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# Recurrence and transience for long range reversible random walks on a random point process*

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**Abstract**

We consider reversible random walks in random environment obtained from symmetric long–range jump rates on a random point process. We prove almost sure transience and recurrence results under suitable assumptions on the point process and the jump rate function. For recurrent models we obtain almost sure estimates on effective resistances in finite boxes. For transient models we construct explicit fluxes with finite energy on the associated electrical network.

**Key words:** random walk in random environment, recurrence, transience, point process, electrical network.

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# 1 Introduction and results

We consider random walks in random environment obtained as random perturbations of long–range random walks in deterministic environment. Namely, let $S$ be a locally finite subset of $\mathbb{R}^d$, $d \geqslant 1$ and call $X_n$ the discrete time Markov chain with state space $S$ that jumps from a site $x$ to another site $y$ with a probability $p(x,y)$ that is proportional to $\varphi(|x-y|)$, where $\varphi : (0,\infty) \to (0,1]$ is a positive bounded measurable function and $|x|$ stands for the Euclidean norm of $x \in \mathbb{R}^d$. We write $P$ for the law of $X_n$, so that for $x \neq y \in S$:

$$P(X_{n+1} = y \,|\, X_n = x) = p(x,y) := \frac{\varphi(|y-x|)}{w_S(x)}\,,$$

where $w_S(x) := \sum_{z\in S:\, z\neq x} \varphi(|z-x|)$. Note that the random walk $X_n$ is well defined as soon as $w_S(x) \in (0,\infty)$ for every $x \in S$. In this case, $w_S = \{w_S(x)\,,\ x \in S\}$ is a reversible measure:

$$w_S(x)p(x,y) = w_S(y)p(y,x).$$

Since the random walk is irreducible due to the strict positivity of $\varphi$, $w_S$ is the unique invariant measure up to a multiplicative constant. We shall often speak of the random walk $(S,\varphi)$ when we need to emphasize the dependence on the state space $S$ and the function $\varphi$. Typical special cases of functions $\varphi$ will be the polynomially decaying function $\varphi_{\mathrm{p},\alpha}(t) := 1 \wedge t^{-d-\alpha}$, $\alpha > 0$ and the stretched exponential function $\varphi_{\mathrm{e},\beta}(t) := \exp(-t^\beta)$, $\beta > 0$. We investigate here the transience and recurrence of the random walk $X_n$. We recall that $X_n$ is said to be *recurrent* if for some $x \in S$, the walk started at $X_0 = x$ returns to $x$ infinitely many times with probability one. Because of irreducibility if this happens at some $x \in S$ then it must happen at all $x \in S$. $X_n$ is said to be *transient* if it is not recurrent. If we fix $S = \mathbb{Z}^d$, we obtain standard homogeneous lattice walks. Transience and recurrence properties of these walks can be obtained by classical harmonic analysis, as extensively discussed e.g. in Spitzer's book [25] (see also Appendix B). For instance, it is well known that for dimension $d \geqslant 3$ both $(\mathbb{Z}^d, \varphi_{\mathrm{e},\beta})$ and $(\mathbb{Z}^d, \varphi_{\mathrm{p},\alpha})$ are transient for all $\beta > 0$ and $\alpha > 0$ while for $d = 1,2$, $(\mathbb{Z}^d, \varphi_{\mathrm{e},\beta})$ is recurrent for all $\beta > 0$ and $(\mathbb{Z}^d, \varphi_{\mathrm{p},\alpha})$ is transient iff $0 < \alpha < d$.

We shall be interested in the case where $S$ is a locally finite *random* subset of $\mathbb{R}^d$, i.e. the realization of a simple point process on $\mathbb{R}^d$. We denote by $\mathbb{P}$ the law of the point process. For this model to be well defined for $\mathbb{P}$–almost all $S$ we shall require that, given the choice of $\varphi$:

$$\mathbb{P}\left(w_S(x) \in (0,\infty)\,,\ \text{for all } x \in S\right) = 1\,. \tag{1.1}$$

If we look at the set $S$ as a random perturbation of the regular lattice $\mathbb{Z}^d$, the first natural question is to find conditions on the law of the point process $\mathbb{P}$ and the function $\varphi$ such that $(S,\varphi)$ is $\mathbb{P}$–a.s. transient (recurrent) iff $(\mathbb{Z}^d,\varphi)$ is transient (recurrent). In this case we say that the random walks $(S,\varphi)$ and $(\mathbb{Z}^d,\varphi)$ have a.s. the same *type*. A second question we shall address in this paper is that of establishing almost sure bounds on finite volume effective resistances in the case of certain recurrent random walks $(S,\varphi)$. Before going to a description of our main results we discuss the main examples of point processes we have in mind. In what follows we shall use the notation $S(\Lambda)$ for the number of points of $S$ in any given bounded Borel set $\Lambda \subset \mathbb{R}^d$. For any $t > 0$ and $x \in \mathbb{R}^d$ we write

$$Q_{x,t} := x + \left[-\frac{t}{2}, \frac{t}{2}\right]^d, \quad B_{x,t} = \{y \in \mathbb{R}^d :\ |y-x| < t\}\,,$$

for the cube with side $t$ and the open ball of radius $t$ around $x$. To check that the models $(S,\varphi)$ are well defined, i.e. (1.1) is satisfied, in all the examples described below the following simple criterion will be sufficient. We write $\Phi_d$, for the class of functions $\varphi:(0,\infty)\to(0,1]$ such that $\int_0^\infty t^{d-1}\varphi(t)dt<\infty$. Suppose the law of the point process $\mathbb{P}$ is such that

$$\sup_{x\in\mathbb{Z}^d}\mathbb{E}[S(Q_{x,1})]<\infty\,. \tag{1.2}$$

Then it is immediate to check that $(S,\varphi)$ satisfies (1.1) for any $\varphi\in\Phi_d$.

### 1.1 Examples

The main example we have in mind is the case when $\mathbb{P}$ is a *homogeneous Poisson point process* (PPP) on $\mathbb{R}^d$. In this case we shall show that $(S,\varphi)$ and $(\mathbb{Z}^d,\varphi)$ have a.s. the same type, at least for the standard choices $\varphi=\varphi_{\mathrm{p},\alpha},\varphi_{\mathrm{e},\beta}$. Besides its intrinsic interest as random perturbation of lattice walks we point out that the Poisson point process model arises naturally in statistical physics in the study of the low-temperature conductivity of disordered systems. In this context, the $(S,\varphi_{\mathrm{e},\beta})$ model with $\beta=1$ is a variant of the well known Mott variable–range hopping model, see [12] for more details. The original variable–range hopping model comes with an environment of energy marks on top of the Poisson point process that we neglect here since it does not interfere with the recurrence or transience of the walk. It will be clear that, by elementary domination arguments, all the results we state for homogeneous PPP actually apply to non–homogeneous PPP with an intensity function that is uniformly bounded from above and away from zero.

Motivated by the variable–range hopping problem one could consider point fields obtained from a crystal by dilution and spatial randomization. By crystal we mean any locally finite set $\Gamma\subset\mathbb{R}^d$ such that, for a suitable basis $v_1,v_2,\dots,v_d$ of $\mathbb{R}^d$, one has

$$\Gamma-x=\Gamma\qquad\forall x\in G:=\{z_1v_1+z_2v_2+\cdots+z_dv_d\,:\,z_i\in\mathbb{Z}\ \ \forall i\}\,. \tag{1.3}$$

The *spatially randomized and p–diluted crystal* is obtained from $\Gamma$ by first translating $\Gamma$ by a random vector $V$ chosen with uniform distribution in the elementary cell

$$\Delta=\{t_1v_1+t_2v_2+\cdots+t_dv_d\,:\,0\leqslant t_i<1\ \ \forall i\}\,,$$

and then erasing each point with probability $1-p$, independently from the others. One can check that the above construction depends only on $\Gamma$ and not on the particular $G$ and $\Delta$ chosen. In the case of spatially randomized and $p$–diluted crystals, $\mathbb{P}$ is a stationary point process, i.e. it is invariant w.r.t. spatial translations. It is not hard to check that all the results we state for PPP hold for any of these processes as well for the associated Palm distributions (see [12] for a discussion on the Palm distribution and its relation to Mott variable–range hopping). Therefore, we shall not explicitly mention in the sequel, to avoid lengthy repetitions, the application of our estimates to these cases. We shall also comment on applications of our results to two other classes of point processes: *percolation clusters* and *determinantal point processes*. We say that $S$ is a percolation cluster when $\mathbb{P}$ is the law of the infinite cluster in super–critical Bernoulli site–percolation on $\mathbb{Z}^d$. For simplicity we shall restrict to site–percolation but nothing changes here if one considers bond–percolation instead. The percolation cluster model has been extensively studied in the case of nearest–neighbor walks, see, e.g., [15; 5]. In particular, it is well known that the simple random walk on the percolation cluster has almost surely the same type as the simple random walk on $\mathbb{Z}^d$. Our results will allow to prove that if $S$ is the percolation cluster on $\mathbb{Z}^d$ then $(S,\varphi)$ has a.s. the same type as $(\mathbb{Z}^d,\varphi)$, at least for the standard choices $\varphi=\varphi_{\mathrm{p},\alpha},\varphi_{\mathrm{e},\beta}$. Determinantal point processes (DPP) on the other hand are defined as follows, see [24; 4] for recent insightful reviews on DPP. Let $\mathscr{K}$ be a locally trace class self–adjoint operator on $L^2(\mathbb{R}^d,dx)$. If, in addition, $\mathscr{K}$ satisfies $0\leqslant\mathscr{K}\leqslant 1$ we can speak of the DPP associated with $\mathscr{K}$. Let $\mathbb{P},\mathbb{E}$ denote the associated law and expectation. It is always possible to associate with $\mathscr{K}$ a kernel $K(x,y)$ such that, for any bounded measurable set $B\subset\mathbb{R}^d$, one has

$$\mathbb{E}[S(B)]=\mathrm{tr}(\mathscr{K}1_B)=\int_B K(x,x)dx<\infty \tag{1.4}$$

where $S(B)$ is the number of points in the set $B$ and $1_B$ stands for multiplication by the indicator function of the set $B$, see [24]. Moreover, for any family of mutually disjoint subsets $D_1,D_2,\dots,D_k\subset\mathbb{R}^d$ one has

$$\mathbb{E}\left[\prod_{i=1}^k S(D_i)\right]=\int_{\prod_i D_i}\rho_k(x_1,x_2,\dots,x_k)dx_1dx_2\dots dx_k\,, \tag{1.5}$$

where the $k$–correlation function $\rho_k$ satisfies

$$\rho_k(x_1,x_2,\dots,x_k)=\det\Big(K(x_i,x_j)\Big)_{1\leqslant i,j\leqslant k}\,.$$

Roughly speaking, these processes are characterized by a tendency towards repulsion between points, and if we consider a *stationary* DPP, i.e. the case where the kernel satisfies $K(x,y)=K(0,y-x)$, then the repulsive character forces points to be more regularly spaced than in the Poissonian case. A standard example is the sine kernel in $d=1$, where $K(x,y)=\frac{\sin(\pi(x-y))}{\pi(x-y)}$. Our results will imply for instance that for stationary DPP $(S,\varphi)$ and $(\mathbb{Z}^d,\varphi)$ have a.s. the same type if $\varphi=\varphi_{\mathrm{p},\alpha}$ (any $\alpha>0$) or $\varphi=\varphi_{\mathrm{e},\beta}$ with $\beta<d$.

## 1.2 Random resistor networks

Our analysis of the transience and recurrence of the random walk $X_n$ will be based on the well known resistor network representation of probabilistic quantities associated to reversible random walks on graphs, an extensive discussion of which is found e.g. in the monographs [9; 20]. For the moment let us recall a few basic ingredients of the electrical network analogy. We think of $(S,\varphi)$ as an undirected weighted graph with vertex set $S$ and complete edge set $\{\{x,y\},\ x\neq y\}$, every edge $\{x,y\}$ having weight $\varphi(|x-y|)$. The equivalent electrical network is obtained by connecting each pair of nodes $\{x,y\}$ by a resistor of magnitude $r(x,y):=\varphi(|x-y|)^{-1}$, i.e. by a conductance of magnitude $\varphi(|x-y|)$. We point out that other long–range reversible random walks have already been studied (see for example [2], [3], [17], [22] and references therein), but since the resistor networks associated to these random walks are locally finite and not complete as in our case, the techniques and estimates required here are very different.

One can characterize the transience or recurrence of $X_n$ in terms of the associated resistor network. Let $\{S_n\}_{n\geqslant 1}$ be an increasing sequence of subsets $S_n\subset S$ such that $S=\cup_{n\geqslant 1}S_n$ and let $(S,\varphi)_n$ denote the network obtained by collapsing all sites in $S_n^c=S\backslash S_n$ into a single site $z_n$ (this corresponds to the network where all resistors between nodes in $S_n^c$ are replaced by infinitely conducting wires but all other wires connecting $S_n$ with $S_n$ and $S_n$ with $S_n^c$ are left unchanged). For $x\in S$ and $n$ large enough such that $x \in S_n$, let $R_n(x)$ denote the effective resistance between the nodes $x$ and $z_n$ in the network $(S,\varphi)_n$. We recall that $R_n(x)$ equals the inverse of the effective conductance $C_n(x)$, defined as the current flowing in the network when a unit voltage is applied across the nodes $x$ and $z_n$. On the other hand it is well known that $w_S(x)R_n(x)$ equals the expected number of visits to $x$ before exiting the set $S_n$ for our original random walk $(S,\varphi)$ started at $x$ (see, e.g., (4.28) in [13]). The sequence $R_n(x)$ is non–decreasing and its limit $R(x)$ is called the effective resistance of the resistor network $(S,\varphi)$ between $x$ and $\infty$. Then, $w_S(x)R(x) = \lim_{n\to\infty} w_S(x)R_n(x)$ equals the expected number of visits to $x$ for the walk $(S,\varphi)$ started in $x$, and the walk $(S,\varphi)$ is recurrent iff $R_n(x) \to \infty$ for some (and therefore any) $x \in S$. In the light of this, we shall investigate the rate of divergence of $R_n(x)$ for specific recurrent models. Lower bounds on $R_n(x)$ can be obtained by the following variational characterization of the effective conductance $C_n(x)$:

$$C_n(x) = \inf_{\substack{h:S\to[0,1]\\ h(x)=0,\, h\equiv 1 \text{ on } S_n^c}} \frac{1}{2}\sum_{\substack{y,z\in S\\ y\neq z}} \varphi(|y-z|)\big(h(y)-h(z)\big)^2 . \tag{1.6}$$

The infimum above is attained when $h$ equals the electrical potential, set to be zero on $x$ and 1 on $S_n^c$. From (1.6) one derives Rayleigh's monotonicity principle: the effective conductance $C_n(x)$ decreases whenever $\varphi$ is replaced by $\varphi'$ satisfying $\varphi'(t) \leqslant \varphi(t)$ for all $t > 0$. Upper bounds on $R_n(x)$ can be obtained by means of fluxes. We recall that, given a point $x \in S$ and a subset $B \subset S$ not containing $x$, a unit flux from $x$ to $B$ is any antisymmetric function $f : S \times S \to \mathbb{R}$ such that

$$\mathrm{div} f(y) := \sum_{z\in S} f(y,z) \begin{cases} = 1 & \text{if } y = x\,, \\ = 0 & \text{if } y \neq x \text{ and } y \notin B\,, \\ \leqslant 0 & \text{if } y \in B\,. \end{cases}$$

If $B = \emptyset$ then $f$ is said to be a unit flux from $x$ to $\infty$. The energy $\mathscr{E}(f)$ dissipated by the flux $f$ is defined as

$$\mathscr{E}(f) = \frac{1}{2}\sum_{\substack{y,z\in S\\ y\neq z}} r(y,z) f(y,z)^2 . \tag{1.7}$$

To emphasize the dependence on $S$ and $\varphi$ we shall often call $\mathscr{E}(f)$ the $(S,\varphi)$–energy. Finally, $R_n(x)$, $R(x)$ can be shown to satisfy the following variational principles:

$$R_n(x) = \inf\left\{\mathscr{E}(f) \,:\, f \text{ unit flux from } x \text{ to } S_n^c\right\}, \tag{1.8}$$

$$R(x) = \inf\left\{\mathscr{E}(f) \,:\, f \text{ unit flux from } x \text{ to } \infty\right\}. \tag{1.9}$$

In particular, one has the so called Royden–Lyons criterion [19] for reversible random walks: the random walk $X_n$ is transient if and only if there exists a unit flux on the resistor network from some point $x \in S$ to $\infty$ having finite energy. An immediate consequence of these facts is the following comparison tool, which we shall often use in the sequel.

