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EXISTENCE, UNIQUENESS AND REGULARITY OF PARABOLIC SPDES DRIVEN BY POISSON RANDOM MEASURE

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Abstract. In this paper we investigate SPDEs in certain Banach spaces driven by a Poisson random measure. We show existence and uniqueness of the solution, investigate certain integrability properties and verify the càdlàg property.

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1. Introduction

The classical Itô stochastic integral with respect to Brownian motion has been generalized in several directions. One of these directions is to define the stochastic integral in the L^p–mean, $1 < p \le 2$ (see Bichteler [6, 7]). This can be done easily if the integrator is of integrable p–variation. The works of Dettweiler [19], Giné and Marcus [24], Dang Hung Thang [44] and Neidhardt [36] have extended this procedure to certain types of Banach spaces in which most of the probabilistic theorems necessary for defining a stochastic integral are valid.

If the integrator is a Poisson random measure it is quite natural to define the stochastic integral in L^p-mean. In the case of SPDEs, examples can be covered which are not easily treated using the Hilbert space or M type 2 Banach space theory (see Example 2.2 and Example 2.3).

Let E be a separable Banach space of M type p, $1 < p \le 2$, with Borel σ-algebra $\mathcal{E}$. Let A be an infinitesimal generator of an analytic semigroup $(S_t)_{t\ge 0}$ on E. Let Z be a Banach space and $\mathcal{Z}$ be the Borel σ–algebra on Z. Let η be a Poisson random measure defined on $(Z, \mathcal{Z})$ with symmetric Lévy measure $\nu : \mathcal{Z} \to \mathbb{R}^+$ satisfying certain moment conditions. Assume that $f : E \to E$, and $g : E \times Z \to E$ are Lipschitz continuous and measurable functions. We consider the following SPDE written in the Itô-form

$$\begin{cases} du(t) &= (Au(t-) + f(u(t-)))\ dt + \int_Z g(u(t-); z)\eta(dz; dt), \\ u(0) &= u_0. \end{cases} \tag{1}$$

Under a mild solution of equation (1) we understand a predictable càdlàg process u taking values in a certain Banach space and satisfying the integral equation

$$u(t) = S_t u_0 + \int_{0+}^t S_{t-s} f(u(s-))\, ds + \int_{0+}^t \int_Z S_{t-s} g(u(s-); z)\ \eta(ds, dz), \text{ a.s. }, \quad t \ge 0.$$

In Theorem 2.1 we find conditions which guarantee existence and uniqueness of the solution u of problem (1).

In contrast to the Wiener process, the Lévy process

$$L(t) = \int_{0+}^t \int_Z z\ \eta(dz; ds), \quad t \ge,$$

itself is a.s. discontinuous, however $L = \{L(t), t \ge 0\}$ is càdlàg.

Parabolic SPDEs driven by the Gaussian white noise were initially introduced and discussed by Walsh [46, 47], where he also mentioned as an example the cable equation driven by a Poisson random measure. Kallianpur and Xiong [28, 29] showed existence and uniqueness for equation (1) in the space of distributions, while Albeverio, Wu and Zhang [1] investigated equation (1) and showed existence and uniqueness in Hilbert spaces under the L^2–integrability condition of the Poisson random measure under suitable hypothesis. Existence and uniqueness of SPDEs driven by time and space Poisson random measure were considered by, among others, Applebaum and Wu [2], Bié [8], Knoche [30], Mueller [34], and Mytnik [35]. Stochastic integration in Banach spaces has been discussed e.g. by Brooks and Dinculeanu [9], Dettweiler [19], Kussmaul [31], Neidhardt [36], Rüdiger [42], in which article Banach spaces of M type 1 and M type 2 were considered. SPDEs in M type 2 Banach spaces driven by Wiener noise have been investigated by Brzeźniak [10, 11], Elworthy and Brzeźniak [12].

We extend the Itô stochastic integral in the L^p–mean, $1 \leq p \leq 2$ to a wide class of Banach spaces. This allows us to investigate the existence and uniqueness of the solution to (1) a large class of Banach spaces (see e.g. Example 2.2).

This article is organized as follows: In section two we state our main theorem and some examples. In section three we recall some results about Poisson random measures, Lévy processes, and stochastic integration. We then prove our results in sections four and five.

2. The main result

There exist many connections between the validity of certain theorems for Banach space valued processes and the geometric structure of the underlying Banach space. We have omitted a detailed introduction to this topic as this would exceed the scope of the paper. A short summary of stochastic integration in Banach spaces is given in Chapter 3.2.

Definition 2.1. *(see Pisier* [39]*) Let $1 \leq p \leq 2$. A Banach space E is of M type p, iff there exists a constant $C = C(E,p) > 0$ such that for any E-valued discrete martingale $(M_0, M_1, M_2, \ldots)$ with $M_0 = 0$ the following inequality holds*

$$\sup_{n\geq 1} \mathbb{E}|M_n|^p \leq C \sum_{n\geq 1} \mathbb{E}|M_n - M_{n-1}|^p.$$

Remark 2.1. *This definition is equivalent to uniform p-smoothability. For literature on this subject see e.g. Brzeźniak* [10]*, Burkholder* [13]*, Dettweiler* [18, 19]*, Pisier* [39] *and Woyczyński* [48, 49]*.*

Example 2.1. *Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^d$ and $p > 1$. Then $L^p(\mathcal{O})$ is of M type $p \wedge 2$ (see e.g.* [48, Chapter 2, Example 2.2]*).*

Definition 2.2. *(see Linde* [32, Chapter 5.4]*) Let E be a separable Banach space and E' be the topological dual of E. Let $\mathcal{E}$ be the Borel-σ-algebra of E. A Borel-measure $\nu : \mathcal{E} \to \mathbb{R}^+$ is called a Lévy measure if it is σ–finite, $\nu(\{0\}) = 0$, and the function*

$$E' \ni a \mapsto \exp\left(\int_E (e^{i\langle x,a\rangle} - 1)\; \nu(dx) \right) \in \mathbb{C}$$

is a characteristic function of a Radon measure on E. If in addition $\nu(A) = \nu(-A)$ for all $A \in \mathcal{E}$, then ν is called symmetric Lévy measure. $\mathcal{L}^{sym}(E)$ denotes the set of all symmetric Lévy measures on $(E, \mathcal{E})$.

Moreover, for $\delta \in \mathbb{R}$ we denote by V_δ the domain of the fractional power of $-A$ (for the exact definition we refer to Appendix B). Now we can formulate our main result.

Theorem 2.1. *Let $1 < p \le 2$ and E and Z be separable Banach spaces, where E is of M type p and let $\mathcal{E}$ and $\mathcal{Z}$ be theirs Borel σ–algebras. Let A be an infinitesimal generator of a compact, analytic semigroup on E. Let $(\Omega, \mathcal{F}, P)$ be a probability space with given right-continuous filtration $(\mathcal{F}_t)_{t\ge 0}$ and $\eta : \mathcal{Z}\hat{\times}\mathcal{B}(\mathbb{R}^+) \to \mathbb{R}$ be a Poisson random measure with characteristic measure $\nu \in \mathcal{L}^{sym}(Z)$ over $(\Omega, \mathcal{F}, \mathbb{P})$. Let $p, q \in [1, \infty)$ be two constants such that $1 < p \le 2$ and $p < q$. Let δ_f and δ_g be two constants and $f : E \to V_{-\delta_f}$ and $g : E \to L(Z, V_{-\delta_g})$ two mappings satisfying the following global Lipschitz conditions*

$$|f(x) - f(y)|_{-\delta_f} \le C_1|x - y|, \quad x, y \in E, \tag{2}$$

$$\int_Z |g(x; z) - g(y; z)|^{p^l}_{-\delta_g} \nu(dz) \le C_2\, |x - y|^{p^l}, \quad x, y \in E, \quad l = 1, \dots, n, \tag{3}$$

where C_1 and C_2 are some constants. Let $\gamma \ge 0$ be fixed. Then the following holds:

- a.) *Let $q < \infty$. Assume that the constants δ_f, δ_g, δ and γ satisfy the following conditions*
  - (i) $\delta_g q < 1$, *and* $\delta_f < 1$,
  - (ii) $(\delta_g - \gamma)p < 1 - \frac{1}{q}$, *and* $\delta_f - \gamma < 1 - \frac{1}{q}$,
  - (iii) $\gamma < \frac{1}{q}$,
  - (iv) $\delta > \max(\frac{1}{q} + \delta_g, \delta_f - 1 + \frac{1}{q}, \frac{1}{q})$.

  If the initial condition u_0 satisfies $\mathbb{E}|u_0|^q_{-\gamma} < \infty$, then there exists a unique mild solution to Problem (1) such that for any $T > 0$

  $$\int_0^T \mathbb{E}|u(s)|^q \, ds < \infty, \qquad \sup_{0\le s\le T} \mathbb{E}\, |u(s)|^p_{-\gamma} < \infty$$

  and

  $$u \in L^0(\Omega; \mathbb{D}([0, T]; V_{-\delta})).$$

- b.) *Let $q < \infty$. Assume that the constants δ_f, δ_g, δ and γ satisfy the following conditions*
  - (i) $\delta_g p < 1$, *and* $\delta_f < 1$,
  - (ii) $(\delta_g - \gamma)p < 1 - \frac{1}{q}$, *and* $\delta_f - \gamma < 1 - \frac{1}{q}$,
  - (iii) $\gamma < \frac{1}{q}$,
  - (iv) $\delta > \max(\frac{1}{q} + \delta_g, \delta_f - 1 + \frac{1}{q}, \frac{1}{q})$.

  If the initial condition u_0 satisfies $\mathbb{E}|u_0|^p_{-\gamma} < \infty$, then there exists a unique mild solution to Problem (1), such that for any $T > 0$

  $$\int_0^T (\mathbb{E}|u(s)|^p)^{\frac{q}{p}} \, ds < \infty, \qquad \sup_{0\le s\le T} \mathbb{E}|u(s)|^p_{-\gamma} < \infty$$

  and

  $$u \in L^0(\Omega; \mathbb{D}([0, T]; V_{-\delta})).$$

- c.) *Let $q = \infty$ and $\gamma = 0$ and $\delta > \max(\delta_f, \delta_g)$. Assume that $\delta_g p < 1$ and $\delta_f < 1$. If initial condition u_0 satisfies $\mathbb{E}|u_0|^p < \infty$, then there exists a unique mild solution to Problem*

(1), such that for any $T > 0$

$$\sup_{0\le s\le T} \mathbb{E}|u(s)|^p < \infty,$$

and for all $\delta > 0$

$$u \in L^0(\Omega; \mathbb{D}([0,T]; V_{-\delta}))).$$

Remark 2.2. *(see Section 5) Using the same idea as Ikeda and Watanabe* [26, Chapter 4] *and the amalgamation procedure of Elworthy* [21, Chapter III.6], *one can show the càdlàg property for solutions to equation (1) under the hypothesis of Theorem 2.1, with no integrability condition on the jumps size. In particular, conditions (4) can be replaced by*

$$\begin{aligned} \int_{\{z\in Z||z|\le 1\}} |g(x;z)-g(y;z)|^p_{-\delta_g}\,\nu(dz) &\le C_1(|x-y|^p), \quad x,y\in E,\\ |g(x;\cdot)-g(y;\cdot)|_{L(V_{-\delta_g},E)} &\le C_2\,|x-y|, \quad x,y\in E. \end{aligned}$$

where C_1 and C_2 are some generic constants.

Remark 2.3. *Let $1 \le p \le 2$. Let E and Z be two separable Banach spaces, E is of M type p. Let $\nu : \mathcal{B}(Z) \to \mathbb{R}^+$ be a not necessarily symmetric Lévy measure and $\eta : \mathcal{B}(E)\hat{\times}\mathcal{B}(\mathbb{R}^+) \to \mathbb{R}^+$ be a Poisson random measure with characteristic measure ν. Let $\gamma(t) : \mathcal{B}(Z)\hat{\times}\mathcal{B}(\mathbb{R}^+) \to \mathbb{R}^+$ be the compensator of η, i.e. the unique predictable random measure such that*

$$\eta(A\times(0,t]) - \gamma(A,(0,t])$$

is a martingale for each $A \in \mathcal{B}(Z)$. Let $\tilde\eta := \eta - \gamma$ be the compensated Poisson random measure. Then by small modification (see Remark 3.3 and Remark 3.6) it can be shown, that the Theorem 2.1 holds also for the following SPDE

$$\left\{\begin{aligned} u(t)\,dt &= (Au(t-)+f(u(t-)))\;dt + \textstyle\int_Z g(u(t-);z)\tilde\eta(dz;dt),\\ u(0) &= u_0. \end{aligned}\right.$$

Example 2.2. *Let $1 < \alpha < 2$. Take any $p \in (\alpha,\infty)$ and put $E = W^p_{\frac{d}{p}}(\mathcal{O})$, where $\mathcal{O}$ a smooth domain in $\mathbb{R}^d$. Let $\gamma > 0$ and $Z = W^p_\gamma(\mathcal{O})$ and $U = \{x \in Z \mid |x| \le 1\}$. Let $\sigma : \partial U \to \mathbb{R}^+$ be a finite measure. Let $\eta : \mathcal{B}(E)\hat{\times}\mathcal{B}(\mathbb{R}^+) \to \mathcal{B}(E)$ a Poisson random measure with characteristic measure $\nu : \mathcal{B}(\mathbb{E}) \to \mathbb{R}$ given by (see Example 3.1)*

$$\nu(B) = \int_{\mathbb{R}^+}\int_{\partial U} \chi_B(sz)\sigma(dz)\,s^{-1-\alpha}\,ds, \quad B\in\mathcal{B}(E).$$

Then the formula

$$L_t = \int_{0^+}^t\int_E z\,\eta(dz;ds), \quad t\ge 0,$$

defines an Z–valued, α-stable symmetric process. Moreover, the process

$$L_t^U = \int_{0+}^t\int_U z\,\eta(dz;ds), \quad t\ge 0,$$

is of integrable p–variation (see 3.3). Our interest is the existence and uniqueness of the solutions $(u(t))_{t\geq 0}$ *to the following parabolic equation*

$$\text{(4)}\qquad \begin{cases} du(t,\xi) &=& \Delta u(t-,\xi)\,dt + \int_E z(\xi)\,u(t-,\xi)\,\eta(dz;dt), \\ & & \xi\in\mathcal{O}, t\geq 0, \\ u(0,\xi) &=& u_0(\xi) \quad \xi\in\mathcal{O}. \end{cases}$$

First, note that by Examples 3.9 and 3.10 the spaces $W^{\vartheta,p}(\mathcal{O})$, $\vartheta\in\mathbb{R}$ *are of* M *type* p. *By multiplier theorems (see e.g. Runst and Sickel* [43, Theorem 1, p. 190]*) one sees that*

$$|z\,u|_{L^p} \leq C\,|z|_{W^p_\gamma}|u|_{W^p_{\frac{d}{p}}}.$$

If $\frac{d}{p} < 2(1-\frac{1}{\alpha})$ *then Theorem 2.1 and Remark 2.2 give existence and uniqueness of the mild solution of (4) in* $L^0(\mathbb{D}([0,T];L^p(\mathcal{O}))$.

Example 2.3. *Fix* $p\in(1,2]$ *and* $d\in\mathbb{N}$. *Let* $(\Omega;\mathcal{F},\mathbb{P})$ *be a complete measurable probability space with usual filtration* $(\mathcal{F}_t)_{t\geq 0}$. *Let* $\theta:\mathcal{B}(\mathbb{R})\to\mathbb{R}^+$ *be a symmetric Lévy measure such that* $\int_{\mathbb{R}}|z|^p\,\theta(dz)<\infty$ *and*

$$\eta:\mathcal{B}(\mathbb{R}^d)\times\mathcal{B}(\mathbb{R})\times\mathcal{B}(\mathbb{R}^+)\to\mathbb{R}^+$$

be a Poisson random measure over $(\Omega;\mathcal{F},\mathbb{P})$ *with characteristic measure* ν *defined by*

$$\mathcal{B}(\mathbb{R}^d)\times\mathcal{B}(\mathbb{R})\ni(A,B)\mapsto\nu(A,B)=\lambda_d(A)\theta(B)\in\mathbb{R}^+,$$

where λ_d *denotes the Lebegue measure in* $\mathbb{R}^d$. *We consider the following SPDE*

$$\text{(5)}\qquad \begin{cases} du(t,\xi) &=& \Delta u(t,\xi) \\ & + & \int_{\mathbb{R}^d}\int_{\mathbb{R}} g(\xi,u(t,\xi);d\zeta)\eta(d\xi,d\zeta,dt),\ t>0,\ \xi\in\mathbb{R}^d, \\ u(0,\xi) &=& u_0(\xi), \quad \xi\in\mathbb{R}^d, \end{cases}$$

and $\lim_{\xi\to\pm\infty}u(t,\xi)=0$, $t\geq 0$. *We will show, that if* $\alpha<\frac{2}{d}+1$ *is satisfied, then for any* $T>0$ *Theorem 2.1 will give existence and uniqueness for Problem* (5).

Let $E=L^p(\mathbb{R}^d)$, $Z=B^{\frac{d}{p}-d}_{p,\infty}(\mathbb{R}^d)$ *be the Besov space (see e.g. Triebel* [45, Chapter 2.3.2]*) and* $B:=W^{-\gamma,p}(\mathbb{R}^d)$, *where* γ *satisfies* $\gamma>d-\frac{d}{p}$.

Let U *be the unit ball in* Z, *i.e.* $U=\{x\in B\mid|x|\leq 1\}$. *By Runst and Sickel* [43, p.34, Remark 3] *we have* $\delta_\xi\in Z$, *where* δ_ξ *denotes the Delta measure at* $\xi\in\mathbb{R}^d$. *Moreover, there exists a constant* $c>0$ *such that* $|\delta_\xi|_Z=c$ *for all* $\xi\in\mathbb{R}^d$ *and therefore* $\frac{1}{c}\delta_\xi\in U$.

First, note that if $A\in\mathcal{B}(\mathbb{R}^d)$, *then* $\mathcal{A}=\{\delta_\xi\mid\xi\in A\}\in\mathcal{B}(Z)$. *In particular, let us define the function*

$$\begin{aligned} f:\mathbb{R}^d &\to Z \\ \xi &\mapsto c^{-1}\delta_\xi. \end{aligned}$$

The function f *is bounded from* $\mathbb{R}^d$ *into* Z *and is continuous from* $\mathbb{R}^d$ *into* $W^{-\gamma}_p(\mathbb{R}^d)$. *The latter holds, since for some* $\beta>0$, *the dual of* $W^{-\gamma}_p(\mathbb{R}^d)$ *can be continuously embedded in the Hölder space* $\bar{C}^\beta(\mathbb{R}^d)$ *(see e.g. Triebel* [45, Chapter 4.6.1]*). Let* j *be the embedding of* Z *in* $W^{-\gamma}_p(\mathbb{R}^d)$. *Then* $\mathcal{A}=j^{-1}\circ f(A)$ *and therefore* $\mathcal{A}\in\mathcal{B}(Z)$.

Let the measure $\sigma : \partial U \to \mathbb{R}^+$ *be defined by* $\sigma(A) := \lambda(f^{-1}(A))$, $A \in \mathcal{B}(\partial U)$, *and* λ *denotes the Lebesgue measure. Now, the characteristic measure* $\nu : \mathcal{B}(Z) \to \mathbb{R}$ *is defined by*

$$\nu(B) = \int_{\mathbb{R}^+} \int_{\partial U} \chi_B(rx) \sigma(dx) \theta(dr), \quad B \in \mathcal{B}(Z).$$

Let $\eta : \mathcal{B}(B) \hat{\times} \mathcal{B}(\mathbb{R}^+) \to \mathbb{R}^+$ *be the corresponding Poisson random measure. By Linde* [32, Theorem 6.2.8] *the process*

$$L_t = \int_0^t \int_Z z \eta(dx, ds), \quad t \geq 0,$$

is well defined and coincides with the space-time white noise defined above (for definition see e.g. Bié [8]*). We are interested in the problem*

$$\begin{cases} du(t) &= \Delta u(t-)\, dt + \int_Z u(t-)\, z\, \eta(dz, dt), t \geq 0, \\ u(0) &= u_0 \in E, \end{cases} \tag{6}$$

which is equivalent to Problem (5)*. Theorem 2.1 gives existence and uniqueness of the solution* u *to Problem (6) such that*

$$u \in L^0(\Omega; \mathbb{D}((0,T]; W_p^{-\gamma}(\mathbb{R}^d))) \cap C([0,T]; L^p(\Omega; L^p(\mathbb{R}^d))).$$

In fact, one can show that there exists constant C_1 *and* C_2*, such that for all* $u \in L^p(\mathbb{R}^d)$

$$\int_{|z|\leq 1} |uz|_Z^p \nu(dz) = \int_0^1 |r|^p \theta(dr) \int_{\mathbb{R}}^d |u(\xi)|^p d\xi \leq C_1 |u|_{L^p(\mathbb{R}^d)}^p$$

and for all $u, v \in L^p(\mathbb{R}^d)$

$$\int_{|z|\leq 1} |(u-v)z|_Z^p \nu(dz) = \int_0^1 |r|^p \theta(dr) \int_{\mathbb{R}}^d |u(\xi) - v(\xi)|^p d\xi \leq C_2 |u-v|_{L^p(\mathbb{R}^d)}^p.$$

Therefore, the conditions of Theorem 2.1 are satisfied if

$$\frac{1}{2}\left(d - \frac{d}{p}\right) < \frac{1}{p}.$$

A short calculation shows, that above is satisfied, if $p < \frac{2}{d} + 1$*. This is a condition which coincides with the condition of Bié* [8]*. In particular, for all* $\beta > \frac{n}{p}$*, the unique solution* u *to Problem (6) exists and satisfies* $\langle u, \phi \rangle \in \mathbb{D}((0,T]; \mathbb{R})$ *for all* $\phi \in W_p^\beta(\mathbb{R}^d)$*.*

3. Poisson random measures and stochastic integration

3.1. **Poisson random measures.** Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space. Let Z be an arbitrary separable Banach space with σ–algebra $\mathcal{Z}$. A point process with state space $\mathcal{Z}$ is a sequence of $Z \times \mathbb{R}^+$-valued random variables (Z_i, T_i), $i = 1, 2, \ldots$ such that for each i, Z_i is $\mathcal{F}_{T_i}$-measurable. Given a point process, one usually works with the associated random measure η defined by

$$\eta(A \times [0,t])(\omega) = \sum_{T_i \leq t} 1_A(Z_i(\omega)), \quad A \in \mathcal{Z}, t \geq 0, \omega \in \Omega.$$

A point process is called a Poisson point process with characteristic measure ν on $(Z,\mathcal{Z})$, iff for each Borel set $A\in\mathcal{Z}$ with $\nu(A)<\infty$ and for each t the counting process of the set A, i.e. the random variable $N_t(A)=\eta(A\times[0,t])$, has a Poisson distribution with parameter $\nu(A)t$, i.e.

$$\mathbb{P}(N_t(A)=k) \quad = \quad \exp(-t\nu(A))\frac{(\nu(A)t)^k}{k!}. \tag{7}$$

It follows, that the random measure η associated to a Poisson point process has independent increments and that $\eta(A\times[0,t])$ and $\eta(B\times[0,t])$ are independent for all $A,B\in\mathcal{Z}$, $A\cap B=\emptyset$, and $t\geq 0$.

