit mesures $M = \lim_{l\to 0} M_l(dx)$ we define are in the scope of the theory of gaussian multiplcative chaos developed by Kahane in [11] (see [15] for an introduction to this theory). Formally, the measure M is defined by:

$$M(A) = \int_A e^{\gamma X(x) - \frac{\gamma^2 \mathbb{E}[X(x)^2]}{2}}\, dx$$

where X is a centered gaussian field (in fact a random tempered distribution) with correlations given by:

$$\mathbb{E}[X(x)X(y)] = \int_G \ln^+(T/|x_1^m - y_1^m|) H(dm). \tag{4}$$

Let us suppose that $T=1$ for simplicity. Using invariance of the Haar measure H by multiplication, it is plain to see that $\mathbb{E}[X(x)X(y)]$ is of the form $F(|y-x|)$ where $F:\mathbb{R}_+\to\mathbb{R}_+\cup\{\infty\}$. We have the scaling relation $F(ab)=\ln(1/a)+F(b)$ if $a\leq 1$ and $b\leq 1$ (see also lemma 4.8 below) which entails that for $|x|\leq T$:

$$F(|x|)=\ln(1/|x|)-\int_G \ln(|e_1^m|)H(dm).$$

where $e=(1,0,\dots,0)$ is the first vector of the canonical basis. As a corollary, we get the existence of some constant C (take $C=-\int_G \ln(|e_1^m|)H(dm)$) such that $\ln(1/|x|)+C$ is positive definite (as a tempered distribution) in a neighborhood of 0. This easily implies that $\ln(1/|x|)$ is positive definite in a neighborhood of 0: to our knowledge, this result is new. This contrasts with the fact that $\ln^+(1/|x|)$ is positive definite in dimension $d\leq 3$ but is not positive definite for $d\geq 4$ (see [15]).

Remark 3.1. *It is easy to see that* $1-|x|^\alpha$ *is positive definite in a neighborhhod of* 0 *as a function defined in* $\mathbb{R}$ *if* $\alpha\leq 2$. *Therefore, one can consider the isotropic and positive definite function in a neighborhhood of* 0 *in* $\mathbb{R}^d$ *defined by:*

$$F(|x|)=\int_G (1-|x_1^m|^\alpha)H(dm).$$

By scaling, one can see that $F(|x|)=1-C|x|^\alpha$ *for some* $C>0$. *It is easy to see that this entails that* $1-|x|^\alpha$ *is positive definite in a neighborhood* V *of* 0. *Using the main theorem in [13], one can extend* $1-|x|^\alpha$ *(defined in* V*) in an isotropic and positive definite function defined in* $\mathbb{R}^d$. *This is in contrast with the so called Kuttner-Golubov problem which is to determine the* $\alpha,\kappa>0$ *such that* $(1-|x|^\alpha)_+^\kappa$ *is positive definite in* $\mathbb{R}^d$. *It is known (see [8] for instance) that for* $\alpha>0$ *the function* $(1-|x|^\alpha)_+$ *is not positive definite in* $\mathbb{R}^d$ *if* $d\geq 3$.

Finally, let us mention that the field X with correlations given by (4) exhibits long range correlations as soon as $d\geq 2$. More precisely, we have:

Lemma 3.2. *If* $d\geq 2$, *we get the following equivalence:*

$$\mathbb{E}[X(x)X(y)] \underset{|x-y|\to\infty}{\sim} \frac{2\Gamma(d/2)}{\Gamma(1/2)\Gamma((d-1)/2)}\sqrt{\frac{T}{|x-y|}}.$$

where Γ *is the standard gamma function:* $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt$.

4 Proofs

4.1 Proof of lemma 1.1

We will consider the case $d=1$ for simplicity (the general case is an adaptation of the one dimensional case). With no restriction, we can suppose that the α in lemma 1.1 belongs to $]0,1[$. For $s,t>0$, we have the following inequalities by using the assumptions on M and the concavity of

$x \mapsto x^\alpha$:

$$\begin{aligned}\mathbb{E}[M[0,t+s]^\alpha] &= (t+s)^\alpha \mathbb{E}[(\frac{t}{t+s}\frac{M[0,t]}{t} + \frac{s}{t+s}\frac{M[t,t+s]}{s})^\alpha] \\ &\geq (t+s)^\alpha \mathbb{E}[\frac{t}{t+s}(\frac{M[0,t]}{t})^\alpha + \frac{s}{t+s}(\frac{M[t,t+s]}{s})^\alpha] \\ &= (t+s)^\alpha(\frac{t}{t+s}\mathbb{E}[(\frac{M[0,t]}{t})^\alpha] + \frac{s}{t+s}\mathbb{E}[(\frac{M[t,t+s]}{s})^\alpha]) \\ &= (t+s)^\alpha \mathbb{E}[M[0,1]^\alpha] \\ &= \mathbb{E}[M[0,t+s]^\alpha].\end{aligned}$$