**Lemma 1.1.** *Let $\mathbb{P}, \mathbb{P}'$ denote two point processes on $\mathbb{R}^d$ such that $\mathbb{P}$ is stochastically dominated by $\mathbb{P}'$ and let $\varphi, \varphi' : (0,\infty) \to (0,\infty)$ be such that $\varphi \leqslant C\varphi'$ for some constant $C > 0$. Suppose further that (1.1) is satisfied for both $(S,\varphi)$ and $(S',\varphi')$, where $S, S'$ denote the random sets distributed according to $\mathbb{P}$ and $\mathbb{P}'$, respectively. The following holds:*

1. *if $(S,\varphi)$ is transient $\mathbb{P}$–a.s., then $(S',\varphi')$ is transient $\mathbb{P}'$–a.s.*

2. *if $(S',\varphi')$ is recurrent $\mathbb{P}'$–a.s., then $(S,\varphi)$ is recurrent $\mathbb{P}$–a.s.*

*Proof.* The stochastic domination assumption is equivalent to the existence of a coupling of $\mathbb{P}$ and $\mathbb{P}'$ such that, almost surely, $S \subset S'$ (see e.g. [14] for more details). If $(S,\varphi)$ is transient then there exists a flux $f$ on $S$ with finite $(S,\varphi)$–energy from some $x \in S$ to infinity. We can lift $f$ to a flux on $S' \supset S$ (from the same $x$ to infinity) by setting it equal to 0 across pairs $x, y$ where either $x$ or $y$ (or both) are not in $S$. This has finite $(S',\varphi)$-energy, and since $\varphi \leqslant C\varphi'$ it will have finite $(S',\varphi')$–energy. This proves (1). The same argument proves (2) since if $S \subset S'$ were such that $(S,\varphi)$ is transient then $(S',\varphi')$ would be transient and we would have a contradiction. □

### 1.3 General results

Recall the notation $B_{x,t}$ for the open ball in $\mathbb{R}^d$ centered at $x$ with radius $t$ and define the function $\psi : (0,\infty) \to [0,1]$ by

$$\psi(t) := \sup_{x\in\mathbb{Z}^d} \mathbb{P}\big(S\big(B_{x,t}\big) = 0\big)\,. \tag{1.10}$$

**Theorem 1.2.** *(i) Let $d \geqslant 3$ and $\alpha > 0$, or $d = 1,2$ and $0 < \alpha < d$. Suppose that $\varphi \in \Phi_d$ and*

$$\varphi(t) \geqslant c\,\varphi_{p,\alpha}(t)\,, \tag{1.11}$$

$$\psi(t) \leqslant C\,t^{-\gamma}\,, \qquad \forall t > 0\,, \tag{1.12}$$

*for some positive constants $c, C$ and $\gamma > 3d + \alpha$. Then, $\mathbb{P}$–a.s. $(S,\varphi)$ is transient. (ii) Suppose that $d \geqslant 3$ and*

$$\int_0^\infty e^{a\,t^\beta}\psi(t)\,dt < \infty\,, \tag{1.13}$$

*for some $a, \beta > 0$. Then there exists $\delta = \delta(a,\beta) > 0$ such that $(S,\varphi)$ is a.s. transient whenever $\varphi(t) \geqslant c\,e^{-\delta\,t^\beta}$ for some $c > 0$. (iii) Set $d \geqslant 1$ and suppose that*

$$\sup_{x\in\mathbb{Z}^d} \mathbb{E}\left[S(Q_{x,1})^2\right] < \infty\,. \tag{1.14}$$

*Then $(S,\varphi)$ is $\mathbb{P}$–a.s. recurrent whenever $(\mathbb{Z}^d,\varphi_0)$ is recurrent, where $\varphi_0$ is given by*

$$\varphi_0(x,y) := \max_{u\in Q_{x,1},\, v\in Q_{y,1}} \varphi(|v-u|)\,. \tag{1.15}$$

The proofs of these general statements are given in Section 2. It relies on rather elementary arguments not far from the *rough embedding* method described in [20, Chapter 2]. In particular, to prove (i) and (ii) we shall construct a flux on $S$ from a point $x \in S$ to infinity and show that it has finite $(S,\varphi)$–energy under suitable assumptions. The flux will be constructed using comparison with suitable long–range random walks on $\mathbb{Z}^d$. Point (iii) of Theorem 1.2 is obtained by exhibiting a candidate for the electric potential in the network $(S,\varphi)$ that produces a vanishing conductance. Again the construction is achieved using comparison with long–range random walks on $\mathbb{Z}^d$.

Despite the simplicity of the argument, Theorem 1.2 already captures non–trivial facts such as, e.g., the transience of the super–critical percolation cluster in dimension two with $\varphi = \varphi_{\mathrm{p},\alpha}$, $\alpha < 2$. More generally, combining (i) and (iii) of Theorem 1.2 we shall obtain the following corollary.

**Corollary 1.3.** *Fix $d \geqslant 1$. Let $\mathbb{P}$ be one of the following point processes: a homogeneous PPP; the infinite cluster in super–critical Bernoulli site–percolation on $\mathbb{Z}^d$; a stationary DPP on $\mathbb{R}^d$. Then $(S, \varphi_{\mathrm{p},\alpha})$ has a.s. the same type as $(\mathbb{Z}^d, \varphi_{\mathrm{p},\alpha})$, for all $\alpha > 0$.*

We note that for the transience results (i) and (ii) we only need to check the sufficient conditions (1.12) and (1.13) on the function $\psi(t)$. Remarks on how to prove bounds on $\psi(t)$ for various processes are given in Subsection 2.2. Conditions (1.12) and (1.13) in Theorem 1.2 are in general far from being optimal. We shall give a bound that improves point (i) in the case $d = 1$, see Proposition 1.7 below. The limitations of Theorem 1.2 become more important when $\varphi$ is rapidly decaying and $d \geqslant 3$. For instance, if $\mathbb{P}$ is the law of the infinite percolation cluster, then $\psi(t)$ satisfies a bound of the form $e^{-c\,t^{d-1}}$, see Lemma 2.5 below. Thus in this case point (ii) would only allow to conclude that there exists $a = a(p) > 0$ such that, in $d \geqslant 3$, $(S, \varphi)$ is $\mathbb{P}$–a.s. transient if $\varphi(t) \geqslant C e^{-a\,t^{d-1}}$. However, the well known Grimmett–Kesten–Zhang theorem about the transience of *nearest–neighbor* random walk on the infinite cluster in $d \geqslant 3$ ([15], see also [5] for an alternative proof) together with Lemma 1.1 immediately implies that $(S, \varphi)$ is a.s. transient for any $\varphi \in \Phi_d$. Similarly, one can use stochastic domination arguments to improve point (ii) in Theorem 1.2 for other processes. To this end we say that the process $\mathbb{P}$ *dominates (after coarse–graining) super–critical Bernoulli site–percolation* if $\mathbb{P}$ is such that for some $L \in \mathbb{N}$ the random field

$$\sigma = \left(\sigma(x) \,:\, x \in \mathbb{Z}^d\right), \quad \sigma(x) := \chi\left(S(Q_{Lx,L}) \geqslant 1\right), \tag{1.16}$$

stochastically dominates the i.i.d. Bernoulli field on $\mathbb{Z}^d$ with some super–critical parameter $p$. Here $\chi(\cdot)$ stands for the indicator function of an event. In particular, it is easily checked that any homogeneous PPP dominates super–critical Bernoulli site–percolation. For DPP defined on $\mathbb{Z}^d$, stochastic domination w.r.t. Bernoulli can be obtained under suitable hypotheses on the kernel $K$, see [21]. We are not aware of analogous conditions in the continuum that would imply that DPP dominates super–critical Bernoulli site–percolation. In the latter cases we have to content ourselves with point (ii) of Theorem 1.2 (which implies point 3 in Corollary 1.4 below). We summarize our conclusions for $\varphi = \varphi_{\mathrm{e},\beta}$ in the following

**Corollary 1.4.** *1. Let $\mathbb{P}$ be any of the processes considered in Corollary 1.3. Then, for any $\beta > 0$, $(S, \varphi_{\mathrm{e},\beta})$ is a.s. recurrent when $d = 1, 2$. 2. Let $\mathbb{P}$ be the law of the infinite cluster in super–critical Bernoulli site–percolation on $\mathbb{Z}^d$ or a homogeneous PPP or any other process that dominates super–critical Bernoulli site–percolation. Then, for any $\beta > 0$, $(S, \varphi_{\mathrm{e},\beta})$ is a.s. transient when $d \geqslant 3$. 3. Let $\mathbb{P}$ be any stationary DPP. Then, for any $\beta \in (0, d)$, $(S, \varphi_{\mathrm{e},\beta})$ is $\mathbb{P}$–a.s. transient when $d \geqslant 3$*

We point out that, by the same proof, statement 2) above remains true if $(S, \varphi_{\mathrm{e},\beta})$ is replaced by $(S, \varphi)$, $\varphi \in \Phi_d$.

## 1.4 Bounds on finite volume effective resistances

When a network $(S, \varphi)$ is recurrent the effective resistances $R_n(x)$ associated with the finite sets $S_n := S \cap [-n, n]^d$ diverge, see (1.8), and we may be interested in obtaining quantitative information on their growth with $n$. We shall consider, in particular, the case of point processes in dimension $d = 1$, with $\varphi = \varphi_{\mathrm{p},\alpha}$, $\alpha \in [1, \infty)$, and the case $d = 2$ with $\varphi = \varphi_{\mathrm{p},\alpha}$, $\alpha \in [2, \infty)$. By Rayleigh's monotonicity principle, the bounds given below apply also to $(S, \varphi)$, whenever $\varphi \leqslant C\varphi_{\mathrm{p},\alpha}$. In particular, they cover the stretched exponential case $(S,\varphi_{\mathrm{e},\beta})$. We say that the point process $\mathbb{P}$ is *dominated by an i.i.d. field* if the following condition holds: There exists $L\in\mathbb{N}$ such that the random field

$$N_L=\big(N(v)\,:\,v\in\mathbb{Z}^d\big)\,,\quad N(v):=S(Q_{Lv,L})\,,$$

is stochastically dominated by independent non–negative random variables $\{\Gamma_v,\ v\in\mathbb{Z}^d\}$ with finite expectation. For the results in dimension $d=1$ we shall require the following exponential moment condition on the dominating field $\Gamma$: There exists $\varepsilon>0$ such that

$$\mathbb{E}[e^{\varepsilon\,\Gamma_v}]<\infty\,. \tag{1.17}$$

For the results in dimension $d=2$ it will be sufficient to require the existence of the fourth moment:

$$\mathbb{E}\left[\Gamma_v^4\right]<\infty\,. \tag{1.18}$$

It is immediate to check that any homogeneous PPP is dominated by an i.i.d. field in the sense described above and the dominating field $\Gamma$ satisfies (1.17). Moreover, this continues to hold for non–homogeneous Poisson process with a uniformly bounded intensity function. We refer the reader to [21; 14] for examples of determinantal processes satisfying this domination property.

**Theorem 1.5.** *Set $d=1$, $\varphi=\varphi_{\mathrm{p},\alpha}$ and $\alpha\geqslant 1$. Suppose that the point process $\mathbb{P}$ is dominated by an i.i.d. field satisfying (1.17). Then, for $\mathbb{P}$–a.a. $S$ the network $(S,\varphi)$ satisfies: given $x\in S$ there exists a constant $c>0$ such that*

$$R_n(x)\geqslant c\begin{cases}\log n & \text{if } \alpha=1\,,\\ n^{\alpha-1} & \text{if } 1<\alpha<2\,,\\ n/\log n & \text{if } \alpha=2,,\\ n & \text{if } \alpha>2\,,\end{cases} \tag{1.19}$$

*for all $n\geqslant 2$ such that $x\in S_n$.*

**Theorem 1.6.** *Set $d=2$, $\varphi=\varphi_{\mathrm{p},\alpha}$ and $\alpha\geqslant 2$. Suppose that $\mathbb{P}$ is dominated by an i.i.d. field satisfying (1.18). Then, for $\mathbb{P}$–a.a. $S$ the network $(S,\varphi)$ satisfies: given $x\in S$ there exists a constant $c>0$ such that*

$$R_n(x)\geqslant c\begin{cases}\log n & \text{if } \alpha>2\,,\\ \log(\log n) & \text{if } \alpha=2\,,\end{cases} \tag{1.20}$$

*for all $n\geqslant 2$ such that $x\in S_n$.*

The proofs of Theorem 1.5 and Theorem 1.6 are given in Section 3. The first step is to reduce the network $(S,\varphi)$ to a simpler network by using the domination assumption. In the proof of Theorem 1.5 the effective resistance of this simpler network is then estimated using the variational principle (1.6) with suitable trial functions. In the proof of Theorem 1.6 we are going to exploit a further reduction of the network that ultimately leads to a one–dimensional nearest–neighbor network where effective resistances are easier to estimate. This construction uses an idea that already appeared in [18]. – see also [7] and [1] for recent applications. The construction allows to go from long–range to nearest–neighbor networks, as explained in Section 3. Theorem 1.6 could be also proved using the variational principle (1.6) for suitable choices of the trial function, see the remarks in Section 3. It is worthy of note that the proofs of these results are constructive in the sense that they do not rely on results already known for the corresponding $(\mathbb{Z}^d,\varphi_{\mathrm{p},\alpha})$ network. In particular, the method can be used to obtain quantitative lower bounds on $R_n(x)$ for the deterministic case $S \equiv \mathbb{Z}^d$, which is indeed a special case of the theorems. In the latter case the lower bounds obtained here, as well as suitable upper bounds, are probably well known but we were not able to find references to them in the literature. In appendix B, we show how to bound from above the effective resistance $R_n(x)$ of the network $(\mathbb{Z}^d, \varphi_{\mathrm{p},\alpha})$ by means of harmonic analysis. The resulting upper bounds match the lower bounds of Theorems 1.5 and 1.6, with exception of the case $d = 1$, $\alpha = 2$, where our upper and lower bounds differ by a factor $\sqrt{\log n}$.

### 1.5 Constructive proofs of transience

While the transience criteria summarized in Corollary 1.3 and Corollary 1.4 are based on known results for the deterministic networks $(\mathbb{Z}^d, \varphi)$ obtained by classical harmonic analysis, it is possible to give constructive proofs of these results by exhibiting explicit fluxes with finite energy on the network under consideration. We discuss two results here in this direction. The first gives an improvement over the criterion in Theorem 1.2, part (i), in the case $d = 1$. This can be used, in particular, to give a "flux–proof" of the well known fact that $(\mathbb{Z}, \varphi_{\mathrm{p},\alpha})$ is transient for $\alpha < 1$. The second result gives a constructive proof of transience of a deterministic network, which, in turn, reasoning as in the proof of Theorem 1.2 part (i), gives a flux–proof that $(\mathbb{Z}^2, \varphi_{\mathrm{p},\alpha})$ is transient for $\alpha < 2$. In order to state the one-dimensional result, it is convenient to number the points of $S$ as $S = \{x_i\}_{i \in I}$ where $x_i < x_{i+1}$, $x_{-1} < 0 \leqslant x_0$ and $\mathbb{N} \subset I$ or $-\mathbb{N} \subset I$ (we assume that $|S| = \infty$, $\mathbb{P}$–a.s., since otherwise the network is recurrent). For simplicity of notation we assume below that $\mathbb{N} \subset I$, $\mathbb{P}$–a.s. The following result can be easily extended to the general case by considering separately the conditional probabilities $\mathbb{P}(\cdot|\mathbb{N} \subset I)$ and $\mathbb{P}(\cdot|\mathbb{N} \not\subset I)$, and applying a symmetry argument in the second case.

**Proposition 1.7.** *Take $d = 1$ and $\alpha \in (0,1)$. Suppose that the following holds for some positive constants $c, C$:*

$$\varphi(t) \geqslant c\varphi_{\mathrm{p},\alpha}(t), \qquad t > 0, \tag{1.21}$$

$$\mathbb{E}\big(|x_n - x_k|^{1+\alpha}\big) \leqslant C\,(n-k)^{1+\alpha}, \qquad \forall n > k \geqslant 0. \tag{1.22}$$

*Then $\mathbb{P}$–a.s. $(S, \varphi)$ is transient. In particular, if $\mathbb{P}$ is a renewal point process such that*

$$\mathbb{E}(|x_1 - x_0|^{1+\alpha}) < \infty, \tag{1.23}$$

*then $\mathbb{P}$–a.s. $(S, \varphi)$ is transient.*

Suppose that $\mathbb{P}$ is a renewal point process and write

$$\widetilde{\psi}(t) := \mathbb{P}(x_1 - x_0 \geqslant t).$$

Then (1.23) certainly holds as soon as e.g. $\widetilde{\psi}$ satisfies $\widetilde{\psi}(t) \leqslant C t^{-(1+\alpha+\varepsilon)}$ for some positive constants $C, \varepsilon$. We can check that this improves substantially over the requirement in Theorem 1.2, part (i), since if $\psi$ is defined by (1.10), then we have, for all $t > 1$:

$$\tilde{\psi}(2t) = \mathbb{P}(S \cap B_{x_0+t,t} = \emptyset) \leqslant \psi(t-1).$$

The next result concerns the deterministic two-dimensional network $(S_*, \varphi_{\mathrm{p},\alpha})$ defined as follows.

Identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, and define the set $S_* := \cup_{n=0}^{\infty} C_n$, where

$$C_n := \left\{ n e^{k \frac{2i\pi}{n+1}} \in \mathbb{C} : \ k \in \{0, \ldots, n\} \right\} . \tag{1.24}$$

**Theorem 1.8.** *The network $(S_*, \varphi_{\mathrm{p},\alpha})$ is transient for all $\alpha \in (0,2)$.*

This theorem, together with the comparison techniques developed in the next section (see Lemma 2.1 below), allows to recover by a flux–proof the transience of $(\mathbb{Z}^2, \varphi_{\mathrm{p},\alpha})$ for $\alpha \in (0,2)$. The proofs of Proposition 1.7 and Theorem 1.8 are given in Section 4.1.

