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Escaping the Brownian stalkers

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Abstract

We propose a simple model for the behaviour of longterm investors on a stock market. It consists of three particles that represent the stock's current price and the buyers', respectively sellers', opinion about the right trading price. As time evolves, both groups of traders update their opinions with respect to the current price. The speed of updating is controled by a parameter γ; the price process is described by a geometric Brownian motion. We consider the market's stability in terms of the distance between the buyers' and sellers' opinion, and prove that the distance process is recurrent/transient in dependence on γ.

Key words: financial markets, market stability, stochastic dynamics, recurrence, transience.

AMS 2000 Subject Classification: Primary 60J65; Secondary: 60K10.

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1 Introduction

In this article, we suggest a simple model for the behaviour of longterm investors on a share market. We observe the evolution of three particles. One of them represents the share's current price, the second one the shareholders' opinion about the share's value and the last one the opinion of potential buyers. As longterm investors do not speculate on fast returns, it is reasonable to assume two features: first, the share's value is much higher than the current price in the shareholders' eyes, but it is much lower in the eyes of potential buyers. However, both groups of investors do not wait forever. They modify their opinions in dependence on the price development. Yet, as second feature, the traders adapt to price changes only slowly. As opposed to short-time traders, who gamble on returns on short time intervals, there is no need for longterm investors to react on small fluctuations.

Eventually, as the price changes and the investors adjust their opinions, the price reaches the value that is expected by the traders. We assume a symmetric behaviour of buyers and sellers, and thus, need to consider only what happens if the price reaches the right value in the shareholders' opinion. Because the price has reached a *fair* level, the investors sell their shares. At the very moment, there are new shareholders, namely the buyers of the shares. Eventually, the price drops, and there is again a group of individuals not willing to follow this decrement. This means, although the individuals in the group of longterm investors change in time, the group itself persists. Figure 1 shows an example for the system's evolution on a logarithmic scale. The price is denoted by B, the opinion of buyers by X, and the one of sellers by Y.

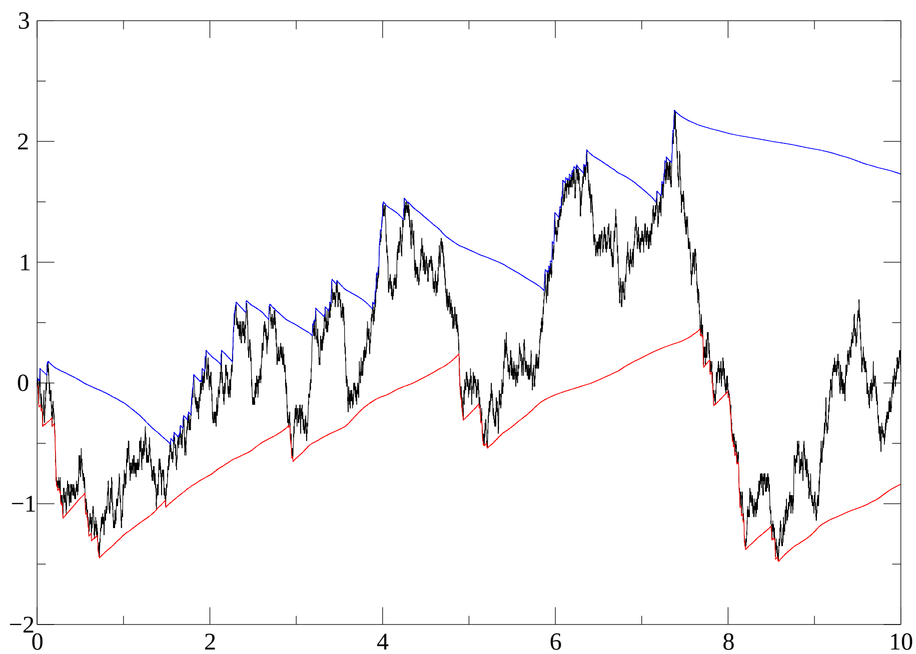

Figure 1: The price B (black), and the opinions X (red) and Y (blue) evolving in time on a logarithmic scale.

We are interested in the evolution of the distance between X and Y. In illiquid markets, *i.e.* in markets with few supply, already smaller demands can be satisfied in connection with a strong price change only. Thus, a large group of traders willing to trade for a certain price provides some resistance against further price evolution into this direction. Consequently, it is of great interest how longterm investors adapt to strong price changes, since they provide resistance on levels which are normally on some distance from the price. If these investors react too slow, the price can fluctuate between these levels without much resistance which leads to strong volatility. The theory of trading strategies on illiquid markets is a very active field of research, and there are many different approaches to model these markets and their reactions on trading [1; 5; 8]. However, the question if large orders on illiquid markets can destabilize the markets seems to be open.

Bovier *et al.* describe in [4] the *opinion game*, a class of Markovian agent-based models for the evolution of a share price. Therein, they present the idea of a *virtual order book*, which keeps track of the traders' opinions about the share's value irrespective of whether the traders have placed an order or not. For practical purposes, the model is stated in a discrete time setting, and in every round, one agent updates his or her opinion. As a main feature, the probability to be chosen depends on the agent's distance to the price. In particular, in a market with N traders and current price p, the probability for agent i with current opinion p_i to be chosen is given by

$$\frac{h(|p_i - p|)}{\sum_{j=0}^{N} h(|p_j - p|)}. \tag{1.1}$$

The function h is assumed to be positive and decreasing. This assumption reflects the idea that traders with opinions far away from the price react to price changes more slowly. The model is stated in a very general setting, but the authors are able to reproduce several qualitative statistical properties of the price process, sometimes called *stylized facts*, by choosing

$$h(x) = \frac{1}{(1+x)^{\gamma}}. \tag{1.2}$$

We pick up on this choice for our model. The logarithmic price process B is a Brownian motion; the opinions of buyers, X, and sellers, Y, are described by ordinary differential equations in dependence on parameter $\gamma > 0$ and the Brownian motion B.

The buyers' opinion at time t is given by the solution of

$$\frac{d}{dt} f(t) = \frac{1}{\left(1 + B_t - f(t)\right)^{\gamma}} \tag{1.3}$$

whenever $X_t < B_t$. By the argumentation above that the individuals within the group may change, but the group itself remains, X can hit B, but it is not allowed to cross it, and thus, it describes the same movement as B until B goes up too fast for X to follow (observe that 1 is an upper bound for the speed of X). This happens immediately after the two processes have met, because B is fluctuating almost everywhere. As soon as the distance is positive, X is driven by (1.3) again. Since B is differentiable almost nowhere, some work is needed to give a rigorous construction of this process.

For the shareholders' opinion, Y, we assume the same construction with a changed sign on the right hand side of (1.3). $-B$ is also a Brownian motion, and thus, we can define

$$Y(B) := -X(-B). \tag{1.4}$$

Notice that the speed of adaption to price fluctuations is governed by the parameter γ. Therefore, we are interested in the longterm behaviour of $Y - X$ as a function of γ. In particular, we would like to know when $Y-X$ is recurrent and when it is transient. A heuristic argument suggests that $\gamma=1$ is a critical value. For a constant $c>0$, we scale time by c^2 and space by c. We denote the processes' scaled versions by adding superscript c. By Brownian scaling, we have that B^c is equal to B in distribution. On the other hand, X^c solves

$$\frac{d}{dt}X_t^c=\frac{c^{1-\gamma}}{\left(1/c+B_t^c-X_t^c\right)^\gamma}. \tag{1.5}$$

If one assumes c to be large and $B_t^c-X_t^c$ to be positive, the slope tends to infinity for $\gamma<1$ and to 0 for $\gamma>1$. This observation suggests that $Y-X$ remains stable for $\gamma<1$ only. In this paper, we show that this first guess is right, and prove a rigorous statement about the stability in dependence on γ.

The remainder of this article is organized as follows: in section 2, we define the particle system formally. X, or Y respectively, are constructed pathwisely as a sequence of processes. The existence of these limits is stated in lemma 2.1, its lengthy proof is given in appendix A. In section 3, we present the main theorem and its proof, and in section 4, we discuss what our results mean for the opinion game from [4].

2 Construction

In this section, we introduce the processes B,X and Y formally. $B:=(B_t)_{t\geq 0}$ is defined to be a Brownian motion on a probability space $\{\Omega,\mathscr{F},(\mathscr{F}_t)_{t\geq 0},P\}$; the construction of X is more complicated, thus, we first give a short summary of the procedure in the next paragraph.

We introduce a sequence of random step functions $B^\epsilon(\omega)$ such that the distance between $B^\epsilon(\omega)$ and $B(\omega)$ is uniformly smaller than, respectively equal to, ϵ. The construction of X^ϵ that is attracted to B^ϵ in the sense as explained in the introduction turns out to be easy. At last, we show that X^ϵ has a limit as ϵ tends to zero, and call this limit process X. The construction of Y follows immediately afterwards. The advantage of a step function approach is the simple transition to a discrete setting that we use extensively in the proof of the main theorem later on.