Therefore, the above inequalities are in fact equalities and, since $x \mapsto x^\alpha$ is strictly concave, this shows that $\frac{M[0,t]}{t}$ and $\frac{M[t,t+s]}{s}$ are equal almost surely. Let us consider the random non decreasing function $f(t) = M[0,t]$. The function f satisfies for all $s,t > 0$:

$$\frac{f(t)}{t} = \frac{f(t+s) - f(t)}{s}.$$

Letting s go to 0 in the above equality, we conclude that f has a derivative f' which satisfies:

$$f'(t) = \frac{f(t)}{t}.$$

Therefore, $f(t)$ is of the form Ct where C is some constant.

4.2 Proof of Theorem 2.7

The relation $\mathbb{E}[e^{\omega_l(x)}] = e^{\psi(1)H\otimes\theta(C_l(x))} = 1$ (and the fact that for $l' < l$, $\omega_{l'}(x) - \omega_l(x)$ is independent of $\mathscr{F}_l$) ensures that for each Borelian subset $A \subset \mathbb{R}^d$, the process $M_l(A)$ is a martingale with respect to $\mathscr{F}_l$. Existence of the MRM then results from [10]. Properties i) and ii) result from Fatou's lemma. □

4.3 Characteristic function of $\omega_l(x)$

As in [1], the crucial point is to compute the characteristic function of $\omega_l(x)$. We consider $(x^1,\dots,x^q) \in (\mathbb{R}^d)^q$ and $(\lambda_1,\dots,\lambda_q) \in \mathbb{R}^q$ and we have to compute

$$\phi(\lambda) = \mathbb{E}\big[e^{i\lambda_1\omega_l(x^1)+\cdots+i\lambda_q\omega_l(x^q)}\big].$$

Let us denote by $\mathscr{S}_q$ the permutation group of the set $\{1;\dots;q\}$. For a generic element $\sigma \in \mathscr{S}_q$, we define

$$B^\sigma = \{m \in G; x_1^{\sigma(1),m} < \cdots < x_1^{\sigma(q),m}\}.$$

Finally, given $x,z \in \mathbb{R}^d$, we define cone like subset product

$$C_l^\sigma(x) = C_l(x) \cap B^\sigma = \{(m,t,y) \in B^\sigma \times S; (t,y) \in A_l(x_1^m)\}$$

and

$$C_l^\sigma(x,z) = \{(m,t,y) \in B^\sigma \times S; (t,y) \in A_l(x_1^m) \cap A_l(z_1^m)\}.$$

Lemma 4.4. *The characteristic function of the vector* $(\omega_l(x^i))_{1\le i\le q}$ *exactly matches*

$$\mathbb{E}\Big[\exp\Big(i\lambda_1\omega_l(x^1)+\cdots+i\lambda_q\omega_l(x^q)\Big)\Big]=\exp\Big(\sum_{\sigma\in\mathscr{S}_q}\sum_{j=1}^{q}\sum_{k=1}^{j}\alpha^\sigma(j,k)\rho_l^\sigma(x^{\sigma(k)}-x^{\sigma(j)})\Big)$$

where $\rho_l^\sigma(x)=H\otimes\theta(C_l^\sigma(0,x))$*, and*

$$\alpha^\sigma(j,k)=\varphi(r^\sigma_{k,j})+\varphi(r^\sigma_{k+1,j-1})-\varphi(r^\sigma_{k,j-1})-\varphi(r^\sigma_{k+1,j})$$
$$r^\sigma_{k,j}=\sum_{i=k}^{j}\lambda_{\sigma(i)}\quad (\text{or } 0 \text{ if } k>j).$$

Moreover

$$\sum_{j=1}^{q}\sum_{k=1}^{j}\alpha^\sigma(j,k)=\varphi\big(\sum_{k=1}^{q}\lambda_k\big).$$