Let Z be a separable Banach space with Borel σ algebra $\mathcal{Z}$. Starting with a measure ν on $\mathcal{Z}$, one may ask under what conditions one can construct a Poisson random measure η on $\mathcal{Z}\hat{\times}\mathcal{B}(\mathbb{R}^+)$, such that ν is the characteristic measure of η. If the measure ν is finite and satisfies $\nu(\{0\})=0$, then the Poisson random measure $\eta:\mathcal{Z}\hat{\times}\mathcal{B}(\mathbb{R}^+)\to\mathbb{R}^+$ exists. Moreover, one can show, that if $\nu\in L^{sym}(Z)$ (see Definition 2.2), then the Poisson random measure $\eta:\mathcal{Z}\hat{\times}\mathcal{B}(\mathbb{R}^+)\to\mathbb{R}^+$ exists.

Some useful properties of symmetric Lévy measures are stated in the following remark.

Remark 3.1. *(See also Linde* [32, Proposition 5.4.5]*) Let Z be a separable Banach space with norm $|\cdot|$, let $\mathcal{Z}$ be the Borel-σ algebra on Z and let Z' be the topological dual of Z. Let $\nu\in\mathcal{L}(Z)$ be a Lévy measure. Then the following holds true.*

- *For each $\delta>0$, $\nu\{|x|>\delta\}<\infty$.*
- $\sup_{|a|\leq 1}\int_{|x|\leq 1}|\langle x,a\rangle|^2\,d\nu(x)<\infty$, *where $\langle a,x\rangle:=a(x)$, $a\in Z'$.*
- *If $\sigma\leq\nu$, then σ is also a Lévy measure.*

A typical example of Poisson random measure is provided by an α–stable Poisson random measure, $0<\alpha<2$.

Definition 3.1. *A probability measure $\mu\in\mathcal{P}(E)$ is said to be stable iff for each $a,b>0$ there exists some $c>0$ and an element $z\in E$, such that for all independent random variables X and Y with law μ we have*

$$\mathcal{L}(aX+bY)=\mathcal{L}(cX+z). \tag{8}$$

The measure μ is called strictly stable if for all $a>0$ and $b>0$ one can choose $z=0$ in (8). Moreover, μ is called α–stable iff (8) holds with

$$c=(a^\alpha+b^\alpha)^{\frac{1}{\alpha}}.$$

Example 3.1. *Let E be a separable Banach space and $U=\{x\in E\mid |x|\leq 1\}$. Let $\sigma:\mathcal{B}(\partial U)\to\mathbb{R}^+$ be an arbitrary finite measure and $\nu:\mathcal{B}(E)\to\mathbb{R}^+$ be defined by*

$$\nu(B)=\int_{\mathbb{R}^+}\int_{\partial U}\chi_B(sz)\sigma(dz)\,s^{-1-\alpha}\,ds,\quad B\in\mathcal{B}(E).$$

Then $\nu : \mathcal{B}(E) \to \mathbb{R}^+$ is a Lévy measure. Let η be the corresponding Poisson random measure. If E is of type p, $p > \alpha$ (for definition see the next Section), then the process $L = (L_t)_{t\ge 0}$ defined by

$$L_t = \int_{0^+}^t \int_E z\, \eta(dz; ds), \quad t \ge 0,$$

is an E–valued, unique, α-stable symmetric Lévy-process. On the other hand it is known from Linde [32, Theorem 6.2.8] *that the Lévy measure of an α-stable random variable is given by its distribution on ∂U, i.e. by the measure $\sigma : \mathcal{B}(\partial U) \to \mathbb{R}^+$.*

3.2. M **type** p **Banach spaces: a short account.** A stochastic integral is defined with respect to its integrator. If the integrator is of finite variation, then the integral is defined as a Stieltjes integral in a pathwise sense. If the integrator is a square integrable martingale, Itô's extension procedure yields to an integral defined on all previsible square integrable processes. Let Z and E be two separable Banach spaces with Borel σ algebras $\mathcal{Z}$ and $\mathcal{E}$. Fix $1 \le p < 2$. Let $\eta : \mathcal{Z}\hat{\times}\mathcal{B}(\mathbb{R}^+) \to \mathbb{R}^+$ be a Poisson random measure with characteristic measure $\nu \in \mathcal{L}^{sym}(Z)$. We will define an integral with respect to ν, i.e.

$$\int_{0^+}^t \int_E h(s,z)\, \eta(dz; ds), \tag{9}$$

with $h : \Omega\times\mathbb{R}^+ \to L(Z,E)$ is a càglàd predictable step function, such that $\int_0^T \int_Z |h(s,z)|^p\, \nu(dz)ds < \infty$. The question which needs to be answered is under which conditions on the underlying Banach space E and on the integrator, the stochastic integral in (9) can be extended to the set of all functions with finite L^p-mean. First, we consider the case where h is a deterministic function, secondly we consider the case, where h is a random function.

Here and hereafter $\{\epsilon_n\}_{n\in\mathbb{N}}$ denotes a sequence of $\{1,-1\}$-valued random variables such that

$$\mathbb{P}(\epsilon_i = \pm 1) = \frac{1}{2}.$$

Definition 3.2. *(Linde* [32]*) Let E be a Banach space and $0 < p \le 2$ be fixed. Let $x = \{x_i\}_{n\in\mathbb{N}}$ be a sequence in E. A Banach space E is of type p (R-type p) iff $x \in l_p(E)$ implies that the series*

$$\sum_{i=1}^\infty \epsilon_i x_i$$

is a.s. convergent in E.

Proposition 3.1. *(see Linde [Proposition 3.5.1]*[32]*) The Banach space E is of type p iff for some (each) $r \in (0,\infty)$ there exists a constant $C = C(E,p) > 0$ such that for all sequences $x = \{x_i\}_{n\in\mathbb{N}}$, for all sequences $\{\epsilon_n\}_{n\in\mathbb{N}}$ and all $N \in \mathbb{N}$*

$$\left\{ \mathbb{E} \left| \sum_{i=1}^N \epsilon_i x_i \right|^r \right\}^{\frac{1}{r}} \le C \left\{ \sum_{i=1}^N |x_i|^p \right\}^{\frac{1}{p}}.$$

Example 3.2. *Let $(M,\mathcal{M},\mathbb{P})$ be a probability space. Then $L^p(M,\mathcal{M},\mathbb{P})$ is of type p, $1\le p\le 2$, see Pisier* [39, p.186].

Example 3.3. *Let E be of type p, $0<p\le 2$. Then E is of type q, $0<q\le p$, see Linde* [32, Chapter 3, Theorem 3.5.2].

Example 3.4. *Assume E is of type p. Then any closed subspace of E is also of type p (this follows from Pisier* [39, Theorem 4.5]*).*

Definition 3.3. *(see Dettweiler* [19]*) Let $1\le p\le 2$ and let E be a separable Banach space. An E-valued process $X=(X(t))_{t\ge 0}$ is said to be of integrable p–variation iff for any $T>0$ one can find a constant $C(T)$ such that for any partition $0=t_0<t_1<\cdots<t_n=T$ one has*

$$\sum_{i=1}^{n}\mathbb{E}\,|X(t_{i+1})-X(t_i)|^p\le C(T).$$

Example 3.5. *Let E be a Hilbert space and $(W(t))_{t\ge 0}$ be an E-valued Wiener process with covariance operator Q. If Q is of trace class, then the Wiener process $(W(t))_{t\ge 0}$ is of integrable 2– variation.*

Example 3.6. *Let E be a Hilbert space and $M=(M_t)_{t\ge 0}$ be an E-valued stationary Lévy process with bounded second moment. Then the Lévy–Khintchine formula implies that M is of integrable 2–variation.*

Example 3.7. *Let E be a Hilbert space. Let $\nu\in\mathcal{L}^{sym}(E)$ such that $\int_E |z|^2\nu(dz)<\infty$. Then one can construct a unique Poisson random measure $\eta:\mathcal{B}(E)\hat{\times}\mathcal{B}([0,T])\to\mathbb{R}^+$ such that ν is the characteristic measure of η. Moreover, the process $L_t=\int_{0+}^t\int_E z\,\eta(dz,ds)$ is of integrable 2-variation.*

A Hilbert space is a Banach space of type 2. Let $p\in(0,2]$. In case the underlying Banach space E is of type p and the Lévy measure is p integrable, the Example 3.7 can be transferred to E.

Example 3.8. *(see Theorem 2.1 in Dettweiler* [17]*, in the proof of the implication of $(ii)\Rightarrow(i)$ on p. 129 of Proposition 2.3,* [19] *or Hamedani and Mandrekar* [25]*) Let E and Z be separable Banach spaces, E of type p, $p\in(1,2]$. Let $\nu\in\mathcal{L}^{sym}(Z,\mathcal{B}(Z))$ and $h\in L(Z,E)$, such that*

$$\int_Z |h(z)|_E^p\,\nu(dz)<\infty.$$

Let η be a Poisson random measure on $\mathcal{B}(Z)\hat{\times}\mathcal{B}(\mathbb{R}^+)$ with characteristic measure ν. Then a Lévy process $L=(L(t))_{t\ge 0}$ exists such that

$$L(t)\overset{d}{=}\int_{0^+}^t\int_Z h(z)\,\eta(dz,ds).$$

Moreover, L is a martingale and is of integrable p–variation.

Remark 3.2. *To be more precise, Dettweiler shows in his paper* [17] *the following: Let E be a separable Banach space of type p, $p \in (1,2]$ with Borel σ–algebra $\mathcal{E}$. Then there exists a constant $C = C(E,p)$ such that for each Poisson random measure $\eta : \mathcal{E}\hat{\times}\mathcal{B}([0,T]) \to \mathbb{R}^+$ with characteristic measure $\nu \in \mathcal{L}^{sym}(E)$ and each $I \in \mathcal{B}([0,T])$ we have*

$$\mathbb{E}|\int_E h(z)\eta(z,I)|^p \le C \int_E |h(z)|_E^p \,\nu(dz)\lambda(I), \tag{10}$$

where λ denotes the Lebesgue measure. If E is not of type p, then the inequality above does not hold necessarily.

Remark 3.3. *Let E be a Banach spaces of type p, $1 \le p \le 2$. In Example 3.2 we assumed that the characteristic measure ν is symmetric. If we consider compensated Poisson random measure, where the characteristic measure is an arbitrary Lévy measure, the inequality* (10) *remains valid. In particular, it follows from the proof of Proposition 2.5 in Dettweiler* [19] *that there exists a constant $C = C(E,p)$ such that for each Poisson random measure η : $\mathcal{B}(E)\hat{\times}\mathcal{B}(\mathbb{R}^+) \to \mathbb{R}^+$ with characteristic measure $\nu : \mathcal{B}(E) \to \mathbb{R}^+$ being a Lévy measure one has*

$$\mathbb{E}\left|\int_0^t \int_E h(z)\,(\eta-\gamma)(ds,dz)\right|^p \le C \int_E |h(z)|^p \,\nu(dz),$$

where γ denotes the compensator of η. Hence, the process

$$t \mapsto \int_0^t \int_E h(z)\,(\eta-\gamma)(ds,dz), \quad t \ge 0$$

is also of integrable p–variation.

If h is a deterministic function, E a Banach space of type p and the integrator of integrable p variation, one can extend the stochastic integral to the class of all $h : [0,T] \to L(Z,E)$ such that $\int_0^T \int_Z |h(s,z)|^p \,\nu(dz)\,ds < \infty$, see e.g. Pisier [39, Proposition 4.3], Rosiński [41]. If h is random, then the underlying Banach space has additionally to be UMD.

Definition 3.4. *A Banach space is said to be UMD (Unconditional Martingale Differences) if for each $1 < r < \infty$ there exists a constant $C(E,r) < \infty$, such that for each E-valued martingale $M = (M_0, M_1, \dots)$, each $\{-1,1\}$–valued sequence $(\epsilon_0, \epsilon_1, \dots)$ and each positive integer $n \in \mathbb{N}$*

$$\mathbb{E}\left|\sum_{k=1}^n \epsilon_k\,(M_k - M_{k-1})\right|_E^r \le C(E,r)\,\mathbb{E}\,|M_n - M_0|_E^r\,.$$

Remark 3.4. *Let $1 < p \le 2$. If a Banach space E is UMD and of type p then E is of M type p (see e.g. Brzeźniak* [11]*).*

Definition 3.5. *(see e.g. Pisier* [39, Chapter 6]*) Let $1 \le p < \infty$. A Banach space E is of M type p, iff there exists a constant $C = C(E;p)$, such that for each discrete E-valued martingale $M = (M_1, M_2, \dots)$ one has*

$$\sup_{n\ge1} \mathbb{E}|M_n|_E^p \le C \sum_{n\ge1} \mathbb{E}|M_n - M_{n-1}|_E^p.$$

Example 3.9. *Let $\mathcal{O}$ be a bounded domain. Then the space $L^p(\mathcal{O})$ is of M type $p\wedge 2$ (see e.g.* [48, Chapter 2]*).*

Example 3.10. *Let $0<p\le 2$. Let E be of M type p and $A:E\to E$ an operator with domain $D(A)$. If A^{-1} is bounded, then $D(A)$ is isomorphic to E and therefore of M type p (Brzeźniak* [10, p. 10]*).*

Example 3.11. *(Brzeźniak* [10, Appendix A, Theorem A.4]*) Assume E_1 and E_2 are a Banach space of M type p, where E_2 is continuously and densely embedded in E_1. Then for any $\vartheta\in(0,1)$ the complex interpolation space $[E_1,E_2]_\vartheta$ and the real interpolation space $(E_1,E_2)_{\vartheta,p}$ are of M type p.*

By means of this inequality one can extend the stochastic integral in p mean (see e.g. Chapter 3.3, or Woyczyński [48, Theorem 2.2], Dettweiler [19]).

To deal with moments of higher order, stronger inequalities are needed. In fact using the techniques described in Burkholder [14] one can prove a generalized version of an inequality of Burkholder type. In particular, let E be a Banach space of M type p, $1\le p\le 2$. Then it can be shown, that there exists a constant $C<\infty$, such that we have for all discrete E-valued martingales $M=(M_1,M_2,\dots)$ and $1\le r<\infty$ (see e.g. Assuad [3], Brzeźniak [10, Proposition 2.1], Pisier [38, Remark 3.3 on page 346] and [39, Chapter 6])

$$\mathbb{E}\sup_{n\ge 1}|M_n|_E^r\le C\mathbb{E}\left[\sum_{n\ge 1}|M_{n-1}-M_n|_E^p\right]^{\frac{r}{p}}. \tag{11}$$

Remark 3.5. *An interested reader can consult the following articles: Burkholder* [13, 15]*, Pisier* [38, 39] *and Woyczyński* [48, 49] *for the connection of Banach spaces of M type p and their geometric properties. Brzeźniak* [10]*, Dettweiler* [19, 20, 17] *Neidhardt* [36] *and Rüdiger* [42] *for the connection between Banach spaces of M type p and stochastic integration. Linde* [32]*, and the articles Dettweiler* [17, 18]*, Gine and Arujo* [24]*, Hamedani and Mandrekar* [25] *for the connection between Lévy processes and Banach spaces of type p.*

3.3. **The Stochastic Integral in M type p Banach spaces.** In the following let $p\in(1,2]$ be fixed. Let Z be a separable Banach space and E be a M type p Banach space. The Borel σ algebras of Z and E are denoted by $\mathcal{Z}$ and $\mathcal{E}$ respectively.

As mentioned in the introduction the stochastic integral will first be defined on the set of predictable and simple function. To be precise a process $h:[0,T]\to L(Z,E)$ is said to be simple predictable if h has a representation

$$h(s,z)=\sum_{i=1}^n 1_{(t_{i-1},t_i]}(s)H_{i-1}(z),\quad s\ge 0,\ z\in Z, \tag{12}$$

where $0=t_0<\cdots<t_n=T$ is a partition of $[0,T]$ consisting of stopping times, and $H_i\in L(Z,E)$, $i=0,\dots,n$, are $\mathcal{F}_{t_i}$-measurable random variables. The collection of simple

predictable processes $h:[0,T]\to L(Z,E)$ is denoted by $\mathbb{S}$. Let $\mathbb{D}:=\mathbb{D}([0,T];E)$ be the Skorohod space of all adapted càdlàg functions $h:[0,T]\to E$, endowed by the Skorohod topology (see Appendix A).

Let $\eta:\mathcal{Z}\hat{\times}\mathcal{B}([0,T])\to\mathbb{R}^+$ be a Poisson random measure with characteristic measure $\nu\in\mathcal{L}^{sym}(Z,\mathcal{Z})$. The stochastic integral with respect to η is a linear operator $I:\mathbb{S}\to\mathbb{D}$ defined by

$$
\begin{aligned}
I(h)(t) = \int_{0+}^t\int_Z h(s,z)\eta(dz,ds) :=& \\
\sum_{i=1}^n \int_Z H_{i-1}(z)\eta(dz\times(t_{i-1}\wedge t, t_i\wedge t]),& \quad t>0, \qquad (13)
\end{aligned}
$$

where $h\in\mathbb{S}$ has the representation (12).

The next step is to extend the stochastic integral to the set $\mathbb{L}_p(\nu)$, where

$$
\mathbb{L}_p(\nu) = \Big\{ h:\Omega\times\mathbb{R}^+\to L(Z,E), \quad h \text{ is a predictable càglàd process such that } \int_0^t\int_Z \mathbb{E}\,|h(\omega,s,z)|^p\,\nu(dz)\,ds<\infty \Big\}
$$

equipped with norm

$$
|u|^p_{\mathbb{L}_p(\nu)} := \int_0^t\int_Z \mathbb{E}\,|h(\omega,s,z)|^p\,\nu(dz).
$$

Since $\mathbb{L}_p(\nu)$ is separable, by standard arguments, one can show, that $\mathbb{S}\cap\mathbb{L}_p(\nu)$ is dense in $\mathbb{L}_p(\nu)$. Let $\mathbb{D}_p$ the Skorohod space equipped with the following norm

$$
|u|^p_{\mathbb{D}_p} := \mathbb{E}\sup_{0\le t\le T}|u(t)|^p, \quad u\in\mathbb{D}_p.
$$

Note, the Skorohod space $\mathbb{D}([0,T];E)$ topologized with uniform convergence is a complete metric space, but not separable. Analysing the proof of completeness of L^p spaces over an arbitrary measurable set, one can see that the space $\mathbb{D}_p$ is a complete normed space. By means of the following proposition, it can be shown that the stochastic integral defined in (13) is a continuous operator from $\mathbb{S}\cap\mathbb{L}_p(\nu)$ into $\mathbb{D}_p$ (see e.g. Woyczyński [48, Theorem 2.2] or Dettweiler [19]).