For any $\epsilon>0$, we define jump times by $\bar{\sigma}_0^\epsilon:=0$ and

$$\bar{\sigma}_i^\epsilon:=\min\left\{t>\bar{\sigma}_{i-1}^\epsilon:\left|B_t-B_{\bar{\sigma}_{i-1}^\epsilon}\right|\geq\epsilon\right\},\ i\in\mathbb{N}, \tag{2.1}$$

neglecting the ϵ-index whenever no confusion is caused. Furthermore, we define step functions $B^\epsilon:[0,\infty)\to\mathbb{R}$ by

$$B_t^\epsilon:=B_{\bar{\sigma}_i}\text{ for }t\in\left[\bar{\sigma}_i,\bar{\sigma}_{i+1}\right). \tag{2.2}$$

Observe that

$$\sup_{t\geq 0}\left|B_t-B_t^\epsilon\right|=\epsilon\text{ a.s.} \tag{2.3}$$

by definition, and thus, B^ϵ converges to B on $[0,\infty)$ in sup-norm. As already mentioned in the introduction, we basically want X to fulfil

$$\frac{d}{dt}X_t=\left(1+B_t-X_t\right)^{-\gamma} \tag{2.4}$$

as long as $X_t < B_t$. If we substitute B by a fixed number $b \geq 0$, the *ode* (2.4) is explicitly solvable. The solution of

$$\frac{d}{dt} f(t) = \left(1 + b - f(t)\right)^{-\gamma}, \; f(0) = 0, \tag{2.5}$$

is

$$\bar{h}(t, b) := b + 1 - \left((b+1)^{\gamma+1} - (\gamma+1)\, t\right)^{\frac{1}{\gamma+1}}. \tag{2.6}$$

We call $\bar{h}(t, b)$ *well-defined* if

$$b \geq 0 \text{ and } t \leq \frac{(b+1)^{\gamma+1} - 1}{\gamma + 1}. \tag{2.7}$$

Observe that the bound on t ensures $\bar{h}(t, b) \leq b$. As we will be mainly interested in the distance from $\bar{h}$ to b at time t, we set

$$h(t, b) := \begin{cases} b - \bar{h}(t, b) & \text{if } \bar{h}(t, b) \text{ is well-defined} \\ 0 & \text{else} \end{cases}. \tag{2.8}$$

We define X^ϵ in the following way: for $t \in [\bar{\sigma}_i, \bar{\sigma}_{i+1})$, $i \in \mathbb{N}_0$, we set

$$X_t^\epsilon := B_{\bar{\sigma}_i}^\epsilon - h(t - \bar{\sigma}_i, B_{\bar{\sigma}_i}^\epsilon - X_{\bar{\sigma}_i -}^\epsilon) \tag{2.9}$$

with $X_{0-}^\epsilon := 0$ (figure 2). This means we first consider $X_{\bar{\sigma}_i -}^\epsilon$ for $t \in [\bar{\sigma}_i, \bar{\sigma}_{i+1})$. If $B_{\bar{\sigma}_i}^\epsilon$ is smaller than this value, we set $X_t^\epsilon := B_{\bar{\sigma}_i}^\epsilon$; else we can apply function $\bar{h}$ to calculate the movement of X^ϵ towards B^ϵ. If X^ϵ reaches B^ϵ before time t, it remains on this level.

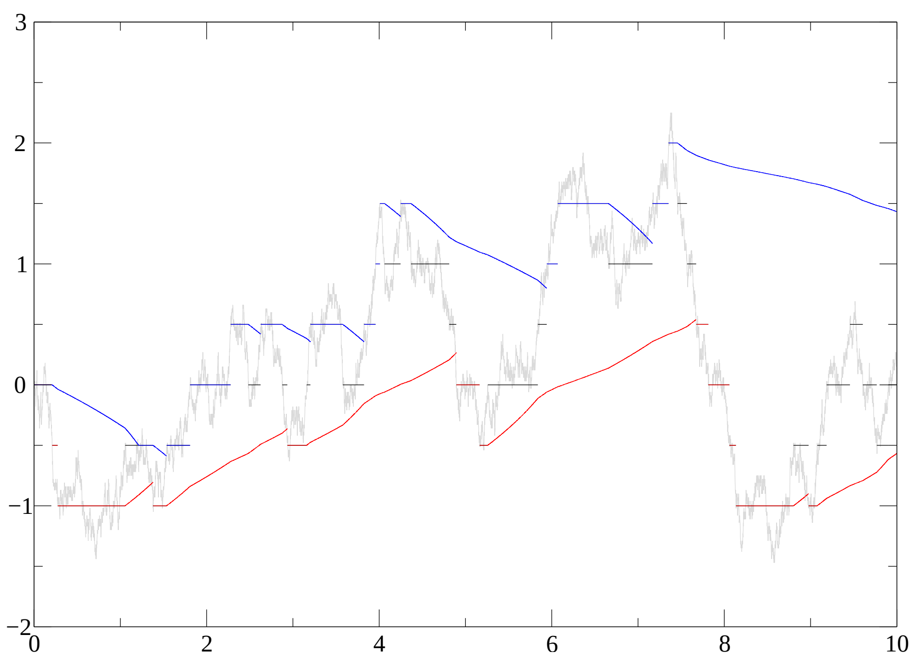

Figure 2: The three processes B^ϵ (black), X^ϵ (red) and Y^ϵ (blue); B is displayed beneath in grey. $\epsilon = 1/2$ in this figure.

Lemma 2.1. *Let $S \subset [0,\infty)$ be a compact set and $\epsilon \ll \exp(-\gamma \cdot \sup S)$. Then*

$$\sup_{t\in S} \left| X_t^{\epsilon'} - X_t^{\epsilon} \right| \leq \epsilon K_S \text{ a.s.} \tag{2.10}$$

with K_S being a finite, deterministic constant that depends on S, and $\epsilon' < \epsilon$.

Proof. See appendix A. □

Lemma 2.1 shows that $(X_t^{\epsilon})_{\epsilon>0}$ is a Cauchy sequence in the set of all bounded functions from S to $\mathbb{R}$ equipped with the sup-norm. As this space is complete, $(X_t^{\epsilon})_{\epsilon>0}$ converges. We denote the limit process by X. Equivalently, we define

$$Y^{\epsilon}(B^{\epsilon}(\omega)) := -X^{\epsilon}(-B^{\epsilon}(\omega)) \text{ and } Y(B(\omega)) := -X(-B(\omega)). \tag{2.11}$$

3 The main theorem

3.1 The theorem

Theorem 3.1. *Let B, X and Y be defined as before, and let*

$$\theta_r := \sup\left\{ t \geq 0 : \left| Y_t - X_t \right| \leq r \right\} \tag{3.1}$$

be the last exit time from an r-ball with respect to the $||\cdot||_1$-norm. Then,

1. *for $\gamma < 1$,*
$$(\forall r > 0)\ \theta_r = \infty \text{ a.s.}, \tag{3.2}$$

2. *and for $\gamma > 1$,*
$$(\forall r > 0)\ \theta_r < \infty \text{ a.s.} \tag{3.3}$$

The theorem confirms our guess that 1 is a critical value for γ. For the critical case, there is no statement at all, but as the proof of transience in the supercritical case seems to be sharp, our conjecture is null-recurrence if $\gamma = 1$.

We prove theorem 3.1 by discretising the process $Y - X$. This results in a Markov chain which we examine in detail in subsection 3.2. In 3.3, we prove the subcritical case by reducing it to a one-dimensional random walk problem. For the transient case ($\gamma > 1$), we basically use that a Markov chain is transient if we can find a bounded subharmonic function with respect to the generator of the chain. The particular theorem and its application in the proof can be found in subsection 3.4.

3.2 Discretising the problem and facts about Markov chains

Let us look at the problem from another perspective. We consider the two-dimensional process $(B^{\epsilon} - X^{\epsilon}, Y^{\epsilon} - B^{\epsilon})$ and interprete it in the following as particle moving in $[0,\infty)^2$. Observe that $Y^{\epsilon} - X^{\epsilon}$ is just the sum of both coordinates. Furthermore, because $Y^{\epsilon} - X^{\epsilon}$ can only increase at the times $\bar{\sigma}_i$ and decreases afterwards, we have

$$\inf_{t\in[\bar{\sigma}_i,\bar{\sigma}_{i+1})} (Y^{\epsilon} - X^{\epsilon})_t = (Y^{\epsilon} - X^{\epsilon})_{\bar{\sigma}_{i+1}-}. \tag{3.4}$$

For all $\epsilon > 0$, we define a two-dimensional Markov chain $\Phi^\epsilon = \Phi(B^\epsilon) := \big(\Phi(B^\epsilon)_i\big)_{i\in\mathbb{N}}$ with state space $[0,\infty)^2$, equipped with the Borel-σ-algebra $\mathfrak{B}([0,\infty)^2)$, by

$$\Phi_i^\epsilon := (B^\epsilon - X^\epsilon, Y^\epsilon - B^\epsilon)_{\bar{\sigma}_i -} \tag{3.5}$$

with $\bar{\sigma}_0 - := 0$. The j-step transition probabilities from $x \in [0,\infty)^2$ to $A \subset [0,\infty)^2$ are denoted by $P_x^j(A)$, but we neglect the index for $j = 1$. The generator L is given by

$$Lg(x) := \int_{[0,\infty)^2} P_x(dy)g(y) - g(x) \tag{3.6}$$

for suitable functions $g : [0,\infty)^2 \to [0,\infty)$.