Proof. Without loss of generality, we assume $x^i\neq x^j$ for $i\neq j$. We point out that the family $(B^\sigma)_{\sigma\in\mathscr{S}_d}$ is a partition of G up to a set of null H-measure. The function ϕ breaks down as

$$\begin{aligned}\phi(\lambda)=&\mathbb{E}\big[e^{i\lambda_1\mu(C_l(x^1))+\cdots+i\lambda_q\mu(C_l(x^q))}\big]\\ =&\mathbb{E}\big[e^{\sum_{\sigma\in\mathscr{S}_d}i\lambda_1\mu(C_l^\sigma(x^1))+\cdots+i\lambda_q\mu(C_l^\sigma(x^q))}\big]\\ =&\prod_{\sigma\in\mathscr{S}_d}\mathbb{E}\big[e^{i\lambda_1\mu(C_l^\sigma(x^1))+\cdots+i\lambda_q\mu(C_l^\sigma(x^q))}\big]\end{aligned}$$

Let us fix $\sigma\in\mathscr{S}_q$. We focus on the term

$$\phi^\sigma(\lambda)=\mathbb{E}\big[e^{i\lambda_1\mu(C_l^\sigma(x^1))+\cdots+i\lambda_q\mu(C_l^\sigma(x^q)))}\big]=\mathbb{E}\big[e^{i\lambda_{\sigma(1)}\mu(C_l^\sigma(x^{\sigma(1)}))+\cdots+i\lambda_{\sigma(q)}\mu(C_l^\sigma(x^{\sigma(q)})))}\big].$$

Given $\sigma\in\mathscr{S}_d$ and $p\le q$, we further define

$$\phi^\sigma(\lambda,p)=\mathbb{E}\big[e^{i\lambda_{\sigma(1)}\mu(C_l^\sigma(x^{\sigma(1)}))+\cdots+i\lambda_{\sigma(p)}\mu(C_l^\sigma(x^{\sigma(p)})))}\big].$$

From now on, we adapt the proof of [1, Lemma 1] and proceed recursively. We define

$$Y_q^\sigma=\sum_{k=1}^{q}\lambda_{\sigma(k)}\mu(C_l^\sigma(x^{\sigma(k)})\setminus C_l(x^{\sigma(q)})),$$

which stands for the contribution of the points of the above sum that do not belong to $C_l(x^{\sigma(q)})$. Moreover, the points in the set $C_l(x^{\sigma(q)})$ can be grouped into the disjoint sets

$$C_l(x^{\sigma(k)},x^{\sigma(q)})\setminus C_l(x^{\sigma(k-1)},x^{\sigma(q)}).$$

We stress that the latter assertion is valid since, for $m\in B^\sigma$, the coordinates are suitably sorted, that is: $x_1^{\sigma(1),m}<x_2^{\sigma(2),m}<\cdots<x_1^{\sigma(q),m}$. We define

$$X^\sigma_{k,q}=\mu\big(C_l(x^{\sigma(k)},x^{\sigma(q)})\setminus C_l(x^{\sigma(k-1)},x^{\sigma(q)})\big)$$

with the convention $C_l(x^{\sigma(k)}, x^{\sigma(0)}) = C_l(x^{\sigma(0)}, x^{\sigma(k)}) = \emptyset$, in such a way that one has

$$\lambda_{\sigma(1)}\mu(C_l^\sigma(x^{\sigma(1)})) + \cdots + \lambda_{\sigma(q)}\mu(C_l^\sigma(x^{\sigma(q)})) = Y_q^\sigma + \sum_{k=1}^q r_{k,q}^\sigma X_{k,q}^\sigma.$$

Furthermore, since the variable Y_q and $(X_{k,q}^\sigma)_k$ are mutually independent, we get the following decomposition:

$$\phi^\sigma(\lambda) = \mathbb{E}\big[e^{iY_q^\sigma}\big]\prod_{k=1}^q \mathbb{E}\big[e^{ir_{k,q}^\sigma X_{k,q}^\sigma}\big]. \tag{5}$$

Similarly, one can prove

$$\phi^\sigma(\lambda, q-1) = \mathbb{E}\big[e^{iY_q^\sigma}\big]\prod_{k=1}^q \mathbb{E}\big[e^{ir_{k,q-1}^\sigma X_{k,q}^\sigma}\big]. \tag{6}$$

Gathering (5) and (6) yields

$$\phi^\sigma(\lambda, q) = \phi^\sigma(\lambda, q-1)\prod_{k=1}^q \frac{\mathbb{E}\big[e^{ir_{k,q}^\sigma X_{k,q}^\sigma}\big]}{\mathbb{E}\big[e^{ir_{k,q-1}^\sigma X_{k,q}^\sigma}\big]}.$$