Proposition 3.2. *Let $1<p\le 2$. Assume Z and E are separable Banach spaces, E is of M type p. Let $\mathcal{Z}$ and $\mathcal{E}$ be the Borel σ–algebras. Then there exists some constant $C=C(p,E)<\infty$ such that for all Poisson random measures η on $\mathcal{Z}\hat{\times}\mathcal{B}([0,T])$ with characteristic measure $\nu\in\mathcal{L}^{sym}(Z)$ and all functions $h:\Omega\times[0,T]\to L(Z,E)$ belonging to $\mathbb{S}$ with representation (12)* *we have*

$$\mathbb{E} \sup_{0<t\le T} \left| \int_{0+}^t \int_Z h(\sigma, z)\, \eta(dz; d\sigma) \right|^r \le \\ C\, \mathbb{E} \left(\int_0^T \int_Z |h(s,z)|^p \;\; \eta(dz; ds) \right)^{\frac{r}{p}}, \quad 0 < r < \infty,$$

and

$$\mathbb{E} \sup_{0<t\le T} \left| \int_{0+}^t \int_Z h(\sigma, z)\, \eta(dz; d\sigma) \right|^r \le \\ C\, \left(\int_0^T \int_Z \mathbb{E}\, |h(s,z)|^p \;\; \nu(dz)\; ds \right)^{\frac{r}{p}}, \quad 0 < r \le p. \tag{14}$$

Proof. Let $h = \{h(s,z), 0 \le s \le T, z \in Z\}$ be a simple, predictable process written in the form of (12). Now, the stochastic integral is defined by the Riemann integral (see (13))

$$\int_{0+}^t \int_Z h(s,z) \eta(dz, ds) \;\; = \;\; \sum_{i=1}^n \int_Z H_{i-1}(z) \eta(dz \times (t_{i-1} \wedge t, t_i \wedge t]).$$

First, let us fix $r \le p$. The sequence $\{\int_Z H_{i-1}(z)\eta(dz; (t_{i-1} \wedge t, t_i \wedge t])\}_{i=0}^n$ is a sequence of martingale differences. Since ν is symmetric and H_{i-1} is $\mathcal{F}_{t_{i-1}}$–measurable and linear for $i = 0, \dots n$, it follows that

$$\mathbb{E} \int_Z H_{i-1}(z) \eta(dz; (t_{i-1} \wedge t, t_i \wedge t]) = 0.$$

Remark 3.2 implies

$$\mathbb{E} \left| \int_Z H_{i-1}(z) \eta(dz; (t_{i-1} \wedge t, t_i \wedge t]) \right|^p = \\ C\, (t_i \wedge t - t_{i-1} \wedge t) \int_Z \mathbb{E} |H_{i-1}(z)|^p\, \nu(dz). \tag{15}$$

Thus, we can apply the generalized Burkholder inequality (11) to get

$$\mathbb{E} \sup_{1\le j\le n} \left| \sum_{i=1}^j \int_Z H_{i-1}(z) \eta(dz; (t_{i-1} \wedge t, t_i \wedge t]) \right|^r \le \\ C\, \mathbb{E} \left| \sum_{i=1}^n \left| \int_Z H_{i-1}(z) \eta(dz; (t_{i-1} \wedge t, t_i \wedge t]) \right|^p \right|^{\frac{r}{p}}.$$

Moreover, the process defined by

$$t \mapsto \sum_{i=1}^n \int_Z H_{i-1}(z) \eta(dz; (t_{i-1} \wedge t, t_i \wedge t]), \quad 0 \le t \le T,$$

is a martingale. In fact the integrability conditions are given by (15). Further, let $0 \leq s < t \leq T$ and let $j \in \{1, \ldots, n\}$ be this index for which $t_j \leq s < t_{j+1}$ holds. Then

$$\begin{aligned}
&\mathbb{E}\left[\sum_{i=1}^{n} \int_Z H_{i-1}(z)\eta(dz;(t_{i-1}\wedge t, t_i \wedge t]) \mid \mathcal{F}_s\right] = \\
&= \mathbb{E}\left[\sum_{i=1}^{j} H_{i-1} \int_Z z\eta(dz;(t_{i-1}, t_i]) \mid \mathcal{F}_s\right] + \\
&\quad \mathbb{E}\left[\sum_{i=j+1}^{n} \int_Z H_{i-1} z\eta(dz;(t_{i-1}\wedge t, t_i \wedge t]) \mid \mathcal{F}_s\right].
\end{aligned}$$

Since for $k \leq j$ the random variable H_k is $\mathcal{F}_{t_j}$-measurable, H_j is linear and η is independently scattered, we get

$$\begin{aligned}
&\mathbb{E}\left[\sum_{i=1}^{n} \int_Z H_{i-1}(z)\eta(dz;(t_{i-1}\wedge t, t_i \wedge t]) \mid \mathcal{F}_s\right] = \\
&= \sum_{i=1}^{j} H_{i-1} \int_Z z\eta(dz;(t_{i-1}, t_i]) + H_j \mathbb{E}\left[\int_Z z\eta(dz;(t_j, s]) \mid \mathcal{F}_s\right] \\
&\quad + H_j \mathbb{E}\left[\int_Z z\eta(dz;(s, t_{j+1}\wedge t]) \mid \mathcal{F}_s\right] + \\
&\quad \mathbb{E}\left[\sum_{i=j+2}^{n} \int_Z z\eta(dz;(t_i\wedge t, t_i \wedge t]) \mid \mathcal{F}_s\right].
\end{aligned}$$

Since $\eta(dz;(t_{i-1}\wedge s, t_i \wedge s])$ is $\mathcal{F}_s$-measurable one obtains

$$\begin{aligned}
&\mathbb{E}\left[\sum_{i=1}^{n} \int_Z H_{i-1}(z)\eta(dz;(t_{i-1}\wedge t, t_i \wedge t]) \mid \mathcal{F}_s\right] = \\
&= \sum_{i=1}^{j+1} H_{i-1} \int_Z z\eta(dz;(t_{i-1}\wedge s, t_i \wedge s]) + \\
&\quad \mathbb{E}\left[\int_Z H_j z\eta(dz;(s, t_{j+1}\wedge t]) + \sum_{i=j+2}^{n} \int_Z z\eta(dz;(t_i \vee s, t_i \wedge t]) \mid \mathcal{F}_s\right].
\end{aligned}$$

But ν is symmetric. Thus it follows

$$\begin{aligned}
&\mathbb{E}\left[\sum_{i=1}^{n} \int_Z H_{i-1}(z)\eta(dz;(t_{i-1}\wedge t, t_i \wedge t]) \mid \mathcal{F}_s\right] = \\
&= \sum_{i=1}^{n} H_{i-1} \int_Z z\eta(dz;(t_{i-1}\wedge s, t_i \wedge s]),
\end{aligned}$$

and we can apply the Doob's maximal inequality for martingales to get

$$\mathbb{E}\sup_{0\le t\le T}\left|\sum_{i=1}^n\int_Z H_{i-1}(z)\eta(dz;(t_{i-1}\wedge t,t_i\wedge t])\right|^r \le$$
$$C\,\mathbb{E}\left|\sum_{i=1}^n\left|\int_Z H_{i-1}(z)\eta(dz;(t_{i-1}\wedge t,t_i\wedge t])\right|^p\right|^{\frac{r}{p}}.$$

The Jensen inequality yields

$$\mathbb{E}\sup_{0\le t\le T}\left|\sum_{i=1}^n\int_Z H_{i-1}(z)\eta(dz;(t_{i-1}\wedge t,t_i\wedge t])\right|^r \le$$
$$\le\ C\ \left(\sum_{i=1}^n\mathbb{E}\ \left|\int_Z H_{i-1}(z)\eta(dz;(t_{i-1}\wedge t,t_i\wedge t])\right|^p\right)^{\frac{r}{p}}. \tag{16}$$

The tower property of the conditional expectation leads to

$$\mathbb{E}\sup_{0\le t\le T}\left|\sum_{i=1}^n\int_Z H_{i-1}(z)\eta(dz;(t_{i-1}\wedge t,t_i\wedge t])\right|^r \le$$
$$\le\ C\ \left(\sum_{i=1}^n\mathbb{E}\left[\ \mathbb{E}\left[\left|\int_Z H_{i-1}(z)\eta(dz;(t_{i-1}\wedge t,t_i\wedge t])\right|^p\mid\mathcal{F}_{t_{i-1}}\right]\right]\right)^{\frac{r}{p}}. \tag{17}$$

The random variable H_{i-1} is $\mathcal{F}_{t_{i-1}}$-measurable. Therefore by Remark 3.2 we have

$$\mathbb{E}\left[\left|\int_Z H_{i-1}(z)\eta(dz;(t_{i-1}\wedge t,t_i\wedge t])\right|^p\mid\mathcal{F}_{t_{i-1}}\right]=$$
$$\int_Z|H_{i-1}(z)|^p\ \nu(dz)\ (t_i\wedge t-t_{i-1}\wedge t). \tag{18}$$

Inserting (18) in equation (17) yields

$$\mathbb{E}\sup_{0\le t\le T}\left|\sum_{i=1}^n\int_Z H_{i-1}(z)\eta(dz;(t_{i-1}\wedge t,t_i\wedge t])\right|^r$$
$$\le\ \left(\sum_{i=1}^n\mathbb{E}\ \left|\int_Z H_i(z)\eta(dz\times(t_{i-1}\wedge t,t_i\wedge t])\right|^p\right)^{\frac{r}{p}}\le$$
$$\le\ C\ \left(\sum_{i=1}^n\int_Z(t_i-t_{i-1})\ \mathbb{E}\ \int_Z|H_{i-1}(z)|^p\nu(dz)\right)^{\frac{r}{p}}, \tag{19}$$

which can be written as

$$\left(\int_0^t\int_Z\mathbb{E}\,|h(s,z)|^p\ \nu(dz)\ ds\right)^{\frac{p}{r}}.$$

In case of $r>p$, we can only apply the martingale inequality (11) and have to stop before equation (16). □

By means of Proposition 3.2 the stochastic integral is a continuous operator from $\mathbb{S}\cap\mathbb{L}_p(\nu)$ into $\mathbb{D}_p$. The next point is to reassure, that $\mathbb{S}\cap\mathbb{L}_p(\nu)$ is dense in $\mathbb{L}_p(\nu)$. But in fact $\mathbb{L}_p(\nu)$ equipped with the predictable σ algebra

$$\mathcal{P} := \sigma\left(h : [0,T]\times Z \to E, h \text{ is } (\mathcal{F}_t)\text{–adapted, and càdlàg}\right)$$

is a measurable separable metric space and by a modification of the Proof of Proposition I.4.7, Da Prato and Zabczyk [16] one can show that $\mathbb{S}\cap\mathbb{L}_p(\nu)$ is dense in $\mathbb{L}_p(\nu)$. Since $\mathbb{D}_p$ is complete, there exists a bounded linear operator $I : \mathbb{L}_p(\nu) \to \mathbb{D}_p$, which is an extension of the operator introduced in (13). In the next Proposition we will show, that the inequalities in Proposition 3.2 are preserved.

Proposition 3.3. *(see e.g. Dettweiler* [19, Theorem 3.1]*) Let $1 < p \le 2$. Assume Z and E are a separable Banach spaces, E is of M type p. Let $\mathcal{Z}$ and $\mathcal{E}$ be the Borel σ–algebras. Then there exists some constant $C = C(p,E) < \infty$ such that for all Poisson random measures η on $\mathcal{Z}\times\mathcal{B}(\mathbb{R}^+)$ with characteristic measure $\nu \in \mathcal{L}^{sym}(Z)$ and all functions $h : \Omega\times\mathbb{R}^+\times Z \to E$ belonging to $\mathbb{L}_p(\nu)$ we have*

$$\mathbb{E}\sup_{0<s\le t}\left|\int_{0^+}^s\int_Z h(\sigma,z)\,\eta(dz;d\sigma)\right|^r \le C\,\mathbb{E}\left(\int_0^t\int_Z |h(s,z)|^p\ \eta(dz;ds)\right)^{\frac{r}{p}}, \quad 0<r<\infty,$$

and

$$\mathbb{E}\sup_{0<s\le t}\left|\int_{0^+}^t\int_Z h(\sigma,z)\,\eta(dz;d\sigma)\right|^r \le C\ \left(\int_0^t\int_Z \mathbb{E}\,|h(s,z)|^p\ \nu(dz)\ ds\right)^{\frac{r}{p}}, \quad 0<r\le p. \tag{20}$$

Proof. To show that $\mathbb{S}\cap\mathbb{L}_p(\nu)$ is dense in $\mathbb{L}^p(\nu)$, one can modify the Proof of Proposition I.4.7 of Da Prato and Zabczyk [16]. Thus there exists a sequence $\{h^j(s,z) \mid j\in\mathbb{N}\}$ in $\mathbb{S}\cap\mathbb{L}^p(\nu)$, such that

$$\int_{0+}^T\int_Z \mathbb{E}\left|h^j(s,z)-h(s,z)\right|^p\ \nu(dz)\,ds \longrightarrow 0 \text{ as } j\to\infty,$$

where the step functions h^j can be written as a sum of the following type

$$h^j(s,z) = \sum_{i=1}^{n^j} 1_{(t^j_{i-1},t^j_i]}(s)H^j_{i-1}(z), \quad s\ge 0,\ z\in Z$$

where $0 = t^j_1 < \cdots < t^j_{n^j} = T$ is a sequence of partitions of $[0,T]$, such that the maximal diameter $|\pi^j| := \max\{t^j_i - t^j_{i-1}\}$ tends to zero as $j\to\infty$ and $H^j_i : Z\to E$, $i = 0,\dots,n^j$ are $\mathcal{F}_{t^j_i}$-measurable random variables. The stochastic integral for h^j defined in (13) is given by

$$\int_{0^+}^t\int_Z h^j(s,z)\eta(dz,ds) \ = \ \sum_{i=1}^{n^j}\int_Z H^j_{i-1}(z)\eta(dz\times(t^j_{i-1}\wedge t, t^j_i\wedge t]).$$

Proposition 3.2 yields

$$\mathbb{E} \sup_{0\le t\le T} \left| \int_{0+}^t h^j(s,z)\eta(dz;ds)\right|^r \le C\, \left(\int_0^t \int_Z \mathbb{E}\,|h^j(s,z)|^p\, \nu(dz)\; ds\right)^{\frac{p}{r}}.$$

Since

$$\begin{aligned}\mathbb{E} \sup_{0\le t\le T} \left| \int_{0+}^t h^j(s,z)\eta(dz;ds)\right|^r &\le C\, \left(\int_0^t \int_Z \mathbb{E}\,|h^j(s,z)|^p\, \nu(dz)\, ds\right)^{\frac{p}{r}} \\ &\le C\, \left(\int_0^t \int_Z \mathbb{E}\,|h(s,z)|^p\, \nu(dz)\; ds\right)^{\frac{p}{r}}\end{aligned}$$

it follows by the Lebesgue's dominated convergence theorem, that for all $t > 0$

$$\int_{0+}^t h^j(s,z)\eta(dz;ds) \longrightarrow \int_{0+}^t h(s,z)\eta(dz;ds) \quad \text{as } j\to\infty$$

and

$$\mathbb{E} \sup_{0\le t\le T} \left| \int_{0+}^t h(s,z)\eta(dz;ds)\right|^p \le C\, \left(\int_0^t \int_Z \mathbb{E}\,|h(s,z)|^p\, \nu(dz)\; ds\right)^{\frac{p}{r}}.$$

□

The inequalities of Proposition (3.3) can be extended to higher order moments.

Corollary 3.1. *Let $1 \le p \le 2$. Let E and Z be two separable Banach spaces, E be of M type p. Let $\mathcal{Z}$ and $\mathcal{E}$ be the Borel σ-algebras on Z and E. Let η be Poisson random measure on Z with characteristic measure $\nu \in \mathcal{L}^{sym}(Z,\mathcal{Z})$. Let $h \in \mathbb{L}_p(\nu)$. Let $q = p^n$ for some $n \in \mathbb{N}$ and suppose, that*

$$\mathbb{E} \left(\int_0^T \int_Z |h(s,z)|^p\, \nu(dz)\, ds\right)^{\frac{q}{p}} < \infty, \tag{21}$$

and

$$\int_0^T \int_Z \mathbb{E}|h(s,z)|^q\, \nu(dz)\, ds < \infty \tag{22}$$

holds. Then there exists some constant $C < \infty$ such that

$$\begin{aligned}&\mathbb{E} \sup_{0< s\le t} \left| \int_{0^+}^t \int_Z h(s,z)\eta(dz;ds)\right|^q \le \\ &\qquad C\, \sum_{l=1}^n \left(\int_0^t \int_Z \mathbb{E}|h(s,z)|^{p^l}\, \nu(dz)\; ds\right)^{p^{n-l}}.\end{aligned}$$

Proof. The proof is a generalization of the proofs of Bass and Cranston [4, Lemma 5.2] or Protter and Talay [40, Lemma 4.1]. By Proposition 3.3 and inequality (14) we have

$$\mathbb{E} \sup_{0\le s\le t} \left| \int_{0+}^s \int_Z h(r,z)\eta(dz;dr)\right|^q \le C\, \mathbb{E} \left(\int_{0+}^t \int_Z |h(s,z)|^p\, \eta(dz;ds)\right)^{\frac{q}{p}}.$$

Let us define

$$X_t := \int_{0+}^t \int_Z |h(s,z)|^p \, \eta(dz;ds), \quad t \geq 0.$$

The process X_t is a real valued semimartingale and by condition (21) of finite variation. Moreover, it only has positive jumps, i.e. it is a subordinator. The associated random measure $\eta^X : \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}^+) \to \mathbb{R}^+$ is given by (see Jacod and Shiryaev [27, Proposition 2.1.16])

$$\eta^X(\omega; dt; dx) \quad = \quad \sum_s 1_{\{\Delta X_s(\omega) \neq 0\}} \delta_{(s, \Delta X_s(\omega))}(dx; dt).$$

Let γ^X be the compensator of η^X, i.e. the unique predictable random measure on $\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}^+)$, such that for all $A \in \mathcal{B}(\mathbb{R})$ the process $\int_{0+}^t (\eta^X - \gamma^X)(A, ds)$ is a local martingale. Let us define

$$L(t)^{(0)} := \int_{0+}^t \int_{\mathbb{R}} z \; (\eta^X - \gamma^X)(dz; ds), \quad t \geq 0.$$

Since (21) holds, $L(t)^{(0)}$ is a real–valued martingale, only has positive jumps and is of finite variation. Note, $L_t^{(0)}$ has a characteristic function of the following form $\mathbb{E} \exp\left(i\xi L_t^{(0)}\right) = \exp(\psi_t(\xi))$, where

$$\psi_t(\xi) = \int_{\mathbb{R}} \left(e^{i\xi\zeta} - 1 - i\xi\right) \nu_t^X(d\xi),$$

where $\nu_t : \mathcal{B}(\mathbb{R}) \to \mathbb{R}^+$. The uniqueness of the characteristic function gives for $A \in \mathcal{B}(\mathbb{R})$

$$\nu_t^X(A) = \nu \left\{ z \in E \mid h(t,z) \in A \right\}.$$

Since η and η^X are Poisson random measures and by the uniqueness of the compensator we infer that

$$\gamma^X(dt, ds) \quad = \quad \nu_t^X(ds) \times dt. \tag{23}$$

Since E is of M type p, it follows from Proposition (3.3) for some constant $C < \infty$

$$\mathbb{E} \sup_{0 < s \leq t} \left| \int_{0^+}^t \int_Z h(s;z) \eta(dz;ds) \right|^{p^n} \quad \leq \quad C \, \mathbb{E} \left(\int_{0^+}^t \int_Z |h(s;z)|^p \; \eta(dz;ds) \right)^{p^{n-1}}.$$

Simple calculations leads to

$$\begin{aligned}
\mathbb{E} \sup_{0 < s \leq t} \left| \int_{0^+}^t \int_Z h(s;z) \eta(dz;ds) \right|^{p^n} &\leq C \, \mathbb{E} \left(\int_{0^+}^t \int_{\mathbb{R}} z \; \eta_X(dz;ds) \right)^{p^{n-1}} \\
&\leq \; C \; \left(\mathbb{E} \left(\int_{0^+}^t \int_{\mathbb{R}} z \; (\eta^X - \gamma^X)(dz;ds) \right)^{p^{n-1}} + \mathbb{E} \left(\int_{0^+}^t \int_{\mathbb{R}} z \; \gamma^X(dz;ds) \right)^{p^{n-1}} \right) \\
&\leq \; C \; \left(\mathbb{E} |L(t)^{(0)}|^{p^{n-1}} + \mathbb{E} \left(\int_{0^+}^t \int_{\mathbb{R}} z \; \nu_t^X(dz) \; ds \right)^{p^{n-1}} \right) \\
&\leq \; C \; \left(\mathbb{E} |L(t)^{(0)}|^{p^{n-1}} + \mathbb{E} \left(\int_{0^+}^t \int_Z |h(s;z)|^p \; \nu(dz) ds \right)^{p^{n-1}} \right).
\end{aligned}$$

In case $n = 2$, we have

$$
\begin{aligned}
\mathbb{E}\left|L(t)^{(0)}\right|^p &= \mathbb{E}\left|\int_{0+}^t \int_{\mathbb{R}} z\,(\eta^X - \gamma^X)(dz;ds)\right|^p \\
&= \mathbb{E}\int_{0+}^t \int_{\mathbb{R}} |z|^p\,\eta^X(dz;ds).
\end{aligned}
$$

By the definition of the compensator, in particular since $\int_{0+}^t \left(\eta^X - \gamma^X\right)(A, ds)$ is a martingale, we can continue

$$
\begin{aligned}
\mathbb{E}\left|L(t)^{(0)}\right|^p &\leq \mathbb{E}\int_{0+}^t \int_{\mathbb{R}} |z|^p\,\gamma^X(dz;ds) \\
&= \mathbb{E}\int_{0+}^t \int_{\mathbb{R}} |z|^p\,\nu_t^X(dz)\,ds.
\end{aligned}
$$

By the discussion above, i.e. relation (23) we have

$$
\mathbb{E}\left|L(t)^{(0)}\right|^p \leq \mathbb{E}\int_{0+}^t \int |z|^{p^2}\,\nu(dz)\,ds.
$$