## 2 Recurrence and transience by comparison methods

Let $S_0$ be a given locally finite subset of $\mathbb{R}^d$ and let $(S_0, \varphi_0)$ be a random walk on $S_0$. We assume that $w_{S_0}(x) < \infty$ for all $x \in S_0$ and that $\varphi_0(t) > 0$ for all $t > 0$. Recall that in the resistor network picture every node $\{x, y\}$ is given the resistance $r_0(x,y) := \varphi_0(|x-y|)^{-1}$. To fix ideas we may think of $S_0 = \mathbb{Z}^d$ and either $\varphi_0 = \varphi_{\mathrm{p},\alpha}$ or $\varphi_0 = \varphi_{\mathrm{e},\beta}$. $(S_0, \varphi_0)$ will play the role of the deterministic background network.

Let $\mathbb{P}$ denote a simple point process on $\mathbb{R}^d$, i.e. a probability measure on the set $\Omega$ of locally finite subsets $S$ of $\mathbb{R}^d$, endowed with the $\sigma$–algebra $\mathscr{F}$ generated by the counting maps $N_\Lambda : \Omega \to \mathbb{N} \cup \{0\}$, where $N_\Lambda(S) = S(\Lambda)$ is the number of points of $S$ that belong to $\Lambda$ and $\Lambda$ is a bounded Borel subset of $\mathbb{R}^d$. We shall use $S$ to denote a generic random configuration of points distributed according to $\mathbb{P}$. We assume that $\mathbb{P}$ and $\varphi$ are such that (1.1) holds. Next, we introduce a map $\phi : S_0 \to S$, from our reference set $S_0$ to the random set $S$. For any $x \in S_0$ we write $\phi(x) = \phi(S,x)$ for the closest point in $S$ according to Euclidean distance. If the Euclidean distance from $x$ to $S$ is minimized by more than one point in $S$ then choose $\phi(x)$ to be the lowest of these points according to lexicographic order. This defines a measurable map $\Omega \ni S \mapsto \phi(S,x) \in \mathbb{R}^d$ for every $x \in S_0$. For any point $u \in S$ define the *cell*

$$V_u := \{x \in S_0 \ : \ u = \phi(x)\} .$$

By construction $\{V_u \, , \ u \in S\}$ determines a partition of the original vertex set $S_0$. Clearly, some of the $V_u$ may be empty, while some may be large (if $S$ has large "holes" with respect to $S_0$). Let $N(u)$ denote the number of points (of $S_0$) in the cell $V_u$. We denote by $\mathbb{E}$ the expectation with respect to $\mathbb{P}$.

**Lemma 2.1.** *Suppose $(S_0, \varphi_0)$ is transient. If there exists $C < \infty$ such that, for all $x \neq y$ in $S_0$,*

$$\mathbb{E}\left[ N(\phi(x)) N(\phi(y))\, r(\phi(x), \phi(y)) \right] \leqslant C\, r_0(x,y) , \tag{2.1}$$

*then $(S, \varphi)$ is $\mathbb{P}$–a.s. transient.*

*Proof.* Without loss of generality we shall assume that $0 \in S_0$. Since $(S_0, \varphi_0)$ is transient, from the Royden–Lyons criterion recalled in Subsection 1.2, we know that there exists a unit flux $f : S_0 \times S_0 \to \mathbb{R}$ from $0 \in S_0$ to $\infty$ with finite $(S_0, \varphi_0)$-energy. By the same criterion, in order to prove the transience of $(S, \varphi)$ we only need to exhibit a unit flux from some point of $S$ to $\infty$ with finite $(S, \varphi)$-energy. To this end, for any $u, v \in S$ we define

$$\theta(u,v) = \sum_{x \in V_u} \sum_{y \in V_v} f(x,y) .$$

If either $V_u$ or $V_v$ are empty we set $\theta(u,v)=0$. Note that the above sum is finite for all $u,v\in S$, $\mathbb{P}$–a.s. Indeed condition (2.1) implies that $N(\phi(x))<\infty$ for all $x\in S_0$, $\mathbb{P}$–a.s. Thus, $\theta$ defines a unit flux from $\phi(0)$ to infinity on $(S,\varphi)$. Indeed, for every $u,v\in S$ we have $\theta(u,v)=-\theta(v,u)$ and for every $u\neq\phi(0)$ we have $\sum_{v\in S}\theta(u,v)=0$. Moreover,

$$\sum_{v\in S}\theta(\phi(0),v)=\sum_{x\in V_{\phi(0)}}\sum_{y\in S_0}f(x,y)=\sum_{\substack{x\in V_{\phi(0)}:\\ x\neq 0}}\sum_{y\in S_0}f(x,y)+\sum_{y\in S_0}f(0,y)=0+1=1\,.$$

The energy of the flux $\theta$ is given by

$$\mathscr{E}(\theta):=\frac{1}{2}\sum_{u\in S}\sum_{v\in S}\theta(u,v)^2r(u,v)\,. \tag{2.2}$$

From Schwarz' inequality

$$\theta(u,v)^2\leqslant N(u)N(v)\sum_{x\in V_u}\sum_{y\in V_v}f(x,y)^2\,.$$

It follows that

$$\mathscr{E}(\theta)\leqslant\frac{1}{2}\sum_{x\in S_0}\sum_{y\in S_0}f(x,y)^2N(\phi(x))N(\phi(y))\,r(\phi(x),\phi(y))\,. \tag{2.3}$$

Since $f$ has finite energy on $(S_0,\varphi_0)$ we see that condition (2.1) implies $\mathbb{E}[\mathscr{E}(\theta)]<\infty$. In particular, this shows that $\mathbb{P}$–a.s. there exists a unit flux $\theta$ from some point $u_0\in S$ to $\infty$ with finite $(S,\varphi)$-energy. $\square$To produce an analogue of Lemma 2.1 in the recurrent case we introduce the set $\widetilde{S}=S\cup S_0$ and consider the network $(\widetilde{S},\varphi)$. From monotonicity of resistor networks, recurrence of $(S,\varphi)$ is implied by recurrence of $(\widetilde{S},\varphi)$. We define the map $\phi':\widetilde{S}\to S_0$, from $\widetilde{S}$ to the reference set $S_0$ as the map $\phi$ introduced before, only with $S_0$ replaced by $\widetilde{S}$ and $S$ replaced by $S_0$. Namely, given $x\in\widetilde{S}$ we define $\phi'(x)$ as the closest point in $S_0$ according to Euclidean distance (when there is more than one minimizing point, we take the lowest of these points according to lexicographic order). Similarly, for any point $x\in S_0$ we define

$$V'_x:=\{u\in\widetilde{S}\,:\;x=\phi'(u)\}\,.$$

Thus $\{V'_x,\;x\in S_0\}$ determines a partition of $\widetilde{S}$. Note that in this case all $V'_x$ are non–empty ($V'_x$ contains $x\in\widetilde{S}$). As an example, if $S_0=\mathbb{Z}^d$ and $x\in\widetilde{S}$, then $\phi'(x)$ is the only point $y$ in $\mathbb{Z}^d$ such that $x\in y+(-1/2,1/2]^d$, while $V'_x=\widetilde{S}\cap(x+(-\frac{1}{2},\frac{1}{2}]^d)$ for any $x\in\mathbb{Z}^d$.

**Lemma 2.2.** *Suppose that $(S_0,\varphi_0)$ is recurrent and that $\mathbb{P}$–a.s. $V'_x$ is finite for all $x\in S_0$. If there exists $C<\infty$ such that, for all $x\neq y$ in $S_0$,*

$$\mathbb{E}\Big[\sum_{u\in V'_x}\sum_{v\in V'_y}\varphi(|u-v|)\Big]\leqslant C\,\varphi_0(|x-y|)\,, \tag{2.4}$$

*then $(S,\varphi)$ is $\mathbb{P}$–a.s. recurrent.*

*Proof.* Without loss of generality we shall assume that $0\in S_0$. Set $S_{0,n}=S_0\cap[-n,n]^d$, collapse all sites in $S_{0,n}^c=S_0\setminus S_{0,n}$ into a single site $z_n$ and call $c(S_{0,n})$ the effective conductance between 0 and

$z_n$, i.e. the net current flowing in the network when a unit voltage is applied across 0 and $z_n$. Since $(S_0, \varphi_0)$ is recurrent we know that $c(S_{0,n}) \to 0$, $n \to \infty$. Recall that $c(S_{0,n})$ satisfies

$$c(S_{0,n}) = \frac{1}{2} \sum_{x,y \in S_0} \varphi_0(|x-y|)(\psi_n(x) - \psi_n(y))^2 \,, \tag{2.5}$$

where $\psi_n$ is the electric potential, i.e. the unique function on $S_0$ that is harmonic in $S_{0,n}$, takes the value 1 at 0 and vanishes outside of $S_{0,n}$. Given $S \in \Omega$, set

$$\widetilde{S}_n = \cup_{x \in S_{0,n}} V'_x \,.$$

Note that $\widetilde{S}_n$ is an increasing sequence of finite sets, covering all $S$. Collapse all sites in $(\widetilde{S}_n)^c$ into a single site $\widetilde{z}_n$ and call $c(\widetilde{S}_n)$ the effective conductance between 0 and $\widetilde{z}_n$ (by construction $0 \in \widetilde{S}_n$). From the Dirichlet principle (1.6) we have

$$c(\widetilde{S}_n) \leqslant \frac{1}{2} \sum_{u,v \in \widetilde{S}} \varphi(|u-v|)(g(u) - g(v))^2 \,,$$

for any $g : \widetilde{S} \to [0,1]$ such that $g(0) = 1$ and $g = 0$ on $(\widetilde{S}_n)^c$. Choosing $g(u) = \psi_n(\phi'(u))$ we obtain

$$c(\widetilde{S}_n) \leqslant \frac{1}{2} \sum_{x,y \in S_0} (\psi_n(x) - \psi_n(y))^2 \sum_{u \in V'_x} \sum_{v \in V'_y} \varphi(|u-v|) \,.$$

From the assumption (2.4) and the recurrence of $(S_0, \varphi_0)$ implying that (2.5) goes to zero, we deduce that $\mathbb{E}[c(\widetilde{S}_n)] \to 0$, $n \to \infty$. Since $c(\widetilde{S}_n)$ is monotone decreasing we deduce that $c(\widetilde{S}_n) \to 0$, $\mathbb{P}$–a.s. This implies the $\mathbb{P}$–a.s. recurrence of $(\widetilde{S}, \varphi)$ and the claim follows. □

### 2.1 Proof of Theorem 1.2

We first prove part (i) of the theorem, by applying the general statement derived in Lemma 2.1 in the case $S_0 = \mathbb{Z}^d$ and $\varphi_0 = \varphi_{\mathrm{p},\alpha}$. Since $(S_0, \varphi_{\mathrm{p},\alpha})$ is transient whenever $d \geqslant 3$, or $d = 1,2$ and $0 < \alpha < d$ (the classical proof of these facts through harmonic analysis is recalled in Appendix B), we only need to verify condition (2.1). For the moment we only suppose that $\psi(t) \leqslant C' t^{-\gamma}$ for some $\gamma > 0$. Let us fix $p, q > 1$ s.t. $1/p + 1/q = 1$. Using Hölder's inequality and then Schwarz' inequality (or simply using Hölder inequality with the triple $(1/2q, 1/2q, 1/p)$), we obtain

$$\begin{aligned} &\mathbb{E}\left[N(\phi(x))N(\phi(y))\, r(\phi(x),\phi(y))\right] \\ &\qquad \leqslant \mathbb{E}\left[N(\phi(x))^{2q}\right]^{\frac{1}{2q}} \mathbb{E}\left[N(\phi(y))^{2q}\right]^{\frac{1}{2q}} \mathbb{E}\left[r(\phi(x),\phi(y))^p\right]^{\frac{1}{p}} \end{aligned} \tag{2.6}$$

for any $x \neq y$ in $\mathbb{Z}^d$. By assumption (1.11) we know that

$$r(\phi(x),\phi(y))^p \leqslant c\, r_{p,\alpha}(\phi(x),\phi(y))^p := c\left(1 \vee |\phi(x)-\phi(y)|^{p(d+\alpha)}\right). \tag{2.7}$$

We shall use $c_1, c_2, \dots$ to denote constants independent of $x$ and $y$ below. From Jensen's inequality

$$|\phi(x)-\phi(y)|^{p(d+\alpha)} \leqslant c_1 \left(|\phi(x)-x|^{p(d+\alpha)} + |x-y|^{p(d+\alpha)} + |\phi(y)-y|^{p(d+\alpha)}\right) \,.$$

From (2.7) and the fact that $|x-y|\geqslant 1$ we derive that

$$\mathbb{E}\left[r(\phi(x),\phi(y))^p\right] \leqslant c_2 \sup_{z\in\mathbb{Z}^d} \mathbb{E}\left[|\phi(z)-z|^{p(d+\alpha)}\right] + c_2|x-y|^{p(d+\alpha)}. \tag{2.8}$$

Now we observe that $|\phi(z)-z|\geqslant t$ if and only if $B_{z,t}\cap S=\emptyset$. Hence we can estimate

$$\mathbb{E}\left[|\phi(z)-z|^{p(d+\alpha)}\right] \leqslant 1+\int_1^\infty \psi\left(t^{\frac{1}{p(d+\alpha)}}\right)dt \leqslant 1+C\int_1^\infty t^{-\frac{\gamma}{p(d+\alpha)}}dt \leqslant c_3\,,$$

provided that $\gamma > p(d+\alpha)$. Therefore, using $|x-y|\geqslant 1$, from (2.8) we see that for any $x\neq y$ in $\mathbb{Z}^d$:

$$\mathbb{E}\left[r(\phi(x),\phi(y))^p\right]^{\frac{1}{p}} \leqslant c_4\, r_{p,\alpha}(x,y)\,. \tag{2.9}$$

Next, we estimate the expectation $\mathbb{E}\left[N(\phi(x))^{2q}\right]$ from above, uniformly in $x\in\mathbb{Z}^d$. To this end we shall need the following simple geometric lemma.

**Lemma 2.3.** *Let $E(x,t)$ be the event that $S\cap B(x,t)\neq\emptyset$ and $S\cap B(x\pm 3\sqrt{d}\,te_i,t)\neq\emptyset$, where $\{e_i : 1\leqslant i\leqslant d\}$ is the canonical basis of $\mathbb{R}^d$. Then, on the event $E(x,t)$ we have $\phi(x)\in B(x,t)$, i.e. $|\phi(x)-x|<t$, and $z\notin V_{\phi(x)}$ for all $z\in\mathbb{R}^d$ such that $|z-x|>9d\sqrt{d}\,t$*

Assuming for a moment the validity of Lemma 2.3 the proof continues as follows. From Lemma 2.3 we see that, for a suitable constant $c_5$, the event $N(\phi(x))>c_5t^d$ implies that at least one of the $2d+1$ balls $B(x,t)$, $B(x\pm 3\sqrt{d}te_i,t)$ must have empty intersection with $S$. Since $B(x\pm\lfloor 3\sqrt{d}t\rfloor e_i, t-1)\subset B(x\pm 3\sqrt{d}te_i,t)$ for $t\geqslant 1$, we conclude that

$$\mathbb{P}\left[N(\phi(x))>c_5t^d\right] \leqslant (2d+1)\psi(t-1)\,, \qquad t\geqslant 1.$$

Taking $c_6$ such that $c_5^{-\frac{1}{d}}c_6^{\frac{1}{2qd}}=2$, it follows that

$$\begin{aligned}\mathbb{E}\left[N(\phi(x))^{2q}\right] &= \int_0^\infty \mathbb{P}(N(\phi(x))^{2q}>t)dt \\ &\leqslant c_6+(2d+1)\int_{c_6}^\infty \psi\left(c_5^{-\frac{1}{d}}t^{\frac{1}{2qd}}-1\right)dt \leqslant c_6+c_7\int_1^\infty t^{-\frac{\gamma}{2qd}}dt \leqslant c_8\,, \end{aligned}\tag{2.10}$$

as soon as $\gamma>2qd$. Due to (2.6), (2.9) and (2.10), the hypothesis (2.1) of Lemma 2.1 is satisfied when $\psi(t)\leqslant C\,t^{-\gamma}$ for all $t\geqslant 1$, where $\gamma$ is a constant satisfying

$$\gamma > p(d+\alpha)\,, \qquad \gamma > 2qd = \frac{2pd}{p-1}\,.$$

We observe that the functions $(1,\infty)\ni p\mapsto p(d+\alpha)$ and $(1,\infty)\ni p\mapsto \frac{2pd}{p-1}$ are respectively increasing and decreasing and intersect in only one point $p_*=\frac{3d+\alpha}{d+\alpha}$. Hence, optimizing over $p$, it is enough to require that

$$\gamma > \inf_{p>1}\max\{p(d+\alpha), \frac{2pd}{p-1}\} = p_*(d+\alpha) = 3d+\alpha\,.$$