For any $m \in B^\sigma$, one has $x_1^{\sigma(k-1),m} < x_1^{\sigma(k),m} < x_1^{\sigma(q),m}$ and therefore

$$\begin{aligned}\mathbb{E}\big[e^{i\alpha X_{k,q}^\sigma}\big] &= e^{\varphi(\alpha)H\otimes\theta\left(C_l(x^{\sigma(k)},x^{\sigma(q)})\setminus C_l(x^{\sigma(k-1)},x^{\sigma(q)})\right)}\\ &= e^{\varphi(\alpha)\left(H\otimes\theta(C_l(x^{\sigma(k)},x^{\sigma(q)}))-H\otimes\theta(C_l(x^{\sigma(k-1)},x^{\sigma(q)}))\right)}\end{aligned}$$

Note that

$$\begin{aligned}H\otimes\theta(C_l(x^{\sigma(i)}, x^{\sigma(j)})) &= \int_{B^\sigma}\theta\big(A_l(x_1^{\sigma(i),m})\cap A_l(x_1^{\sigma(j),m})\big)H(dm)\\ &= \int_{B^\sigma}\theta\big(A_l(0)\cap A_l(x_1^{\sigma(i),m} - x_1^{\sigma(j),m})\big)H(dm)\\ &= \rho_l^\sigma(x^{\sigma(i)} - x^{\sigma(j)})\end{aligned}$$

The proof can now be completed recursively. For further details, the reader is referred to [1]. □

4.5 Homogenity and isotropy

Lemma 4.4 is useful to prove the main properties of the MMRM. For instance, to prove the invariance of the law of the MMRM under translations, it suffices to prove that the law of ω_l is itself invariant. This results from Lemma 4.4 since each term $\rho_l^\sigma(x^{\sigma(k)} - x^{\sigma(i)})$ is invariant under translations, that is ρ_l^σ remains unchanged when you replace $x^1, \ldots, x^q$ by $x^1 + z, \ldots, x^q + z$ for a given $z \in \mathbb{R}^d$. However, Lemma 4.4 may not be adapted to prove the isotropy of the MMRM so that we give a proof by a direct approach:

Lemma 4.6. Proof of the isotropy of the MMRM *The measure M is isotrop, that is:*

$$\forall m \in G, \quad (M(mA))_{A\subset B(0,T)} \overset{law}{=} (M(A))_{A\subset B(0,T)}.$$

Proof. Once again, it is sufficient to prove that the characteristic function of ω_l is invariant under G. This time, we compute that characteristic function in a more direct way. We consider $x = (x^1, \dots, x^q) \in (\mathbb{R}^d)^q$, $\lambda_1, \dots, \lambda_q \in \mathbb{R}$ and $m_0 \in G$, and define the function

$$f_x(m,t,y) = \sum_{k=1}^{q} \lambda_k \mathbb{1}_{C_l(x^k)}.$$

For $m \in G$, we define mx as $(mx^1, \dots, mx^q)$. We have, using the right translation invariance of the Haar measure and the fact that $f_{m_0 x}(m,t,y) = f_x(mm_0,t,y)$:

$$\begin{aligned}
\mathbb{E}\Big[\exp\Big(i\lambda_1\omega_l(m_0x^1) + \cdots + i\lambda_q\omega_l(m_0x^q)\Big)\Big] =& \mathbb{E}\Big[\exp\Big(i\int f_{m_0x}(m,t,y)\,\mu(dm,dt,dy)\Big)\Big] \\
=& \mathbb{E}\Big[\exp\Big(i\int f_x(mm_0,t,y)\,\mu(dm,dt,dy)\Big)\Big] \\
=& \exp\Big(\int \varphi \circ f_x(mm_0,t,y) H(dm)\theta(dt,dy)\Big) \\
=& \exp\Big(\int \varphi \circ f_x(m,t,y) H(dm)\theta(dt,dy)\Big) \\
=& \mathbb{E}\Big[\exp\Big(i\lambda_1\omega_l(x^1) + \cdots + i\lambda_q\omega_l(x^q)\Big)\Big].
\end{aligned}$$

The isotropy follows. □

4.7 Exact scaling and stochastic scale invariance

Lemma 4.8. Exact scaling of $M_l(dx)$. *Given $\forall\lambda \in]0,1]$, $\forall x^1, \dots, x^q \in B(0,T/2)$, the functions ρ_l^σ satisfy the exact scaling relation*

$$\sum_{\sigma\in\mathscr{S}_q}\sum_{j=1}^{q}\sum_{k=1}^{j}\alpha^\sigma(j,k)\rho_{\lambda l}^\sigma(\lambda x^{\sigma(k)} - \lambda x^{\sigma(j)}) = -\ln(\lambda) + \sum_{\sigma\in\mathscr{S}_q}\sum_{j=1}^{q}\sum_{k=1}^{j}\alpha^\sigma(j,k)\rho_l^\sigma(x^{\sigma(k)} - x^{\sigma(j)}). \quad (7)$$