Thus, the proposition is proven, provided $n = 2$. In case $n > 2$, we have to continue. In particular, let

$$
L(t)^{(r)} := \int_{0+}^t \int_{\mathbb{R}} z^{p^r}\,(\eta^X - \gamma^X)(dz;ds) \qquad \text{for } r = 1, \ldots, n.
$$

Since (21) and (22) holds, $L(t)^{(r)}$ is a real–valued martingale, has only positive jumps and is of finite variation for $r = 1, \ldots, n$. Moreover, since $\mathbb{R}$ is of M type 2, $\mathbb{R}$ is also of M type p for all $p \in [1, 2]$. Using inequality (14) we have

$$
\begin{aligned}
\mathbb{E}\left|L(t)^{(r)}\right|^{p^m} &\leq C\,\mathbb{E}\left(\int_0^t \int_{\mathbb{R}} z^{p^r}\,(\eta_X - \gamma_X)\,(dz;ds)\right)^{p^m} \\
&\leq C\,\mathbb{E}\left(\int_0^t \int_{\mathbb{R}} z^{p^{r+1}} \eta^X(dz;ds)\right)^{p^{m-1}}.
\end{aligned}
$$

Using simple calculations we get

$$\begin{aligned}
\mathbb{E}\left|L(t)^{(r)}\right|^{p^m} \leq & \\
& C\,\mathbb{E}\left(\int_0^t\int_{\mathbb{R}} z^{p^{r+1}}(\eta^X-\gamma^X)(dz;ds)+\int_0^t\int_{\mathbb{R}} z^{p^{r+1}}\gamma^X(dz;ds)\right)^{p^{m-1}}\\
\leq\ & C_1\,\mathbb{E}\left(\int_0^t\int_{\mathbb{R}} z^{p^{r+1}}(\eta^X-\gamma^X)(dz;ds)\right)^{p^{m-1}}\\
& +C_2\,\mathbb{E}\left(\int_0^t\int_{\mathbb{R}} z^{p^{r+1}}\gamma^X(dz;ds)\right)^{p^{m-1}}\\
\leq\ & C_1\,\mathbb{E}\left|L(t)^{(r+1)}\right|^{p^{m-1}}+C_2\,\mathbb{E}\left(\int_0^t\int_{\mathbb{R}} z^{p^{r+1}}\nu_t^X(dz)\,ds\right)^{p^{m-1}}\\
\leq\ & C_1\,\mathbb{E}\left|L(t)^{(r+1)}\right|^{p^{m-1}}+C_2\mathbb{E}\left(\int_{0^+}^t\int_E |h(s;z)|^{p^{r+2}}\,\nu(dz)ds\right)^{p^{m-1}}.
\end{aligned}$$

That means we have

$$\begin{aligned}
\mathbb{E}\left|L(t)^{(r)}\right|^{p^m} \leq & \\
& C_1\,\mathbb{E}\left|L(t)^{(r+1)}\right|^{p^{m-1}}+C_2\mathbb{E}\left(\int_{0^+}^t\int_E |h(s;z)|^{p^{r+2}}\,\nu(dz)ds\right)^{p^{m-1}}.
\end{aligned}\tag{24}$$

Note, since $\mathbb{R}$ is of M type p, we have by Proposition (3.3)

$$\begin{aligned}
\mathbb{E}\left|L(t)^{(r)}\right|^{p} &= \mathbb{E}\left|\int_{0^+}^t\int_{\mathbb{R}} z^{p^r}\,(\eta^X-\gamma^X)(dz;ds)\right|^p\\
&\leq C\,\mathbb{E}\int_{0^+}^t\int_{\mathbb{R}} z^{p^{r+1}}\eta^X(dz;ds).
\end{aligned}\tag{25}$$

Note, that $\mathbb{R}$ is also of type 1, that means of M type 1. Thus we have

$$\begin{aligned}
\mathbb{E}\int_{0^+}^t\int_{\mathbb{R}} z^{p^{r+1}}\eta^X(dz;ds) &\leq \mathbb{E}\int_{0^+}^t\int_{\mathbb{R}} z^{p^{r+1}}\nu_t^X(dz)\,ds\\
&\leq \int_{0^+}^t\int_{\mathbb{R}}\mathbb{E}\,|h(s;z)|^{p^{r+2}}\,\nu(dz)\,ds.
\end{aligned}\tag{26}$$

Substitution of (26) to (25) gives

$$\begin{aligned}
\mathbb{E}\left|L(t)^{(r)}\right|^{p} &\leq C\,\mathbb{E}\int_{0^+}^t\int_Z |h(s,z)|^{p^{r+2}}\,\nu(dz)ds\\
&\leq C\,\int_0^t\int_Z\mathbb{E}\,|h(s,z)|^{p^{r+2}}\,\nu(dz)ds.
\end{aligned}\tag{27}$$

Iteration of the calculation (24) and substitution of (27) leads to

$$\begin{aligned}
\mathbb{E}\left|L(t)^{(r)}\right|^{p^m} &\le \\
& C_0\,\mathbb{E}\left|L(t)^{(r+1)}\right|^{p^{m-2}} + \sum_{i=1}^{2} C_i \mathbb{E}\left(\int_{0^+}^t \int_E |h(s;z)|^{p^{r+i}}\,\nu(dz)ds\right)^{p^{m-i+1}} \\
&\le \; C \sum_{k=1}^{m} \mathbb{E}\left(\int_{0^+}^t \int_Z |h(s,z)|^{p^{r+1+k}}\,\nu(z)ds\right)^{m-k}
\end{aligned}$$

Assumption (22) and interpolation yields

$$\mathbb{E}\left|L(t)^{(r)}\right|^{p^m} < \infty.$$

□

Remark 3.6. *By Remark 3.3 it follows, that Proposition 3.3 and Proposition 3.1 remain valid also if η is a compensated Poisson random measure with characteristic measure ν such that ν is a Lévy measure on $(\mathcal{Z}, Z)$.*

4. Proof of Existence and Uniqueness of Solutions

The proof of existence and uniqueness is based on the Banach fixed point theorem for contractions (see e.g. Zeidler [50, Theorem 1.A, Chapter 1.1]). We will first prove part (a) and then secondly sketch the proof of part (b) and (c).

4.1. **Proof of Part (a) of Theorem 2.1.** We denote by $\mathcal{V}_{q,q}(T)$ the Banach space of all $V_{-\delta}$-valued predictable processes u defined on the time interval $[0,T]$ equipped with the norm

$$|||u|||_{q,q} = \left[\int_0^T \mathbb{E}|u(s)|^q\,ds\right]^{\frac{1}{q}} < \infty, \quad u \in \mathcal{V}_{q,q}(T). \tag{28}$$

For $p \le r < \infty$ and $\delta \in \mathbb{R}$ let

$$\begin{aligned}
\mathbb{V}_{r,\delta} := \Big\{\varphi \in L^0(\Omega; V_\delta),\ &\text{such that} \\
&\varphi \text{ is } \mathcal{F}_0\text{–measurable and } \mathbb{E}|\varphi|_\delta^r < \infty\Big\},
\end{aligned} \tag{29}$$

equipped with norm

$$\|\varphi\|_{r,\delta} := \left(\mathbb{E}|\varphi|_\delta^r\right)^{\frac{1}{r}}, \quad \varphi \in \mathbb{V}_{r,\delta}.$$

Let δ_f and δ_g be given. We say the constants δ_f, δ_g, δ and γ satisfy the assumption

(A) iff $\delta_g q < 1$ and $\delta_f < 1$,
(B) iff $(\delta_g - \gamma)p < 1 - \frac{1}{q}$ and $\delta_f - \gamma < 1 - \frac{1}{q}$,
(C) iff $0 \le \gamma q < 1$
(D) iff $\delta > \max(\delta_f - 1 + \frac{1}{q}, \delta_g + \frac{1}{q}, \frac{1}{q})$.

Furthermore, let

$$\mathcal{V}^{\mathbb{D}}_{q,q}(T) := \mathcal{V}_{q,q}(T) \cap L^0\left(\Omega; \mathbb{D}([0,T]; V_{-\delta})\right).$$

Since $V_{-\delta}$ is a Banach space, $\mathbb{D}([0,T]; V_{-\delta})$ is a metrizable topological space and the resulting metric is complete (see e.g. [23, Theorem 5.6, Chapter 3]). Moreover, by Theorem 1.7, Chapter 3 in [23], and since $V_{-\delta}$ is separable, $L^0\left(\Omega; \mathbb{D}([0,T]; V_{-\delta})\right)$ is also a complete separable metric space with respect to the Prohorov metric (for the definition see e.g. [23, Chapter 3.1] or the Appendix, Chapter A.1). Further, the completion of $\mathcal{V}^{\mathbb{D}}_{q,q}(T)$ with respect to the norm given in (28) is $\mathcal{V}_{q,q}(T)$. Note, since $\mathbb{D}([0,T]; V_{-\delta})$ is not a topological vector space $\mathcal{V}_{q,q}(T) \supset \mathcal{V}^{\mathbb{D}}_{q,q}(T)$ but not necessarily $\mathcal{V}_{q,q}(T) = \mathcal{V}^{\mathbb{D}}_{q,q}(T)$. Let $\mathcal{K}_\varphi : \mathcal{V}^{\mathbb{D}}_{q,q}(T) \to \mathcal{V}^{\mathbb{D}}_{q,q}(T)$ be the following transformation

$$\begin{aligned} (\mathcal{K}_\varphi u)(t) &= S_t\varphi + \int_{0+}^t S_{t-s} f(u(s-))\, ds + \\ & \qquad \int_{0+}^t \int_Z S_{t-s} g(u(s-); z)\, \eta(dz; dz) \\ &:= S_t\varphi + \mathcal{K}_1 u + \mathcal{K}_2 u, \quad t \in [0,T], \quad u \in \mathcal{V}^{\mathbb{D}}_{q,q}(T). \end{aligned} \tag{30}$$

The proof of existence and uniqueness of the solution under the assumptions (A), (B), (C) and (D) is divided into the following steps:

(1) Firstly we will show that for all $\varphi \in \mathbb{V}_{q,-\gamma}$ the operator $\mathcal{K}_\varphi$ maps $\mathcal{V}^{\mathbb{D}}_{q,q}(T)$ into $\mathcal{V}^{\mathbb{D}}_{q,q}(T)$. Thus, we show that for all $\varphi \in \mathbb{V}_{q,-\gamma}$
   - (i) $\mathcal{K}_\varphi$ maps $\mathcal{V}^{\mathbb{D}}_{q,q}(T)$ into $\mathcal{V}_{q,q}(T)$.
   - (ii) $\mathcal{K}_\varphi$ maps $\mathcal{V}^{\mathbb{D}}_{q,q}(T)$ into $L^0\left(\Omega; \mathbb{D}([0,T]; V_{-\delta})\right)$.

(2) Secondly, we will show that there exists a constant $\bar{T} > 0$ such that for all $\varphi \in \mathbb{V}_{q,-\gamma}$ the operator $\mathcal{K}_\varphi$ has a unique fixed point $x^\star$ in $\mathcal{V}_{q,q}(\bar{T})$ and $x^\star \in L^0(\Omega; \mathbb{D}([0,\bar{T}]; V_{-\delta})$. This will be shown in the following two substeps:
   - (i) there exists constants $\bar{T} > 0$ and $0 < k < 1$, such that for all $\varphi \in \mathbb{V}_{q,-\gamma}$ the operator $\mathcal{K}_\varphi : \mathcal{V}^{\mathbb{D}}_{q,q}(\bar{T}) \to \mathcal{V}_{q,q}(\bar{T})$ is Lipschitz continuous with respect to the norm in $\mathcal{V}_{q,q}(T)$ with Lipschitz constant k.
   - (ii) For all $\varphi \in \mathbb{V}_{q,-\gamma}$ the sequence $\left\{x^{(n)}\right\}_{n\in\mathbb{N}}$ defined by $x^{(n)} = \mathcal{K}_\varphi x^{(n-1)}$, $n \geq 1$ and $x^{(0)}(t) = S_t\varphi$, is tight in $L^0(\Omega; \mathbb{D}([0,\bar{T}]; V_{-\delta})$.

   Applying the Banach fixed point theorem (see e.g. Zeidler [50, Theorem 1.A]) for all $\varphi \in \mathbb{V}_{q,-\gamma}$ there exists a unique $x^\star \in \mathcal{V}_{q,q}(\bar{T})$, such that $\mathcal{K}_\varphi x^\star = x^\star$ and ${\mathcal{K}_\varphi}^{(n)} y \to x^\star$ for all $y \in \mathcal{V}_{q,q}(\bar{T})$. From (2)-(ii), it follows $x^\star \in L^0(\Omega; \mathbb{D}([0,\bar{T}]; V_{-\delta})$, in particular $x^\star \in \mathcal{V}^{\mathbb{D}}_{q,q}(\bar{T})$.

(3) By step (2) there exists a unique fixed point $u^0 \in \mathcal{V}^{\mathbb{D}}_{q,q}(\bar{T})$ for the operator $\mathcal{K}_{u_0}$. Using the same argument as in step (2) on $\mathcal{V}^{\mathbb{D}}_{q,q}(\bar{T})$, there exists a unique fixed point $u^1 \in \mathcal{V}^{\mathbb{D}}_{q,q}(\bar{T})$ for $\mathcal{K}_{u^0(\bar{T})}$ provided $u^0(\bar{T}) \in \mathbb{V}_{q,-\gamma}$. The solution, i.e. the fixed point $u^1 : [0,\bar{T}] \to V_{-\delta}$, can be shifted to the time interval $[\bar{T}, 2\bar{T}]$ by defining a new function $\bar{u}^1(t) := u^1(t-\bar{T})$, $t \in [\bar{T}, 2\bar{T}]$. This argument can be repeated a finite number of times. Since $\bar{T}$ does not depend on φ, we can find solutions u^0, u^1, …, u^m, respectively,

on $[0,\bar T],[\bar T,2\bar T],\ldots,[m\bar T,T\wedge(m+1)\bar T]$, respectively, provided $u^{i-1}(i\bar T)\in\mathbb{V}_{q,-\gamma}$ for $i=1,\ldots,m$. Finally, we have to glue together all the solutions.

Claim 4.1. *Under Assumptions (A), (B) and (C) and for all $\varphi\in\mathbb{V}_{q,-\gamma}$ the operator $\mathcal{K}_\varphi: V^{\mathbf{D}}_{q,q}\to V_{p,\infty,-\gamma}$ is bounded and Lipschitz continuous. In particular, there exists some constant $C_1,C_2<\infty$, such that*

$$
\begin{aligned}
&|||\mathcal{K}_\varphi u|||_{p,\infty,-\gamma}\le\\
&\qquad C_1\,|||u|||_{q,q,}+C_2\,\|\varphi\|_{p,-\gamma},\quad u\in\mathcal{V}^{\mathbf{D}}_{q,q}(T),\quad \varphi\in\mathbb{V}_{p,-\gamma},
\end{aligned}
\tag{31}
$$

and there exists a constant $C<\infty$, such that

$$
\begin{aligned}
&|||\mathcal{K}_\varphi u-\mathcal{K}_\varphi v|||_{p,\infty,-\gamma}\le\\
&\qquad C\,|||u-v|||_{q,q,},\quad u\in\mathcal{V}^{\mathbf{D}}_{q,q}(T),\quad \varphi\in\mathbb{V}_{p,-\gamma}.
\end{aligned}
\tag{32}
$$

Proof. Inequality (31) follows from a sequence of calculations:

$$
\begin{aligned}
\mathbb{E}\,|\mathcal{K}_\varphi u(t)|^p_{-\gamma}&\le \mathbb{E}|S_t\varphi|^p_{-\gamma}+\mathbb{E}\int_0^t|S_{t-s}f(u(s-))\,ds|^p_{-\gamma}\\
&+\mathbb{E}\left|\int_{0+}^t\int_Z S_{t-s}g(u(s-);z)\,\eta(ds,dz)\right|^p_{-\gamma}
\end{aligned}
$$

The generalized Burkholder inequality implies

$$
\begin{aligned}
\mathbb{E}\,|\mathcal{K}_\varphi u(t)|^p_{-\gamma}&\le C\,\mathbb{E}|\varphi|^p_{-\gamma}+\mathbb{E}\left(\int_{0+}^t|S_{t-s}f(u(s-)|_{-\gamma}\,\,ds\right)^p\\
&+\mathbb{E}\int_{0+}^t\int_Z|S_{t-s}g(u(s-);z)|^p_{-\gamma}\,\,\nu(dz)\,ds\\
&\le\quad C\,\mathbb{E}|\varphi|^p_{-\gamma}+\mathbb{E}\left(\int_{0+}^t|S_{t-s}|_{L(V_{-\delta_f},V_{-\gamma})}\,|f(u(s-))|_{-\delta_f}\,\,ds\right)^p\\
&+\mathbb{E}\int_{0+}^t\int_Z|S_{t-s}|^p_{L(V_{-\delta_g},V_{-\gamma})}\,|g(u(s-);z)|^p_{-\delta_g}\,\,\nu(dz)\,ds
\end{aligned}
$$

$$
\begin{aligned}
&\le\quad C\,\mathbb{E}|\varphi|^p_{-\gamma}+\mathbb{E}\left(\int_{0+}^t(t-s)^{-\delta_f+\gamma}\,|f(u(s-))|_{-\delta_f}\,\,ds\right)^p\,dt\\
&+\mathbb{E}\int_{0+}^t\int_Z(t-s)^{(-\delta_g+\gamma)p}\,|g(u(s-);z)|^p_{-\delta_g}\,\,\nu(dz)\,ds.
\end{aligned}
$$

The Lipschitz continuity, the Hölder inequality and the Jensen inequality give

$$\begin{aligned}
\mathbb{E}\,|\mathcal{K}_\varphi u(t)|^p &\le C\,\mathbb{E}|\varphi|^p_{-\gamma} + \mathbb{E}\left(\int_{0+}^t (t-s)^{-\delta_f+\gamma}\,|u(s-)|\;ds\right)^p \\
&\quad +\mathbb{E}\int_{0+}^t (t-s)^{(\gamma-\delta_g)p}\,|u(s-)|^p\;\;ds \\
&\le\;\; C\,\mathbb{E}|\varphi|^p_{-\gamma} + C\,t^{p(1-\frac{1}{r}-(\delta_f-\gamma))}\left(\mathbb{E}\int_{0+}^t |u(s-)|^r\;ds\right)^{\frac{p}{r}} \\
&\quad + t^{p(1-\frac{1}{r}-(\delta_g-\gamma)p)}\left(\mathbb{E}\int_{0+}^t |u(s-)|^r\;\;ds\right) \\
&\le\;\; C(t)|||u|||^p_{r,r} + C\,\mathbb{E}|\varphi|^p_{-\gamma},
\end{aligned}$$

where r satisfies $(\delta_g-\gamma)p < 1-\frac{1}{r}$ and $\delta_f-\gamma < 1-\frac{1}{r}$. Since q satisfies the assumptions above, inequality (31) follows. Inequality (32) follows from similar calculation:

$$\begin{aligned}
\mathbb{E}\,|\mathcal{K}_\varphi u(t)-\mathcal{K}_\varphi v(t)|^p_{-\gamma} &\le \mathbb{E}\int_0^t |S_{t-s}\left(f(u(s-)-f(v(s-))\;\;ds\right|^p_{-\gamma} \\
&\quad + \mathbb{E}\left|\int_{0+}^t\int_Z S_{t-s}\left(g(u(s-);z)-g(v(s-);z)\right)\;\eta(ds,dz)\right|^p_{-\gamma} \\
&\le\;\; \mathbb{E}\left(\int_{0+}^t |S_{t-s}\left(f(u(s-)-f(v(s-))\right)|_{-\gamma}\;\;ds\right)^p \\
&\quad + \mathbb{E}\int_{0+}^t\int_Z |S_{t-s}\left(g(u(s-);z)-g(v(s-);z)\right)|^p_{-\gamma}\;\;\nu(dz)\,ds
\end{aligned}$$

$$\begin{aligned}
&\le\;\; \mathbb{E}\left(\int_{0+}^t |S_{t-s}|_{L(V_{-\delta_f},V_{-\gamma})}\,|f(u(s-)-f(v(s-)|_{-\delta_f}\;\;ds\right)^p \\
&\quad + \mathbb{E}\int_{0+}^t\int_Z |S_{t-s}|^p_{L(V_{-\delta_g},V_{-\gamma})}\,|g(u(s-);z)-g(v(s-);z)|^p_{-\delta_g}\;\;\nu(dz)\,ds
\end{aligned}$$

$$\begin{aligned}
&\le\;\; \mathbb{E}\left(\int_{0+}^t (t-s)^{-\delta_f+\gamma}\,|f(u(s-)-f(v(s-)|_{-\delta_f}\;\;ds\right)^p\;dt \\
&\quad + \mathbb{E}\int_{0+}^t\int_Z (t-s)^{(-\delta_g+\gamma)p}\,|g(u(s-);z)-g(v(s-);z)|^p_{-\delta_g}\;\;\nu(dz)\,ds.
\end{aligned}$$