In the following, it is of great importance to understand how the particle moves exactly while $\Phi_i^\epsilon = (x,y)$ jumps to Φ_{i+1}^ϵ (figure 3). At first, a jump of size ϵ happens at time $\bar{\sigma}_i$. The position afterwards is either $(x+\epsilon,\ (y-\epsilon) \vee 0)$ or $((x-\epsilon) \vee 0,\ y+\epsilon)$ with probability $1/2$ each. Let us call this new position (x', y'). Before the next jump happens at time $\bar{\sigma}_{i+1}$, the particle drifts into the origin's direction. If it reaches one of the axes, it remains there and only drifts towards the other axis until it reaches $(0,0)$. Thus, the coordinates of Φ_{i+1}^ϵ are given by $(h(\bar{\sigma}_{i+1} - \bar{\sigma}_i, x'),\ h(\bar{\sigma}_{i+1} - \bar{\sigma}_i, y'))$. Observe that Φ^ϵ can only increase (in $||\cdot||_1$-sense) on the axes.

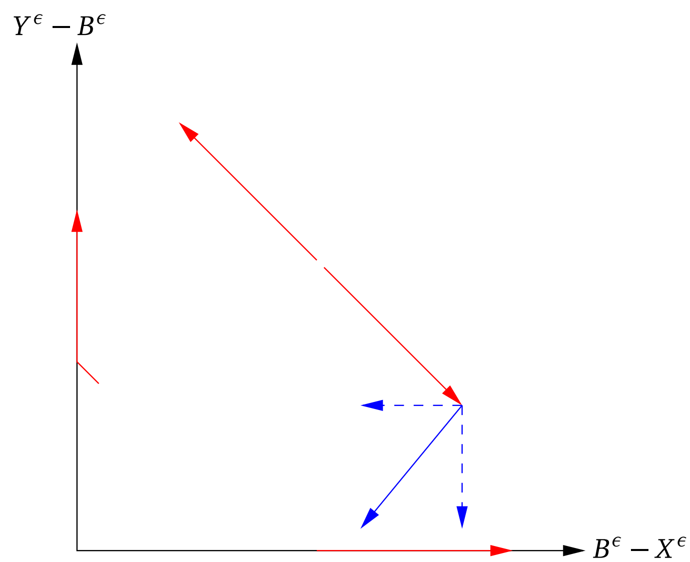

Figure 3: The particle's jumps (red arrows) are parallel to the level lines of the $||\cdot||_1$-norm. In the sense of this norm, it can increase on the axes only. The drift consists of two independent components (blue dashed arrows), which are orthogonal to the axes. The resulting drift is illustratetd by the solid blue arrow.

Next, we need to understand the distribution of $\bar{\sigma}_{i+1} - \bar{\sigma}_i$. Thus, we set

$$\sigma_i := \bar{\sigma}_{i+1} - \bar{\sigma}_i \stackrel{d}{=} \inf\{t > 0 : B_t = \epsilon\}. \tag{3.7}$$

As already suggested in the equation above, all σ_i are *i.i.d.* with support on $(0,\infty)$ and $\mathbb{E}\sigma = \epsilon^2$. The distribution is not known explicitly, but it can be expressed as a series with alternating summands with decreasing absolute values (refer to section C.2 in [2]). Calculating the first two summands results in

$$\frac{4}{\pi}e^{-\pi^2/(8\epsilon)}\left(1-\frac{1}{3}e^{-\pi^2/\epsilon}\right) \tag{3.8}$$

$$\leq \quad P(\sigma>\epsilon) \;=\; P\left(\sup_{0\leq s\leq\epsilon}\left|B_s\right|<\epsilon\right) \tag{3.9}$$

$$\leq \quad \frac{4}{\pi}e^{-\pi^2/(8\epsilon)}. \tag{3.10}$$

For our purposes, it is sufficient to know that both bounds are of order $\exp(-1/\epsilon)$.

As we are operating on a continuous state space, the question for irreducibilty is a question for reaching sets instead of single states. Formally, Φ^ϵ is called *φ-irreducible* if there exists a measure φ on $\mathfrak{B}([0,\infty)^2)$ such that

$$\varphi(A)>0 \Rightarrow P_x\left(\Phi^\epsilon \text{ ever reaches } A\right)>0 \text{ for all } x\in[0,\infty)^2. \tag{3.11}$$

In our case,

$$P_x\left(\{\mathbf{0}\}\right)>0 \text{ for all } x\in[0,\infty)^2, \tag{3.12}$$

because the support of σ's density function is unbounded. Thus, Φ^ϵ is $\delta_{\mathbf{0}}$-irreducible. The existence of an irreducibility measure ensures that there is also a *maximal irreducibility measure* Ψ (compare with [7], proposition 4.2.2) on $\mathfrak{B}([0,\infty)^2)$ with the properties:

1. Ψ is a probability measure.
2. Φ^ϵ is Ψ-irreducible.
3. Φ^ϵ is φ'-irreducible iff $\Psi \succ \varphi'$ (i.e. $\Psi(A)=0 \Rightarrow \varphi'(A)=0$).
4. $\Psi(A)=0 \;\Rightarrow\; \Psi\left(\left\{x : P_x(\Phi^\epsilon \text{ ever enters } A)\right\}\right)=0$.
5. Here, Ψ is equivalent to

$$\Psi'(A)=\sum_{j=0}^{\infty}P_0^j(A)2^{-j}. \tag{3.13}$$

We denote the set of measurable, Ψ-irreducible sets by

$$\mathfrak{B}^+([0,\infty)^2):=\{A\in\mathfrak{B}([0,\infty)^2):\Psi(A)>0\}. \tag{3.14}$$

Because the density of $\bar{\sigma}_{i+1}-\bar{\sigma}_i$ has support on $(0,\infty)$, it is easy to see that

$$\mu(A):=Leb(A)+\delta_{\mathbf{0}}(A)\neq 0 \;\Rightarrow\; \Psi(A)\neq 0, \tag{3.15}$$

and therefore, $\Psi \succ \mu$ with Leb denoting the Lebesgue measure.

Since Φ^ϵ is a Markov chain on the *(possible) local minima* of $Y^\epsilon - X^\epsilon$ in the sense of (3.4), it is obvious that transience of Φ^ϵ implies transience of $Y^\epsilon - X^\epsilon$. On the other hand, $||\Phi^\epsilon||_1$ can only increase by ϵ, at most, in every step. Thus,

$$\sup_{t\in[\bar{\sigma}_i,\bar{\sigma}_{i+1})}(Y^\epsilon-X^\epsilon)_t \leq \left|\left|\Phi_i^\epsilon\right|\right|_1+\epsilon, \tag{3.16}$$

and recurrence of Φ^ϵ also implies recurrence of $Y^\epsilon - X^\epsilon$. However, observe that the proof of recurrence/transience for $Y^\epsilon - X^\epsilon$, $\epsilon > 0$, does not directly imply recurrence/transience for the limit process, $Y - X$, because we have convergence on compact sets only. Thus, we explain in the end of both parts of the proof shortly how to deduce the desired result.

3.3 Proof of the subcritical case: $\gamma < 1$

For the subcritical case, we reduce the movement of Φ^ϵ to a nearest neighbour random walk on certain level sets

$$M(k) := \left\{ (x,y) \in [0,\infty)^2 \,\middle|\, x + y = 4^k \right\},\ k \in \mathbb{Z}, \tag{3.17}$$

of $||\cdot||_1 : [0,\infty)^2 \to [0,\infty)$, and show that the probability to jump to $M(k-1)$ is larger than $1/2+\delta$, $\delta > 0$, for small ϵ and all $k \geq k^*$ for a $k^* \in \mathbb{Z}$. Then it is well-known that $||\Phi^\epsilon_\cdot||_1 < 4^{k^*}$ infinitely often. Recurrence for Φ^ϵ follows from irreducibility.

In particular, we introduce

$$M^-(k) := \left\{ (x,y) \in [0,\infty)^2 \,\middle|\, x + y \leq 4^{k-1} \right\}, \tag{3.18}$$

$$M^+(k) := \left\{ (x,y) \in [0,\infty)^2 \,\middle|\, x + y \geq 4^{k+1} \right\} \tag{3.19}$$

for $k \in \mathbb{Z}$, and the hitting time of Φ^ϵ

$$\tau^\epsilon_M := \min\left\{ i : \Phi^\epsilon_i \in M \right\} \tag{3.20}$$

for a set $M \subseteq [0,\infty)^2$; we neglect the ϵ whenver possible. Then we have to show

$$\left(\exists k^*\right)\left(\forall k \geq k^*\right) \lim_{\epsilon\to 0} \inf_{m\in M(k)} P_m\left(\tau^\epsilon_{M^-(k)} < \tau^\epsilon_{M^+(k)}\right) > 1/2 + \delta,\ \delta > 0. \tag{3.21}$$

The proof works in four steps (figure 4).

1. We show that $P_m(\tau_{M^-(k)} < \tau_{M^+(k)})$ is minimized for $m^* \in \{(4^k,0),(0,4^k)\}$. As the model is symmetric, we may assume $m^* = (0,4^k)$ without loss of generality.

2. We show
$$P_{m^*}\left(\tau_{\{(x,y):x=y\}} < \tau_{M^+(k)}\right) > 1 - e^{-6/7} \approx 0.576 \tag{3.22}$$
as ϵ tends to 0.