Proof. We remind that for x real we have $\int_{A_l(0)\cap A_l(x)} \theta(dt,dy) = g_l(|x|)$. Given $B \subset G$ and $x \in \mathbb{R}^d$, we define:

$$\rho_l^B(x) = \int \mathbb{1}_B(m)\mathbb{1}_{\{(t,y)\in A_l(0)\cap A_l(x_1^m)\}}\theta(dt,dy)H(dm).$$

Then we can compute the function ρ_l^B:

$$\begin{aligned}
\rho_l^B(x) =& \int \mathbb{1}_{\{(t,y)\in A_l(0)\cap A_l(x_1^m)\}} H(dm)\theta(dt,dy) = \int_B \Big(\int_{A_l(0)\cap A_l(x_1^m)} \theta(dt,dy)\Big) H(dm) \\
=& \int_B \Big(\big(\ln(T/l) + 1 - |x_1^m|/l\big)\mathbb{1}_{|x_1^m|\le l} + \ln(T/|x_1^m|)\mathbb{1}_{l\le |x_1^m| \le T}\Big) H(dm)
\end{aligned}$$

Given $\lambda \in]0,1]$ and $x \in B(0,T)$

$$\begin{aligned}\rho^B_{\lambda l}(\lambda x) =& \int_B \Big(\big(\ln(\frac{T}{\lambda l}) + 1 - \frac{|\lambda x^m_1|}{\lambda l} \big) 1\!\!1_{|\lambda x^m_1| \leq \lambda l} + \ln(\frac{T}{\lambda |x^m_1|}) 1\!\!1_{\lambda l \leq |\lambda x^m_1| \leq T} \Big) H(dm) \\ =& \int_B \Big(\big(\ln(\frac{T}{l}) + 1 - \frac{|x^m_1|}{l} \big) 1\!\!1_{|x^m_1| \leq l} + \ln(\frac{T}{|x^m_1|}) 1\!\!1_{\lambda l \leq |\lambda x^m_1| \leq T} \Big) H(dm) \\ & - \ln(\lambda) \int_B 1\!\!1_{|x^m_1| \leq l} + 1\!\!1_{l \leq |x^m_1| \leq T} H(dm) \\ =& \rho^B_l(x) - \ln(\lambda) H(B)\end{aligned}$$

We therefore obtain

$$\begin{aligned}& \sum_{\sigma \in \mathscr{S}_q} \sum_{j=1}^{q} \sum_{k=1}^{j} \alpha^\sigma(j,k) \rho^\sigma_{\lambda l}(\lambda x^{\sigma(k)} - \lambda x^{\sigma(j)}) \\ &= \sum_{\sigma \in \mathscr{S}_q} \sum_{j=1}^{q} \sum_{k=1}^{j} \alpha^\sigma(j,k) \rho^\sigma_{l}(x^{\sigma(k)} - x^{\sigma(j)}) - \ln(\lambda) \sum_{\sigma \in \mathscr{S}_q} \sum_{j=1}^{q} \sum_{k=1}^{j} \alpha^\sigma(j,k) H(B^\sigma) \\ &= \sum_{\sigma \in \mathscr{S}_q} \sum_{j=1}^{q} \sum_{k=1}^{j} \alpha^\sigma(j,k) \rho^\sigma_{l}(x^{\sigma(k)} - x^{\sigma(j)}) - \ln(\lambda) \sum_{\sigma \in \mathscr{S}_q} \varphi(\sum_{k=1}^{q} \lambda_k) H(B^\sigma) \\ &= \sum_{\sigma \in \mathscr{S}_q} \sum_{j=1}^{q} \sum_{k=1}^{j} \alpha^\sigma(j,k) \rho^\sigma_{l}(x^{\sigma(k)} - x^{\sigma(j)}) - \ln(\lambda) \varphi(\sum_{k=1}^{q} \lambda_k) \quad \square\end{aligned}$$

From Lemma 4.4, we deduce that, for any $\lambda \in]0,1]$, there exists a random variable C_λ such that $(\omega_{\lambda l}(\lambda x))_{x \in B(0,T/2)} \overset{law}{=} (C_\lambda + \omega_l(x))_{x \in B(0,T/2)}$ and such that C_λ is independent of $(\omega_l(x))_{x \in B(0,T/2)}$ and its characteristic function is given by $\mathbb{E}[e^{iqC_\lambda}] = \lambda^{-\varphi(q)}$.