Again, the Lipschitz continuity, the Hölder inequality and the Jensen inequality give

$$\begin{aligned}\mathbb{E}\,|\mathcal{K}_\varphi u(t)|^p \le \mathbb{E}\left(\int_{0+}^t (t-s)^{-\delta_f+\gamma}\,|u(s-)-v(s-)|\;ds\right)^p \\ +\mathbb{E}\int_{0+}^t (t-s)^{(\gamma-\delta_g)p}\,|u(s-)-v(s-)|^p\;\;ds \\ \le\;\; C\,t^{p(1-\frac{1}{r}-(\delta_f-\gamma))}\left(\mathbb{E}\int_{0+}^t |u(s-)-v(s-)|^r\;ds\right)^{\frac{p}{r}} \\ +\,t^{p(1-\frac{1}{r}-(\delta_g-\gamma)p)}\left(\mathbb{E}\int_{0+}^t |u(s-)-v(s-)|^r\;\;ds\right) \\ \le\;\; C(t)|||u-v|||_{r,r}^p,\end{aligned}$$

where r satisfies $(\delta_g-\gamma)p<1-\frac{1}{r}$ and $\delta_f-\gamma<1-\frac{1}{r}$. Since q satisfies the assumptions above, (32) follows. □

Claim 4.2. *Assume the conditions (A), (B), (C) and (D) are fulfilled. Then, there exists a constant $C_1,C_2,C_3<\infty$ such that for all $\varphi\in\mathbb{V}_{q,-\gamma}$ and all $u\in\mathcal{V}^{\mathbf{D}}_{q,q}(T)$ we have*

$$\mathbb{E}\left[\sup_{0\le t\le T}|\mathcal{K}_\varphi u(t)|_{-\delta}\right]\le C_1\,\|\phi\|_{q,-\gamma}+C_2\,|||u|||_{p,p}+C_3\,|||u|||_{q,q}.$$

Proof. A short calculation shows,

$$\int_0^t\int_Z S_{t-s}g(u(s-);z)\eta(dz;ds)=\int_0^t\int_Z g(u(s-);z)\eta(dz;ds)+ \\ \int_0^t S_{t-s}A\int_0^s\int_Z g(u(r-);z)\eta(dz;dr)\,ds.$$

Thus, we have

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le t\le T}|\mathcal{K}_\varphi u(t)|_{-\delta}\right]=\mathbb{E}\sup_{0\le t\le T}|S_t\varphi|_{-\delta}+ \\ \mathbb{E}\sup_{0\le t\le T}\left|\int_0^t S_{t-s}f(u(s-))\,ds\right|_{-\delta}+\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t\int_Z g(u(s-);z)\eta(dz;ds)\right|_{-\delta} \\ +\,\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t S_{t-s}A\int_0^s\int_Z g(u(r-);z)\eta(dz;dr)\,ds\right|_{-\delta} \\ =:\;\; I+II+III+IV.\end{aligned}$$

Since $\gamma<\delta$, we have for the first term

$$I\;\;\le\;\;C\,\mathbb{E}\sup_{0\le t\le T}|\varphi|_{-\delta}\le\mathbb{E}\,|\varphi|_{-\gamma}\,.$$

The Minkowski inequality yields for $\epsilon=\delta-\delta_f$

$$\begin{aligned}II\;\;&\le\;\;\mathbb{E}\sup_{0\le t\le T}\int_0^t|S_{t-s}f(u(s-))|_{-\delta}\;ds \\ &\le\;\;\mathbb{E}\sup_{0\le t\le T}\int_0^t(t-s)^{-\epsilon}\,|f(u(s-))|_{-\delta-\epsilon}\;ds.\end{aligned}$$

The Lipschitz property of f and the Hölder inequality lead to

$$\begin{aligned} II &\leq \mathbb{E} \sup_{0\leq t\leq T} \int_0^t (t-s)^{-\epsilon} \left|u(s-)\right|_{-\delta-\epsilon+\delta_f} \, ds \\ &\leq \mathbb{E} \sup_{0\leq t\leq T} C\, T^{1-\frac{1}{q}-\epsilon} \left(\int_0^t \left|u(s-)\right|^q_{-\delta-\epsilon+\delta_f} \, ds \right)^{\frac{1}{q}} \\ &\leq C\, T^{1-\frac{1}{q}-\epsilon} \mathbb{E} \left(\int_0^T \left|u(s-)\right|^q_{-\delta-\epsilon+\delta_f} \, ds \right)^{\frac{1}{q}}. \end{aligned}$$

The Jensen inequality and the fact that $\delta > \delta_f - 1 + \frac{1}{q}$ give

$$\begin{aligned} II &\leq C\, T^{1-\frac{1}{q}-\epsilon} \left(\mathbb{E} \int_0^T \left|u(s-)\right|^q_{-\delta-\epsilon+\delta_f} \, ds \right)^{\frac{1}{q}} \\ &\leq C\, T^{1-\frac{1}{q}-\epsilon} |||u|||_{q,q}. \end{aligned}$$

The Burkholder inequality gives for the third term

$$III \leq \left(\int_0^T \int_Z \mathbb{E} \left|g(u(s-);z)\right|^p_{-\delta} \nu(dz)\, ds \right)^{\frac{1}{p}}.$$

The Lipschitz property of g gives

$$III \leq \mathbb{E} \left(\int_0^T \left|u(s-;z)\right|^p_{-\delta+\delta_g} \, ds \right)^{\frac{1}{p}} \leq |||u|||_{p,p}.$$

Let $\epsilon = 1 + \delta_g - \delta$. The Minkowski inequality gives for the fourth term

$$\begin{aligned} IV &\leq \mathbb{E} \sup_{0\leq t\leq T} \int_0^t \left| S_{t-s} A \int_0^s \int_Z g(u(r-);z)\eta(dz;dr) \right|_{-\delta} ds \\ &\leq \mathbb{E} \sup_{0\leq t\leq T} \int_0^t (t-s)^{-\epsilon} \left| A \int_0^s \int_Z g(u(r-);z)\eta(dz;dr) \right|_{-\delta-\epsilon} ds. \end{aligned}$$

The Hölder inequality gives for $q' = \frac{q}{q-1}$

$$\begin{aligned} &IV \leq \\ &\quad \mathbb{E} \sup_{0\leq t\leq T} \left(\int_0^t (t-s)^{-q'\epsilon} \right)^{\frac{1}{q'}} \left(\int_0^t \left| \int_0^s \int_Z g(u(r-);z)\eta(dz;dr) \right|^q_{-\delta+1-\epsilon} ds \right)^{\frac{1}{q}}. \end{aligned}$$

The Jensen and the Burkholder inequality give

$$\begin{aligned} IV &\leq \\ &\quad C\, T^{1-\frac{1}{q}-\epsilon} \mathbb{E} \left(\int_0^T \left| \int_0^s \int_Z g(u(r-);z)\eta(dz;dr) \right|^q_{-\delta+1-\epsilon} ds \right)^{\frac{1}{q}} \\ &\leq C\, T^{1-\frac{1}{q}-\epsilon} \left(\int_0^T \int_0^s \int_Z \mathbb{E} \left|g(u(r-);z)\right|^q_{-\delta+1-\epsilon} \nu(dz) dr\, ds \right)^{\frac{1}{q}}. \end{aligned}$$

The Lipschitz property yields

$$IV \leq$$
$$C\,T^{1-\epsilon}\left(\int_0^T \mathbb{E}\,|u(r-)|^q_{-\delta+1+\delta_g-\epsilon}\,dr\,ds\right)^{\frac{1}{q}} \leq C\,T^{1-\epsilon}|||u|||_{q,q}.$$

Collecting all together gives the assertion. □

Proof of step (1)-(i): In this section we will show, that under assumptions (A), (B) and (C) and for all $\varphi \in \mathbb{V}_{q,-\gamma}$ the operator $\mathcal{K}_\varphi$ maps $\mathcal{V}^{\mathbf{D}}_{q,q}(T)$ into $\mathcal{V}_{q,q}(T)$. Note that for $0<\gamma<\frac{1}{q}$

$$|||S_t\varphi|||^q_{q,q} = \int_0^T \mathbb{E}|S_t\varphi|^q \;\;\leq\;\; \int_0^T |S_t|^q_{L(V_{-\gamma},E)}\,dt\,|\varphi|^q_{-\gamma}.$$

Using Remark B.1 we have

$$\begin{aligned} |||S_t\varphi|||^q_{q,q} &\leq \int_0^T C\,t^{-q\gamma}\,dt\,|\varphi|^q_{-\gamma} \\ &\leq C\,T^{1-q\gamma}\,\mathbb{E}\,|\varphi|^q_{-\gamma}. \end{aligned}$$

Furthermore we have

$$\begin{aligned} |||\mathcal{K}_1 u|||^q_{q,q} &\leq \mathbb{E}\left[\int_0^T \left|\int_{0+}^t S_{t-s}f(u(s-))\,ds\right|^q\,dt\right] \\ &\leq \mathbb{E}\int_0^T \left(\int_{0+}^t |S_{t-s}f(u(s-))|\,\,ds\right)^q\,dt \\ &\leq \mathbb{E}\int_0^T \left(\int_{0+}^t |S_{t-s}|_{L(V_{-\delta_f},E)}\,|f(u(s-))|_{-\delta_f}\,\,ds\right)^q\,dt \\ &\leq \mathbb{E}\int_0^T \left(\int_{0+}^t (t-s)^{-\delta_f}\,|f(u(s-))|_{-\delta_f}\,\,ds\right)^q\,dt. \end{aligned} \tag{33}$$

Therefore, by the Young inequality we infer that

$$|||\mathcal{K}_1 u|||^q_{q,q} \;\;\leq\;\; \left(\int_{0+}^T (T-t)^{-\delta_f}\,dt\right)^q\,\mathbb{E}\int_{0+}^T |f(u(t-))|^q_{-\delta_f}\,\,dt. \tag{34}$$

Next, the Lipschitz condition of f, i.e. (2), implies that

$$\begin{aligned} |||\mathcal{K}_1 u|||^q_{q,q} &\leq \frac{C\,T^{(1-\delta_f)q}}{(1-\delta_f)^{\frac{1}{q'}}}\,\mathbb{E}\int_{0+}^T |u(t-)|^q\,\,dt \\ &\leq \frac{C\,T^{\frac{1-\delta_f}{q'}}}{(1-\delta_f)^q}\,|||u|||^q_{q,q}. \end{aligned} \tag{35}$$

By Proposition 3.1 we get

$$\begin{aligned}|||\mathcal{K}_2 u|||_{q,q}^q &= \mathbb{E}\int_0^T \left|\int_{0+}^t\int_Z S_{t-s}g(u(s-);z)\,\eta(ds,dz)\right|^q\,dt\\ &= \int_0^T \mathbb{E}\left|\int_{0+}^t\int_Z S_{t-s}g(u(s-);z)\,\eta(ds,dz)\right|^q\,dt\\ &\le C\int_0^T\sum_{l=1}^n\left(\mathbb{E}\int_{0+}^t\int_Z |S_{t-s}g(u(s-);z)|^{p^l}\nu(dz)\,ds\right)^{p^{n-l}}\,dt\\ &\le C\int_0^T\sum_{l=1}^n\left(\mathbb{E}\int_{0+}^t\int_Z (t-s)^{-\delta_g p^l}|g(u(s-);z)|_{-\delta_g}^{p^l}\nu(dz)\,ds\right)^{p^{n-l}}\,dt. \end{aligned}\tag{36}$$

By the Lipschitz condition of g, i.e. (4), we infer that

$$|||\mathcal{K}_2 u|||_{q,q}^q \le C\int_0^T\sum_{l=1}^n\left(\mathbb{E}\int_{0+}^t (t-s)^{-\delta_g p^l}|u(s-))|^{p^l}ds\right)^{p^{n-l}}.\tag{37}$$

Again, the Young inequality yields

$$|||\mathcal{K}_2 u|||_{q,q}^q \le C\sum_{l=1}^n\left(\int_0^T t^{-\delta_g p^l}\,dt\right)^{p^{n-l}}\mathbb{E}\int_0^t |u(s-)|^q ds.$$

Summing up, we have proved that the operator $\mathcal{K}_\varphi$ maps $\mathcal{V}_{q,q}(T)$ into $\mathcal{V}_{q,q}(T)$.

Proof of step (1)-(ii). In order to show $\mathcal{K}_\varphi u$ belongs to $\mathbb{D}([0,\infty);V_{-\delta})$, we will show, that the set $\{\mathcal{K}_\varphi u\}$ satisfies the Aldou's condition (see Definition A.2) and the compact containment condition (see Definition A.3).

Proof that the Aldou's condition is satisfied: We will show, that $\{\mathcal{K}_\varphi u\}$ satisfies the assumptions of Theorem A.1 with $\beta = p$. In particular, we will show that there exists some $\rho > 0$ such that for all $t \in [0,T]$ and all $\theta > 0$ we have

$$\begin{aligned}\mathbb{E}\left[|\mathcal{K}_\varphi u(t+\theta) - \mathcal{K}_\varphi u(t)|_{-\gamma}^p \mid \mathcal{F}_t\right] &\le \theta^\rho |||1_{(t,t+\theta]}u|||_{q,q,-\gamma}^p + \\ & C_2\theta^{p\delta}|||1_{(0,t]}\mathcal{K}_\varphi u|||_{p,\infty,-\gamma}^p, \quad \mathbb{V}_{p,-\gamma}u \in \mathcal{V}_{q,q}^{\mathbb{D}}, \quad \varphi \in \mathbb{V}_{p,-\gamma}.\end{aligned}\tag{38}$$

By (31) there exists some $C_1, C_2 < \infty$ such that

$$|||1_{(0,t]}\mathcal{K}_\varphi u|||_{p,\infty,-\gamma} \le |||\mathcal{K}_\varphi u|||_{p,\infty,-\gamma} \le C_1\,|||u|||_{q,q} + C_2\,\|\phi\|_{p,-\gamma}.$$

Therefore there exists some constant $C < \infty$ such that $C\,\left(\theta^\rho + \theta^\delta\right)$ is an upper bound for the RHS of (38). Moreover $C\,\left(\theta^\rho + \theta^\delta\right) \to 0$ as $\theta \to 0$ and the assumptions of Lemma A.1 are satisfied.

Proof. To prove inequality (38), we first note that

$$
\begin{aligned}
\mathcal{K}_\varphi u(t+\theta) - \mathcal{K}_\varphi u(t) &= (S_{t+\theta} - S_t)\,\varphi + \\
&\int_{0+}^{t+\theta} S_{t+\theta-\sigma} f(u(\sigma-))\,d\sigma + \int_{0+}^{t+\theta} S_{t+\theta-\sigma} g(u(\sigma-);z)\,\eta(dz;d\sigma) \\
&- \int_{0+}^{t} S_{t-\sigma} f(u(\sigma-))\,d\sigma - \int_{0+}^{t} S_{t-\sigma} g(u(\sigma-);z)\,\eta(dz;d\sigma) \\
= \ & (S_\theta - I)\,S_t\varphi + (S_\theta - I)\int_{0+}^{t+\theta} S_{t-\sigma} f(u(\sigma-))\,d\sigma \\
+ \ & (S_\theta - I)\int_{0+}^{t+\theta} S_{t-\sigma} g(u(\sigma-);z)\,\eta(dz;d\sigma) \\
+ \ & \int_{t+}^{t+\theta} S_{t+\theta-\sigma} f(u(\sigma-))\,d\sigma + \int_{t+}^{t+\theta} S_{t+\theta-\sigma} g(u(\sigma-);z)\,\eta(dz;d\sigma) \\
= \ & (S_\theta - I)\,S_t\varphi + (S_\theta - I)\,(\mathcal{K}_\varphi u)\,(t+\theta) \\
+ \ & \underbrace{\int_{t+}^{t+\theta} S_{t+\theta-\sigma} f(u(\sigma-))\,d\sigma}_{=:f_1(t,\theta)} + \underbrace{\int_{t+}^{t+\theta} S_{t+\theta-\sigma} g(u(\sigma-);z)\,\eta(dz;d\sigma)}_{=:f_2(t,\theta)}.
\end{aligned}
$$

Hence $S_\theta - I = \int_0^\theta AS_r\,dr$ (see Pazy [37, Theorem 1.2.4-(b)]) and Remark B.1 we have

$$
\begin{aligned}
|(S_\theta - I)x|_{-(\gamma+\delta)} &\leq \int_0^\theta |AS_r x|_{-(\gamma+\delta)}\,dr \\
\leq C \int_0^\theta r^{-1+\delta}\,|x|_{-\gamma}\,dr &\leq C\,\theta^\delta\,|x|_{-\gamma}.
\end{aligned}
$$

Therefore

$$
|(S_\theta - I)\,S_t\varphi|_{-(\gamma+\delta)} \leq C\,\theta^\delta\,|S_t\varphi|_{-\gamma} \leq C\,\theta^\delta\,|\varphi|_{-\gamma} \tag{39}
$$

and

$$
\mathbb{E}\,|(S_\theta - I)\,(\mathcal{K}_\varphi u)\,(t+\theta)|^p_{-(\gamma+\delta)} \leq \theta^{p\delta}\mathbb{E}\,|(\mathcal{K}_\varphi u)\,(t+\theta)|^p_{-\gamma}. \tag{40}
$$

Using the same calculations as in (33), (34) and (35) we obtain

$$
\begin{aligned}
\mathbb{E}\,|f_1(t,\theta)|^p_{-(\gamma+\delta)} = C\,\mathbb{E}\left|\int_{0+}^\theta S_{\theta-\sigma} f(u(t+\sigma-))\,d\sigma\right|^p_{-(\gamma+\delta)} &\leq \\
C\,\mathbb{E}\left(\int_{0+}^\theta (\theta-\sigma)^{-(\delta_f-(\gamma+\delta))}\,|f(u(t+\sigma-))|_{-\delta_f}\,d\sigma\right)^p& \\
\leq C\,\theta^{p\left(1-\frac{1}{q}-(\delta_f-\gamma-\delta)\right)} \times \left\{\int_0^\theta \mathbb{E}\,|u(t+\sigma)|^q\,d\sigma\right\}^{\frac{p}{q}}&.
\end{aligned}
$$

Note, that

$$
\mathbb{E}\,|f_2(t,\theta)|^p_{-(\gamma+\delta)} \leq C\,\mathbb{E}\left|\int_{0+}^\theta \int_Z S_{\theta-\sigma} g(u((\sigma-)+t);z)\,\eta\circ\theta_t(dz;d\sigma)\right|^p_{-(\gamma+\delta)},
$$

where $\eta \circ \theta_t(\omega) := \eta(\theta_t \circ \omega)$, and θ_t is the usual shift operator defined by $\theta_t\omega(s) := \omega(t+s)$. Following (36) and (37) we get

$$\mathbb{E}\,|f_2(t,\theta)|^p_{-(\gamma+\delta)} \le C\,\theta^{1-\frac{p}{q}-p(\delta_g-\gamma-\delta)} \left\{ \int_0^\theta \mathbb{E}\,|u(t+\sigma)|^q \;d\sigma \right\}^{\frac{p}{q}}.$$

□

Proof that the compact containment condition is satisfied: We will show A.3 by Lemma A.1. By Remark B.2 the space $V_{-\tilde{\delta}}$ is compactly embedded in $V_{-\delta}$ for all $\tilde{\delta} < \delta$. Let $\tilde{\delta} > 0$ be chosen accordingly

$$\delta > \tilde{\delta} > \max(\delta_f - 1 + \frac{1}{q}, \delta_g + \frac{1}{q}, \frac{1}{q}).$$

Moreover, by Claim 4.2 there exists some constant $C_1, C_2 < \infty$ such that

$$\mathbb{E}\left|\sup_{0\le t\le T} \mathcal{K}_\varphi u(t)\right|_{-\tilde{\delta}} \le C_1|||u|||_{q,q} + C_2\|\varphi\|_{p,-\gamma}, \quad u \in \mathcal{V}^{\mathbf{D}}_{q,q}, \quad \varphi \in \mathbb{V}_{p,-\gamma}.$$

By Lemma A.1 the compact containment condition follows.
Proof of step (2)-(i) Next, we will show, that there exists some function $C : [0,1] \to \mathbb{R}^+$ such that

$$\begin{aligned} &|||\mathcal{K}_\varphi u - \mathcal{K}_\varphi v|||^q_{q,q} \le \\ &\qquad C(T)\;|||u-v|||^q_{q,q}, \quad u,v \in \mathcal{V}^{\mathbf{D}}_{q,q}(T), \quad \varphi \in \mathbb{V}_{q,-\gamma}, \end{aligned} \tag{41}$$

and $C(T) \to 0$ as $T \to 0$. Thus, we can find a $\bar{T}$, such that

$$C(\bar{T}) < 1.$$

In particular, there exists a constant $0 < k < 1$ such that for all $\varphi \in \mathbb{V}_{q,-\gamma}$ the operator $\mathcal{K}_\varphi : \mathcal{V}^{\mathbf{D}}_{q,q}(\bar{T}) \to \mathcal{V}^{\mathbf{D}}_{q,q}(\bar{T})$ is Lipschitz continuous with constant k.