3. We assume the particle has been successful in the last step and has reached $(x,x) \notin M^+(k)$. In the worst case, it is at position $(2\cdot 4^k, 2\cdot 4^k)$ or arbitrarily close to it (as ϵ becomes small). Since the jumps' directions and the drift times σ_i are mutually indpendent, we can treat the jumps and drift phases independently. We use this feature to determine the diameter of a tube around the bisector. As long as the particle is located within this area, it does not drift to the axes *too fast*. When we know the diameter, we can calculate the probability that the jumps do not take the particle out of the tube within a certain time period. Knowing this time and the speed torwards the origin, we can calculate how close it gets to the origin before hitting the axes.

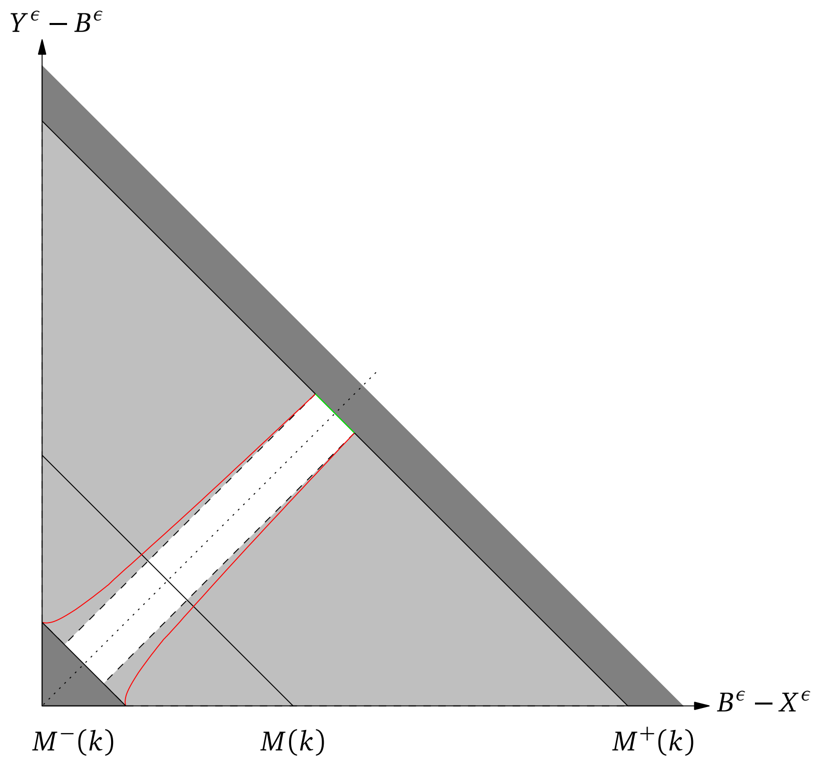

Figure 4: The idea of the proof: the particle starts in $M(k)$ (black line). We show that the probability to get to $M^-(k)$ (left dark grey area) before it gets to $M^+(k)$ (right dark grey area) is larger than $1/2$. This probability is bounded from below by the product of the probability to reach the bisector (dotted line) before reaching $M^+(k)$ and the probability to get back to $M^-(k)$ before hitting the axes. In particular, we calculate the probability of a random walk with step size $\sqrt{2}\epsilon$ to stay in the white slot around the bisector. Its diameter (green line), $diam(A_{4^{k+1}})$, is a lower bound for the diameter of the area enclosed by $g(x)$ and $g^{-1}(x)$ (red lines).

4. Finally, we combine steps 2 and 3. It will turn out that the probability to stay in the tube for a certain time (step 3) can be chosen large enough such that it is still strictly larger than $1/2$ if multiplied with the probability to reach the bisector (step 2). On the other hand, the time Φ^ϵ stays in the tube is sufficient to reach $M^-(k)$.

For step 1, we consider a realisation $B^\epsilon(\omega)$ of the Brownian step function, $X^\epsilon(B^\epsilon(\omega))$ with starting distance $|B_0^\epsilon - X_0^\epsilon| = d$, and $\bar{X}^\epsilon(B^\epsilon(\omega))$ that is constructed like X^ϵ and attracted to the same realisation, but starts with initial distance $|B_0^\epsilon - \bar{X}_0^\epsilon| = \bar{d}$, $\bar{d} > d$. Since X^ϵ is Markovian, we can extend our construction of X^ϵ to initial values different from 0 easily. Then

$$(\forall t \geq 0)\left[\left(B^\epsilon(\omega) - \bar{X}^\epsilon\right)_t \geq \left(B^\epsilon(\omega) - X^\epsilon\right)_t\right] \tag{3.23}$$

with equality for all $t \geq r \geq 0$ whereby r fulfils

$$\left(B^\epsilon(\omega) - \bar{X}^\epsilon\right)_r = 0 = \left(B^\epsilon(\omega) - X^\epsilon\right)_r. \tag{3.24}$$

The respective statement holds also for $Y^\epsilon - B^\epsilon$ due to symmetry. Thus, if $\Phi_i^\epsilon(\omega)$ is smaller than or equal to a copy $\bar{\Phi}_i^\epsilon(\omega)$ in both coordinates for some time i, this (in)equality remains for all times afterwards. We can conclude that

$$P_{(x,0)}\left(\tau_{M^-(k)} < \tau_{M^+(k)}\right) \geq P_{(x',0)}\left(\tau_{M^-(k)} < \tau_{M^+(k)}\right) \tag{3.25}$$

for $x < x'$, because every realisation of B^ϵ fulfiling the event on the right side also fulfils the one on the left side.

As Φ^ϵ can only increase on the axes, starting it from a point inside the quadrant results in a decrease of both coordinates until one of the axes is hit. But then (3.25) applies, and therefore step 1 is proven.

In step 2, we show

$$P_{m^*}\left(\tau_{\{(x,y):x=y\}} > \tau_{M^+(k)}\right) < e^{-6/7}. \tag{3.26}$$

We assume $m^* = (0, 4^k)$. The particle has two possibilities: either it jumps upwards the axis to $(0, 4^k + \epsilon)$, or it jumps into the quadrant to $(\epsilon, 4^k - \epsilon)$; afterwards, it drifts. In this step, we ignore the drift phase for two reasons. First, the change of position by jumping is of order ϵ, but it is of order ϵ^2 by drifting, because $\mathbb{E}\sigma = \epsilon^2$. Furthermore, the drift direction is different from the jump direction, and for every change of position in jump direction by drifting, there is also a drift *down*, orthogonal to the jump direction, by the same amount at least. Thus, considering the drift would help us in reaching our aim to drift down.

We introduce the following *game*: sitting on the axis, the particle can either reach the bisector, or it can move up the axis by ϵ. As the particle needs $4^k/(2\epsilon)$[1] steps to reach the bisector but only one step to go up, the success probability is small. If we do not success, we have another, smaller, chance at $(0, 4^k + \epsilon)$ and so on until we reach $M^+(k)$. It is well known that the probability of an one-dimensional, symmetric random walk to reach -1 before reaching $k \in \mathbb{N}$ is given by $k/(k+1)$ if the random walk is started in 0. Thus,

$$\begin{aligned}
P_{m^*}\left(\tau_{\{(x,y):x=y\}} > \tau_{M^+(k)}\right) &= \prod_{i=0}^{\left(4^{k+1}-4^k\right)/\epsilon-1} \frac{(4^k/2 + i\epsilon)/\epsilon}{(4^k/2+i\epsilon)/\epsilon + 1} && (3.27)\\
&= \prod_{i=0}^{3\cdot 4^k/\epsilon - 1} \frac{4^k/2 + i\epsilon}{4^k/2 + i\epsilon + \epsilon} && (3.28)\\
&< \left(\frac{4^k/2 + 3\cdot 4^k - \epsilon}{4^k/2 + 3\cdot 4^k}\right)^{3\cdot 4^k/\epsilon} && (3.29)\\
&= \left(1 - \frac{2\epsilon}{7\cdot 4^k}\right)^{3\cdot 4^k/\epsilon} && (3.30)\\
&\to e^{-6/7} \text{ as } \epsilon \text{ tends to 0.} && (3.31)
\end{aligned}$$

For part 3, we assume that the particle has reached the bisector and is at position $(2\cdot 4^k, 2\cdot 4^k)$. First,

[1] Here we neglect that the expression is meaningful for integers only, because the difference does not play a role as ϵ tends to zero.

we are interested in the particle's speed when it drifts. In particular, we are looking for a uniform lower bound for the speed orthogonal to the $||\cdot||_1$-level sets on $[0,\infty)^2\backslash M^+(k)$. If we denote the particle's current position by (x,y), its speed in x-direction is given by $(1+x)^{-\gamma}$ and in y-direction by $(1+y)^{-\gamma}$ because of equation (2.5). Thus, the speed orthogonal to the level sets is given by

$$v_{(x,y)} := \sqrt{(1+x)^{-2\gamma} + \left(1+y\right)^{-2\gamma}}. \tag{3.32}$$

Differentiation of v shows that the speed has a minimum at $(2\cdot 4^k, 2\cdot 4^k)$ on the set $\{(x,y) : x+y \leq 4^{k+1}\}$. This minimum amounts

$$v_{\min} := \sqrt{2}\left(1+2\cdot 4^k\right)^{-\gamma}. \tag{3.33}$$

Next, let us take a closer look at the drifting particle's movement. Observe that the paths of two drifting particles started in different positions can meet on the axes only. This follows from our argumentation in step 1. Let us assume that $x \leq y$; the other case will follow by symmetry. If $x \leq y$, the particle hits the x-axis first, which happens at time

$$t_x := \min\{t : h(x,t) = 0\} = \frac{(x+1)^{\gamma+1} - 1}{\gamma+1} \tag{3.34}$$

following from the definition of h in (2.8). What constraints must hold for y such that the particle will hit the axes in $M^-(k)$? Clearly, y must fulfil

$$h(y,t_x) \leq 4^{k-1} \text{ or, equivalently,} \tag{3.35}$$

$$y \leq \left(\left(4^{k-1}+1\right)^{\gamma+1} + (x+1)^{\gamma+1} - 1\right)^{1/(\gamma+1)} - 1. \tag{3.36}$$

Let us denote the right side of inequality (3.36) by $g(x)$. By differentiation, we see that $g(x) - x$ is a positive, strictly decreasing function tending to 0 as x becomes large. On the other hand, if we consider the starting position (x,y), $y \leq x$, the calculation is the same with exchanged roles of y and x, and we end up with $g(y)$. Thus, as long as the particle starts in

$$(x,y) \in A := \left\{(x,y) : \left(x+y \leq 4^{k+1}\right) \wedge \left(g^{-1}(x) \leq y \leq g(x)\right)\right\}, \tag{3.37}$$

it reaches $M^-(k)$ first and hit the axes only afterwards. Thus, as long as the particle jumps to positions $(x,y) \in A$ only, we do not have to worry that the particle will reach the axes before reaching $M^-(k)$.