This concludes the proof of Theorem 1.2 (i). *Proof of lemma 2.3.* The first claim is trivial since $S\cap B(x,t)\neq\emptyset$ implies $\phi(x)\in B(x,t)$. In order to prove the second one we proceed as follows. For simplicity of notation we set $m:=3\sqrt{d}$ and $k:=9d$. Let us take $z\in\mathbb{R}^d$ with $|z-x|>k\sqrt{d}\,t$. Without loss of generality, we can suppose that $x=0$, $z_1>0$ and $z_1\geqslant|z_i|$ for all $i=2,3,\dots,d$. Note that this implies that $k\sqrt{d}\,t<|z|\leqslant\sqrt{d}z_1$, hence $z_1>kt$. Since

$$\min_{u\in B(0,t)}|z-u|=|z|-t$$
$$\max_{u\in B(mte_1,t)}|z-u|\leqslant|z-mte_1|+t\,,$$

if we prove that

$$|z|-|z-mte_1|>2t\,, \tag{2.11}$$

we are sure that the distance from $z$ to each point in $S\cap B(0,t)$ is larger than the distance from $z$ to each point of $S\cap B(mte_1,t)$. Hence it cannot be that $z\in V_{\phi(0)}$. In order to prove (2.11), we first observe that the map $(0,\infty)\ni x\mapsto\sqrt{x+a}-\sqrt{x+b}\in(0,\infty)$ is decreasing for $a>b$. Hence we obtain that

$$|z|-|z-mte_1|\geqslant\sqrt{d}z_1-\sqrt{(z_1-mt)^2+(d-1)z_1^2}\,.$$

Therefore, setting $x:=z_1/t$, we only need to prove that

$$\sqrt{d}x-\sqrt{(x-m)^2+(d-1)x^2}>2\,,\qquad\forall x>k\,.$$

By the mean value theorem applied to the function $f(x)=\sqrt{x}$

$$\sqrt{d}x-\sqrt{(x-m)^2+(d-1)x^2}\geqslant\frac{1}{2\sqrt{d}x}\left(dx^2-(x-m)^2-(d-1)x^2\right)=$$
$$\frac{2xm-m^2}{2\sqrt{d}x}>\frac{m}{\sqrt{d}}-\frac{m^2}{k}=2\,.$$

This completes the proof of (2.11). □

*Proof of Theorem 1.2 Part (ii).* We use the same approach as in Part (i) above. We start again our estimate from (2.6). Moreover, as in the proof of (2.10) it is clear that hypothesis (1.13) implies $\mathbb{E}[N(\phi(x))^{2q}]<\infty$ for any $q>1$, uniformly in $x\in\mathbb{Z}^d$. Therefore it remains to check that

$$r_0(x,y):=\mathbb{E}\left[r(\phi(x),\phi(y))^p\right]^{\frac{1}{p}}\,, \tag{2.12}$$

defines a transient resistor network on $\mathbb{Z}^d$, for any $d\geqslant3$, under the assumption that

$$r(\phi(x),\phi(y))\leqslant C\,e^{\delta|\phi(x)-\phi(y)|^\beta}\,.$$

For any $\beta>0$ we can find a constant $c_1=c_1(\beta)$ such that

$$r(\phi(x),\phi(y))\leqslant C\,\exp\left(\delta\,c_1\left[|\phi(x)-x|^\beta+|x-y|^\beta+|\phi(y)-y|^\beta\right]\right).$$

Therefore, using Schwarz' inequality we have

$$\mathbb{E}\left[r(\phi(x),\phi(y))^p\right]^{\frac{1}{p}}$$
$$\leqslant c_2\exp\left(\delta\,c_2|x-y|^\beta\right)\mathbb{E}\left[\exp\left(\delta\,c_2|\phi(x)-x|^\beta\right)\right]^{\frac{1}{2}}\mathbb{E}\left[\exp\left(\delta\,c_2|\phi(y)-y|^\beta\right)\right]^{\frac{1}{2}}\,.$$

For $\gamma > 0$

$$\mathbb{E}\left[\exp\left(\gamma|\phi(x)-x|^{\beta}\right)\right] \leqslant 1+\int_1^\infty \psi\left(\left(\frac{1}{\gamma}\log t\right)^{\frac{1}{\beta}}\right)dt$$
$$= 1+\gamma\beta\int_0^\infty \psi(s)\,e^{\gamma s^\beta}\,s^{\beta-1}\,ds\,,$$

where, using (1.13), the last integral is finite for $\gamma < a$. Taking $\gamma = \delta\, c_2$ and $\delta$ sufficiently smal we arrive at the conclusion that uniformly in $x$,

$$\mathbb{E}\left[r(\phi(x),\phi(y))^p\right]^{\frac{1}{p}} \leqslant c_3\exp\left(c_3|x-y|^\beta\right) =: \widetilde{r}_0(x,y)\,.$$

Clearly, $\widetilde{r}_0(x,y)$ defines a transient resistor network on $\mathbb{Z}^d$ and the same claim for $r_0(x,y)$ follows from monotonicity. This ends the proof of Part (ii) of Theorem 1.2. $\square$

*Proof of Theorem 1.2 Part (iii).* Here we use the criterion given in Lemma 2.2 with $S_0 = \mathbb{Z}^d$. With this choice of $S_0$ we have that $V'_x \subset \{x\}\cup(S\cap Q_{x,1})$, $x\in\mathbb{Z}^d$. Recalling definition (1.15) we see that for all $x\neq y$ in $\mathbb{Z}^d$,

$$\mathbb{E}\left[\sum_{u\in V'_x}\sum_{v\in V'_y}\varphi(|u-v|)\right] \leqslant c_1\,\varphi_0(x,y)\,\mathbb{E}\left[(1+S(Q_{x,1}))(1+S(Q_{y,1}))\right]\,.$$

Using Schwarz' inequality and condition (1.14) the last expression is bounded above by $c_2\,\varphi_0(x,y)$. This implies condition (2.4) and therefore the a.s. recurrence of $(S,\varphi)$. $\square$

## 2.2 Proof of Corollary 1.3

We start with some estimates on the function $\psi(t)$. Observe that, for Poisson point processes PPP($\lambda$), one has $\psi(t) = e^{-\lambda\, t^d}$. A similar estimate holds for a stationary DPP. More generally, for DPP we shall use the following facts.

**Lemma 2.4.** *Let $\mathbb{P}$ be a determinantal point process on $\mathbb{R}^d$ with kernel $K$. Then the function $\psi(t)$ defined in (1.10) equals*

$$\psi(t) = \sup_{x\in\mathbb{Z}^d}\prod_i\left(1-\lambda_i(B(x,t))\right), \tag{2.13}$$

*where $\lambda_i(B)$ denote the eigenvalues of $\mathscr{K}1_B$ for any bounded Borel set $B\subset\mathbb{R}^d$. In particular, condition (1.12) is satisfied if*

$$\exp\left\{-\int_{B(x,t)}K(u,u)du\right\} \leqslant C\,t^{-\gamma}\,, \qquad t>0\,,\ x\in\mathbb{Z}^d\,, \tag{2.14}$$

*for some constants $C>0$ and $\gamma > 3d+\alpha$. If $\mathbb{P}$ is a stationary DPP then $\psi(t)\leqslant e^{-\delta\, t^d}$, $t>0$, for some $\delta>0$. Finally, condition (1.14) reads*

$$\sup_{x\in\mathbb{Z}^d}\left\{\sum_i\lambda_i(Q(x,1))+\sum_i\sum_{j:\,j\neq i}\lambda_i(Q(x,1))\lambda_j(Q(x,1))\right\} < \infty\,. \tag{2.15}$$

*In particular, condition (1.14) always holds if $\mathbb{P}$ is a stationary DPP.*

*Proof.* It is known (see [4] and [24]) that, for each bounded Borel set $B \subset \mathbb{R}^d$, the number of points $S(B)$ has the same law as the sum $\sum_i B_i$, $B_i$'s being independent Bernoulli random variables with parameters $\lambda_i(B)$. This implies identity (2.13). Since $1 - x \leqslant e^{-x}$, $x \geqslant 0$, we can bound the r.h.s. of (1.10) by $\prod_i e^{-\lambda_i(B)} = e^{-Tr(\mathscr{K} 1_B)}$. This identity and (1.4) imply (2.14). If the DPP is stationary then $K(u,u) \equiv K(0) > 0$ and therefore $\psi(t) \leqslant e^{-K(0)\,t^d}$. Finally, (2.15) follows from the identity $\mathbb{E}[S(Q(x,1))^2] = \sum_i \lambda_i(Q(x,1)) + \sum_i \sum_{j:\, j\neq i} \lambda_i(Q(x,1))\lambda_j(Q(x,1))$, again a consequence of the fact that $S(Q(x,1))$ is the sum of independent Bernoulli random variables with parameters $\lambda_i(Q(x,1))$. Since the sum of the $\lambda_i(Q(x,1))$'s is finite, $\mathbb{E}[S(Q(x,1))^2] < \infty$ for any $x \in \mathbb{Z}^d$. If the DPP is stationary it is uniformly finite. □The next lemma allows to estimate $\psi(t)$ in the case of percolation clusters.

**Lemma 2.5.** *Let $\mathbb{P}$ be the law of the infinite cluster in super–critical Bernoulli site (or bond) percolation in $\mathbb{Z}^d$, $d \geqslant 2$. Then there exist constants $k, \delta > 0$ such that*

$$e^{-\delta^{-1} n^{d-1}} \leqslant \psi(n) \leqslant k\, e^{-\delta\, n^{d-1}}, \quad n \in \mathbb{N}.$$

*Proof.* The lower bound follows easily by considering the event that e.g. the cube centered at the origin with side $n$ has all the boundary sites (or bonds) unoccupied. To prove the upper bound one can proceed as follows. Let $K_n(\gamma)$, $\gamma > 0$, denote the event that there exists an open cluster $C$ inside the box $B(n) = [-n,n]^d \cap \mathbb{Z}^d$ such that $|C| \geqslant \gamma\, n^d$. Known estimates (see e.g. Lemma (11.22) in Grimmett's book [16] for the case $d = 2$ and Theorem 1.2 of Pisztora's [23] for $d \geqslant 3$) imply that there exist constants $k_1, \delta_1, \gamma > 0$ such that

$$\mathbb{P}(K_n(\gamma)^c) \leqslant k_1\, e^{-\delta_1\, n^{d-1}}. \tag{2.16}$$

On the other hand, let $C_x$ denote the open cluster at $x \in \mathbb{Z}^d$ and write $\mathscr{C}_\infty$ for the infinite open cluster. From [16, Theorem (8.65)] we have that there exist constants $k_2, \delta_2$ such that for any $x \in \mathbb{Z}^d$ and for any $\gamma > 0$:

$$\mathbb{P}(\gamma\, n^d \leqslant |C_x| < \infty) \leqslant k_2\, e^{-\delta_2\, n^{d-1}}. \tag{2.17}$$

Now we can combine (2.16) and (2.17) to prove the desired estimate. For any $n$ we write

$$\mathbb{P}(B(n) \cap \mathscr{C}_\infty = \emptyset) \leqslant \mathbb{P}(B(n) \cap \mathscr{C}_\infty = \emptyset\,;\, K_n(\gamma)) + \mathbb{P}(K_n(\gamma)^c).$$

The last term in this expression is bounded using (2.16). The first term is bounded by

$$\mathbb{P}(\exists x \in B(n)\,:\, \gamma\, n^d \leqslant |C_x| < \infty) \leqslant \sum_{x \in B(n)} \mathbb{P}(\gamma\, n^d \leqslant |C_x| < \infty).$$

Using (2.17) we arrive at $\mathbb{P}(B(n) \cap \mathscr{C}_\infty = \emptyset) \leqslant k\, e^{-\delta\, n^{d-1}}$ for suitable constants $k, \delta > 0$. □We are now ready to finish the proof of Corollary 1.3. It is clear from the previous lemmas that in all cases we have both conditions (1.12) and (1.14). Moreover it is easily verified that $(\mathbb{Z}^d, \varphi_0)$ and $(\mathbb{Z}^d, \varphi_{\mathrm{p},\alpha})$ have the same type, when $\varphi_0$ is defined by (1.15) with $\varphi = \varphi_{\mathrm{p},\alpha}$. This ends the proof of Corollary 1.3.

### 2.3 Proof of Corollary 1.4

It is easily verified that $(\mathbb{Z}^d, \varphi_0)$ and $(\mathbb{Z}^d, \varphi_{e,\beta})$ have the same type, when $\varphi_0$ is defined by (1.15) with $\varphi = \varphi_{e,\beta}$. Therefore the statement about recurrence follows immediately from Theorem 1.2, Part (iii) and the fact that in all cases (1.14) is satisfied (see the previous Subsection).

To prove the second statement we recall that our domination assumption and Strassen's theorem imply that on a suitable probability space $(\Omega, \mathscr{P})$ one can define the random field $(\sigma_1, \sigma_2) \in \{0,1\}^{\mathbb{Z}^d} \times \{0,1\}^{\mathbb{Z}^d}$ such that $\sigma_1$ has the same law as the infinite cluster in a super–critical Bernoulli site–percolation on $\mathbb{Z}^d$, $\sigma_2$ has the same law as the random field $\sigma$ defined in (1.16) and $\sigma_1 \leqslant \sigma_2$, $\mathscr{P}$–a.s., i.e. $\sigma_1(x) \leqslant \sigma_2(x)$ for all $x \in \mathbb{Z}^d$, $\mathscr{P}$–a.s. With each $\sigma_1$ we associate the nearest–neighbor resistor network $\mathscr{N}_1$ with nodes $\{x \in \mathbb{Z}^d \,:\, \sigma_1(x) = 1\}$ such that nearest–neighbor nodes are connected by a conductance of value $c_0 = c_0(L) > 0$ to be determined below. We know from [15] that $\mathscr{N}_1$ is transient a.s. Now, for each cube $Q_{Lx,L}$ intersecting $S$ we fix a point $\bar{x} \in Q_{Lx,L} \cap S$ (say the lowest one according to lexicographic order). If we keep all points $\bar{x}$'s belonging to the infinite cluster in $\sigma_2$ and neglect all other points of $S$ we obtain a subnetwork $(\widetilde{S}, \varphi)$ of $(S, \varphi)$. If $c_0$ is sufficiently small we have $\varphi(y,z) \geqslant c_0$ for all $y, z \in S$ such that $y \in Q_{Lx_1,L}$, $z \in Q_{Lx_2,L}$, $x_1, x_2 \in \mathbb{Z}^d$, $|x_1 - x_2| = 1$. Reasoning as in the proof of Lemma 1.1 then immediately implies the a.s. transience of $(S, \varphi)$. Note that this actually works for any $\varphi \in \Phi_d$.

To prove the third statement we observe that, for stationary DPP, $\psi(t) \leqslant e^{-\delta\, t^d}$, see Lemma 2.4. Therefore the claim follows from Theorem 1.2, Part (ii).

## 3 Lower bounds on the effective resistance

Assume that $\mathbb{P}$ is dominated by an i.i.d. field $\Gamma$ as stated before Theorem 1.5 and suppose the domination property holds with some fixed $L \in \mathbb{N}$. We shall write $Q_v$ for the cube $Q_{vL,L}$. To prove Theorem 1.5 and Theorem 1.6 we only need to show that, given $v_0 \in \mathbb{Z}^d$, for $\mathbb{P}$–a.a. $S$ there exists a positive constant $c$ such that, for all $x \in S \cap Q_{v_0 L,L}$, the lower bounds (1.19) and (1.20) on $R_n(x)$ hold. We restrict to $v_0 = 0$, since the general case can be treated similarly.

We start by making a first reduction of the network, which uses the stochastic domination assumption. This procedure works in any dimension $d$. First, we note that it is sufficient to prove the bounds in the two theorems for the quantity $\hat{R}_n(x)$, defined as the effective resistance from $x$ to $Q^c_{0,2Ln}$, instead of $R_n(x)$, which is the effective resistance from $x$ to $Q^c_{0,2n}$. In particular, there is no loss of generality in taking $L = 1$, in which case $\hat{R}_n(x) = R_n(x)$. The next observation is that, by monotonicity, $R_n(x)$ is larger than the same quantity computed in the network obtained by collapsing into a single node $v$ all points in each cube $Q_v$, $v \in \mathbb{Z}^d$. We now have a network with nodes on the points of $\mathbb{Z}^d$ (although some of them may be empty). Note that across two nodes $u, v$ we have $N_u N_v$ wires each with a resistance bounded from below by

$$\rho_{u,v} := c\, |u - v|^{d+\alpha}$$

for a suitable (non–random) constant $c > 0$. Moreover, using the stochastic domination assumption we know that $N_u \leqslant \Gamma_u$ for all $u \in \mathbb{Z}^d$, and we can further lower the resistance by considering the network where each pair of nodes $u, v$ is connected by $\Gamma_u \Gamma_v$ wires each with the resistance $\rho_{u,v}$. Moreover, we can further lower the resistance by adding a point to the origin. Hence, from now on, we understand that $\Gamma_u$ is replaced by $\Gamma_u + 1$ if $u = 0$. We call $(\Gamma, \rho)$ this new network. Thus the results will follow once we prove that, for $(\Gamma,\rho)$, the effective resistance from 0 to $Q^c_{0,2n} = \{u \in \mathbb{Z}^d : \|u\|_\infty > n\}$ satisfies the desired bounds. From now on we consider the cases $d = 1$ and $d = 2$ separately.