Proof. Similarly to calculations (33), (34) and (35) the Lipschitz continuity of f, i.e. (2), leads to

$$|||\mathcal{K}_1 u - \mathcal{K}_1 v|||^q_{q,q} \le C(q,T)\;|||u-v|||^q_{q,q}, \quad u,v \in \mathcal{V}^{\mathbf{D}}_{q,q}(T),$$

where the constant $C(q,T)$ tends to zero as $T\to 0$. Next we have

$$
\begin{aligned}
|||\mathcal{K}_2u-\mathcal{K}_2v|||_{q,q}^q &= \mathbb{E}\int_0^T |\mathcal{K}_2u(t)-\mathcal{K}_2v(t)|^q\,dt\\
&\le C\int_0^T\left|\int_0^t\int_Z S_{t-s}\left(g(u(s-);z)-g(v(s-);z)\right)\,\eta(dz;ds)\right|^q\,dt\\
&\le C\int_0^T\mathbb{E}\sum_{l=1}^n\\
&\quad\left(\int_{0+}^t\int_Z |S_{t-s}\left(g(u(s-);z)-g(v(s-);z)\right)|^{p^l}\nu(dz)\,ds\right)^{p^{n-l}}\,dt\\
&\le C\int_0^T\mathbb{E}\sum_{l=1}^n\\
&\quad\left(\int_{0+}^t\int_Z (t-s)^{-\delta_g p^l}|g(u(s-);z)-g(v(s-);z)|_{-\delta_g}^{p^l}\nu(dz)\,ds\right)^{p^{n-l}}\,dt\\
&\le C\,\mathbb{E}\sum_{l=1}^n\int_0^T\\
&\quad\left(\int_{0+}^t\int_Z (t-s)^{-\delta_g p^l}|u(s-)-v(s-)|^{p^l}h(z)\,\nu(dz)\,ds\right)^{p^{n-l}}\,dt\\
&\le C\,\mathbb{E}\sum_{l=1}^n\int_0^T\\
&\quad\left(\int_{0+}^t\int_Z (t-s)^{-\delta_g p^l}|u(s-)-v(s-)|^{p^l}\nu(dz)\,ds\right)^{p^{n-l}}\,dt.
\end{aligned}
$$

Again, the Young inequality yields

$$
|||\mathcal{K}_2u-\mathcal{K}_2v|||_{q,q}^q \le \frac{C\,T^{(1-\delta_g q)q}}{(1-\delta_g q)^q}\,\mathbb{E}\int_{0+}^T |u(s-)-v(s-)|^q\,ds.
$$

Summing up we have proved

$$
|||\mathcal{K}_\varphi u-\mathcal{K}_\varphi v|||_{q,q}\le C(T)\,|||u-v|||_{q,q},
$$

where $C(T)$ tends to zero as $T\to 0$.

□

Proof of step (2)-(ii) Let $\{u^{(n)}\}_{n\ge 0}$ be the sequence defined by $u^{(n)}:=\mathcal{K}_\varphi u^{(n-1)}, n\ge 1$ and $u^{(0)}(t)=S(t)\varphi$ and let $u^\star$ the fixed point of $\mathcal{K}_\varphi$. We have to show, that $\{u^{(n)}\}_{n\ge 1}$ is tight in $L^0(\Omega;\mathbb{D}([0,\bar T];V_{-\delta}))$. But first we will show the following.

- Under the assumptions (A) and (C) the set $\{u^{(n)}\}_{n\ge 0}$ is bounded in $\mathcal{V}_{q,q}(\bar T)$. By (2)-(i) we know, that $\mathcal{K}_\varphi:\mathcal{V}_{q,q}(\bar T)\to\mathcal{V}_{q,q}(\bar T)$ is a strict contraction. Therefore there exists a constant $0 < k < 1$, such that

$$\sup_{n\in\mathbb{N}} |||u^{(n)}|||_{q,q} \le \frac{C\,\bar{T}^{\frac{1}{q}-\gamma}}{1-k}\|\varphi\|_{q,-\gamma}.$$

- If the assumptions (A), (B) and (C) are satisfied then it follows from (31) that

$$\begin{aligned}\sup_{n\ge 1} |||u^{(n)}|||_{p,\infty,-\delta} &\le \sup_{n\ge 1} |||u^{(n)}|||_{p,\infty,-\gamma}\\ &\le \sup_{n\ge 1} |||\mathcal{K}_\varphi u^{(n-1)}|||_{p,\infty,-\gamma}\\ &\le C_1 \sup_{n\ge 0} |||u^{(n)}|||_{q,q} + C_2\|\varphi\|_{p,-\gamma}.\end{aligned}$$

- Since $|||u^{(n)} - u^\star|||_{q,q} \to 0$ as $n\to\infty$ it follows from (32) that the sequence $\{\mathcal{K}_\varphi u^{(n)}\}_{n\ge 1}$ converges to $\mathcal{K}_\varphi u^\star = u^\star$ in $\mathcal{V}_{p,\infty,-\gamma}(\bar{T})$, i.e.

$$\sup_{0\le t\le \bar{T}} \mathbb{E}\left|u^{(n)}(t) - u(t)\right|_{-\delta}^p \to 0$$

as $n\to\infty$. Therefore, the Chebyscheff inequality shows, that for any finite set $\{t_1,\ldots,t_k\}\subset[0,\bar{T}]$, we have

$$\Big(u^{(n)}(t_1),\ldots,u^{(n)}(t_k)\Big) \to \Big(u(t_1),\ldots,u(t_k)\Big),$$

as $n\to\infty$.

By Theorem 3.7.8 of Ethier and Kurtz [23] it remains to show, that the set $\{u^{(n)}, n\in\mathbb{N}\}$ is tight in $L^0(\Omega;\mathbb{D}([0,\bar{T}];V_{-\delta}))$. This implies that $u^{(n)}$ converges in distribution to $u^\star$ and $u^\star\in L^0(\Omega;\mathbb{D}([0,\bar{T}];V_{-\delta}))$.

Proof that the Aldou's condition is satisfied: Tracing the calculations in (1)-(ii), we see that there exists some $\rho>0$ such that for all $n\in\mathbb{N}$

$$\mathbb{E}\left[\left|u^{(n)}(t+\theta) - u^{(n)}(t)\right|_{-(\gamma+\delta)}^p \mid \mathcal{F}_t^{(n)}\right] \le$$
$$\theta^\rho |||1_{(t,t+\theta]}u^{(n-1)}|||_{q,q,-\gamma}^p + C_2\theta^{p\delta}|||1_{(0,t]}\mathcal{K}_\varphi u^{(n-1)}|||_{p,\infty,-\gamma}^p, \quad 0\le t\le\bar{T}.$$

But from the consideration before we know

$$\sup_{n\ge 1}|||u^{(n)}|||_{q,q} < \infty$$

and

$$\sup_{n\ge 1}|||\mathcal{K}_\varphi u^{(n)}|||_{p,\infty,-\gamma} < \infty$$

Thus, there exists some constants $C<\infty$ and $\rho>0$ such that

$$\mathbb{E}\left[\left|u^{(n)}(t+\theta) - u^{(n)}(t)\right|_{-(\gamma+\delta)}^p \mid \mathcal{F}_t^{(n)}\right] \le C\left(\theta^\rho + \theta^{p\delta}\right), \quad 0\le t\le\bar{T}, \quad n\in\mathbb{N}.$$

This proves that the assumptions of Lemma A.1 are satisfied.

Proof that the compact containment condition is satisfied: The embedding $V_{\tilde{\delta}} \hookrightarrow V_{-\delta}$

is compact for all $\tilde{\delta} < \delta$ (see Remark B.2). Moreover, by Claim 4.2 it follows

$$\sup_{n\ge 1}\mathbb{E}|\sup_{0\le s\le \bar{T}} u^{(n)}(s)|_{-\tilde{\delta}} \quad \le \quad C\,\left(|||u^{(n-1)}|||_{q,q}^q + \|\varphi\|_{q,-\gamma}\right) \quad n\in\mathbb{N}.$$

Since the set $\{u^{(n)} \mid n\in\mathbb{N}\}$ is bounded in $\mathcal{V}_{q,q}(T)$, the compact containment condition follows.
Proof of step (3) Let $\bar{T}$ be so small that $C(\bar{T}) < 1$, where $C:[0,1]\to\mathbb{R}^+$ is the constant from (41). Then for all $\varphi\in\mathbb{V}_{q,-\gamma}$ there exists a unique fixed point $u^\star\in\mathcal{V}_{q,q}(\bar{T})$ such that $\mathcal{K}_\varphi u^\star = u^\star$. Let u^0 be the fixed point of $\mathcal{K}_{u_0}$ in $\mathcal{V}_{q,q}(\bar{T})$. By (2)-(ii), the fixed point belongs to $\mathcal{V}^{\mathbf{D}}_{q,q}(\bar{T})$. If

$$\mathbb{E}|u^0(\bar{T})|_{-\gamma}^q < \infty, \tag{42}$$

then Step (2) can be repeated. But, by the following calculation we can show (42):

$$\begin{aligned}\mathbb{E}|u^0(\bar{T})|_{-\gamma}^q \le C\Bigg(& |S_{\bar{T}}\varphi|_{-\gamma}^q \\ & +\mathbb{E}\left(\int_0^{\bar{T}} \left|S_{\bar{T}-s} f(u^0(s-))\right|_{-\gamma}\, ds\right)^q + \\ & \sum_{l=1}^n \mathbb{E}\left(\int_{0+}^{\bar{T}}\int_Z |S_{\bar{T}-s} g(u^0(s-);z)|_{-\gamma}^{p^l}\,\nu(dz;ds)\right)^{p^{n-l}}\Bigg).\end{aligned}$$

The first summand is obviously finite, because $\varphi\in\mathbb{V}_{q,-\gamma}$. The second summand is finite, because of (B). In particular

$$\begin{aligned}&\mathbb{E}\left(\int_0^{\bar{T}} \left|S_{\bar{T}-s} f(u^0(s-))\right|_{-\gamma}\, ds\right)^q \le \\ \le\ & C\,\mathbb{E}\left(\int_0^{\bar{T}} (\bar{T}-s)^{-(\delta_f-\gamma)}\left|f(u^0(s-))\right|_{-\delta_f}\, ds\right)^q \\ \le\ & C\left(\int_0^{\bar{T}} (\bar{T}-s)^{-(\delta_f-\gamma)q''}\, ds\right)^{\frac{q}{q''}}\left(1+\mathbb{E}\int_0^{\bar{T}}\left|u^0(s-)\right|^q\, ds\right) \\ \le\ & C(\bar{T})\ \left(1+|||u^0|||_{q,q}^q\right).\end{aligned}$$

In the case of the third summand, we study only the worst case if $l=n$. By the Lipschitz condition of g, i.e. (4), we have

$$\begin{aligned}&\mathbb{E}\int_{0+}^{T_1}\int_Z \left|S_{\bar{T}-s} g(u^0(s-);z)\right|_{-\gamma}^q\,\nu(dz)ds = \int_0^{\bar{T}}\int_Z \mathbb{E}\left|S_{\bar{T}-s} g(u^0(s-);z)\right|_{-\gamma}^q\,\nu(dz)ds \\ \le\ & C\int_0^{\bar{T}} |S_{\bar{T}-s}|^q_{L(V_{-\delta_g},V_{-\gamma})}\,\mathbb{E}\int_Z |g(u^0(s-);z)|_{-\delta_g}^q\,\nu(dz)\,ds \\ \le\ & C\int_0^{\bar{T}} |S_{\bar{T}-s}|^q_{L(V_{-\delta_g},V_{-\gamma})}\,\left(1+\mathbb{E}|u^0(s-)|^q\right)\,ds.\end{aligned}$$

Because of (A), we have $\gamma \geq \max(\delta_f - 1 + \frac{1}{q}, \delta_g)$ and we can write

$$\int_0^{\bar{T}} \int_Z \mathbb{E}|S_{\bar{T}-s} g(u^0(s-); z)|_{-\gamma}^q \, \nu(dz) ds \leq C(\bar{T}) \left(1 + \int_0^{\bar{T}} \mathbb{E}|u^0(s-)|^q \, ds\right).$$

Thus, we have

$$\mathbb{E}|u^0(\bar{T})|_{-\gamma}^q \leq C(\bar{T}) \left(1 + |||u^0|||_{q,q}^q\right).$$

Therefore, there exists a unique fixed point $u^\star$ of $\mathcal{K}_{u^0(\bar{T})}$ in $V_{q,q}^{\mathbf{D}}(\bar{T})$. Let $u^1 := u^\star$. Using the same calculations as above we can show that $\mathbb{E}|u^1(\bar{T})|_{-\gamma}^q < \infty$. Hence, we can repeat the same argument. Since $\bar{T}$ only depends on δ_g, δ_f, p and q, we only need to repeat the procedure a finite number of times. In particular, there exists a constant m, such that $T \leq \bar{T}m$. Let u^i be the unique fixed point of $\mathcal{K}_{u^{i-1}(\bar{T})}$ in $V_{q,q}^{\mathbf{D}}(\bar{T})$. Let

$$u(t) := 1_{t=0}\varphi + \sum_{i=0}^{(m-1)} 1_{(i\bar{T},(i+1)\bar{T}]}(t) u^i(t - i\bar{T}).$$

Since $u^i \in \mathbb{D}([(i-1)\bar{T}, i\bar{T}]; V_{-\delta})$ for all $i = 1, \ldots m-1$, it follows that $u(t) \in \mathbb{D}([0,T]; V_{-\delta})$. Since $\mathcal{K}_{u^{i-1}(\bar{T})} u^i = u^i$ on $[(i-1)\bar{T}, i\bar{T}]$ and $u^i(i\bar{T}) = u^{i+1}(0)$, it follows that $\mathcal{K}_{u_0} u = u$ and therefore that u is a solution of Problem (1).

4.2. **Proof of Part (b) in Theorem 2.1.** Tracing the proof of part a, part b can be shown. Let us define the space for $p < q < \infty$

$$\mathcal{V}_{p,q}(T) := \Big\{ u : \Omega \times [0,T] \to E, \; u \text{ is an adapted process,} \quad \text{such that } \int_0^T \left(\mathbb{E}|u(s)|^p\right)^{\frac{q}{p}} \, ds < \infty \Big\},$$

and for $q = \infty$

$$\mathcal{V}_{p,\infty}(T) := \Big\{ u : \Omega \times [0,T] \to E, \; u \text{ is an adapted process, } u(0) = \varphi, \quad \text{such that } \sup_{0 \leq s \leq T} \mathbb{E}|u(s)|^q < \infty \Big\},$$

equipped with norm

$$|||u|||_{p,q} := \begin{cases} \left[\int_0^T \left(\mathbb{E}|u(s)|^p\right)^{\frac{q}{p}} \, ds\right]^{\frac{1}{q}} & \text{if} \quad p < q < \infty, \\ \sup_{0 \leq s \leq T} \left[\mathbb{E}|u(s)|^q\right]^{\frac{1}{q}} & \text{if} \quad q = \infty. \end{cases}$$

For $p \leq r < \infty$ let $\mathbb{V}_{r,-\gamma}$ be given by definition (29). In the proof we will consider the case $q < \infty$, the case $q = \infty$ is similar. To be precise Part (c) can be proven by tracing the proof of part (b), setting $q = \infty$ and $\gamma = 0$. Let $\delta > 0$ be arbitrary. Let

$$\mathcal{V}_{p,q}^{\mathbf{D}}(T) := \mathcal{V}_{p,q}(T) \cap L^0\left(\Omega; \mathbb{D}([0,T]; V_{-\delta})\right).$$

For $\varphi \in \mathbb{V}_{p,-\gamma}$ let $\mathcal{K}_\varphi : \mathcal{V}^{\mathbf{D}}_{p,q}(T) \to \mathcal{V}^{\mathbf{D}}_{p,q}(T)$ be the same operator as in (30), in particular

$$\begin{aligned}(\mathcal{K}_\varphi u)(t) &= S_t\varphi + \int_{0+}^t S_{t-s} f(u(s-))\, ds + \\ &\quad \int_{0+}^t \int_Z S_{t-s} g(u(s-);z)\, \eta(dz;dz) \\ &:= S_t\varphi + \mathcal{K}_1 u + \mathcal{K}_2 u, \quad t \in [0,T], \quad u \in \mathcal{V}_{p,q}(T).\end{aligned}$$

As before existence and uniqueness of the solution to (1) will be shown by fixed point arguments. But first, we will introduce the conditions on δ_f, δ_σ, δ and γ.

(A) iff $\delta_g p < 1$ and $\delta_f < 1$,
(B) iff $\delta_g p < 1 - \frac{1}{q}$ and $\delta_f < 1 - \frac{1}{q}$,
(C) iff $\gamma < \frac{1}{q}$,
(D) iff $\delta > \max(\delta_f - 1 + \frac{1}{q}, \delta_g + \frac{1}{q}, \frac{1}{q})$.

Analogously to part (a), the proof of part (b) will rely on the following steps:

(1) Firstly, we will show, that the operator $\mathcal{K}_\varphi$ maps $\mathcal{V}^{\mathbf{D}}_{p,q}(T)$ into $\mathcal{V}^{\mathbf{D}}_{p,q}(T)$, in particular we will show that .
   (i) $\mathcal{K}_\varphi$ maps $\mathcal{V}^{\mathbf{D}}_{p,q}(T)$ into $\mathcal{V}_{p,q}(T)$.
   (ii) $\mathcal{K}_\varphi$ maps $\mathcal{V}^{\mathbf{D}}_{q,q}(T)$ into $L^0(\Omega; \mathbb{D}([0,T]; V_{-\delta}))$.
(2) Secondly, we will show that there exists some constant $\bar{T} > 0$, such that for all $\varphi \in \mathbb{V}_{p,-\gamma}$ the operator $\mathcal{K}_\varphi$ has a unique fixed point $x^\star$ in $\bar{\mathcal{V}}^{\mathbf{D}}_{p,q}(\bar{T})$ and $x^\star$ belongs to $L^0(\Omega; \mathbb{D}([0,\bar{T}]; V_{-\delta})$. This will be shown in the following two steps:
   (i) there exists a constant $\bar{T} > 0$ and a constant $0 < k < 1$, such that for all $\varphi \in \mathbb{V}_{p,-\gamma}$ the operator $\mathcal{K}_\varphi : \mathcal{V}_{p,q}(\bar{T}) \to \mathcal{V}_{p,q}(\bar{T})$ is Lipschitz continuous with constant k.
   (ii) For all $\varphi \in \mathbb{V}_{p,-\gamma}$ the sequence $\{x^{(n)}\}_{n\in\mathbf{N}}$, defined by $x^{(n)} = \mathcal{K}_\varphi x^{(n-1)}$, $n \geq 1$ and $x^{(0)}(t) = S_t\varphi$, is tight in $L^0(\Omega; \mathbb{D}([0,\bar{T}]; V_{-\delta})$.

   Applying the Banach fixed point theorem, there exists a unique $x^* \in \mathcal{V}_{p,q}(\bar{T})$, such that $\mathcal{K}_\varphi x^* = x^*$ and ${\mathcal{K}_\varphi}^{(n)} y \to x^*$ for all $y \in \mathcal{V}_{q,q}(\bar{T})$. From (2)-(ii), it follows that the fixed point x^* belongs to $L^0(\Omega; \mathbb{D}([0,\bar{T}]; V_{-\delta})$, i.e. to $\mathcal{V}^{\mathbf{D}}_{p,q}(\bar{T})$.
(3) Using the same arguments as in (a)-(3), step (2) can be repeated to get a solution on the whole interval $[0,T]$.