By integrating the previous relation, we obtain the relation

$$(M_{\lambda l}(\lambda A))_{A \subset B(0,T/2)} \overset{law}{=} W_\lambda (M_l(A))_{A \subset B(0,T/2)}$$

where $W_\lambda = \lambda^d e^{C_\lambda}$ is a random variable independent of $(M_l(A))_{A \subset B(0,T/2)}$.

4.9 Non-triviality of the MMRM

Proof of Proposition 2.9 (items 1. and 2.) Suppose we can find a "cube" $C_R = [0;R]^d$ and $q > 1$ such that

$$\mathbb{E}\big[M(C_R)^q\big] < +\infty.$$

Then we can find $n \in \mathbb{N}$ such that $[0;2^{-n}R]^d \subset B(0,T/2)$. We split the cube C_R into 2^{nd} smaller cubes

$$C^{k,n} = \prod_{i=1}^{d} [k_i 2^{-n} R; (k_i+1) 2^{-n} R[,$$

where $k=(k_1,\dots,k_d)\in N_d^n \overset{def}{=} \mathbb{N}^d\cap[0;2^n-1]^d$. For each fixed value of n, the cubes $(C^{k,n})_k$, where the index k varies in N_d^n form a partition of C_T. Thus, by using the super-additivity of the function $x\mapsto x^q$, we have:

$$\begin{aligned}\mathbb{E}\big[M(C_T)^q\big] &= \mathbb{E}\Big[\Big(\sum_{k\in N_d^n} M(C^{k,n})\Big)^q\Big]\\ &\geq \sum_{k\in N_d^n}\mathbb{E}\big[\big(M(C^{k,n})\big)^q\big]\end{aligned}$$

By using the translation invariance and the scale invariance property of the MMRM, we deduce:

$$\mathbb{E}\big[\big(M(C^{k,n})\big)^q\big]=\mathbb{E}\big[\big(M(C^{0,n})\big)^q\big]=\mathbb{E}\big[\big(M(2^{-n}C_R)\big)^q\big]=2^{-n\zeta(q)}\mathbb{E}\big[\big(C_R)\big)^q\big].$$

Finally, gathering the previous inequalities yields:

$$\mathbb{E}\big[\big(C_R)\big)^q\big]\geq 2^{nd-n\zeta(q)}\mathbb{E}\big[\big(C_R)\big)^q\big]$$

in such a way that, necessarily, $\zeta(q)\geq d$. □

The proof of 1. and 2. is then a consequence of the following lemma:

Lemma 4.10. *Let $q>1$ and consider the unique $n\in\mathbb{N}$ such that $n<q\leq n+1$. If $\zeta(q)>d$ and $\psi(n+1)<\infty$, then we can find a constant C such that:*

$$\sup_l \mathbb{E}\big[M_l([0,T[^d)^q\big]\leq C.$$

Proof. The proof is an adaptation of the one in [1] (which is itself an adaptation of the corresponding result in [2]). Unfortunately, the multi-dimensional setting is bit more complicated because there is no strict decorrelation property similar to the one dimensional setting. With no restriction, we can suppose that $T=1$ and $d=2$. We consider the following dyadic partition of the cube $[0,1[^2$:

$$[0,1[^2=\bigcup_{0\leq i,j\leq 2^m-1} I_{i,j}^{(m)},$$

where $I_{i,j}^{(m)}=[\frac{i}{2^m},\frac{i+1}{2^m}[\times[\frac{j}{2^m},\frac{j+1}{2^m}[$. Let us write the above decompostion in the following form:

$$[0,1[^2=C_1\cup C_2\cup C_3\cup C_4.$$

where

$$C_1=\bigcup_{i\ and\ j\ even} I_{i,j}^{(m)},\quad C_2=\bigcup_{i\ and\ j\ odd} I_{i,j}^{(m)}$$

and

$$C_3=\bigcup_{i\ odd\ and\ j\ even} I_{i,j}^{(m)},\quad C_2=\bigcup_{i\ even\ and\ j\ odd} I_{i,j}^{(m)}.$$

Since the measure M_l is homogeneous, we get:

$$\begin{aligned}\mathbb{E}\big[M_l([0,1[^2)^q\big] &\leq 4^{q-1}\sum_{i=1}^{4}\mathbb{E}\big[M_l(C_i)^q\big]\\ &\leq 4^q\mathbb{E}\big[M_l(C_1)^q\big].\end{aligned}$$