### 3.1 Proof of Theorem 1.5

Set $d = 1$. We further reduce the network $(\Gamma,\rho)$ introduced above by collapsing into a single node $\widetilde{v}$ each pair $\{v,-v\}$. This gives a network on $\{0,1,2,\dots\}$ where across each pair $0 \leqslant i < j$ there are now $\widetilde{\Gamma}_i\widetilde{\Gamma}_j$ wires, where $\widetilde{\Gamma}_i := \Gamma_i + \Gamma_{-i}$ $(i \neq 0)$ and $\widetilde{\Gamma}_0 := \Gamma_0$ (recall that by $\Gamma_0$ we now mean the original $\Gamma_0$ plus 1). Each of these wires has a resistance at least $\rho_{i,j}$ and thus we further reduce the network by assigning each wire the same resistance $\rho_{i,j}$. We shall call $(\widetilde{\Gamma},\rho)$ this new network and $\widetilde{R}_n(0)$ its effective resistance from 0 to $Q^c_{0,2n}$. An application of the variational formula (1.6) to the network $(\widetilde{\Gamma},\rho)$ yields the upper bound

$$\widetilde{R}_n(0)^{-1} = \widetilde{C}_n(0) \leqslant \frac{1}{f_n^2}\sum_{i=0}^{n}\sum_{j=i+1}^{\infty}\widetilde{\Gamma}_i\widetilde{\Gamma}_j\,(j-i)^{-1-\alpha}(f_j-f_i)^2\,, \tag{3.1}$$

for any sequence $\{f_i\}_{i\geqslant 0}$ such that $f_i = f(i)$, $f$ being a non–decreasing function on $[0,\infty)$ taking the value 0 at the origin only. Next, we choose $f$ as

$$f(x) := \int_0^x g_\alpha(t)dt, \quad g_\alpha(t) := \left(1+\int_0^t\left(1\wedge\frac{s^2}{s^{1+\alpha}}\right)ds\right)^{-1}. \tag{3.2}$$

Note that $f$ satisfies the differential equation

$$f'(t)^2\left(1+\int_0^t\left(1\wedge\frac{s^2}{s^{1+\alpha}}\right)ds\right) = f'(t)\,. \tag{3.3}$$

Moreover, $f$ is increasing on $[0,\infty)$, $f(0) = 0$ and $f_k = f(k)$ behaves as

$$f_k \sim \begin{cases} \log k & \text{if } \alpha = 1\,,\\ k^{\alpha-1} & \text{if } 1<\alpha<2\,,\\ k/\log k & \text{if } \alpha = 2\,,\\ k & \text{if } \alpha > 2\,.\end{cases} \tag{3.4}$$

Here $f_k \sim a_k$ means that there is a constant $C \geqslant 1$ such that $C^{-1}\,a_k \leqslant f_k \leqslant C\,a_k$, for all $k \geqslant C$. Since $g_\alpha$ is non–increasing we have the concavity bounds

$$f_j - f_i \leqslant g_\alpha(i)(j-i)\,, \qquad j \geqslant i \geqslant 0\,. \tag{3.5}$$

Let us first prove the theorem for the easier case $\alpha > 2$. We point out that here we do not need condition (1.17) and a finite first moment condition suffices. Indeed, set $\xi_i := \sum_{j>i}(j-i)^{1-\alpha}\widetilde{\Gamma}_j$. These random variables are identically distributed and have finite first moment since $\alpha > 2$. Note that $\widetilde{\Gamma}_i$ and $\xi_i$ are independent so that $\mathbb{E}[\Gamma_i\,\xi_i] < \infty$. From the ergodic theorem it follows that there exists a constant $C$ such that $\mathbb{P}$–a.s.

$$\sum_{i=0}^{n}\widetilde{\Gamma}_i\,\xi_i \leqslant C\,n\,,$$

for all $n$ sufficiently large. Due to (3.1), we conclude that $\widetilde{C}_n(0) \leqslant n^{-2} \sum_{i=0}^{n} \widetilde{\Gamma}_i \, \xi_i \leqslant C\, n^{-1}$ and the desired bound $\widetilde{R}_n(0) \geqslant c\, n$ follows. The case $1 \leqslant \alpha \leqslant 2$ requires more work. Thanks to our choice of $f$, we shall prove the following deterministic estimate.

**Lemma 3.1.** *There exists a constant $C < \infty$ such that for any $\alpha \geqslant 1$*

$$X_i := \sum_{j=i+1}^{\infty} (j-i)^{-1-\alpha} (f_j - f_i)^2 \leqslant C\, g_\alpha(i), \quad i \in \mathbb{N}. \tag{3.6}$$

Let us assume the validity of Lemma 3.1 for the moment and define the random variables

$$\xi_i := \sum_{j=i+1}^{\infty} \widetilde{\Gamma}_j \, (j-i)^{-1-\alpha} (f_j - f_i)^2 .$$

Let us show that $\mathbb{P}$-a.s. $\xi_i$ satisfies the same bound as $X_i$ in Lemma 3.1. Set $\Lambda(\lambda) := \log \mathbb{E}[e^{\lambda \widetilde{\Gamma}_i}]$. From assumption (1.17) we know $\Lambda(\lambda) < \infty$ for all $\lambda \leqslant \varepsilon$ for some $\varepsilon > 0$. Moreover, $\Lambda(\lambda)$ is convex and $\Lambda(\lambda) \leqslant c\, \lambda$ for some constant $c$, for all $\lambda \leqslant \varepsilon$. Therefore, using Lemma 3.1 we have, for some new constant $C$:

$$\begin{aligned}
\mathbb{E}[e^{a_i \xi_i}] &= \prod_{j>i} \exp\left[\Lambda(a_i (j-i)^{-1-\alpha} (f_j - f_i)^2)\right] \\
&\leqslant \prod_{j>i} \exp\left[c\, a_i (j-i)^{-1-\alpha} (f_j - f_i)^2\right] \\
&= \exp\left[c\, a_i X_i\right] \leqslant \exp\left[C\, a_i\, g_\alpha(i)\right],
\end{aligned} \tag{3.7}$$

provided the numbers $a_i > 0$ satisfy $a_i (j-i)^{-1-\alpha} (f_j - f_i)^2 \leqslant \varepsilon$ for all $j > i > 0$. Note that the last requirement is satisfied by the choice $a_i := \varepsilon / g_\alpha(i)^2$ since, using (3.5):

$$a_i (j-i)^{-1-\alpha} (f_j - f_i)^2 \leqslant a_i (j-i)^{1-\alpha} g_\alpha(i)^2 \leqslant a_i\, g_\alpha(i)^2 ,$$

for $j > i$, $\alpha \geqslant 1$. This will be our choice of $a_i$ for $1 \leqslant \alpha \leqslant 2$. From (3.7) we have

$$\begin{aligned}
\mathbb{P}(\xi_i > 2 c_1\, \varepsilon^{-1}\, g_\alpha(i)) &\leqslant \exp\left(-2 a_i c_1 \varepsilon^{-1} g_\alpha(i)\right) \mathbb{E}[e^{a_i \xi_i}] \\
&\leqslant \exp\left(-2 c_1\, \varepsilon^{-1}\, a_i\, g_\alpha(i)\right) \exp\left(C\, a_i\, g_\alpha(i)\right) \leqslant e^{-c_1\, \varepsilon\, g_\alpha(i)^{-1}} ,
\end{aligned} \tag{3.8}$$

if $c_1$ is large enough. Clearly, $g_\alpha(i) \leqslant C (\log i)^{-1}$ for $1 \leqslant \alpha \leqslant 2$ and $i$ large enough. Therefore, if $c_1$ is sufficiently large, the left hand side in (3.8) is summable in $i \in \mathbb{N}$ and the Borel Cantelli lemma implies that $\mathbb{P}$–a.s. we have $\xi_i \leqslant c_2\, g_\alpha(i)$, $c_2 := 2 c_1 \varepsilon^{-1}$, for all $i \geqslant i_0$, where $i_0$ is an a.s. finite random number. Next, we write

$$\sum_{i=0}^{n} \sum_{j=i+1}^{\infty} \widetilde{\Gamma}_i \widetilde{\Gamma}_j \, (j-i)^{-1-\alpha} (f_j - f_i)^2 \leqslant \sum_{i=0}^{i_0} \widetilde{\Gamma}_i \xi_i + c_2 \sum_{i=1}^{n} \widetilde{\Gamma}_i \, g_\alpha(i) . \tag{3.9}$$

The first term is an a.s. finite random number. The second term is estimated as follows. First, note that

$$\sum_{i=1}^{n} g_\alpha(i) \leqslant f_n , \tag{3.10}$$

since by concavity

$$f_n = \sum_{j=0}^{n-1} (f_{j+1} - f_j) \geqslant \sum_{j=0}^{n-1} g_\alpha(j+1) = \sum_{j=1}^{n} g_\alpha(i).$$

Then we estimate

$$\mathbb{P}(\sum_{i=1}^{n} \widetilde{\Gamma}_i \, g_\alpha(i) > 2c_3 \, f_n) \leqslant e^{-2c_3 f_n} \prod_{i=1}^{n} \mathbb{E}[e^{\widetilde{\Gamma}_i \, g_\alpha(i)}] = e^{-2c_3 f_n} \, e^{\sum_{i=1}^{n} \Lambda(g_\alpha(i))}.$$

If $i$ is large enough (so that $g_\alpha(i) \leqslant \varepsilon$) we can estimate $\Lambda(g_\alpha(i)) \leqslant c \, g_\alpha(i)$. Using (3.10) we then have, for $c_3$ large enough

$$\mathbb{P}(\sum_{i=1}^{n} \widetilde{\Gamma}_i \, g_\alpha(i) > 2c_3 \, f_n) \leqslant e^{-c_3 f_n}.$$

Since $f_n \geqslant \log n$ for all $\alpha \geqslant 1$ we see that, if $c_3$ is sufficiently large, the Borel Cantelli lemma implies that the second term in (3.9) is $\mathbb{P}$–a.s. bounded by $2c_3 \, f_n$ for all $n \geqslant n_0$ for some a.s. finite random number $n_0$. Recalling (3.1) we see that there exists an a.s. positive constant $c > 0$ such that $\widetilde{R}_n(0) \geqslant c \, f_n$. The proof of Theorem 1.5 is thus complete once we prove the deterministic estimate in Lemma 3.1.

*Proof of Lemma 3.1.* We only need to consider the cases $\alpha \in [1,2]$. We write

$$X_i = \sum_{j=i+1}^{2i} \frac{(f_j - f_i)^2}{(j-i)^{1+\alpha}} + \sum_{j>2i} \frac{(f_j - f_i)^2}{(j-i)^{1+\alpha}}. \tag{3.11}$$

We can estimate the first term of the right–hand side by using the concavity of $f$ and equation (3.3):

$$\sum_{j=i+1}^{2i} \frac{(f_j - f_i)^2}{(j-i)^{1+\alpha}} \leqslant g_\alpha^2(i) \sum_{k=1}^{i} \frac{k^2}{k^{1+\alpha}} \leqslant C \, g_\alpha(i) \tag{3.12}$$

for some positive constant $C$. As far as the second term is concerned, first observe that $(f_j - f_i)/(j-i)$ is non-increasing in $j$ (by the concavity of $f$) and so is the general term of the series. As a consequence

$$\sum_{j>2i} \frac{(f_j - f_i)^2}{(j-i)^{1+\alpha}} \leqslant \int_{2i}^{+\infty} \frac{(f(x) - f(i))^2}{(x-i)^{1+\alpha}} dx. \tag{3.13}$$

In the case $\alpha = 1$ we get, for any $i \geqslant 1$,

$$\sum_{j>2i} \frac{(f_j - f_i)^2}{(j-i)^{1+\alpha}} \leqslant \int_{2i}^{+\infty} \left( \frac{1}{x-i} \ln \frac{1+x}{1+i} \right)^2 dx \leqslant$$

$$\frac{1}{i} \int_{2i}^{+\infty} \left( \frac{1}{\frac{x}{i} - 1} \ln \frac{x}{i} \right)^2 \frac{dx}{i} \leqslant 2 g_\alpha(i) \int_{2}^{+\infty} \left( \frac{\ln t}{t-1} \right)^2 dt. \tag{3.14}$$

In the case $\alpha > 1$ we have,

$$\sum_{j>2i} \frac{(f_j - f_i)^2}{(j-i)^{1+\alpha}} \leqslant 2^{1+\alpha} \int_{2i}^{+\infty} \frac{f^2(x)}{x^{1+\alpha}} dx, \tag{3.15}$$

so that, for $1 < \alpha < 2$, there are two positive constants $C$ and $C'$ such that

$$\sum_{j>2i} \frac{(f_j - f_i)^2}{(j-i)^{1+\alpha}} \leqslant 2^{1+\alpha} \int_{2i}^{+\infty} C^2 \frac{x^{2\alpha-2}}{x^{1+\alpha}} dx \leqslant \frac{2^{1+\alpha} C^2}{2-\alpha} (2i)^{\alpha-2} \leqslant C' g_\alpha(i) \tag{3.16}$$

and, for $\alpha = 2$, there are two positive constants $C$ and $C'$ such that

$$\sum_{j>2i} \frac{(f_j - f_i)^2}{(j-i)^{1+\alpha}} \leqslant 8 \int_{2i}^{+\infty} \frac{C^2}{x \log^2 x} dx \leqslant \frac{8C^2}{\log 2i} \leqslant C' g_\alpha(i) . \tag{3.17}$$

□

### 3.2 Proof of Theorem 1.6

To prove Theorem 1.6 we shall make a series of network reductions that allow us to arrive at a nearest–neighbor, one–dimensional problem. We start from the network $(\Gamma, \rho)$ defined at the beginning of this section. We write $F_a = \{u \in \mathbb{Z}^2 : \|u\|_\infty = a\}$, $a \in \mathbb{N}$. The next reduction is obtained by collapsing all nodes $u \in F_a$ into a single node for each $a \in \mathbb{N}$. Once all nodes in each $F_a$ are identified we are left with a one-dimensional network with nodes $a \in \{0, 1, \dots\}$. Between nodes $a$ and $b$ we have a total of $\sum_{u \in F_a} \Gamma_u \sum_{v \in F_b} \Gamma_v$ wires, with a wire of resistance $\rho_{u,v}$ for each $u \in F_a$ and $v \in F_b$. Finally, we perform a last reduction that brings us to a nearest–neighbor one–dimensional network. To this end we consider a single wire with resistance $\rho_{u,v}$ between node $a$ and node $b$, with $a < b-1$. This wire is equivalent to a series of $(b-a)$ wires, each with resistance $\rho_{u,v}/(b-a)$. That is we can add $(b-a-1)$ fictitious points to our network in such a way that the effective resistance does not change. Moreover the effective resistance decreases if each added point in the series is attached to its corresponding node $a+i$, $i = 1, \dots, b-a-1$, in the network. If we repeat this procedure for each wire across every pair of nodes $a < b-1$ then we obtain a nearest–neighbor network where there are infinitely many wires in parallel across any two consecutive nodes. In this new network, across the pair $i-1, i$ we have a resistance $R_{i-1,i}$ such that

$$\phi_i := R_{i-1,i}^{-1} = \sum_{a<i} \sum_{b \geqslant i} \sum_{u \in F_a} \sum_{v \in F_b} (b-a) \Gamma_u \Gamma_v \rho_{u,v}^{-1} . \tag{3.18}$$

Moreover, the reductions described above show that

$$R_n(x) \geqslant \sum_{i=1}^{n+1} R_{i-1,i} .$$

Therefore Theorem 1.6 now follows from the estimates on $R_{i-1,i}$ given in the next lemma.

**Lemma 3.2.** *There exists a positive constant $c$ such that $\mathbb{P}$–almost surely, for $i$ sufficiently large*

$$R_{i,i+1} \geqslant c \begin{cases} i^{-1} & \textit{if } \alpha > 2 , \\ (i \log i)^{-1} & \textit{if } \alpha = 2 . \end{cases} \tag{3.19}$$

*Proof.* We first show that $\mathbb{E}(\phi_i) \leqslant C\omega_i$, where $\omega_i = i$ if $\alpha > 2$ and $\omega_i = i \log i$ if $\alpha = 2$, where $\mathbb{E}$ denotes expectation w.r.t. the field $\{\Gamma_u,\, u \in \mathbb{Z}^2\}$.