Claim 4.3. *Under assumptions (A), (B) and (C) and for all $\varphi \in \mathbb{V}_{p,-\gamma}$ the operator $\mathcal{K}_\varphi$: $V^{\mathbf{D}}_{p,q} \to V_{p,\infty,-\gamma}$ is bounded and Lipschitz continuous. In particular, there exists some constant $C_1, C_2 < \infty$, such that*

$$\begin{aligned}&|||\mathcal{K}_\varphi u|||_{p,\infty,-\gamma} \leq \\ &\qquad C_1\, |||u|||_{p,q} + C_2\, \|\varphi\|_{p,-\gamma}, \quad u \in \mathcal{V}^{\mathbf{D}}_{p,q}(T), \varphi \in \mathbb{V}_{p,-\gamma},\end{aligned} \tag{43}$$

and there exists a constant $C<\infty$, such that

$$|||\mathcal{K}_\varphi u-\mathcal{K}_\varphi v|||_{p,\infty,-\gamma}\le \quad C\,|||u-v|||_{p,q},\quad u\in\mathcal{V}^{\mathbf{D}}_{p,q}(T),\varphi\in\mathbb{V}_{p,-\gamma}. \tag{44}$$

Proof. Inequality (43) follows from the same sequence of calculations as (31). In particular,

$$\begin{aligned}\mathbb{E}\,|\mathcal{K}_\varphi u(t)|^p_{-\gamma} &\le \mathbb{E}|S_t\varphi|^p_{-\gamma}+\mathbb{E}\int_0^t|S_{t-s}f(u(s-))\,ds|^p_{-\gamma}\\ &\quad+\mathbb{E}\left|\int_{0+}^t\int_Z S_{t-s}g(u(s-);z)\,\eta(ds,dz)\right|^p_{-\gamma}\\ &\le C\,\mathbb{E}|\varphi|^p_{-\gamma}+\mathbb{E}\left(\int_{0+}^t(t-s)^{-\delta_f+\gamma}\,|f(u(s-))|_{-\delta_f}\,ds\right)^p\,dt\\ &\quad+\int_{0+}^t\int_Z(t-s)^{(-\delta_g+\gamma)p}\mathbb{E}\,|g(u(s-);z)|^p_{-\delta_g}\;\nu(dz)\,ds\\ &\le C\,\mathbb{E}|\varphi|^p_{-\gamma}+\mathbb{E}\left(\int_{0+}^t(t-s)^{-\delta_f+\gamma}\,|u(s-)|\;ds\right)^p\,dt\\ &\quad+\int_{0+}^t(t-s)^{(-\delta_g+\gamma)p}\mathbb{E}\,|u(s-)|\;ds.\end{aligned}$$

The Hölder inequality gives

$$\begin{aligned}\mathbb{E}\,|\mathcal{K}_\varphi u(t)|^p &\le C\,\mathbb{E}|\varphi|^p_{-\gamma}+\left(\int_{0+}^t(t-s)^{-p'(\delta_f-\gamma)}\,ds\right)^{\frac{p}{p'}}\int_0^t\mathbb{E}\,|u(s-)|^p\;ds\\ &\quad+\int_0^t(t-s)^{(\gamma-\delta_g)p}\mathbb{E}\,|u(s-)|^p\;\;ds\\ &\le C\,\mathbb{E}|\varphi|^p_{-\gamma}+C\,t^{p(1-\frac1r-(\delta_f-\gamma))}\left(\int_0^t(\mathbb{E}\,|u(s-)|^p)^{\frac{q}{p}}\;ds\right)^{\frac{p}{q}}\\ &\quad+t^{p(1-\frac1r-(\delta_g-\gamma)p)}\left(\int_{0+}^t(\mathbb{E}\,|u(s-)|^p)^{\frac{q}{p}}\;ds\right)\\ &\le C(t)|||u|||^p_{q,q}+C\,\mathbb{E}|\varphi|^p_{-\gamma},\end{aligned}$$

where q satisfies $(\delta_g-\gamma)p<1-\frac1q$ and $\delta_f-\gamma<1-\frac1q$. Inequality (44) follows from similar calculations. □

Claim 4.4. *Assume the conditions (A), (B), (C) and (D) are fulfilled. Then, there exists a constant $C_1,C_2,C_3<\infty$ such that for all $\varphi\in\mathbb{V}_{q,-\gamma}$ and all $u\in\mathcal{V}_{q,q}(T)$ we have*

$$\mathbb{E}\left[\sup_{0\le t\le T}|\mathcal{K}_\varphi u(t)|_{-\delta}\right]\le C_1\,\|\phi\|_{q,-\gamma}+C_2\,|||u|||_{p,p}+C_3\,|||u|||_{q,q}.$$

Proof. The proof is similar to the proof of Claim 4.2. We have

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le t\le T}|\mathcal{K}_\varphi u(t)|_{-\delta}\right] &= \mathbb{E}\sup_{0\le t\le T}|S_t\varphi|_{-\delta}+\\ &\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t S_{t-s}f(u(s-))\,ds\right|_{-\delta}+\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t\int_Z g(u(s-);z)\eta(dz;ds)\right|_{-\delta}\\ &+\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t S_{t-s}A\int_0^s\int_Z g(u(r-);z)\eta(dz;dr)\,ds\right|_{-\delta}\\ =:&\quad I+II+III+IV.\end{aligned}$$

Since $\gamma<\delta$, we have for the first term

$$I \quad\le\quad C\,\mathbb{E}\sup_{0\le t\le T}|\varphi|_{-\delta}\le\mathbb{E}\,|\varphi|_{-\gamma}\,.$$

The Minkowski inequality yields for $\epsilon=\delta-\delta_f$

$$\begin{aligned}II &\le \mathbb{E}\sup_{0\le t\le T}\int_0^t|S_{t-s}f(u(s-))|_{-\delta}\;ds\\ &\le \mathbb{E}\sup_{0\le t\le T}\int_0^t(t-s)^{-\epsilon}\,|f(u(s-))|_{-\delta-\epsilon}\;ds.\end{aligned}$$

Let $\epsilon_1,\epsilon_2>0$ and $\epsilon_1+\epsilon_2=\epsilon$. Then, the Lipschitz property of f and the Hölder inequality lead to

$$\begin{aligned}II &\le \mathbb{E}\sup_{0\le t\le T}\int_0^t(t-s)^{-\epsilon}\,|u(s-)|_{-\delta-\epsilon+\delta_f}\;ds\\ &\le \mathbb{E}\sup_{0\le t\le T}C\,T^{1-\frac1p-\epsilon_1}\left(\int_0^t(t-s)^{-\epsilon_1 p}\,|u(s-)|^q_{-\delta-\epsilon+\delta_f}\;ds\right)^{\frac1p}\\ &\le C\,T^{1-\frac1p-\epsilon_1}\mathbb{E}\left(\int_0^T(t-s)^{-\epsilon_1 p}\,|u(s-)|^p_{-\delta-\epsilon+\delta_f}\;ds\right)^{\frac1q}.\end{aligned}$$

The Jensen inequality, the Hölder inequality and the fact that $\delta>\delta_f-1+\frac1q$ give

$$\begin{aligned}II &\le C\,T^{1-\frac1p-\epsilon_1}\left(\int_0^T(t-s)^{-\epsilon_1 p}\mathbb{E}\,|u(s-)|^q_{-\delta-\epsilon+\delta_f}\;ds\right)^{\frac1p}\\ &\le C\,T^{1-\frac1q-\epsilon}\left(\int_0^T\left(\mathbb{E}\,|u(s-)|^p_{-\delta-\epsilon+\delta_f}\right)^{\frac qp}\;ds\right)^{\frac1q}\le C\,T^{1-\frac1q-\epsilon}\,|||u|||_{p,q}.\end{aligned}$$

The Burkholder inequality gives for the third term

$$III \quad\le\quad \left(\int_0^T\int_Z\mathbb{E}\,|g(u(s-);z)|^p_{-\delta}\,\nu(dz)\,ds\right)^{\frac1p}.$$

The Lipschitz property of g gives

$$III \quad\le\quad \mathbb{E}\left(\int_0^T|u(s-;z)|^p_{-\delta+\delta_g}\;ds\right)^{\frac1p}\le C\,|||u|||_{p,p}\le C\,|||u|||_{p,q}.$$

Let $\epsilon = 1 + \delta_g - \delta$. The Minkowski inequality gives for the fourth term

$$\begin{aligned} IV &\leq \mathbb{E} \sup_{0\le t\le T} \int_0^t \left| S_{t-s} A \int_0^s \int_Z g(u(r-);z)\eta(dz;dr) \right|_{-\delta} ds \\ &\leq \mathbb{E} \sup_{0\le t\le T} \int_0^t (t-s)^{-\epsilon} \left| A \int_0^s \int_Z g(u(r-);z)\eta(dz;dr) \right|_{-\delta-\epsilon} ds \end{aligned}$$

Let $\epsilon_1, \epsilon_2 > 0$ and $\epsilon_1 + \epsilon_2 = \epsilon$. The Hölder inequality gives for $p' = \frac{p}{p-1}$

$$IV \leq \mathbb{E} \sup_{0\le t\le T} \left(\int_0^t (t-s)^{-p'\epsilon_1} \right)^{\frac{1}{q'}} \left(\int_0^t (t-s)^{-p\epsilon_2} \left| \int_0^s \int_Z g(u(r-);z)\eta(dz;dr) \right|^p_{-\delta+1-\epsilon} ds \right)^{\frac{1}{p}}.$$

The Jensen, the Burkholder inequality and the Lipschitz property of g give

$$\begin{aligned} IV \leq & \\ & C\, T^{1-\frac{1}{p}-\epsilon_1} \mathbb{E} \left(\int_0^T (T-s)^{-p\epsilon_2} \left| \int_0^s \int_Z g(u(r-);z)\eta(dz;dr) \right|^p_{-\delta+1-\epsilon} ds \right)^{\frac{1}{p}} \\ \leq\ & C\, T^{1-\frac{1}{p}-\epsilon_1} \left(\int_0^T (T-s)^{-p\epsilon_2} \int_0^s \int_Z \mathbb{E}\, |g(u(r-);z)|^p_{-\delta+1-\epsilon}\, \nu(dz) dr\, ds \right)^{\frac{1}{p}}. \\ \leq\ & C\, T^{1-\frac{1}{p}-\epsilon_1} \left(\int_0^T (T-s)^{-p\epsilon_2} \int_0^s \mathbb{E}\, |u(r-)|^p_{-\delta+1-\epsilon+\delta_g}\, dr\, ds \right)^{\frac{1}{p}}. \end{aligned}$$

Again the Hölder inequality yields to

$$\begin{aligned} IV \leq & \\ \leq\ & C\, T^{1-\frac{1}{q}-\epsilon} \left(\int_0^T \int_0^s \left(\mathbb{E}\, |u(r-)|^p_{-\delta+1-\epsilon+\delta_g} \right)^{\frac{q}{p}} dr\, ds \right)^{\frac{1}{q}} \leq C\, |||u|||_{p,q}. \end{aligned}$$

Collecting all together gives the assertion. □

Proof of part (1)-(i): In this section we will show, that under conditions (A) and (C) and for all $\varphi \in \mathbb{V}_{p,-\gamma}$ the operator $\mathcal{K}_\varphi : \mathcal{V}^{\mathbf{D}}_{p,q}(T) \to \mathcal{V}_{p,q}(T)$ is a bounded operator. Note we have for $0 < \gamma < \frac{1}{q}$

$$|||S_t\varphi|||^q_{p,q} = \int_0^T \left[\mathbb{E}|S_t\varphi|^p\right]^{\frac{q}{p}} \leq \int_0^T |S_t|^q_{L(V_{-\gamma},E)}\, dt\, \left[\mathbb{E}|\varphi|^p_{-\gamma}\right]^{\frac{q}{p}}.$$

Using Remark B.1 we have

$$\begin{aligned} |||S_t\varphi|||^q_{p,q} &\leq \int_0^T C\, t^{-q\gamma}\, dt\, \left[\mathbb{E}|\varphi|^p_{-\gamma}\right]^{\frac{q}{p}}. \\ &\leq C\, T^{1-q\gamma} \left[\mathbb{E}|\varphi|^p_{-\gamma}\right]^{\frac{q}{p}}. \end{aligned} \tag{45}$$

Let $c_1, c_2 \geq 0$ such that $c_1 + c_2 = 1$ and p' conjugate to p. Then we have by the Hölder inequality

$$
\begin{aligned}
|||\mathcal{K}_1 u|||_{p,q}^q \leq & \int_0^T \left[\mathbb{E} \left| \int_{0+}^t S_{t-s} f(u(s-))\, ds \right|^p \right]^{\frac{q}{p}} dt \\
\leq & \int_0^T \left[\mathbb{E} \left(\int_{0+}^t |S_{t-s} f(u(s-))|\, ds \right)^p \right]^{\frac{q}{p}} dt \\
\leq & \int_0^T \left[\mathbb{E} \left(\int_{0+}^t (t-s)^{-(c_1+c_2)\delta_f} |f(u(s-))|_{-\delta_f}\, ds \right)^p \right]^{\frac{p}{q}} dt \\
\leq & \int_0^T \left[\left(\int_0^t (t-s)^{-c_1 p' \delta_f}\, ds \right)^{\frac{p}{p'}} \mathbb{E} \left(\int_0^t (t-s)^{-c_2 \delta_f p} |f(u(s-))|_{-\delta_f}^p\, ds \right) \right]^{\frac{q}{p}} dt.
\end{aligned}
$$

A second application of the Hölder inequality yields

$$
\begin{aligned}
& |||\mathcal{K}_1 u|||_{p,q}^q \leq \\
& \leq C\, T^{\left(1-c_1\delta_f p'\right)\frac{q}{p'}} \int_0^T \left[\left(\int_0^t (t-s)^{-c_2\delta_f p} \mathbb{E}\, |f(u(s-))|_{-\delta_f}^p\, ds \right) \right]^{\frac{q}{p}} dt.
\end{aligned}
$$

Therefore, by the Young inequality we infer

$$
\begin{aligned}
& |||\mathcal{K}_1 u|||_{p,q}^q \leq \\
& \leq C\, T^{\left(1-c_1\delta_f p'\right)\frac{q}{p'}} \left(\int_0^T (T-s)^{-c_2\delta_f p}\, ds \right)^{\frac{q}{p}} \left(\int_0^T \left[\mathbb{E}\, |f(u(s-))|_{-\delta_f}^p \right]^{\frac{q}{p}} ds \right) \\
& \leq C\, T^{\left(\frac{1}{p'}+\frac{1}{p}-(c_1+c_2)\delta_f\right) q} \left(\int_0^T \left[(1+\mathbb{E}\, |u(s-))|^p) \right]^{\frac{q}{p}} ds \right).
\end{aligned}
$$

Next, the Lipschitz condition of f, i.e. (2), implies that

$$
|||\mathcal{K}_1 u|||_{p,q}^q \leq \;\leq\; C\, T^{\left(1-\delta_f\right) q} \left(|||u|||_{p,q}^q \right). \tag{46}
$$

By Proposition 3.1 we get

$$
\begin{aligned}
|||\mathcal{K}_2 u|||_{p,q}^q = & \int_0^T \left[\mathbb{E} \left| \int_{0+}^t \int_Z S_{t-s} g(u(s-); z)\, \eta(ds, dz) \right|^p \right]^{\frac{q}{p}} dt \\
\leq & \; C \int_0^T \left[\int_0^t \int_Z \mathbb{E} |S_{t-s} g(u(s-); z)|^p \nu(dz)\, ds \right]^{\frac{q}{p}} dt
\end{aligned}
$$

Remark B.1 yields to

$$
|||\mathcal{K}_2 u|||_{p,q}^q \;\leq\; C \int_0^T \left[\int_0^t (t-s)^{-\delta_g p} \int_Z \mathbb{E} |g(u(s-); z)|_{-\delta_g}^p \nu(dz)\, ds \right]^{\frac{q}{p}} dt.
$$

Again, the Young inequality yields

$$
|||\mathcal{K}_2 u|||_{p,q}^q \;\leq\; C \left(\int_0^T s^{-\delta_g p}\, ds \right)^{\frac{q}{p}} \left(\int_0^T \left(\int_Z \mathbb{E} |g(u(s-); z)|_{-\delta_g}^p \nu(dz) \right)^{\frac{q}{p}} ds \right).
$$

The Lipschitz condition of g, i.e. (4), leads to

$$
\begin{aligned}
|||\mathcal{K}_2 u|||_{p,q}^q &\leq C\, T^{(1-p\delta_g)\frac{q}{p}} \left(\int_0^T (1+\mathbb{E}|u(s)|^p)^{\frac{q}{p}}\; ds \right) \\
&\leq C\, T^{(1-p\delta_g)\frac{q}{p}} \left(1+|||u|||_{p,q}^q\right). \qquad (47)
\end{aligned}
$$

Summing up, we have proved that the operator maps $\mathcal{V}_{p,q}(T)$ into $\mathcal{V}_{p,q}(T)$, i.e.

$$
|||\mathcal{K}_\varphi u|||_{p,q} \leq C_1(T)|u_0|_{-\gamma} + C_2(T)|||u|||_{q,q} + C_3(T). \qquad (48)
$$

Remark 4.1. *Note, that using the Lipschitz conditions (2) and (4) the equation (48) can be written as*

$$
|||\mathcal{K}_\varphi u|||_{p,q} \leq C_1\, T^{\frac{1}{q}-\gamma}\mathbb{E}|u_0|_{-\gamma} + C_2\, T^\rho |||u|||_{q,q}.
$$

where $\rho = \min(1-\delta_f, \frac{1}{p}-\delta_g)$.

Proof of part (1)-(ii). Fix $u \in \mathcal{V}_{p,q}^{\mathbf{D}}(T)$. We have to show that $\mathcal{K}_\varphi u$ belongs to the Skorohod space $\mathbb{D}([0,T];V_{-\delta})$, in particular that $\{\mathcal{K}_\varphi u\}$ satisfies the Aldou's condition A.2 and the compact containment condition A.3 in Appendix A. Since $V_{-\tilde{\delta}}$ is compactly embedded in $V_{-\delta}$ for all $\tilde{\delta} < \delta$, Claim (4.4) implies the compact containment condition. The Aldou's condition will be shown by proving the assumptions of Theorem A.1 with $\beta = p$. In particular, we will prove that there exists some $\rho > 0$ such that for all $0 \leq t \leq T$ and all θ we have

$$
\begin{aligned}
&\mathbb{E}\left[|\mathcal{K}_\varphi u(t+\theta) - \mathcal{K}_\varphi u(t)|_{-(\gamma+\delta)}^p \mid \mathcal{F}_t \right] \leq \\
&\qquad \theta^\rho |||1_{(t,t+\theta]} u|||_{p,q}^p + C_2 \theta^{p\delta} |||1_{(0,t]}\mathcal{K}_\varphi u|||_{p,\infty}^p. \qquad (49)
\end{aligned}
$$

Since (43) holds, there exists some $C < \infty$ such that $2C\, \left(\theta^\rho + \theta^\delta\right)$ is an upper bound for the RHS of (49). Moreover, $2C\, \left(\theta^\rho + \theta^\delta\right) \to 0$ as $\theta \to 0$, the assumptions of Lemma A.1 are satisfied.

To prove inequality in (49), we first note that for $r < q$ such that $1-\delta_f \leq \frac{1}{r}$ and $\frac{1}{p} - \delta_g \leq \frac{1}{r}$ we have

$$
\begin{aligned}
&\mathcal{K}_\varphi u(t+\theta) - \mathcal{K}_\varphi u(t) = (S_\theta - I)\left(\mathcal{K}_\varphi u\right)(t+\theta) \\
&\quad + \underbrace{\int_t^{t+\theta} S_{t+\theta-\sigma} f(u(\sigma-))\, d\sigma}_{=:f_1(t,\theta)} + \underbrace{\int_t^{t+\theta} S_{t+\theta-\sigma} g(u(\sigma-);z)\, \eta(dz;d\sigma)}_{=:f_2(t,\theta)}.
\end{aligned}
$$

Using the same calculation as for (40) we have

$$
\mathbb{E}\left|(S_\theta - I)\left(\mathcal{K}_\varphi u\right)(t+\theta)\right|_{-(\gamma+\delta)}^p \leq \theta^{p\delta}\mathbb{E}\left|\left(\mathcal{K}_\varphi u\right)(t+\theta)\right|_{-\gamma}^p.
$$

Let $c_1, c_2 \geq 0$, $c_1 p'(\delta_f - \gamma) < 1$ and p' conjugate to p. Then, we obtain by the Hölder inequality

$$\begin{aligned}
\mathbb{E}\,|f_1(t,\theta)|^p_{-(\gamma+\delta)} &\leq \\
& C\,\mathbb{E}\left(\int_0^\theta (\theta-\sigma)^{-(\delta_f-(\gamma+\delta))}\,|f(u(t+\sigma-))|_{-\delta_f}\ d\sigma\right)^p \\
\leq\ & C\ \left(\int_0^\theta (\theta-\sigma)^{-c_1 p'(\delta_f-(\gamma+\delta))}\,d\sigma\right)^{\frac{p}{p'}} \\
& \times\left(\int_0^\theta (\theta-\sigma)^{-c_2 p(\delta_f-(\gamma+\delta))}\,\mathbb{E}\,|f(u(t+\sigma))|^p_{-\delta_f}\ d\sigma\right).
\end{aligned}$$

The Hölder inequality yields for r' conjugate to $\frac{q}{r}$

$$\begin{aligned}
\mathbb{E}\,|f_1(t,\theta)|^p_{-(\gamma+\delta)} &\leq C\,\theta^{p(\frac{1}{p'}-c_1(\delta_f-(\gamma+\delta)))}\,\theta^{\frac{1}{r'}-c_2 p(\delta_f-(\gamma+\delta))} \\
&\times\left[\int_0^\theta \left(\mathbb{E}\,|f(u(t+\sigma))|^p_{-\delta_f}\right)^{\frac{r}{p}}\ d\sigma\right]^{\frac{p}{r}}.
\end{aligned}$$

The growth condition on f, i.e. (2) yields

$$\begin{aligned}
\mathbb{E}\,|f_1(t,\theta)|^p_{-(\gamma+\delta)} &\leq C\,\theta^{p(1-\frac{1}{r}(\delta_f-(\gamma+\delta)))}\left[\int_0^\theta \left(1+\mathbb{E}\,|u(t+\sigma)|^p\right)^{\frac{r}{p}}\ d\sigma\right]^{\frac{p}{r}} \\
&\leq C\,\theta^{p(1-\frac{1}{r}(\delta_f-(\gamma+\delta)))}\left[1+|||u1_{[t,t+\theta]}|||^p_{p,r}\right].
\end{aligned}$$

Analogously to the proof of (a)-(1)-(ii) we have

$$\mathbb{E}\,|f_2(t,\theta)|^p_{-(\gamma+\delta)} \leq C\,\mathbb{E}\left|\int_0^\theta\int_Z S_{\theta-\sigma}g(u((\sigma-)+t);z)\,\eta\circ\theta_t(dz;d\sigma)\right|^p_{-(\gamma+\delta)},$$

where $\eta\circ\theta_t(\omega) := \eta(\theta_t\circ\omega)$, and θ_t is the usual shift operator defined by $\theta_t\omega(s) := \omega(t+s)$. Following the calculations of (b)-(1)-(i) we get

$$\mathbb{E}\,|f_2(t,\theta)|^p_{-(\gamma+\delta)} \leq C\,\theta^{p\left(\frac{1}{p}-\frac{1}{r}-(\delta_g-(\gamma+\delta))\right)}\left[1+|||u1_{[t,t+\theta]}|||^p_{p,r}\right].$$

Proof of part (2)-(i) Next, we will show that there exists some function $C:[0,1]\to\mathbb{R}^+$ such that

$$|||\mathcal{K}_\varphi u-\mathcal{K}_\varphi v|||_{p,q} \leq C(T)\,|||u-v|||_{p,q},\quad u,v\in\mathcal{V}^{\mathbb{D}}_{p,q}(T),\quad \varphi\in\mathbb{V}_{p,-\gamma}$$

and $C(T)\to 0$ as $T\to 0$. Thus, we can find a $\bar{T}>0$ such that $C(\bar{T})<1$. Thanks to remark (4.1) we have

$$|||\mathcal{K}_\varphi u|||_{p,q} \leq C_1\,T^{\frac{1}{q}-\gamma}|\varphi|_{-\gamma}+C_2\,T^\rho|||u|||_{q,q}.$$

where $\rho=\min(1-\delta_f,\frac{1}{p}-\delta_g)$ and the assertion follows.