Now we get the following by subadditivity of $x \to x^{q/(n+1)}$:

$$\begin{aligned}
\mathbb{E}\big[M_l(C_1)^q\big] &= \mathbb{E}\big[(\sum_{0\le i,j\le 2^{m-1}-1} M_l(I^{(m)}_{2i,2j}))^q\big] \\
&= \mathbb{E}\big[((\sum_{i,j} M_l(I^{(m)}_{2i,2j}))^{n+1})^{q/(n+1)}\big] \\
&= \mathbb{E}\big[(\sum_{i_1,j_1,\dots,i_{n+1},j_{n+1}} \prod_{k=1}^{n} M_l(I^{(m)}_{2i_k,2j_k}))^{q/(n+1)}\big] \\
&\le \mathbb{E}\big[\sum_{i_1,j_1,\dots,i_{n+1},j_{n+1}} (\prod_{k=1}^{n} M_l(I^{(m)}_{2i_k,2j_k}))^{q/(n+1)}\big] \\
&= 2^{2(m-1)}\mathbb{E}\big[M_l(I^{(m)}_{0,0})^q\big] + \sideset{}{^*}\sum_{i_1,j_1,\dots,i_{n+1},j_{n+1}} \mathbb{E}\big[\prod_{k=1}^{n} M_l(I^{(m)}_{2i_k,2j_k})^{q/(n+1)}\big] \\
&\le 2^{2(m-1)}\mathbb{E}\big[M_l(I^{(m)}_{0,0})^q\big] + \sideset{}{^*}\sum_{i_1,j_1,\dots,i_{n+1},j_{n+1}} \mathbb{E}\big[\prod_{k=1}^{n} M_l(I^{(m)}_{2i_k,2j_k})\big]^{q/(n+1)}
\end{aligned}$$

where $\sum^*_{i_1,j_1,\dots i_{n+1},j_{n+1}}$ is a sum over indices $i_1, j_1, \dots i_{n+1}, j_{n+1}$ which are not all equal and the last inequality is a consequence of Jensen's inequality. Therefore each term in the above sum is of the form:

$$\mathbb{E}\big[\prod_{r=1}^{k} M_l(I^{(m)}_{2i_r,2j_r})^{n_r}\big]^{q/(n+1)} \tag{8}$$

where the sequence of positive integers $(n_r)_{1\le r\le k}$ satisfies $\sum_{r=1}^{k} n_r + 1$ and the $I_{2i_r,2j_r}$ are disjoint intervals wich lie at a distance of at least $\frac{1}{2^m}$. We want to show that each term of the form (8) is bounded by some quantity C_m independent of l. We get the following computation using Fubini:

$$\mathbb{E}\big[\prod_{r=1}^{k} M_l(I^{(m)}_{2i_r,2j_r})^{n_r}\big] = \int_{I^{n_1}_{2i_1,2j_1}\times\cdots\times I^{n_k}_{2i_k,2j_k}} \mathbb{E}\big[e^{\omega_l(x^1)+\cdots+\omega_l(x^{n+1})}\big]dx^1\dots dx^{n+1}.$$

We define $N_r = n_1+\cdots+n_r$ for r in $[1,k]$ and we introduce the following set $\mathscr{A}_l = \mathscr{A}_l(x^1,\dots,x^{n+1})$:

$$\mathscr{A}_l = \cup_{r<r'}(\cup_{N_r\le i\le N_{r+1}-1} C_l(x^i)) \cap (\cup_{N_{r'}\le j\le N_{r'+1}-1} C_l(x^j)).$$

By construction of $\mathscr{A}_l$, if x^i and x^j are in two different $I_{2i_r,2j_r}$ then $\mu(C_l(x^i)\setminus\mathscr{A}_l)$ and $\mu(C_l(x^j)\setminus\mathscr{A}_l)$ are independent. Therefore we get the following factorization:

$$\begin{aligned}
\mathbb{E}\big[e^{\omega_l(x^1)+\cdots+\omega_l(x^{n+1})}\big] &= \mathbb{E}\big[e^{\psi(n+1)H\otimes\theta(\mathscr{A}_l)}\big]\mathbb{E}\big[e^{\mu(C_l(x^1)\setminus\mathscr{A}_l)+\cdots+\mu(C_l(x^{n+1})\setminus\mathscr{A}_l)}\big] \\
&= \mathbb{E}\big[e^{\psi(n+1)H\otimes\theta(\mathscr{A}_l)}\big]\prod_{r=1}^{k}\mathbb{E}\big[e^{\sum_{N_r\le i\le N_{r+1}-1}\mu(C_l(x^i)\setminus\mathscr{A}_l)}\big] \\
&= \frac{\mathbb{E}\big[e^{\psi(n+1)H\otimes\theta(\mathscr{A}_l)}\big]}{\prod_{r=1}^{k}\mathbb{E}\big[e^{\psi(n_r)H\otimes\theta(\mathscr{A}_l)}\big]}\prod_{r=1}^{k}\mathbb{E}\big[e^{\sum_{N_r\le i\le N_{r+1}-1}\omega_l(x^i)}\big].
\end{aligned}$$