Thanks to Lemma A.1 given in the Appendix, from (3.18) we have

$$\mathbb{E}(\phi_i) \leqslant c_1 \sum_{a<i} \sum_{b \geqslant i} a(b-a)^{-\alpha}. \tag{3.20}$$

Next we estimate $\sum_{b \geqslant i}(b-a)^{-\alpha} \leqslant c_2\,(i-a)^{1-\alpha}$, so that, using the Riemann integral, we obtain

$$\begin{aligned}\mathbb{E}(\phi_i) &\leqslant c_2 \sum_{a<i} a(i-a)^{1-\alpha} = c_2 i^{2-\alpha} \sum_{a<i} \frac{a}{i}\left(1-\frac{a}{i}\right)^{1-\alpha} \\ &\leqslant c_3 i^{3-\alpha} \int_0^{1-\frac{1}{i}} y(1-y)^{1-\alpha} dy \leqslant c_3 i^{3-\alpha} \int_{1/i}^{1} y^{1-\alpha} dy \leqslant c_4\, \omega_i\,.\end{aligned}$$

Hence, for $C$ large we can estimate

$$\mathbb{P}(\phi_i \geqslant 2C\omega_i) \leqslant \mathbb{P}(\phi_i - \mathbb{E}(\phi_i) \geqslant C\omega_i) \leqslant (C\,\omega_i)^{-4} \mathbb{E}\left[\left(\phi_i - \mathbb{E}(\phi_i)\right)^4\right], \tag{3.21}$$

where we use $\mathbb{P}$ to denote the law of the variables $\{\Gamma_u\}$. The proof then follows from the Borel–Cantelli Lemma and the following estimate to be established below: There exists $C < \infty$ such that, for all $i \in \mathbb{N}$,

$$\mathbb{E}\left[\left(\phi_i - \mathbb{E}(\phi_i)\right)^4\right] \leqslant C\, i^2. \tag{3.22}$$

To prove (3.22) we write

$$\mathbb{E}\left[\left(\phi_i - \mathbb{E}(\phi_i)\right)^4\right] = \sum_{\mathbf{a}} \sum_{\mathbf{b}} \sum_{\mathbf{u}\sim\mathbf{a}} \sum_{\mathbf{v}\sim\mathbf{b}} \Phi(\mathbf{u},\mathbf{v})\, G(\mathbf{u},\mathbf{v}), \tag{3.23}$$

where the sums are over $\mathbf{a} = (a_1, \dots, a_4)$, $\mathbf{b} = (b_1, \dots, b_4)$ such that $a_k < i \leqslant b_k$, $\mathbf{u} \sim \mathbf{a}$ stands for the set of $\mathbf{u} = (u_1, \dots, u_4)$ such that $u_k \in F_{a_k}$, and we have defined, for $\mathbf{u} \sim \mathbf{a}$, $\mathbf{v} \sim \mathbf{b}$:

$$\Phi(\mathbf{u},\mathbf{v}) = \prod_{k=1}^{4} (b_k - a_k) \rho_{u_k, v_k}, \quad G(\mathbf{u},\mathbf{v}) = \prod_{k=1}^{4} \left(\Gamma_{u_k}\Gamma_{v_k} - \mathbb{E}[\Gamma_{u_k}\Gamma_{v_k}]\right).$$

From the independence assumption on the field $\{\Gamma_u\}$ we know that $G(\mathbf{u},\mathbf{v}) = 0$ unless for every $k = 1, \dots, 4$ there exists a $k' = 1, \dots, 4$ with $k \neq k'$ and $\{u_k, v_k\} \cap \{u_{k'}, v_{k'}\} \neq \emptyset$. Moreover, when this condition is satisfied using (1.18) we can easily bound $G(\mathbf{u},\mathbf{v})$ from above by constant $C$. By symmetry we may then estimate

$$\begin{aligned}&\sum_{\mathbf{a}} \sum_{\mathbf{b}} \sum_{\mathbf{u}\sim\mathbf{a}} \sum_{\mathbf{v}\sim\mathbf{b}} \Phi(\mathbf{u},\mathbf{v})\, G(\mathbf{u},\mathbf{v}) \\ &\leqslant C \sum_{\mathbf{a}} \sum_{\mathbf{b}} \sum_{\mathbf{u}\sim\mathbf{a}} \sum_{\mathbf{v}\sim\mathbf{b}} \Phi(\mathbf{u},\mathbf{v})\, \chi\left(\forall k\, \exists k' \neq k :\ \{u_k, v_k\} \cap \{u_{k'}, v_{k'}\} \neq \emptyset\right) \\ &\leqslant 3\,C \sum_{\mathbf{a}} \sum_{\mathbf{b}} \sum_{\mathbf{u}\sim\mathbf{a}} \sum_{\mathbf{v}\sim\mathbf{b}} \Phi(\mathbf{u},\mathbf{v}) \Big[\chi\left(u_1 = u_2\,;\ u_3 = u_4\right) + \\ &\qquad\qquad + \chi\left(u_1 = u_2\,;\ v_3 = v_4\right) + \chi\left(v_1 = v_2\,;\ v_3 = v_4\right)\Big].\end{aligned} \tag{3.24}$$

We claim that each of the three terms in the summation above is of order $i^2$ as $i$ grows. This will prove the desired estimate (3.22). The first term in (3.24) satisfies

$$\sum_{\mathbf{a}}\sum_{\mathbf{b}}\sum_{\mathbf{u}\sim\mathbf{a}}\sum_{\mathbf{v}\sim\mathbf{b}}\Phi(\mathbf{u},\mathbf{v})\,\chi\left(u_1=u_2\,;\,u_3=u_4\right)\leqslant A(i)^2\,, \tag{3.25}$$

where

$$A(i):=\sum_{a_1<i}\sum_{b_1\geqslant i}\sum_{b_2\geqslant i}(b_1-a_1)(b_2-a_1)\sum_{u_1\in F_{a_1}}\sum_{v_1\in F_{b_1}}\sum_{v_2\in F_{b_2}}\rho_{u_1,v_1}\rho_{u_1,v_2}\,.$$

Similarly the third term in (3.24) is bounded above by $B(i)^2$, with

$$B(i):=\sum_{a_1<i}\sum_{a_2<i}\sum_{b_1\geqslant i}(b_1-a_1)(b_1-a_2)\sum_{u_1\in F_{a_1}}\sum_{u_2\in F_{a_2}}\sum_{v_1\in F_{b_1}}\rho_{u_1,v_1}\rho_{u_2,v_1}\,.$$

Finally, the middle term in (3.24) is less than or equal to the product $A(i)B(i)$. Therefore, to prove (3.22) it suffices to show that $A(i)\leqslant C\,i$ and $B(i)\leqslant C\,i$. Using Lemma A.1 we see that

$$\sum_{u_1\in F_{a_1}}\sum_{v_1\in F_{b_1}}\sum_{v_2\in F_{b_2}}\rho_{u_1,v_1}\rho_{u_1,v_2}\leqslant C\,a_1\,(b_1-a_1)^{-1-\alpha}\,(b_2-a_1)^{-1-\alpha}\,.$$

This bound yields

$$A(i)\leqslant C\sum_{a_1<i}a_1(i-a_1)^{2-2\alpha}\,,$$

for some new constant $C$, where we have used the fact that

$$\sum_{b_2\geqslant i}(b_2-a_1)^{-\alpha}\leqslant C(i-a_1)^{1-\alpha}\,.$$

Using the Riemann integral we obtain

$$\begin{aligned}\sum_{a_1<i}a_1(i-a_1)^{2-2\alpha}&\leqslant C\,i^{4-2\alpha}\int_0^{1-1/i}x(1-x)^{2-2\alpha}dx\\&\leqslant C\,i^{4-2\alpha}\int_{1/i}^{1}x^{2-2\alpha}dx\leqslant(2\alpha-3)C\,i\,.\end{aligned}$$

This proves that $A(i)=O(i)$. Similarly, from Lemma A.1 we see that

$$\sum_{u_1\in F_{a_1}}\sum_{u_2\in F_{a_2}}\sum_{v_1\in F_{b_1}}\rho_{u_1,v_1}\rho_{u_2,v_1}\leqslant C\,b_1^{-1}\,a_1\,a_2\,(b_1-a_1)^{-1-\alpha}\,(b_1-a_2)^{-1-\alpha}\,.$$

Therefore

$$\begin{aligned}B(i)&\leqslant C\sum_{b_1\geqslant i}b_1^{-1}\left[\sum_{a_1<i}a_1(b_1-a_1)^{-\alpha}\right]^2\\&\leqslant C'\,i^2\sum_{b_1\geqslant i}b_1^{-1}(b_1-i+1)^{2-2\alpha}\leqslant C''\,i\,,\end{aligned}$$

where we have used the estimate

$$\sum_{a_1<i} a_1(b_1-a_1)^{-\alpha} \leqslant i \sum_{a_1<i}(b_1-a_1)^{-\alpha} \leqslant C\, i\, (b_1-i+1)^{1-\alpha}\,,$$

and the fact that, for $\alpha \geqslant 2$, we have

$$\sum_{b_1 \geqslant i} b_1^{-1}(b_1-i+1)^{2-2\alpha} \leqslant i^{-1}\sum_{k=1}^{\infty} k^{-2} = C/i\,.$$

□

We remark that a proof of Theorem 1.6 could be obtained by application of the variational principle (1.6) as in the proof of Theorem 1.5. To see this one can start from the network $(\Gamma,\rho)$ introduced at the beginning of this section and choose a trial function that is constant in each $F_a$. Then, for any non–decreasing sequence $(f_0, f_1, \dots)$ such that $f_0 = 0$ and $f_k > 0$ eventually, one has $R_n(x) \geqslant A_n(f)$ where

$$A_n(f) = f_n^{-2}\sum_{a=0}^{n}\sum_{b=a+1}^{\infty}(f_b-f_a)^2\sum_{u\in F_a}\Gamma_u\sum_{v\in F_b}\Gamma_v\,|v-u|^{-2-\alpha}\,. \tag{3.26}$$

One then choose $f_k = \log(1+k)$ for $\alpha > 2$ and $f_k = \log(\log(e+k))$ for $\alpha = 2$ and the desired conclusions will follow from suitable control of the fluctuations of the random sum appearing in (3.26). Here the analysis is slightly more involved than that in the proof of Theorem 1.5 and it requires estimates as in (3.24) above. Moreover, one needs a fifth moment assumption with this approach instead of the fourth moment condition (1.18). Under this assumption, and using Lemma A.1, it is possible to show that there a.s. exists a constant $c$ such that

$$\sum_{u\in F_a}\Gamma_u\sum_{v\in F_b}\Gamma_v\,|v-u|^{-2-\alpha} \leqslant c\, a\,(b-a)^{-1-\alpha}\,, \quad a<b\,. \tag{3.27}$$

Once this estimate is available the proof follows from simple calculations.

# 4 Proof of Proposition 1.7 and Theorem 1.8

## 4.1 Proof of Proposition 1.7

The proof of Proposition 1.7 is based on the following technical lemma related to renewal theory:

**Lemma 4.1.** *Given $\delta > 1$, define the probability kernel*

$$q_k = c(\delta)k^{-\delta} \qquad k\in\mathbb{N}\,, \tag{4.1}$$

*($c(\delta)$ being the normalizing constant $1/\sum_{k\geqslant 1}k^{-\delta}$) and define recursively the sequence $f(n)$ as*

$$\begin{cases} f(0)=1\,,\\ f(n)=\sum_{k=0}^{n-1}f(k)q_{n-k}\,, \qquad n\in\mathbb{N}\,.\end{cases}$$

*If $1<\delta<2$, then*

$$\lim_{n\uparrow\infty} n^{2-\delta} f(n) = \frac{\Gamma(2-\delta)}{\Gamma(\delta-1)}\,. \tag{4.2}$$

*Proof.* Let $\{X_i\}_{i\geqslant 1}$ be a family of IID random variables with $P(X_i = k) = q_k$, $k \in \mathbb{N}$. Observe now that $P(X_i \geqslant k) = \sum_{s=k}^{\infty} q_s \sim c\,k^{1-\delta}$ since $\delta > 1$. In particular, if $1 < \delta < 2$ we can use Theorem B of [10] and get (4.2) with $u(n)$ instead $f(n)$, where $u(n)$ is defined as follows: Consider the random walk $S_n$ on the set $\mathbb{N}\cup\{0\}$, starting at 0, $S_0 = 0$, and defined as $S_n = X_1 + X_2 + \cdots + X_n$ for $n \geqslant 1$. Given $n \in \mathbb{N}$ define $u(n)$ as

$$u(n) := \mathbb{E}\left[\Big|\{m \geqslant 0 \,:\, S_m = n\}\Big|\right] = \sum_{m=0}^{\infty} P(S_m = n)\,.$$

Trivially $u(0) = 1$, while the Markov property of the random walk $S_n$ gives for $n \geqslant 1$ that

$$u(n) = \sum_{m=1}^{\infty}\sum_{k=0}^{n-1} P(S_{m-1} = k,\ S_m = n) = \sum_{m=1}^{\infty}\sum_{k=0}^{n-1} P(S_{m-1} = k) q_{n-k} = \sum_{k=0}^{n-1} u(k) q_{n-k}\,.$$

Hence, $f(n)$ and $u(n)$ satisfy the same system of recursive identities and coincide for $n = 0$, thus implying that $f(n) = u(n)$ for each $n \in \mathbb{N}$. □

We have now all the tools in order to prove Proposition 1.7:

*Proof of Proposition 1.7.* We shall exhibit a finite energy unit flux $f(\cdot,\cdot)$ from $x_0$ to infinity in the network $(S,\varphi)$. To this end we define $f(\cdot,\cdot)$ as follows

$$f(x_i, x_k) = \begin{cases} f(i) q_{k-i} & \text{if } 0 \leqslant i < k\,, \\ -f(x_k, x_i) & \text{if } 0 \leqslant k < i\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{4.3}$$

where $f(m), q_m$ are defined as in the previous lemma for some $\delta \in (1,2)$ that will be fixed below. Since $\varphi \geqslant C\varphi_{\mathrm{p},\alpha}$, the energy $\mathscr{E}(f)$ dissipated by the flux $f(\cdot,\cdot)$ is

$$\mathscr{E}(f) = \sum_{n=0}^{\infty}\sum_{k=n+1}^{\infty} \frac{f(x_n, x_k)^2}{\varphi(|x_n - x_k|)} \leqslant c \sum_{n=0}^{\infty}\sum_{k=n+1}^{\infty} r_{p,\alpha}(x_k, x_n)(f_n q_{k-n})^2\,, \tag{4.4}$$

where $r_{p,\alpha}(x,y) := 1/\varphi_{\mathrm{p},\alpha}(|x-y|)$. Hence, due to the previous lemma we obtain that

$$\mathscr{E}(f) \leqslant c \sum_{n=0}^{\infty}\sum_{k=n+1}^{\infty} r_{p,\alpha}(x_k, x_n)(1+n)^{2\delta-4}(k-n)^{-2\delta}\,.$$

In order to prove that the energy $\mathscr{E}(f)$ is finite $\mathbb{P}$–a.s., it is enough to show that $\mathbb{E}(\mathscr{E}(f))$ is finite for some $\delta \in (1,2)$. To this end we observe that, due to assumption (1.22) and since $r_{p,\alpha}(x_k, x_n) = 1 \vee (x_n - x_k)^{1+\alpha}$, for suitable constants $c_1, c_2$,

$$\begin{aligned} \mathbb{E}(\mathscr{E}(f)) &\leqslant c_1 \sum_{n=0}^{\infty} (1+n)^{2\delta-4} \sum_{u=1}^{\infty} \left[1 + \mathbb{E}\big(|x_u - x_0|^{1+\alpha}\big)\right] u^{-2\delta} \\ &\leqslant c_2 \Big(\sum_{n=1}^{\infty} (1+n)^{2\delta-4}\Big)\Big(\sum_{u=1}^{\infty} u^{1+\alpha-2\delta}\Big)\,. \end{aligned} \tag{4.5}$$

Hence, the mean energy is finite if $2\delta-4<-1$ and $1+\alpha-2\delta<-1$. In particular, for each $\alpha\in(0,1)$ one can fix $\delta\in(1,2)$ satisfying the conditions above. This concludes the proof of the transience of $(S,\varphi)$ for $\mathbb{P}$–a.a. $S$. It remains to verify assumption (1.22) whenever $\mathbb{P}$ is a renewal point process such that $\mathbb{E}((x_1-x_0)^{1+\alpha})<\infty$. To this end we observe that, by convexity,

$$(x_u-x_0)^{1+\alpha}=u^{1+\alpha}\Big(\frac{1}{u}\sum_{k=0}^{u-1}(x_{k+1}-x_k)\Big)^{1+\alpha}\leqslant u^{1+\alpha}\Big(\frac{1}{u}\sum_{k=0}^{u-1}(x_{k+1}-x_k)^{1+\alpha}\Big).$$

Since by the renewal property $(x_{k+1}-x_k)_{k\geqslant 0}$ is a sequence of i.i.d. random variables, the mean of the last expression equals $u^{1+\alpha}\mathbb{E}((x_1-x_0)^{1+\alpha})=cu^{1+\alpha}$. Therefore, (1.22) is satisfied. □