Let us define the level sets of A by

$$A_l := A \cap \left\{(x,y) : x+y = l\right\}. \tag{3.38}$$

We can interprete A_l as an one-dimensional interval or a piece of a line, and because $g(x) - x$ and $g^{-1}(x) - x$ are tending to zero, the length of this interval, denoted by $diam(A_l)$, decreases as l increases. We need to know $diam(A_{4^{k+1}})$, as it is a lower bound for all l we are interested in. Because the particle's jump direction is parallel to A_l, we can estimate afterwards how much time the particle will spend in A when performing jumps. However, it is not possible to calculate $diam(A_{4^{k+1}})$ explicitly, but by the Pythagorean Theorem, the symmetry of $g(x)$ and $g^{-1}(x)$, and the decrement of $g(x) - x$, we have

$$diam(A_{4^{k+1}}) \geq \sqrt{2}\left(g(2\cdot 4^k) - 2\cdot 4^k\right) =: d_k. \tag{3.39}$$

Our ansatz is

$$d_k \geq D4^k \tag{3.40}$$

for a constant D independent of k provided k is large enough. Notice that function g as defined in (3.36) is basically the $||\cdot||_{\gamma+1}$-norm of $(4^{k-1}, x)$ and decreases in γ. We may assume that $\gamma = 1$ without loss of generality.

$$\sqrt{2}\left(\left(\left(4^{k-1}+1\right)^2+\left(2\cdot 4^k+1\right)^2-1\right)^{1/2}-1-2\cdot 4^k\right)\geq D4^k \tag{3.41}$$

transforms to

$$\sqrt{2}D \leq \left(\frac{\sqrt{65}-8}{2} - O(4^{-k})\right). \tag{3.42}$$

Finally, we have to answer the question how long we remain in an interval of diameter $\sqrt{2}D4^k$ when we start in the centre and perfom a random walk with step size $\sqrt{2}\epsilon$. Denoting a standard random walk with step size 1 by R, we are looking for the hitting time

$$\xi^\epsilon(k) := \min\left\{n : R_n \notin (-D4^k/\epsilon, D4^k/\epsilon)\right\}. \tag{3.43}$$

It is well known that

$$\mathbb{E}\xi^\epsilon(k) = \left(\frac{D4^k}{\epsilon}\right)^2. \tag{3.44}$$

We would like to have a lower bound for the probability to stay in the interval for $c\mathbb{E}\xi^\epsilon(k)$ steps at least with $c \in (0,1)$ being arbitrary small. For our purposes, it is sufficient to show that this probability tends to 1 if c goes to 0. As ϵ tends to zero, Donsker's principle (see chapter 2.4.D of [6]) tells us

$$\lim_{\epsilon\to 0}\frac{D4^k}{\epsilon}\tilde{R}_{(D4^k/\epsilon)^2 t} \stackrel{d}{=} B_t \tag{3.45}$$

with $\tilde{R}$ being the linear interpolation of R. We define the exit time of a Brownian motion B from $(-1,1)$ by

$$\bar{\xi} := \inf\left\{t : B_t \notin (-1,1)\right\}. \tag{3.46}$$

For ϵ tending to zero and a constant $\alpha > 0$, we use Donsker's principle to get

$$\begin{aligned}
P\left(\xi^\epsilon(k) < c\mathbb{E}\xi^\epsilon(k)\right) &= P\left(\bar{\xi} < c\right) && (3.47)\\
&= P\left(\exp\left(-\alpha\bar{\xi}\right) > \exp(-\alpha c)\right) && (3.48)\\
&< \frac{\mathbb{E}e^{-\alpha\bar{\xi}}}{e^{-\alpha c}} && (3.49)\\
&= \frac{e^{\alpha c}}{\cosh\left(\sqrt{2\alpha}\right)}. && (3.50)
\end{aligned}$$

In line (3.49), we use the Markov inequality, in line (3.50) the explicit formula for the Laplace transform of $\bar{\xi}$ (refer to formula 3.0.1 in [3]). As α is chosen arbitrarily, we would like to minimize line (3.50) as a function of α. Differentiation shows that the optimizing α fulfils

$$\cosh\left(\sqrt{2\alpha}\right) = \frac{\sinh\left(\sqrt{2\alpha}\right)}{c\sqrt{2\alpha}} \tag{3.51}$$

Using equality (3.51) in (3.50) results in

$$P\left(\xi^\epsilon(k) < c\mathbb{E}\xi^\epsilon(k)\right) < \frac{c\sqrt{2\alpha}e^{\alpha c}}{\sinh\left(\sqrt{2\alpha}\right)} \tag{3.52}$$

which tends to zero when c tends to zero. Let us call

$$p_c := P\left(\xi^\epsilon(k) \geq c\mathbb{E}\xi^\epsilon(k)\right), \tag{3.53}$$

and observe that one can choose c such that p_c is arbitrarily close to one.

In step 4, we summarise the results from the steps before. When the particle starts in $M(k)$, the probability to reach the bisector before reaching $M^+(k)$ is larger than $1-\exp(-6/7)$ by steps 1 and 2. By step 3, we can find a $c^* > 0$ such that $(1-\exp(-6/7))p_{c^*} > 1/2$. This means we stay within A for $c^*(D4^k/\epsilon)^2$ steps at least. As the particle drifts with the minimal speed $v_{\min}$, defined in (3.33), its distance to the origin in terms of the $||\cdot||_1$-norm decreases by

$$\begin{aligned} & \sum_{i=1}^{c^*(D4^k/\epsilon)^2} \sqrt{2}\left(1+2\cdot 4^k\right)^{-\gamma}\sigma_i && (3.54)\\ = \;& c^*D^2 4^{2k}\sqrt{2}\left(1+2\cdot 4^k\right)^{-\gamma} && (3.55)\\ = \;& O\left(4^{(2-\gamma)k}\right) && (3.56)\end{aligned}$$

for ϵ tending to zero. In line (3.55), we use the LLN for the *i.i.d.* σ_i, which have expectation ϵ^2. Thus, the distance covered by the particle is of order $4^{(2-\gamma)k}$. On the other hand, the distance that the particle has to cover to get to $M^-(k)$ is smaller than or equal to

$$4^{k+1}-4^{k-1} \in O\left(4^k\right) \tag{3.57}$$

by the proof's construction. Obviously, (3.56) dominates (3.57) for $\gamma < 1$ such that the proof in the subcritical case is finished for $Y^\epsilon - X^\epsilon$.

To see that the result transfers to $Y-X$, we consider the process $\tilde{X}^\epsilon$ constructed like X^ϵ but with the modified *ode*

$$\frac{d}{dt}f(t) = \left((1+2\epsilon)+b-f(t)\right)^{-\gamma};\; f(0)=0 \tag{3.58}$$

instead of the original *ode* (2.5). Equivalently, we define $\tilde{Y}^\epsilon(B^\epsilon) := -\tilde{X}^\epsilon(-B^\epsilon)$. The proof shows easily that the change of the constant from 1 to $1+2\epsilon$ in (3.58) does not change the calculations or the result in an essential way. Thus, $\tilde{Y}^\epsilon - \tilde{X}^\epsilon$ is also recurrent for $\gamma < 1$. The crucial observation is that these *auxiliary* processes sandwich the original processes:

$$X_t^{\epsilon'} \geq \tilde{X}_t^\epsilon - \epsilon \text{ and } Y_t^{\epsilon'} \leq \tilde{Y}_t^\epsilon + \epsilon \tag{3.59}$$

for all $\epsilon' < \epsilon$. This holds due to the fact that $|X^{\epsilon'}_{\bar{\sigma}_1^\epsilon} - X^\epsilon_{\bar{\sigma}_1^\epsilon}| < \epsilon$ and $|B^\epsilon - B^{\epsilon'}| < \epsilon$. Thus, the difference in speed cannot be larger than 2ϵ. This argument extends to all later times $\bar{\sigma}_i$ inductively. It follows

$$\begin{aligned} Y_t - X_t \;&=\; \lim_{\epsilon\to 0}(Y_t^\epsilon - X_t^\epsilon) && (3.60)\\ &\leq\; \tilde{Y}_t^\epsilon - \tilde{X}_t^\epsilon + 2\epsilon && (3.61)\end{aligned}$$

which proves recurrence for $Y-X$.