We have the following inequality:

$$H\otimes\theta(\mathscr{A}_l)\leq\sum_{r<r'}\sum_{\substack{N_r\leq i\leq N_{r+1}-1\\ N_{r'}\leq j\leq N_{r'+1}-1}}H\otimes\theta(C_l(x^i)\cap C_l(x^j)).$$

Note that for each x^i, x^j in the above sum we have $|x^i-x^j|\geq\frac{1}{2^m}$ and therefore $H\otimes\theta(C_l(x^i)\cap C_l(x^j))$ is bounded by some constant depending on m but independent of l. Indeed, using the notation of section 3 for F, we get:

$$\begin{aligned}H\otimes\theta(C_l(x^i)\cap C_l(x^j))&=\int_G g_l(|(x^i)_1^m-(x^j)_1^m|)H(dm)\\&\leq F(|x^i-x^j|)\\&\leq F(\frac{1}{2^m}).\end{aligned}$$

In conclusion, we get the existence of some constant C_m such that:

$$\mathbb{E}\big[e^{\omega_l(x^1)+\cdots+\omega_l(x^{n+1})}\big]\leq C_m\prod_{r=1}^k\mathbb{E}\big[e^{\sum_{N_r\leq i\leq N_{r+1}-1}\omega_l(x^i)}\big].$$

Thus, we get by integrating the above relation:

$$\mathbb{E}\big[\prod_{r=1}^k M_l(I^{(m)}_{2i_r,2j_r})^{n_r}\big]\leq C_m\prod_{r=1}^k\mathbb{E}\big[M_l(I^{(m)}_{2i_r,2j_r})^{n_r}\big].$$

Since each n_r is less or equal to n, we get by induction that $\mathbb{E}\big[M_l(I^{(m)}_{2i_r,2j_r})^{n_r}\big]$ is bounded independently of l and so is the above product. In conclusion, we get the existence of C_m such that we have:

$$\mathbb{E}\big[M_l([0,1]^2)^q\big]\leq 4^{q-1}2^{2m}\mathbb{E}\big[M_l(I^{(m)}_{0,0})^q\big]+C_m.$$

Using stochastic scale invariance, we get that:

$$\begin{aligned}\mathbb{E}\big[M_l([0,1]^2)^q\big]&\leq 4^{q-1}\frac{2^{2m}}{2^{m\zeta(q)}}\mathbb{E}\big[M_{l2^m}([0,1]^2)^q\big]+C_m\\&\leq 4^{q-1}\frac{2^{2m}}{2^{m\zeta(q)}}\mathbb{E}\big[M_l([0,1]^2)^q\big]+C_m.\end{aligned}$$

Since $\zeta(q)>2$, we can choose m such that $4^{q-1}\frac{2^{2m}}{2^{m\zeta(q)}}<1$ and therefore we get:

$$\mathbb{E}\big[M_l([0,1]^2)^q\big]\leq\frac{C_m}{1-4^{q-1}\frac{2^{2m}}{2^{m\zeta(q)}}}$$

which entails the result.

4.11 Proof of lemma 3.2

By homogenity and isotropy, we can suppose that $y = 0$ and $x = |x|e$ where e is the first vector of the canonical basis. In this case, we get (see relation (4.2.18) in [9]):

$$\mathbb{E}[X(x)X(0)] = \frac{\Gamma(d/2)}{2\Gamma(1/2)\Gamma((d-1)/2)} \int_0^1 \ln^+(T/(v|x|))(1-v)^{d/2-3/2} \frac{dv}{\sqrt{v}}$$

where Γ is the standard gamma function: $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$. By making the change of variable $u = \frac{v|x|}{T}$ in the above integral, one easily deduces that:

$$\mathbb{E}[X(x)X(0)] \underset{|x|\to\infty}{\sim} C\sqrt{\frac{T}{|x|}},$$

where C is given by $C = \frac{\Gamma(d/2)}{2\Gamma(1/2)\Gamma((d-1)/2)} \int_0^1 \ln(1/u)\frac{du}{\sqrt{u}}$. Integrating by parts, we find $\int_0^1 \ln(1/u)\frac{du}{\sqrt{u}} = 4$ and thus $C = \frac{2\Gamma(d/2)}{\Gamma(1/2)\Gamma((d-1)/2)}$.

Acknowledgements: The authors would like to thank E. Bacry, J.F. Muzy and R. Robert for interesting discussions.

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