## 4.2 Proof of Theorem 1.8

We recall that $S_*=\cup_{n\geqslant 0}C_n$, where

$$C_n:=\Big\{ne^{k\frac{2i\pi}{n+1}}\in\mathbb{C}:\ k\in\{0,\dots,n\}\Big\} \tag{4.6}$$

and $\mathbb{C}$ is identified with $\mathbb{R}^2$. In order to introduce more symmetries we consider a family of "rotations" of the $C_n$'s: given a sequence $\theta=(\theta_n)_{n\geqslant 0}$ of independent random variables with uniform law on $(-\frac{\pi}{n+1},+\frac{\pi}{n+1})$ we define

$$C_n^\theta:=e^{i\theta_n}C_n \tag{4.7}$$

and for $x$ in $C_n$ we use the notation

$$x^\theta:=e^{i\theta_n}x\in C_n^\theta \tag{4.8}$$

We will construct a unit flow $f^\theta$ from 0 to infinity on $S_*^\theta:=\cup_n C_n^\theta$ and will make an average over $\theta$ to build a new flow $f$ on $S$. In order to describe the flow $f^\theta$, we consider the probability kernel $q_k=c(\delta)k^{-\delta}$, $\delta\in(1,2)$, introduced in Lemma 4.1. The value of $\delta$ will be chosen at the end. We build $f^\theta$ driving a fraction $q_{n-m}$ of the total flow arriving in a site $x^\theta\in C_m^\theta$ to each $C_n^\theta$ with $n>m$, in such a way that, for each site $y\in C_n^\theta$, the flow received from $x^\theta$ is proportional to $\varphi_{\mathrm{p},\alpha}(x^\theta,y^\theta)$. We have then, for all $n>m$, $x\in C_m$ and $y\in C_n$

$$f^\theta(x^\theta,y^\theta)=q_{n-m}\frac{\varphi_{\mathrm{p},\alpha}(x^\theta,y^\theta)}{Z_n^\theta(x^\theta)}f^\theta(x^\theta) \tag{4.9}$$

with

$$Z_n^\theta(x^\theta):=\sum_{y\in C_n}\varphi_{\mathrm{p},\alpha}(x^\theta,y^\theta) \tag{4.10}$$

and $f^\theta(\cdot)$ defined recursively as

$$f^\theta(y^\theta)=\begin{cases}1 & \text{if } y=0,\\ \sum_{m<n}\sum_{x\in C_m}q_{n-m}\frac{\varphi_{\mathrm{p},\alpha}(x^\theta,y^\theta)}{Z_n^\theta(x^\theta)}f^\theta(x^\theta) & \text{if } y\in C_n,\ n>0.\end{cases} \tag{4.11}$$

Note that the quantity

$$f_n:=\sum_{y\in C_n}f^\theta(y^\theta) \tag{4.12}$$

is independent of $\theta$ and it is defined recursively by

$$\begin{cases} f_0 = 1 \\ f_n = \sum_{m<n} q_{n-m} f_m \,, \; n > 0 \,. \end{cases}$$

By Lemma 4.1 and the condition $\delta \in (1,2)$ we have

$$f_n \sim c\, n^{\delta-2} \,, \qquad n \geqslant 1 \,. \tag{4.13}$$

We can now define our flow $f$ on $(S_*, \varphi_{\mathrm{p},\alpha})$. For all $m < n$, $x \in C_m$ and $y \in C_n$ we set

$$f(x,y) := \mathbb{E}\left[ f^\theta(x^\theta, y^\theta) \right] \,, \tag{4.14}$$

where the expectation is w.r.t. $\theta$. Taking the conditional expectation in (4.9) we get

$$\mathbb{E}\left[ f^\theta(x^\theta, y^\theta) \middle| \theta_m, \theta_n \right] = q_{n-m} \frac{\varphi_{\mathrm{p},\alpha}(x^\theta, y^\theta)}{Z_n^\theta(x^\theta)} \mathbb{E}\left[ f^\theta(x^\theta) \middle| \theta_m \right] \,. \tag{4.15}$$

By radial symmetry the last factor does not depends on $x$ or $\theta_m$ and taking the conditional expectation in (4.12) we get

$$\mathbb{E}\left[ f^\theta(x^\theta) \middle| \theta_m \right] = \frac{f_m}{m+1} \,. \tag{4.16}$$

Recallin (4.14) and taking the expectation of both sides of (4.15), we see that

$$f(x,y) = q_{n-m} \mathbb{E}\left[ \frac{\varphi(x^\theta, y^\theta)}{Z_n^\theta(x^\theta)} \right] \frac{f_m}{m+1} \,. \tag{4.17}$$

By means of this formula it is simple to estimate the energy $\mathscr{E}(f)$ dissipated by the flux $f(\cdot,\cdot)$ in the network. Indeed, we can write

$$\begin{aligned} \mathscr{E}(f) &= \sum_{m<n} \sum_{x \in C_m} \sum_{y \in C_n} \frac{f^2(x,y)}{\varphi_{\mathrm{p},\alpha}(x,y)} \\ &= \sum_{m<n} \sum_{x \in C_m} \sum_{y \in C_n} \frac{q_{n-m}^2}{\varphi_{\mathrm{p},\alpha}(x,y)} \mathbb{E}\left[ \frac{\varphi_{\mathrm{p},\alpha}(x^\theta, y^\theta)}{Z_n^\theta(x^\theta)} \right]^2 \frac{f_m^2}{(m+1)^2} \,. \end{aligned}$$

Now we observe that

$$|x - x^\theta| \leqslant \pi \,, \qquad \forall x \in C_n \tag{4.18}$$

thus implying that one can find $a > 1$ such that, for all $x \neq y$ in $S_*$,

$$a^{-1} \varphi_{\mathrm{p},\alpha}(x,y) \leqslant \varphi_{\mathrm{p},\alpha}(x^\theta, y^\theta) \leqslant a \varphi_{\mathrm{p},\alpha}(x,y) \tag{4.19}$$

As a consequence, setting

$$Z_n(x) := \sum_{y \in C_n} \varphi_{\mathrm{p},\alpha}(x,y) \tag{4.20}$$

we get

$$\mathcal{E}(f) \leqslant \sum_{m<n}\sum_{x\in C_m}\sum_{y\in C_n} \frac{a^4 q_{n-m}^2}{\varphi_{\mathrm{p},\alpha}(x,y)} \frac{\varphi_{\mathrm{p},\alpha}^2(x,y)}{Z_n^2(x)} \frac{f_m^2}{(m+1)^2} \tag{4.21}$$

$$= \sum_{m<n}\sum_{x\in C_m} \frac{a^4 q_{n-m}^2}{Z_n(x)} \frac{f_m^2}{(m+1)^2}\,. \tag{4.22}$$

By Lemma A.1, there exists a constant $c>0$ such that, for all $x\in C_m$ and $n>m$,

$$Z_n(x) \geqslant \frac{c}{(n-m)^{1+\alpha}}\,. \tag{4.23}$$

Hence, we can estimate $\mathcal{E}(f)$ from above as

$$\mathcal{E}(f) \leqslant c\sum_{k>0} k^{1+\alpha} q_k^2 \sum_{m\,\geqslant\,0} \frac{f_m^2}{m+1} \tag{4.24}$$

By (4.13) this is a finite upper bound when

$$\begin{cases} 1+\alpha-2\delta<-1 \\ 2\delta-4-1<-1 \end{cases} \Leftrightarrow \begin{cases} 2\delta>2+\alpha \\ 2\delta<4 \end{cases}$$

We can choose $\delta\in(1,2)$ to have these relations satisfied as soon as $\alpha<2$. This implies the transience of $(S_*,\varphi_{\mathrm{p},\alpha})$. □

## A Some deterministic bounds

We consider here the following subsets of $\mathbb{R}^2$:

$$C_n = \left\{ n e^{k\frac{2i\pi}{n+1}} \in \mathbb{C} :\ k\in\{0,\dots,n\} \right\}$$
$$D_n = \left\{ z\in\mathbb{Z}^2 :\ \|z\|_\infty = n \right\}$$

where $n\in\mathbb{N}$ and the complex plane $\mathbb{C}$ is identified with $\mathbb{R}^2$.

**Lemma A.1.** *Set the sequence $(E_n)_{n\,\geqslant\,0}$ to be equal to $(C_n)_{n\,\geqslant\,0}$ or $(D_n)_{n\,\geqslant\,0}$, and define*

$$Z_n(x) = \sum_{y\in E_n} \frac{1}{|y-x|^{2+\alpha}} \tag{A.1}$$

*for any $m,n\in\mathbb{N}$ with $m\neq n$ and for any $x\in E_m$. Then, there exists a constant $a>1$ depending only on $\alpha$ such that*

$$\frac{a^{-1}}{(n-m)^{1+\alpha}} \leqslant Z_n(x) \leqslant \frac{a}{(n-m)^{1+\alpha}}\,, \qquad \text{if } m<n\,, \tag{A.2}$$

$$\frac{a^{-1}n}{m(m-n)^{1+\alpha}} \leqslant Z_n(x) \leqslant \frac{an}{m(m-n)^{1+\alpha}}\,, \qquad \text{if } m>n\,. \tag{A.3}$$

*Proof.* We start with the proof of (A.2) in the case $E_n = C_n$. Given $r > 0$, we set $\mathscr{C}_r = \{z \in \mathbb{R}^2 : |z| = r\}$. Since the points of $C_n$ are regularly distributed along the circle $\mathscr{C}_n$ and since their number is asymptotically proportional to the perimeter of $\mathscr{C}_n$, it is enough to find $a > 1$ such that

$$\frac{a^{-1}}{(r-|x|)^{1+\alpha}} \leqslant I_r(x) := \oint_{\mathscr{C}_r} \frac{|dz|}{|z-x|^{2+\alpha}} \leqslant \frac{a}{(r-|x|)^{1+\alpha}} \tag{A.4}$$

for all $r > 0$ and for all $x$ in the open ball $B(0,r)$ centered at 0 with radius $r$. Without loss of generality we can assume $x \in \mathbb{R}_+$. Then, by the change of variable $z \to z/(r-x)$, we obtain that

$$I_r(x) = \frac{I_s(s-1)}{(r-x)^{1+\alpha}}, \qquad s := r/(r-x) > 1\,.$$

In order to conclude, we only need to show that there exists a positive constant $c$ such that $c^{-1} \leqslant I_s(s-1) \leqslant c$ for all $s > 1$. We observe that

$$g(s) := I_s(s-1) = \int_{-\pi}^{\pi} \frac{s\,d\theta}{\left[s^2+(s-1)^2-2s(s-1)\cos\theta\right]^{1+\frac{\alpha}{2}}} =$$
$$\int_{-\pi s}^{\pi s} \frac{dy}{\left[1+2s(s-1)(1-\cos\frac{y}{s})\right]^{1+\frac{\alpha}{2}}}\,. \tag{A.5}$$

The last equality follows from the change of variable $\theta \to y := s\theta$. Since $g$ is a continuous positive function converging to $2\pi$ as $s \downarrow 1$, all we have to do is to prove that

$$0 < \liminf_{s\to+\infty} g(s) \leqslant \limsup_{s\to+\infty} g(s) < +\infty\,. \tag{A.6}$$

Since there exists $c > 0$ such that $\cos\theta \leqslant 1 - c\theta^2$ for all $\theta \in [-\pi,\pi]$, whenever $s \geqslant 2$ the denominator in the r.h.s. of (A.5) is bounded from below by $[1+cy^2]^{1+\alpha/2}$. Hence, $g(s)$ is the integral on the real line of a function dominated by the integrable function $y \mapsto [1+cy^2]^{-(1+\alpha/2)}$ as soon as $s \geqslant 2$. This allows to apply the Dominated Convergence Theorem, thus implying that

$$\lim_{s\to+\infty} g(s) = \int_{-\infty}^{+\infty} \frac{dy}{\left[1+y^2\right]^{1+\frac{\alpha}{2}}} \in (0,\infty)\,. \tag{A.7}$$

This shows (A.6) and concludes the proof of (A.2) in the case $E_n = C_n$.

We now prove (A.2) in the case $E_n = D_n$. As before it is enough to find $a > 1$ such that, for all $r > 0$ and $x \in B_\infty(0,r) := \{z \in \mathbb{R}^2 : \|z\|_\infty < r\}$,

$$\frac{a^{-1}}{(r-\|x\|_\infty)^{1+\alpha}} \leqslant \tilde{J}_r(x) \leqslant \frac{a}{(r-\|x\|_\infty)^{1+\alpha}} \tag{A.8}$$

where

$$\tilde{J}_r(x) := \oint_{\mathscr{D}_r} \frac{|dz|}{|z-x|^{2+\alpha}}, \qquad \mathscr{D}_r := \{z \in \mathbb{R}^2 : \|z\|_\infty = r\}\,.$$

Since all the norms are equivalent, we just have to prove (A.8) for some $a > 1$ with

$$J_r(x) := \oint_{\mathscr{D}_r} \frac{|dz|}{\|z-x\|_\infty^{2+\alpha}}$$

instead of $\tilde{J}_r(x)$. At this point it is possible to compute explicitly $J_r(x)$ to check (A.8). We proceed in the following way. We call $\mathscr{D}_r(x)$ the union of the orthogonal projections of the square $x+\mathscr{D}_{r-\|x\|_\infty}$ on the four straight lines that contain the four edges of the square $\mathscr{D}_r$. In other words $\mathscr{D}_r(x)$ is the set of the points in $\mathscr{D}_r$ that share at least one coordinate with at least one point in $x+\mathscr{D}_{r-\|x\|_\infty}$. We have

$$J_r(x)=\int_{\mathscr{D}_r\cap\mathscr{D}_r(x)}\frac{|dz|}{\|z-x\|_\infty^{2+\alpha}}+\int_{\mathscr{D}_r\setminus\mathscr{D}_r(x)}\frac{|dz|}{\|z-x\|_\infty^{2+\alpha}}\,. \tag{A.9}$$

Estimating from below the first term in the r.h.s. of (A.9), we get the lower bound $J_r(x)\geqslant 2/(r-\|x\|_\infty)^{1+\alpha}$. On the other hand (A.9) leads to

$$J_r(x)\leqslant 4\left(\frac{2}{(r-\|x\|_\infty)^{1+\alpha}}+2\int_{r-\|x\|_\infty}^{r-\|x\|_\infty+r}\frac{dy}{y^{2+\alpha}}\right)\leqslant$$

$$8\left(\frac{1}{(r-\|x\|_\infty)^{1+\alpha}}+\int_{r-\|x\|_\infty}^{+\infty}\frac{dy}{y^{2+\alpha}}\right)=\frac{8\left(1+\frac{1}{1+\alpha}\right)}{(r-\|x\|_\infty)^{1+\alpha}} \tag{A.10}$$

and this concludes the proof of (A.2) for $E_n=D_n$.

To prove (A.3) we first look at the case $E_n=C_n$. Once again it is enough to find $a>1$ such that, for all $x\in\mathbb{R}^2$ and $r<|x|$,

$$\frac{a^{-1}r}{|x|(|x|-r)^{1+\alpha}}\leqslant I_r(x)\leqslant\frac{ar}{|x|(|x|-r)^{1+\alpha}}\,. \tag{A.11}$$

Since $I_r(x)$ depends on $r$ and $|x|$ only, we have

$$I_r(x)=\frac{1}{2\pi|x|}\oint_{\mathscr{C}_{|x|}}I_r(z)|dz|=\frac{1}{2\pi|x|}\oint_{\mathscr{C}_{|x|}}\left(\oint_{\mathscr{C}_r}\frac{|dy|}{|z-y|^{2+\alpha}}\right)|dz|\,. \tag{A.12}$$

Integrating first in $z$, using (A.4), then integrating in $y$, we get (A.11). Finally, to prove (A.3) in the case $E_n=D_n$ it is enough to find $a>1$ such that, for all $x\in\mathbb{R}^2$ and $r<\|x\|_\infty$,

$$\frac{a^{-1}r}{\|x\|_\infty(\|x\|_\infty-r)^{1+\alpha}}\leqslant J_r(x)\leqslant\frac{ar}{\|x\|_\infty(\|x\|_\infty-r)^{1+\alpha}}\,. \tag{A.13}$$

As before, we define $\mathscr{D}_r(x)$ as the union of the orthogonal projections of the square $x+\mathscr{D}_{r-\|x\|_\infty}$ on the four straight lines that contain the four edges of the square $\mathscr{D}_r$. Note that $\mathscr{D}_r(x)$ is not anymore a subset of $\mathscr{D}_r$ but we still have

$$J_r(x)=\int_{\mathscr{D}_r\cap\mathscr{D}_r(x)}\frac{|dz|}{\|z-x\|_\infty^{2+\alpha}}+\int_{\mathscr{D}_r\setminus\mathscr{D}_r(x)}\frac{|dz|}{\|z-x\|_\infty^{2+\alpha}} \tag{A.14}$$

This implies

$$\begin{aligned}
J_r(x) &\leqslant \frac{2\min(r,\|x\|_\infty - r)}{(\|x\|_\infty - r)^{2+\alpha}} + 8\int_{\|x\|_\infty - r}^{\|x\|_\infty - r + r} \frac{dy}{y^{2+\alpha}} \\
&= \frac{2\min\left(r,\|x\|_\infty - r\right)}{(\|x\|_\infty - r)^{2+\alpha}} + \frac{8}{1+\alpha}\left[\frac{1}{(\|x\|_\infty - r)^{1+\alpha}} - \frac{1}{\|x\|_\infty^{1+\alpha}}\right] \\
&= \frac{2\min\left(r,\|x\|_\infty - r\right)}{(\|x\|_\infty - r)^{2+\alpha}} \\
&\quad + \frac{8(1+\alpha)^{-1}}{(\|x\|_\infty - r)^{1+\alpha}}\left[1 - \left(1 - \frac{r}{\|x\|_\infty}\right)^{1+\alpha}\right]
\end{aligned}$$

The convexity of $y \mapsto (1-y)^{1+\alpha}$ gives

$$1 - (1+\alpha)\frac{r}{\|x\|_\infty} \leqslant \left(1 - \frac{r}{\|x\|_\infty}\right)^{1+\alpha} \leqslant 1 - \frac{r}{\|x\|_\infty} \tag{A.15}$$

and observing that

$$\min\left(r,\|x\|_\infty - r\right) \leqslant \frac{2r(\|x\|_\infty - r)}{\|x\|_\infty} \tag{A.16}$$

we get

$$J_r(x) \leqslant \frac{12r}{\|x\|_\infty(\|x\|_\infty - r)^{1+\alpha}} \tag{A.17}$$

As far as the lower bound is concerned we distinguish two cases. If $\mathscr{D}_r(x)$ does not contain any vertex of the square $\mathscr{D}_r$ then we can estimate $J_r(x)$ from below with the first term in the right-hand side of (A.14):

$$J_r(x) \geqslant \frac{1}{(\|x\|_\infty - r)^{1+\alpha}} \geqslant \frac{r}{\|x\|_\infty(\|x\|_\infty - r)^{1+\alpha}} \tag{A.18}$$

If $\mathscr{D}_r(x)$ does contain some vertex of $\mathscr{D}_r$ we estimate $J_r(x)$ with the second term in the right-hand side of (A.14). Recalling (A.15):

$$J_r(x) \geqslant \int_{\|x\|_\infty - r}^{\|x\|_\infty} \frac{dy}{y^{2+\alpha}} = \frac{(1+\alpha)^{-1}}{(\|x\|_\infty - r)^{1+\alpha}}\left[1 - \left(1 - \frac{r}{\|x\|_\infty}\right)^{1+\alpha}\right] \geqslant \frac{(1+\alpha)^{-1} r}{\|x\|_\infty(\|x\|_\infty - r)^{1+\alpha}}\,.$$

□

# B The random walk $(\mathbb{Z}^d, \varphi_{\mathrm{p},\alpha})$

In this Appendix, we study by harmonic analysis the random walk on $\mathbb{Z}^d$ with polynomially decaying jump rates. Without loss of generality, we slightly modify the function $\varphi_{\mathrm{p},\alpha}$ as $\varphi_{\mathrm{p},\alpha}(r) = (1+r^{d+\alpha})^{-1}$. Hence, we consider jump probabilities

$$p(x,y) = p(y-x), \quad p(x) = c\,(1+|x|^{d+\alpha})^{-1}, \tag{B.1}$$

for $c > 0$ such that $\sum_{x\in\mathbb{Z}^d} p(x) = 1$. The associated homogeneous random walk on $\mathbb{Z}^d$ is denoted $X = \{X_k,\, k \in \mathbb{N}\}$.