3.4 Proof of the supercritical case: $\gamma > 1$

We define first what transience of Markov chains means.

Definition 3.2. For any $A \subset [0,\infty)^2$, let

$$\eta_A := \sum_{i=0}^{\infty} \mathbb{1}_{\{\Phi_i^\epsilon \in A\}} \tag{3.62}$$

be the number of visits of Φ^ϵ in A. A set A is called *uniformly transient* if there exists $M < \infty$ such that $\mathbb{E}_{(x,y)}(\eta_A) \leq M$ for all $(x,y) \in A$. We call Φ^ϵ *transient* if there is a countable cover of $[0,\infty)^2$ with uniformly transient sets.

We use the next theorem to show that Φ^ϵ is transient in the upper sense. It is stated as a more general result in [7], 8.0.2(i).

Theorem 3.3. *The chain Φ^ϵ is transient if and only if there exists a bounded, non-negative function $g : [0,\infty)^2 \to [0,\infty)$ and a set $\mathscr{B} \in \mathfrak{B}^+([0,\infty)^2)$ such that, for all $(\bar{x},\bar{y}) \in [0,\infty)^2 \backslash \mathscr{B}$,*

$$Lg(\bar{x},\bar{y}) = \int_{[0,\infty]^2} P_{(\bar{x},\bar{y})}(d(x,y))g(x,y) \geq g(\bar{x},\bar{y}) \tag{3.63}$$

and

$$D := \left\{ (x,y) \in [0,\infty)^2 \;\middle|\; g(x,y) > \sup_{(\bar{x},\bar{y}) \in \mathscr{B}} g(\bar{x},\bar{y}) \right\} \in \mathfrak{B}^+([0,\infty)^2). \tag{3.64}$$

Basically, we have to find a certain function g such that the particle jumps away from the origin in expectation with respect to g. This must hold outside a compact set $\mathscr{B}$ containing the origin. To find a proper $\mathscr{B}$, we set

$$\mathscr{B}_z := \left\{ (x,y) \in [0,\infty)^2 \;\middle|\; \left|\left|(x+1,y+1)\right|\right|_{\gamma+1} = z \right\} \tag{3.65}$$

for all $z > 0$. For g, we choose

$$g(x,y) := 1 - \left|\left|(x+1,y+1)\right|\right|_{\gamma+1}^{-1}. \tag{3.66}$$

If we can find a $\bar{z}$ remaining finite as ϵ tends to zero such that equation (3.63) holds for all $(x,y) \in \mathscr{B}_z$, $z \geq \bar{z}$, we are done. Recall what happens in one step of Φ^ϵ in the underlying process as described on page 145. Equation (3.63) becomes

$$\begin{aligned} &\frac{1}{2}\int_0^\infty P(\sigma \in dt)\, g(h(t,\bar{x}+\epsilon), h(t,\bar{y}-\epsilon)) \\ +&\frac{1}{2}\int_0^\infty P(\sigma \in dt)\, g(h(t,\bar{x}-\epsilon), h(t,\bar{y}+\epsilon)) \quad \geq \quad g(\bar{x}\bar{y}) \end{aligned} \tag{3.67}$$

with $(\bar{x},\bar{y}) \in \mathscr{B}_{\bar{z}}$. Because of the ϵ-jump of B^ϵ at time $\bar{\sigma}$, the integral splits into two parts. Within both integrals, the only source of randomness is σ. If its value is given, we can calculate the next position of Φ^ϵ by using function h and finally apply g to this value.

Using the definition of g and observing that the integral of the density, $P(\bar{\sigma} \in dt)$, is 1, we transform (3.67) to

$$\begin{aligned} & \frac{1}{2}\int_0^\infty P(\sigma \in dt) \left|\left|(h(t,\bar{x}+\epsilon)+1, h(t,\bar{y}-\epsilon)+1)\right|\right|_{\gamma+1}^{-1} && (3.68) \\ +&\frac{1}{2}\int_0^\infty P(\sigma \in dt) \left|\left|(h(t,\bar{x}-\epsilon)+1, h(t,\bar{y}+\epsilon))+1\right|\right|_{\gamma+1}^{-1} \quad \leq \quad \bar{z}^{-1}. \end{aligned}$$

As already argued, σ is small, or rather we can change the upper bound of the integrals from ∞ to ϵ at the expense of order $\exp(-1/\epsilon)$. Furthermore, let us assume for the moment that $\bar{x}, \bar{y} \geq 2\epsilon$. As jump size and drift time are ϵ at most, and the drift speed is bounded from above by 1, this condition avoids that we have to handle cases in which the axes are reached. Observe that the only special cases to check later on are $(0, \bar{y})$ and $(\bar{x}, 0)$, because we can choose an $\epsilon > 0$ for every pair $\bar{x}, \bar{y} > 0$ such that the condition above is fulfiled, and we let ϵ tend to zero. Now, we can use Taylor approximations for ϵ and t to get

$$\begin{aligned} & \frac{1}{2}\Big(\left|\left|(h(t,\bar{x}+\epsilon)+1, h(t,\bar{y}-\epsilon)+1)\right|\right|_{\gamma+1}^{-1} && (3.69) \\ & \quad + \left|\left|(h(t,\bar{x}-\epsilon)+1, h(t,\bar{y}+\epsilon)+1)\right|\right|_{\gamma+1}^{-1}\Big) \\ = \; & \frac{1}{2}\Big(\left((\bar{x}+\epsilon+1)^{\gamma+1} + (\bar{y}-\epsilon+1)^{\gamma+1} - 2(\gamma+1)t\right)^{-\frac{1}{\gamma+1}} && (3.70) \\ & \quad + \left((\bar{x}+\epsilon+1)^{\gamma+1} + (\bar{y}-\epsilon+1)^{\gamma+1} - 2(\gamma+1)t\right)^{-\frac{1}{\gamma+1}}\Big) \\ = \; & \bar{z}^{-1} + 2\bar{z}^{-(\gamma+2)}t - \frac{\gamma}{2}\left((\bar{x}+1)^{\gamma-1} + (\bar{y}+1)^{\gamma-1}\right)\bar{z}^{-(\gamma+2)}\epsilon^2 && (3.71) \\ & \quad + (1+t)\,O(\bar{z}^{-(2\gamma+3)}\epsilon^2). \end{aligned}$$

Because

$$\int_0^\epsilon P(\sigma \in dt)\, t \leq \mathbb{E}\sigma = \epsilon^2, \tag{3.72}$$

we can rewrite (3.68) as

$$\begin{aligned} & \bar{z}^{-1} + 2\bar{z}^{-(\gamma+2)}\epsilon^2 + O(\bar{z}^{-(2\gamma+3)}\epsilon^2) && (3.73) \\ \leq \; & \bar{z}^{-1} + \frac{\gamma}{2}\left((\bar{x}+1)^{\gamma-1} + (\bar{y}+1)^{\gamma-1}\right)\bar{z}^{-(\gamma+2)}\epsilon^2 \end{aligned}$$

which holds if

$$\gamma\left((\bar{x}+1)^{\gamma-1} + (\bar{y}+1)^{\gamma-1}\right) \geq 4. \tag{3.74}$$

Equation (3.74) is fulfiled for $\bar{z}$ large enough and $\gamma > 1$ only.

It remains to show the special case for $\bar{x}$ or $\bar{y}$ being zero. Because of symmetry, it is sufficient to treat one of these cases. We assume $\bar{x} = 0$, and thus $\bar{y} = (\bar{z}^{\gamma+1} - 1)^{1/(\gamma+1)} - 1$. Then condition (3.67) becomes

$$\begin{aligned} \bar{z}^{-1} \quad \geq \quad & \frac{1}{2}\int_0^\infty P(\bar{\sigma} \in dt) \left|\left|(h(t,\epsilon)+1, h(t,\bar{y}-\epsilon)+1)\right|\right|_{\gamma+1}^{-1} && (3.75) \\ +&\frac{1}{2}\int_0^\infty P(\bar{\sigma} \in dt) \left|\left|(1, h(t,\bar{y}+\epsilon)+1)\right|\right|_{\gamma+1}^{-1}. \end{aligned}$$

Applying Taylor approximation in the same way as above results in

$$\bar{z}^{-1} \geq \bar{z}^{-1} - \bar{z}^{-(\gamma+2)}\epsilon + O(\epsilon^2) \tag{3.76}$$

which is true for all γ and arbitrary $\bar{z}$.

The idea how to transfer the transient result to $Y - X$ is basically equal to the recurrent case on page 152. This time, we consider the process $\hat{X}^\epsilon$ constructed like X^ϵ but with the modified *ode*

$$\frac{d}{dt}f(t) = \big((1-2\epsilon) + b - f(t)\big)^{-\gamma}\,;\ f(0) = 0 \tag{3.77}$$

instead of (2.5). Equivalently, we define $\hat{Y}^\epsilon(B^\epsilon) := -\hat{X}^\epsilon(-B^\epsilon)$. Again, the proof is not essentially changed by these modifications, and thus, $\hat{Y}^\epsilon - \hat{X}^\epsilon$ is also transient for $\gamma > 1$. Observe that the *auxiliary* processes are sandwiched by the original processes:

$$X_t^{\epsilon'} \leq \hat{X}_t^\epsilon + \epsilon \text{ and } Y_t^{\epsilon'} \geq \hat{Y}_t^\epsilon - \epsilon \tag{3.78}$$

for all $\epsilon' < \epsilon$. This follows from the same idea like in the recurrent case. We conclude

$$\begin{aligned} Y_t - X_t &= \lim_{\epsilon \to 0}(Y_t^\epsilon - X_t^\epsilon) && (3.79)\\ &\geq \hat{Y}_t^\epsilon - \hat{X}_t^\epsilon - 2\epsilon && (3.80) \end{aligned}$$

which implies the desired result.