### B.1 Recurrence and transience

It is known that $X$ is transient if $d \geqslant 3$ for any $\alpha > 0$ and in $d = 1,2$ it is transient if and only if $0 < \alpha < d$. Let us briefly recall how this can be derived by simple harmonic analysis. From [25][ Section 8, T1] the random walk is transient if $d \geqslant 3$ for any $\alpha > 0$, and it is recurrent in $d = 1$ for $\alpha > 1$ and in $d = 2$ for $\alpha > 2$. Other cases are not covered by this theorem but one can use the following facts. Define the characteristic function

$$\phi(\theta) = \sum_{x\in\mathbb{Z}^d} p(x) e^{i x\cdot\theta} .$$

Note that $\phi(\theta)$ is real, $-1 \leqslant \phi(\theta) \leqslant 1$, and $\phi(\theta) < 1$ for all $\theta \neq 0$. By the integrability criterion given in [25][Section 8, P1], $X$ is transient if and only if

$$\lim_{t\uparrow 1} \int_{[-\pi,\pi)^d} \frac{1}{1 - t\phi(\theta)} d\theta < \infty . \tag{B.2}$$

If $\alpha \in (0,2)$ we have, for any $d \geqslant 1$:

$$\lim_{|\theta|\to 0} \frac{1-\phi(\theta)}{|\theta|^\alpha} = \kappa_{d,\alpha} \in (0,\infty) . \tag{B.3}$$

The limit (B.3) is proved in [25][ Section 8, E2] in the case $d = 1$ but it can be generalized to any $d \geqslant 1$. Indeed, writing $\theta = \varepsilon\hat\theta$, $|\hat\theta| = 1$:

$$\frac{1-\phi(\theta)}{|\theta|^\alpha} = \varepsilon^d \sum_{x\in\mathbb{Z}^d} (1+|x|)^{d+\alpha} p(x) \frac{1-\cos(\varepsilon x\cdot\hat\theta)}{(\varepsilon + |\varepsilon x|)^{d+\alpha}} , \tag{B.4}$$

and when $\varepsilon \to 0$, using $(1+|x|)^{d+\alpha} p(x) = c$, we have convergence to the integral

$$c\int_{\mathbb{R}^d} f(x)\, dx , \qquad f(x) := \frac{1-\cos(x\cdot\hat\theta)}{|x|^{d+\alpha}} ,$$

where $\hat\theta$ is a unit vector (the integral does not depend on the choice of $\hat\theta$). This integral is positive and finite for $\alpha \in (0,2)$ and (B.3) follows. Using (B.3) the integrability criterion (B.2) implies that for $d = 1$ the RW is transient if and only if $\alpha \in (0,1)$, while for $d = 2$ the RW is transient for any $\alpha \in (0,2)$. The only case remaining is $d = 2, \alpha = 2$. This, apparently, is not covered explicitly in [25]. However, one can modify the argument above to obtain that for any $d \geqslant 1$, $\alpha = 2$:

$$\lim_{|\theta|\to 0} \frac{1-\phi(\theta)}{|\theta|^2 \log(|\theta|^{-1})} = \kappa_{d,2} \in (0,\infty) . \tag{B.5}$$

Thus, using again the integrability criterion (B.2), we see that the case with $\alpha = 2$ and $d = 2$ is recurrent. To prove (B.5) one can write, reasoning as in (B.4): For any $\delta > 0$

$$\begin{aligned} \frac{1-\phi(\theta)}{|\theta|^2 \log(|\theta|^{-1})} &= \frac{c\,\varepsilon^d}{\log(\varepsilon^{-1})} \sum_{x\in\mathbb{Z}^d :\, 1 \leqslant |x| \leqslant \varepsilon^{-1}\delta} \frac{1-\cos(\varepsilon x\cdot\hat\theta)}{(\varepsilon + |\varepsilon x|)^{d+2}} + O\left(1/\log(\varepsilon^{-1})\right) \\ &= \frac{c}{2\log(\varepsilon^{-1})} \int_{\varepsilon \leqslant |x| \leqslant \delta} \frac{x_1^2\, dx}{|x|^{d+2}} + O\left(1/\log(\varepsilon^{-1})\right) , \end{aligned}$$

where $x_1$ is the first coordinate of the vector $x = (x_1,\dots,x_d)$. The integral appearing in the first term above is, apart from a constant factor, $\int_\varepsilon^\delta r^{-1} dr = \log(\varepsilon^{-1}) + \text{const}$. This proves the claim (B.5).

### B.2 Effective resistance estimates

Let $R_n := R_n(0)$ be the effective resistance associated to the box $\{x \in \mathbb{Z}^d\,,\ \|x\|_\infty \leqslant n\}$. As already discussed in the introduction, $\frac{1}{c} R_n$ (where $c > 0$ is the constant in (B.1)) equals the expected number of visits to the origin before visiting the set $\{x \in \mathbb{Z}^d\,,\ \|x\|_\infty > n\}$ for the random walk $X$ with $X_0 = 0$. We are going to give upper bounds on $R_n$ in the recurrent cases $d = 1, 2$, $\alpha \geqslant \min\{d, 2\}$. By comparison with the simple nearest–neighbor random walk we have that (for any $\alpha$) $R_n \leqslant C \log n$ if $d = 2$ and $R_n \leqslant C\, n$ if $d = 1$. Due to Theorems 1.5 and 1.6, this estimate is of the correct order whenever $p(x)$ has finite second moment ($\alpha > 2$). The remaining cases are treated as follows. We claim that for some constant $C$

$$R_n \leqslant C \int_{[-\pi,\pi)^d} \frac{d\theta}{n^{-\alpha} + (1 - \phi(\theta))} \,. \tag{B.6}$$

The proof of (B.6) is given later. Assuming (B.6), we obtain the following bounds:

$$R_n \leqslant C \begin{cases} \log n & d = 1,\ \alpha = 1 \\ n^{\alpha - 1} & d = 1,\ \alpha \in (1, 2) \\ n/\sqrt{\log n} & d = 1,\ \alpha = 2 \\ \log\log n & d = 2,\ \alpha = 2\,. \end{cases} \tag{B.7}$$

With the only exception of the case $d = 1, \alpha = 2$, the upper bounds above and the lower bounds of Theorems 1.5 and 1.6 are of the same order.

The bounds above are easily obtained as follows. For $\alpha \in [1, 2)$, $d = 1$, using the bound $1 - \phi(\theta) \geqslant \lambda|\theta|^\alpha$, cf. (B.3), we see that the first two estimates in (B.7) follow by decomposing the integral in (B.6) in the regions $|\theta| \leqslant n^{-1}$, $|\theta| > n^{-1}$ and then using obvious estimates. For $\alpha = 2$, we decompose the integral in (B.6) in the regions $|\theta| \leqslant \varepsilon$, $|\theta| > \varepsilon$, $\varepsilon := 1/10$. Since $1 - \phi(\theta)$ vanishes only for $\theta = 0$, the integral over the region $|\theta| > \varepsilon$ is of order 1, while we can use the bound $1 - \phi(\theta) \geqslant \lambda|\theta|^2 \log(|\theta|^{-1})$ over the region $|\theta| \leqslant \varepsilon$, cf. (B.5). Hence, we see that for some $C$

$$R_n \leqslant C \int_{[-\varepsilon,\varepsilon)^d} \frac{d\theta}{(n^{-2} + |\theta|^2 \log(|\theta|^{-1}))} \,. \tag{B.8}$$

Then, if $d = 1$ (B.8) yields

$$R_n \leqslant 2C \int_0^{(n\sqrt{\log n})^{-1}} \frac{d\theta}{n^{-2}} + 2C \int_{(n\sqrt{\log n})^{-1}}^{\varepsilon} \frac{d\theta}{\theta^2 \log(\theta^{-1})} \,.$$

The first integral gives $2C\,\frac{n}{\sqrt{\log n}}$. With the change of variables $y = 1/\theta$ the second integral becomes

$$\int_{\varepsilon^{-1}}^{n\sqrt{\log n}} \frac{dy}{\log y} \,.$$

This gives an upper bound $O(n/\sqrt{\log n})$. (Indeed, for $\varepsilon = 1/10$ we have, for every $y \geqslant \varepsilon^{-1}$, $(\log y)^{-1} \leqslant 2[(\log y)^{-1} - (\log y)^{-2}] = 2\frac{d}{dy}\frac{y}{\log y}$, and this implies the claim). Therefore

$R_n \leqslant C\, n/\sqrt{\log n}$ in the case $d = 1, \alpha = 2$. Reasoning as above, if $d = 2$ and $\alpha = 2$ we have, for some $C$:

$$R_n \leqslant C \int_0^{\varepsilon} \frac{\theta\, d\theta}{(n^{-2} + \theta^2 \log(\theta^{-1}))}\,.$$

We divide the integral as before and obtain

$$R_n \leqslant C\, n^2 \int_0^{(n\sqrt{\log n})^{-1}} \theta\, d\theta + C \int_{(n\sqrt{\log n})^{-1}}^{\varepsilon} \frac{d\theta}{\theta \log(\theta^{-1})}\,.$$

The first integral is small and can be neglected. The second integral is the same as

$$\int_{\varepsilon^{-1}}^{n\sqrt{\log n}} \frac{dy}{y \log y} \leqslant C \log\log n\,.$$

This proves that $R_n \leqslant C \log\log n$.

## B.3 Proof of claim (B.6)

To prove (B.6) we introduce the truncated kernel

$$Q_n(x,y) = \mathbb{P}_x(X_1 = y\,; |X_1 - x| \leqslant c_1\, n) = \frac{c}{1 + |y - x|^{d+\alpha}}\, 1_{\{|y-x| \leqslant c_1\, n\}}\,,$$

where $c > 0$ is defined in (B.1) and $c_1 > 0$ is another constant. Clearly, for all sufficiently large $c_1$

$$R_n \leqslant c \sum_{k=0}^{\infty} Q_n^k(0,0)\,,$$

where $Q_n^k(0,0)$ is the probability of returning to the origin after $k$ steps without ever taking a jump of size larger than $c_1\, n$. Note that for any $x$

$$u_n := \sum_{y \in \mathbb{Z}^d} Q_n(x,y) = \mathbb{P}_0(|X_1| \leqslant c_1\, n) = 1 - \gamma_n\,, \quad \gamma_n := \sum_{|x| > c_1\, n} p(x) \sim n^{-\alpha}\,.$$

Let $\hat{Q}_n(x,y)$ denote the kernel of the RW on $\mathbb{Z}^d$ with transition $p(x)$ conditioned to take only jumps of size less than $c_1 n$, so that $\hat{Q}_n(x,y) = u_n^{-1} Q_n(x,y)$. Set

$$\phi_n(\theta) = \sum_{x \in \mathbb{Z}^d} Q_n(0,x) e^{i\theta\cdot x}\,, \quad \hat{\phi}_n(\theta) = \sum_{x \in \mathbb{Z}^d} \hat{Q}_n(0,x) e^{i\theta\cdot x}\,.$$

$\phi_n(\theta) = u_n \hat{\phi}(\theta)$ is real and $e^{i\theta\cdot x}$ can be replaced by $\cos(\theta \cdot x)$ in the above definitions. We can write

$$\sum_{k=0}^{\infty} Q_n^k(0,0) = \sum_{k=0}^{\infty} \frac{1}{(2\pi)^d} \int_{[-\pi,\pi)^d} \phi_n(\theta)^k\, d\theta = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi)^d} \frac{d\theta}{1 - \phi_n(\theta)}\,,$$

where we use the fact that $|\phi_n(\theta)| \leqslant u_n < 1$ for any $n$. Moreover

$$1 - \phi_n(\theta) = 1 - u_n + u_n(1 - \hat{\phi}_n(\theta)) = \gamma_n + u_n(1 - \hat{\phi}_n(\theta))\,.$$

Therefore it is sufficient to prove that

$$u_n(1-\hat{\phi}_n(\theta)) \geqslant \delta\,(1-\phi(\theta)), \tag{B.9}$$

for some constant $\delta > 0$. Let $B(t)$ denote the Euclidean ball of radius $t > 0$. Suppose $\theta = y\,n^{-1}\hat{\theta}$ for some $y > 0$ and a unit vector $\hat{\theta}$. Then,

$$\frac{u_n(1-\hat{\phi}_n(\theta))}{1-\phi(\theta)} = \frac{\sum_{x\in\mathbb{Z}^d\cap B(c_1 n)} \frac{1-\cos(y(x/n)\cdot\hat{\theta})}{(n^{-1}+|(x/n)|)^{d+\alpha}}}{\sum_{x\in\mathbb{Z}^d} \frac{1-\cos(y(x/n)\cdot\hat{\theta})}{(n^{-1}+|(x/n)|)^{d+\alpha}}} \tag{B.10}$$

Reasoning as in the proofs of (B.3) and (B.5) we see that, for all $\alpha \in (0,2]$, the expression (B.10) is bounded away from 0 for $y \in (0,C]$, for $n$ large enough. Indeed, if $\alpha \in (0,2)$ we have convergence, as $n \to \infty$ to

$$\frac{\int_{B(c_1)} \frac{1-\cos(y x\cdot\hat{\theta})}{|x|^{d+\alpha}}\,dx}{\int_{\mathbb{R}^d} \frac{1-\cos(y x\cdot\hat{\theta})}{|x|^{d+\alpha}}\,dx}.$$

On the other hand, for $\alpha = 2$, from the proof of (B.5) we see that (B.10) converges to 1. Therefore, in all cases (B.9) holds for any $|\theta| \leqslant C\,n^{-1}$, for all $n$ sufficiently large. Next, we consider the case $|\theta| > C\,n^{-1}$. For this range of $\theta$ we know that

$$1-\phi(\theta) \geqslant \lambda|\theta|^\alpha \geqslant \lambda\,C^\alpha\,n^{-\alpha},$$

for some $\lambda > 0$. Note that this holds also in the case $\alpha = 2$ according to (B.5). From

$$\phi(\theta) - u_n\hat{\phi}_n(\theta) = \sum_{x:\,|x|>c_1 n} p(x)\cos(\theta\cdot x),$$

we obtain $\phi(\theta) \geqslant u_n\hat{\phi}_n(\theta) - \gamma_n$. Therefore, for $|\theta| > C\,n^{-1}$

$$\begin{aligned} u_n(1-\hat{\phi}_n(\theta)) - \delta\,(1-\phi(\theta)) &\geqslant -2\gamma_n + (1-\delta)(1-\phi(\theta)) \\ &\geqslant -2\gamma_n + (1-\delta)\lambda\,C^\alpha\,n^{-\alpha}. \end{aligned}$$

Taking $C$ large enough and using $\gamma_n = O(n^{-\alpha})$ shows that (B.9) holds. This ends the proof of (B.6).

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