4 Conclusions

In this last section, we describe what our results mean for the opinion game [4]. We begin with a short description of the model. Although it is introduced in great generality in the original article, we adhere to that implementation that has produced *interesting* results in the simulations. For a deeper discussion about the choice of parameters, we refer to the original paper. In the second subsection, we point out the connections between our work and the opinion game.

4.1 The opinion game

Bovier et al. consider a generalised, respectively virtual, order book containing the opinion of each participating agent about the share's value. Here, the notion of *value* is distinguished from the one of *price*. The price is determined by the market and is the same for all agents, but the value is driven by fundamental and speculative considerations, and thus, varies individually. This is a fundamental difference to a classical order book model, because the order book only keeps track of placed orders; the generalised order book *knows* the opinion of all market participants independent on whether they have made them public. The model's dynamics are driven by the change of the agents' opinions.

A market with N traders trading $M < N$ stocks is considered. For simplification, every trader can own at most one share, and furthermore, a discrete time and space setting is assumed. The state of trader i is given by his or her opinion $p_i \in \mathbb{Z}$ and the number of possesed stocks $n_i \in \{0, 1\}$. A trader with one share is called a *buyer*, one without a share is called a *seller*. The order book's state is given by the states of all traders. A state is said to be *stable*, if the traders with the M highest opinions posses shares. In particular, one can describe an order book's stable state just by the traders' opinions $\mathbf{p} := (p_1, \dots, p_N)$. For stable states, one can define an ask price as the lowest opinion of all traders possesing a share:

$$p^a := \min\{p_i : n_i = 1\}; \tag{4.1}$$

the bid price is defined as the highest opinion of all traders without a share:

$$p^b := \max\{p_i : n_i = 0\}. \tag{4.2}$$

The *current (logarithmic) price* of the stock is given by $p := (p^a - p^b)/2$. The updating of the order book's state, $\mathbf{p}$, happens in three steps:

1. At time $(t+1) \in \mathbb{N}_0$, trader i is selected with probability $g(\cdot; \mathbf{p}(t), t)$.
2. The selected trader i changes his or her opinion to $p_i(t) + d$ with $d \in \mathbb{Z}$ having distribution $f(\cdot; \mathbf{p}(t), i, t)$.
3. If $\mathbf{p}' = (p_1(t), \dots, p_i(t) + d, \cdots, p_N(t))$ is stable, then $\mathbf{p}(t+1) = \mathbf{p}'$. Otherwise, trader i exchanges the state of ownership, $n_i(t)$, with the lowest asker, respectively highest bidder, j. Afterwards, to avoid a direct re-trade, both participants change their opinion away from the trading price.

The function g is defined by

$$g(i; \mathbf{p}(t), t) := h(p_i(t) - p(t))/Z_g(\mathbf{p}(t)) \tag{4.3}$$

with

$$h(x) := 1/(1 + |x|)^\gamma, \; \gamma > 0, \tag{4.4}$$

and Z_g normalizing g such that $\sum_{i=1}^N g(i; \mathbf{p}(t), t) = 1$.

The size of d is chosen from the set $\{-l, \dots, l\}$ with probability

$$f(d; \mathbf{p}(t), i, t) := \frac{1}{2l+1} \left(\left(\delta_{p_i, p(t)} \delta_{\text{ext}}(t) \right)^d \wedge 1 \right) \text{ for } d \neq 0 \tag{4.5}$$

and $f(0; \mathbf{p}(t), i, t) = 1 - \sum_{0<|k|\le l} f(k; \mathbf{p}(t), i, t)$. The parameter $\delta_{p_i, p(t)}$ describes the tendency to change the opinion into the price's direction. Thus it is larger than 1 for $p_i < p$ and smaller for $p_i > p$. The second parameter, δ_{ext}, simulates outer influences on the change of opinion, news or rumors for example. This force is the same for all traders but changes its strength in time. Good results have been achieved by taking $l = 4$, $\delta_{p_i, p(t)} = \exp(0.1)$ for buyers, and $\delta_{p_i, p(t)} = \exp(-0.1)$ for sellers. The external influence changes its strength after independent, exponentially distributed times with rate 1/2000 to $\exp(\epsilon_i s_i')$ with ϵ_i being Bernoulli with $P(\epsilon_i = \pm 1) = 1/2$ and s_i' being Exponential with mean 0.12. Observe that, in expectation, the external force is slightly stronger than the drift to the price.

The jump away from the trading price in the last step is implemented by setting

$$p_i(t+1) = p^b(t) - k, \;\; p_j(t+1) = p^b(t) + k \tag{4.6}$$

if trader i sells a stock in this step, and

$$p_i(t+1) = p^a(t) + k, \quad p_j(t+1) = p^a(t) - k \tag{4.7}$$

if he or she buys it. Here, k is a uniformly distributed variable on $\{5, \ldots, 20\}$.

In the simulations, the price was recorded every 100 opinion updates. Thus, if we talk about one simulation step in the next section, we mean 100 steps of the underlying dynamics.

4.2 Our result in context

Simulations show that the price process produced by these dynamics has some interesting properties. At first, the distribution of returns, which is the relative change of the price in one step, has heavy tails. Furthermore, the volatility, which is the average returns' size in some time interval, shows correlations on much larger time scales than the implementation would suggest. For the volatility of an interval of size 100, correlations after 10^4 steps can be observed. This is suprising, because 10^4 recorded steps are equal to 10^6 steps of the dynamics, but the model is Markovian, and even the strength of the external influence changes after $2 \cdot 10^3$ steps only.

The explanation for these observations can be found in two features of the implementation. As alreday suggested, the external force brings excitement into the market; else the traders would basically perform random walks into the direction of the price. The returns would be much smaller; an interesting structure of the volatility would not exist. This coincides with the *Efficient Market Hypothesis*, because in a world without news and rumors there are no reasons for price changes.

But the external force on its own does not explain the system's memory in terms of volatility. This behaviour arises from the slower updating speed of traders far away from the current price. This mechanism ensures that the system *remembers* price changes on large time scales. If we observe an order book state including a group of traders with a large distance to the current price, we can deduce that the price must have been in the traders' region before, as it is very unlikely that the whole group has moved against its drift. Furthermore, after fast price movements, the distance between ask and bid price, called *gap*, is larger than average and needs some time to recover. In these periods, the market is *illiquid*, and a small number of trades can move the price a lot resulting in an increased volatility. Increased volatility after large price movements is a well observed feature of real world markets.

Thus, the connection of the updating speed and the distance to the price is of paramount importance for the model. Indeed, the larger γ is chosen in formula (4.4) the better the just explained phenomena can be observed. However, a larger γ contains the risk of instablity in the system. It turns out that once the gap has exceeded a certain size (depending on γ), it cannot recover anymore, and the two groups, buyers and sellers, drift away from each other. Then the price waves between these groups. In particular, it is driven by two traders - one from each group - that have been able to get away from the other agents, and that move according to the external drift without any resistance by surrounding traders. For $\gamma \geq 1.6$, this happens quite fast, yet, the model has remained stable in simulations over several days for $\gamma = 1.5$ (figure 5). On the other hand, if we start a simulation with a large gap and $\gamma = 1.5$, the system is also not able to recover. As a large gap size is eventually reached by randomness, it is justified to talk about a *metastable* behaviour. In figure 6, we illustrate these statements with a sample. Instead of recording the difference between ask and bid price, we recorded the distance between the 950th and the 1050th trader ordered by their opinions (in other

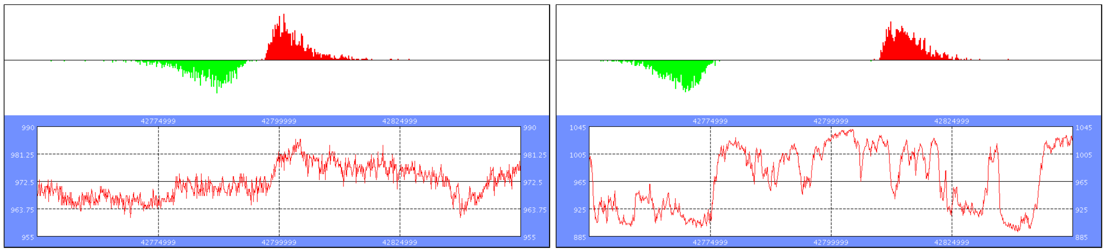

Figure 5: Screenshots of the virtual order books after 428500 simulation steps for $\gamma = 1.5$ (left) and $\gamma = 1.6$ (right) with the same initial conditions and the same realisation of external influences. Observe the different distances between buyers (green) and sellers (red) and the different behaviour of the price processes (blue box).

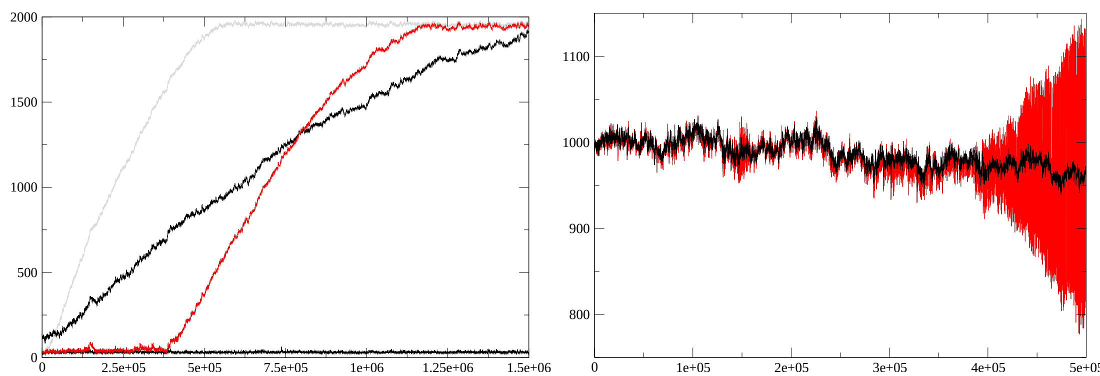

Figure 6: The left graph shows the *gap* of the system for different γ. Being stable for $\gamma = 1.5$ (black lower graph), it increases for $\gamma = 1.6$ (red) and $\gamma = 1.7$ (grey). However, if the system is started with $\gamma = 1.5$ but with an artificially enlarged gap, it also increases (black increasing graph). The convergence to a value below 2000 is due to a restriction of the state space in the numerical simulations. The right graph shows the stable, respectively unstable, behaviour for $\gamma = 1.5$ (black) and $\gamma = 1.6$ (red) in terms of the price process.

words, the buyer with the 50th highest opinion and the seller with the 50th lowest one), because traders close to the price suffer much more fluctuations than agents with some distance.

In the situation when the traders' groups have already a large distance from each other, the two traders in between and also the price perform basically a random walk. Especially, when the two traders are close to the middle in between both groups, their probability to move is almost 1. In this case, our model with a Brownian motion as driving force offers a reasonable approximation for the behaviour of the system. Thus, our results give few hope that any simulation with $\gamma > 1$ remains stable. But for $\gamma < 1$, the memory effect producing all the statistical facts is too small. However, as already mentioned, the model seems to be stable on a large time scale for $\gamma = 1.5$. This and the sharp threshold between 1.5 and 1.6 are not understood. More research is neccessary here.

Besides these findings, the three particle model introduced in this paper has its qualities on its own. As a simple model for longterm investors, the simple setting already exhibits an interesting and non-trivial longterm behaviour. As a next step, it is interesting to see how the results change if we substitute the Brownian motion by a Lévy process, which is more realistic for price process on stock markets.

A Proof of Lemma 2.1

We turn to the proof of Lemma 2.1.

Let $S \subset [0,\infty)$ be a compact set and $\epsilon \ll \exp(-\gamma \cdot \sup S)$. Then

$$\sup_{t\in S} \left| X_t^{\epsilon'} - X_t^{\epsilon} \right| \leq \epsilon K_S \text{ a.s.} \tag{A.1}$$

with K_S being a finite, deterministic constant that depends on S, and $\epsilon' < \epsilon$.

Because S is compact, we may assume $S = [0, t^*]$ for some $0 \leq t^* < \infty$. Remember that the jump times of B^ϵ are denoted by $\bar{\sigma}^\epsilon$ (see (2.1)), and the time between two jumps by σ^ϵ (see (3.7)). Furthermore,

$$\left| B^\epsilon - B^{\epsilon'} \right| < \epsilon. \tag{A.2}$$

We denote the distance of X^ϵ to B^ϵ by

$$d_i := B^\epsilon_{\bar{\sigma}_i} - X^\epsilon_{\bar{\sigma}_i}, \tag{A.3}$$

and the distance to $X^{\epsilon'}$ by

$$\Delta_i := X^\epsilon_{\bar{\sigma}_i} - X^{\epsilon'}_{\bar{\sigma}_i} \tag{A.4}$$

with $\bar{\sigma}$ with respect to ϵ. We would like to maximize Δ_2, thus we assume that B^ϵ has jumped upwards at $\bar{\sigma}_1$. Then $d_1 = \epsilon$ and $|\Delta_1| < \epsilon$. We assume that Δ_1 is positive first. By definition of Δ and of $\bar{h}$ in (2.6),

$$\begin{aligned}
\Delta_2 &= \left(X^\epsilon_{\bar{\sigma}_2} - X^\epsilon_{\bar{\sigma}_1} \right) - \left(X^{\epsilon'}_{\bar{\sigma}_2} - X^{\epsilon'}_{\bar{\sigma}_1} \right) + \left(X^\epsilon_{\bar{\sigma}_1} - X^{\epsilon'}_{\bar{\sigma}_1} \right) && \text{(A.5)}\\
&\overset{(A.2)}{\leq} \bar{h}(\sigma_1, d_1) - \bar{h}(\sigma_1, d_1 + \Delta_1 + \epsilon) + \Delta_1 && \text{(A.6)}\\
&\overset{(2.8)}{=} h(\sigma_1, d_1 + \Delta_1 + \epsilon) - h(\sigma_1, d_1) - \epsilon. && \text{(A.7)}
\end{aligned}$$

Remember that h is *basically* defined as

$$h(t,d) = \left((d+1)^{\gamma+1} - (\gamma+1)\, t \right)^{1/(\gamma+1)} - 1. \tag{A.8}$$

As the distance does not increase anymore, once X^ϵ has hit B^ϵ, we get an upper bound for σ_1:

$$h(\sigma_1, d_1) \geq 0 \iff \sigma_1 \leq \frac{(d_1+1)^{\gamma+1} - 1}{\gamma+1}. \tag{A.9}$$

Because $d_1 = \epsilon$, we have $\sigma_1 \leq \epsilon$. As d_1, Δ_1, ϵ, and σ_1 are small in comparison to 1, we apply Taylor approximations to line (A.7) twice and get

$$\begin{aligned}
\Delta_2 &\leq \left(1 - (\gamma+1)\, \sigma_1 \right)^{-\gamma/(\gamma+1)} (\Delta_1 + \epsilon) - \epsilon && \text{(A.10)}\\
&= \Delta_1 + \gamma\, (\Delta_1 + \epsilon)\, \sigma_1 && \text{(A.11)}\\
&= \Delta_1\, (1 + \gamma\epsilon)\,. && \text{(A.12)}
\end{aligned}$$

With the same argumentation, we can conclude that

$$\Delta_{i+1} \leq \Delta_i \left(1 + \gamma\epsilon\right), \tag{A.13}$$

and thus,

$$\begin{aligned} X^{\epsilon}_{t^*} - X^{\epsilon'}_{t^*} &= \Delta_{t^*/\epsilon} && \text{(A.14)} \\ &\leq \Delta_1 \left(1 + \gamma\epsilon\right)^{t^*/\epsilon} && \text{(A.15)} \\ &\to \epsilon e^{\gamma t^*}. && \text{(A.16)} \end{aligned}$$

On the other hand, if $X^{\epsilon'} > X^{\epsilon}$, the same idea applies: the distance grows the quickest, if one of the processes always stays close to its attracting process such that it has drift speed 1. Now, if $X^{\epsilon'}$ increases with speed 1 (as a worst case assumption), $\sigma^{\epsilon} = \epsilon$, and we end up with the same calculation as before.

It should be mentioned that our estimations are rough, as we do not consider the structure of Brownian paths, but only the worst case of all continous paths. However, uniform convergence on compact intervals is the best one can get, and every improvement would only change the constant K_S.

References

[1] Robert Almgren and Neil Chriss, *Optimal execution of portfolio transactions*, J. Risk **3** (2001), no. 2, 5–39.

[2] Nils Berglund and Barbara Gentz, *Noise-induced phenomena in slow-fast dynamical systems*, 1st ed., Probability and its Applications, vol. XIII, Springer, November 2006. MR2197663

[3] Andrei N. Borodin and Paavo Salminen, *Handbook of brownian motion - facts and formulae*, 1st ed., Probability and Its Applications, vol. XV, Birkhäuser, Basel, November 1996. MR1477407

[4] Anton Bovier, Jiří Černý, and Ostap Hryniv, *The opinion game: Stock price evolution from microscopic market modeling*, Int. J. Theoretical Appl. Finance **9** (2006), no. 1, 91–111. MR2205716

[5] Gur Huberman and Werner Stanzl, *Optimal liquidity trading*, Rev. Finance **9** (2005), no. 2, 165–200.

[6] Ioannis Karatzas and Steven E. Shreve, *Brownian motion and stochastic calculus*, 2nd ed., Graduate Texts in Mathematics, vol. 113, Springer, New York, Heidelberg & Berlin, 1991. MR1121940

[7] Sean P. Meyn and Richard L. Tweedie, *Markov chains and stochastic stability*, 1st ed., Control and Communication Engineering Series, vol. XVI, Springer, March 1996.

[8] Anna Obizhaeva and Jiang Wang, *Optimal trading strategy and supply/demand dynamics*, revised and resubmitted, J. Financ. Mark., 2005.