Proof of part (2)-(ii) Let $\left\{u^{(n)}\right\}_{n\geq 0}$ be the sequence defined by $u^{(n)} := \mathcal{K}_\varphi u^{(n-1)}$ and $u^{(0)}(t)=S(t)\varphi$ and let $u^\star$ the fixed point of $\mathcal{K}_\varphi$. We have to show, that $\left\{u^{(n)}\right\}_{n\geq 1}$ is tight in $L^0(\Omega;\mathbb{D}([0,\bar{T}];V_{-(\gamma+\delta)}))$. But first we show the following:

- Under the assumptions (A) and (C), the set $\left\{u^{(n)}\right\}_{n\geq 0}$ is bounded in $V_{q,q}(\bar{T})$.

- If the assumptions (A), (B) and (C) are satisfied, then it follows from (43) that for all $\varphi \in \mathbb{V}_{p,-\gamma}$

$$\begin{aligned}\sup_{n\ge 1} |||u^{(n)}|||_{p,\infty,-\delta} &\le \sup_{n\ge 1} |||u^{(n)}|||_{p,\infty,-\gamma} \\ &\le \sup_{n\ge 0} |||\mathcal{K}_\varphi u^{(n)}|||_{p,\infty,-\gamma} \\ &\le C \sup_{n\ge 0} |||u^{(n)}|||_{p,q} + \|\varphi\|_{p,-\gamma}.\end{aligned}$$

- Since for all $\varphi \in \mathbb{V}_{p,-\gamma}$ we have $|||u^{(n)} - u^\star|||_{p,q} \to 0$ as $n \to \infty$ it follows from (44) that the sequence $\{\mathcal{K}_\varphi u^{(n)}\}_{n\ge 1}$ converges to $\mathcal{K}_\varphi u^\star = u^\star$ in $\mathcal{V}_{p,\infty,-\gamma}(T)$, i.e.

$$\sup_{0\le t\le T} \mathbb{E}\left|u^{(n)}(t) - u(t)\right|^p_{-\delta} \to 0$$

as $n \to \infty$. Therefore, the Chebyscheff inequality shows, that for any finite set $\{t_1, \ldots, t_k\} \subset [0, \bar{T}]$, we have

$$\left(u^{(n)}(t_1), \ldots, u^{(n)}(t_k)\right) \to \left(u(t_1), \ldots, u(t_k)\right),$$

as $n \to \infty$.

By Theorem 3.7.8 of Ethier and Kurtz [23] it remains to show, that the set $\{u^{(n)}, n \in \mathbb{N}\}$ is tight in $L^0(\Omega; \mathbb{D}([0,\bar{T}]; V_{-\gamma-\delta}))$. This implies that $u^{(n)}$ converges in distribution to $u^\star$ and $u^\star \in L^0(\Omega; \mathbb{D}([0,\bar{T}]; V_{-\gamma-\delta}))$. **Proof that the Aldou's condition is satisfied:** Tracing the calculation in (b)-(1)-(ii), we can see that there exists some $\rho > 0$ such that for all $n \in \mathbb{N}$

$$\mathbb{E}\left[\left|u^{(n)}(t+\theta) - u^{(n)}(t)\right|^p_{-(\gamma+\delta)} \mid \mathcal{F}_t^{(n)}\right] \le$$
$$\theta^\rho |||1_{(t,t+\theta]} u^{(n-1)}|||^p_{p,q} + C_2 \theta^{p\delta} |||1_{(0,t]} \mathcal{K}_\varphi u^{(n-1)}|||^p_{p,\infty,-\gamma},$$

But (43) implies, that there exists some constant $C < \infty$ such that the RHS is bounded by $C\left(\theta^\rho + \theta^{p\delta}\right)$. Therefore, the assumptions of Lemma A.1 are satisfied. **Proof that the compact containment condition is satisfied:** In order to prove the compact containment condition we use again the fact that the embedding $V_{\tilde{\delta}} \hookrightarrow V_{-\delta}$ is compact for all $\tilde{\delta} < \delta$ (see Remark B.2). Claim 4.4 implies

$$\sup_{n\ge 1} \mathbb{E}|\sup_{0\le s\le \bar{T}} u^{(n)}(s)|_{-\tilde{\delta}} \le C\left(|||u^{(n-1)}|||^q_{q,q} + \|\varphi\|_{q,-\gamma}\right) \quad n \in \mathbb{N}.$$

Since the set $\{u^{(n)} \mid n \in \mathbb{N}\}$ is bounded in $\mathcal{V}_{q,q}(T)$, i.e. there exists some $C < \infty$ such that $|||u^{(n)}|||_{q,q} \le C$, the compact containment condition follows.

Proof of part (3) This part can be shown by the same consideration as in (a)-(3).

5. A.s. Regularity Results - Proof of Remark 2.2

Let Z be a separable Banach space, $\mathcal{Z}$ the Borel-σ algebra and $\eta : \mathcal{Z}\hat{\times}\mathcal{B}(\mathbb{R}^+) \to \mathbb{R}^+$ a Poisson random measure on $Z \times \mathbb{R}^+$ with characteristic measure $\nu : \mathcal{Z} \to \mathbb{R}^+ \in L^{sym}(Z)$. Let $N_t(A)$, $A \in \mathcal{Z}$, be the counting process defined on page 1503. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $(\mathcal{F}_t)_{t\ge 0}$ be the right continuous filtration induced by η. That means, the smallest filtration, such that the counting measure $N_t(A)$ is $\mathcal{F}_t$-measurable for all $s \le t \le T$ and $A \in \mathcal{Z}$. Let $Z_0 = \{z \in Z \mid |z| \le 1\}$,

$$\mathcal{F}_t^0 := \sigma\left(N_s(A);\ A \subset Z_0, 0 \le s \le t\right),$$

and

$$\mathcal{F}_t^C := \sigma\left(N_s(A);\ A \subset Z \setminus Z_0, 0 \le s \le t\right).$$

Since for $A, B \in \mathcal{Z}$, $A \cap B = \emptyset$, the random variables $\eta(A \times (s_1, s_2])$ and $\eta(A \times (s_1, s_2])$ are independent for all $0 < s_1 \le s_2 \le T$, the two filtration $\mathcal{F}_t^0$ and $\mathcal{F}_t^C$ are independent. Let us define the two probability spaces $(\Omega_0, \mathcal{F}_0, \{\mathcal{F}_t\}_t, \mathbb{P}_0)$ and $(\Omega_C, \mathcal{F}_C, \{\mathcal{F}_t^C\}_t, \mathbb{P}_C)$, where $\Omega_0 := \Omega$, $\Omega_C := \Omega$, $\mathcal{F}_0 := \wedge_t \mathcal{F}_t^0$, $\mathcal{F}_C := \wedge_t \mathcal{F}_t^C$, $\mathbb{P}_0(\,\cdot\,) := \mathbb{P}(\,\cdot\,|\mathcal{F}_C)$, and $\mathbb{P}_C(\,\cdot\,) := \mathbb{P}(\,\cdot\,|\mathcal{F}_0)$. From the independence of $\mathcal{F}_0$ and $\mathcal{F}_C$ it follows, that $\mathbb{P}$ is the product of $\mathbb{P}_0$ and $\mathbb{P}_C$.

Let E be a separable Banach space of M type p, $1 < p \le 2$. Further we assume that the mapping g and f satisfies the hypothesis of Theorem 2.1-(c) with δ_f and δ_g. We consider the following SPDE

$$\left\{\begin{array}{rcl} u(t)\,dt & = & (Au(t) + f(u(t-)))\ dt \\ & & + \int_{Z^0} g(u(t-); z)\eta(dz; dt), \quad t \ge 0 \\ u(0) & = & u_0 \in E, \end{array}\right. \tag{50}$$

Given $\delta > 0$, by Theorem 2.1–(c) it follows that Problem (50) has a unique solution $u^{(1)}$ belonging to

$$L^0\left(\Omega^0; \mathbb{D}((0,T]; V_{-\delta})\right)$$

such that $\sup_{0\le s\le T} \mathbb{E}|u(s)|^p < \infty$. Let τ_1 be the following $\mathcal{F}_t^C$–stopping time

$$\tau_1 := \inf_{t>0} \{N_t(Z \setminus Z_0) > 0\}$$

and $\bar\tau_1 = \tau_1 \wedge T$. Note, τ_1 is an exponential distributed random variable over $(\Omega_C, \mathcal{F}_C, \mathbb{P}_C)$ with parameter $C_\nu := \nu(Z \setminus Z_0)$, independent from $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$. Let $\bar u^{(1)}(t) = u^{(1)}(t)$ for $0 \le t < \bar\tau_1$ and $u^{(1)}(t) = 0$ for $t \ge \bar\tau_1$. Next, $\mathbb{E}|u^{(1)}(\bar\tau_1)|^p$ can be written as

$$\mathbb{E}|u^{(1)}(\bar\tau_1)| \;=\; \mathbb{E}\left(\mathbb{E}\left[|u^{(1)}(s)|^p \mid \bar\tau_1 = s\right]\right).$$

Since τ_1 is independent from $(\Omega_0; \mathcal{F}_0; \mathbb{P}_0)$ we have

$$\begin{aligned} \mathbb{E}|u^{(1)}(\bar\tau_1)| &= \int_0^{\bar\tau_1} \mathbb{P}^C\left(\bar\tau_1 = s\right) \mathbb{E}_0\, |u^{(1)}(s)|^p\, ds \\ &= \int_0^T C_\nu e^{-sC_\nu}\, ds \sup_{0\le s\le T} \mathbb{E}_0 |u^{(1)}(s)|^p + e^{-TC_\nu}\mathbb{E}^0\, |u^{(1)}(T)|^p, \end{aligned} \tag{51}$$

where $\mathbb{E}_0$ denotes the expectation with respect to the measure $\mathbb{P}_0$. The underlying Poisson random measure is time homogeneous, therefore we can introduce the shift operator. We assume that a quadruple

$$\mathfrak{T} = (\Omega, \mathcal{F}, \mathbb{P}, \vartheta),$$

is given where $(\Omega,\mathcal{F},\mathbb{P},)$ is a probability space and $\vartheta:\mathbb{R}\times\Omega\ni(t,\omega)\mapsto\vartheta_t\omega\in\Omega$ is a measurable map such that for all $t,s\in\mathbb{R}$, $\vartheta_{t+s}=\vartheta_t\circ\vartheta_s$. Let $u^{(2)}$ be a solution to

$$\left\{\begin{array}{rcl} du(t) &=& (Au(t)+f(u(t-)))\ dt+ \\ && \int_{Z^0} g(z;u(t-))\,(\eta\circ\vartheta_{\bar\tau_1})\,(dz;dt),\ t\ge 0, \\ u(0) &=& u^{(1)}(\bar\tau_1-)+\int_Z g(z;u^{(1)}(\bar\tau_1-))\,\eta(dz;\{\bar\tau_1\}). \end{array}\right. \tag{52}$$

By the calculation (51) the assumptions of Theorem 2.1-(c) are satisfied. Therefore the solution $u^{(2)}$ exists, is unique and belongs to

$$L^0(\Omega^0;\mathbb{D}([0,T-\bar\tau_1];V_{-\delta})).$$

Let τ_2 be the $\mathcal{F}_t^C$–stopping time

$$\tau_2:=\inf_{t>\tau_1}\{N_t(Z\setminus Z_0)>1\}$$

and $\bar\tau_2=\tau_2\wedge T$. Let us define $\bar u^{(2)}$ by

$$\bar u^{(2)}(t):=\left\{\begin{array}{rll} \bar u^{(1)}(t), & \text{for} & 0\le t<\bar\tau_1, \\ u^{(2)}(t-\bar\tau_1), & \text{for} & \bar\tau_1\le t<\bar\tau_2. \end{array}\right.$$

Obviously, we have $\bar u^{(2)}\in L^0(\Omega^0;\mathbb{D}([0,\bar\tau_2);V_{-\delta}))$. Repeating the step and taking into account that $\nu(Z\setminus Z_0)<\infty$, we get a countable set of stopping times $\{\bar\tau_n\mid n\in\mathbb{N}\}$, where

$$\tau_n:=\inf_{t>\tau_{n-1}}\{N_t(Z\setminus Z_0)>n-1\}$$

and $\bar\tau_n=\tau_n\wedge T$. Moreover, for each n we can define the process $\bar u^{(n)}\in\mathbb{D}([0,\bar\tau_n,V_\delta)$ by

$$\bar u^{(n)}(t):=\left\{\begin{array}{rll} \bar u^{(n-1)}(t), & \text{for} & 0\le t<\bar\tau_{n-1}, \\ u^{(n)}(t-\tau_{n-1}), & \text{for} & \bar\tau_{n-1}\le t<\bar\tau_n, \end{array}\right.$$

where $u^{(n)}$ is the solution to the Problem

$$\left\{\begin{array}{rcl} du(t) &=& (Au(t)+f(u(t-)))\ dt+ \\ && \int_{Z^0} g(z;u(t-))\left(\eta\circ\vartheta_{\bar\tau_{n-1}}\right)(dz;dt),\ t\ge 0, \\ u(0) &=& u^{(n-1)}(\bar\tau_{n-1}-)+\int_Z g(z;u^{(n-1)}(\bar\tau_{n-1}-))\,\eta(dz;\{\bar\tau_{n-1}\}). \end{array}\right.$$

We have to show, that $\lim_{n\to\infty}\bar u^{(n)}$ exists and belongs to $L^0\left(\Omega;\mathbb{D}([0,T);V_{-\delta})\right)$. This can be done by the amalgamation procedure of Elworthy [21, Chapter III.6]. The family of stopping times is ordered by its natural order. To be precise $\bar\tau_n\le\bar\tau_m$ holds a.s. as $n\le m$. The stopping time T is an upper bound of the family of stopping times, i.e. $\bar\tau_n\le T$ and $\mathbb{P}(T-\bar\tau_n\ge\delta)\to 0$ as $n\to\infty$, because $\nu(Z\setminus Z_0)<\infty$. Moreover, for each $n\in\mathbb{N}$, we have an adapted process $\bar u^{(n)}$ belonging to $L^0(\mathbb{D}([0,\tau_n);V_{-\delta}))$. By Elworthy [21, Chapter III, Lemma 6B, p. 43], the process $\bar u:[0,T)\to V_{-\delta}$ exists and $\bar u|_{[0,\tau_n)}=\bar u^{(n)}$ for all $n\in\mathbb{N}$. Moreover, tracing the proof of Elworthy [21, Chapter III, Lemma 6B, p. 43] one can show, that a version of the process $\bar u$ exists, such that the version belongs to $L^0(\mathbb{D}([0,T);V_{-\delta}))$.

Appendix A. The Skorohod Space

For an introduction to the Skorohod space we refer to Ethier and Kurtz [23] and Jacod and Shiryaev [27]. The results stated in this chapter are taken from Ethier and Kurtz [23].

Let X be a separable Banach space. The space $\mathbb{D}([0,T];X)$ denotes the space of all right continuous functions $x : [0,T] \to X$ with left limits. Since $\mathbb{D}([0,T];X)$ is complete but not separable in the uniform topology, we endow $\mathbb{D}([0,T];X)$ with the Skorohod topology, which is characterized as follows: a sequence $\{x^{(n)} \mid n \in \mathbb{N}\} \subset \mathbb{D}([0,T];X)$ converges to $x \in \mathbb{D}([0,T];X)$ iff there is a sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset \Lambda := \big\{\lambda : \mathbb{R} \to \mathbb{R}, \lambda(0) = 0, \lambda(T) = T$, and λ is strictly increasing $\big\}$, such that

- $\sup_{0\le s\le T} |\lambda_n(s) - s| \to 0$ as $n \to \infty$.
- $\sup_{0\le s\le T} |x^{(n)} \circ \lambda_n(s) - x(s)| \to 0$ as $n \to \infty$.

The Skorohod topology is metrizable and the resulting metric space is separable and complete. Nevertheless, $\mathbb{D}([0,T];X)$ is not a topological vector space (see Jacod and Shiryaev [27, Remark VI.1.22]).

Let $\{x^{(n)} \mid n \in \mathbb{N}\}$ be a sequence of processes defined on the filtered probability space $(\Omega^{(n)}, \mathcal{F}^{(n)}, \left(\mathcal{F}_t^{(n)}\right)_{0\le t\le T}, \mathbb{P}^{(n)})$, such that $x^{(n)} \in \mathbb{D}([0,T];X)$ for all $n \in \mathbb{N}$. Let $L^0(\Omega;\mathbb{D}([0,T];X))$ be the space of all random variables $x : \Omega \to \mathbb{D}([0,T];X)$ topologized by convergence in distribution. Note, if the underlying metric space is separable, convergence in distribution is equivalent to convergence in the Prohorov metric.

Definition A.1. *Let (S,d) be a metric space, $\mathcal{S}$ denoting the Borel σ algebra of S and $\mathcal{P}(S)$ is the family of Borel probability measures on S. The Prohorov ρ metric on $\mathcal{P}(S)$ is given by*

$$\rho(P,Q) := \inf\{\epsilon > 0 : P(F) \le Q(F^\epsilon) + \epsilon \ \textit{for all } F \in \mathcal{C}\},$$

where $\mathcal{C}$ is the collection of closed subsets of S and

$$F^\epsilon = \left\{x \in S : \inf_{y\in F} d(x,y) < \epsilon\right\}.$$

Now, under which conditions does the sequence $\{x^{(n)} \mid n \in \mathbb{N}\}$, $x^{(n)} \in L^0(\Omega^{(n)};\mathbb{D}([0,T];X))$ converge in law to a limit $x \in L^0(\Omega;\mathbb{D}([0,T];X))$? By Theorem 3.7.8–(b) of Ethier and Kurtz [23] it remains to be shown that

- $\{x^{(n)} \mid n \in \mathbb{N}\}$ is tight in $L^0(\Omega;\mathbb{D}([0,T];X))$,
- there exists a dense set $D \subset [0,T]$ such that for any finite set $\{t_1,\dots,t_k\} \subset D$ we have

  $$\big(x^{(n)}(t_1),\dots,x^{(n)}(t_k)\big) \to \big(x(t_1),\dots,x(t_k)\big)$$

  in distribution as $n \to \infty$.

If X is finite dimensional, then tightness of the sequence $\{x^{(n)} \mid n \in \mathbb{N}\}$ can be shown by the Aldou's condition.

Definition A.2. *Let $\{x^{(n)} \mid n \in \mathbb{N}\}$ be a sequence of stochastic processes with sample paths belonging to $\mathbb{D}([0,T];\mathbb{R})$ and $x^{(n)} \in L^0(\Omega^{(n)},\mathcal{F}^{(n)},\mathbb{P}^{(n)})$, $n \in \mathbb{N}$. Let $(\mathcal{F}_t^{(n)})$ be the natural filtration induced by $x^{(n)}$, $n \in \mathbb{N}$. We say the sequence $\{x^{(n)} \mid n \in \mathbb{N}\}$ satisfies the Aldou's condition, iff for all $\epsilon > 0$ and $\delta > 0$ there exist $\theta > 0$ and $n_0 \in \mathbb{N}$ such that for any family $\{\tau_n\}_{n\in\mathbb{N}}$, where τ_n is a $(\mathcal{F}_t^{(n)})$–stopping time on $\Omega^{(n)}$ with $\tau_n \le T$, we have*

$$\mathbb{P}^{(n)}\left(|x^{(n)}_{\tau_n} - x^{(n)}_{(\tau_n+h)\wedge T}| \ge \delta\right) \le \epsilon, \quad 0 < h \le \theta, \quad n \ge n_0.$$

Note, in the definition above, only τ_n, $n \in \mathbb{N}$ are stopping times, θ is a constant. In case of E being infinite dimensional we have to add to the Aldou's condition a compact containment condition (see e.g. Ethier and Kurtz [23, Chapter 3]) .

Definition A.3. *Let $\{x^{(n)} \mid n \in \mathbb{N}\}$ be a sequence of stochastic processes with sample paths belonging to $\mathbb{D}([0,T];\mathbb{R})$ and $x^{(n)} \in L^0(\Omega^{(n)},\mathcal{F}^{(n)},\mathbb{P}^{(n)})$, $n \in \mathbb{N}$. Let $(\mathcal{F}_t^{(n)})$ be the natural filtration induced by $x^{(n)}$, $n \in \mathbb{N}$. We say this sequence $\{x^{(n)} \mid n \in \mathbb{N}\}$ satisfies the compact containment condition, iff for each $\epsilon > 0$ and every rational $t > 0$, there exists a compact subset $K_{\epsilon,t} \subset X$, such that*

$$\mathbb{P}^{(n)}\left(x^{(n)}(s) \in K_{\epsilon,t}\, \forall\, s \in [0,t]\right) \ge 1-\epsilon, \quad n \in \mathbb{N}.$$

Let X be a separable Banach space with norm $|\cdot|$. The Aldou's condition is often difficult to verify directly. Thus, to show that $\{x^{(n)} \mid n \in \mathbb{N}\}$ satisfies the Aldou's condition A.2, one can show certain integrability conditions given the Lemma below (see Ethier and Kurtz [23, Chapter 3.8]).

Theorem A.1. *(see Ethier and Kurtz* [23, Theorem 8.6, Remark 8.7 in Chapter 3]*) Let X be a separable Banach space and let $\{x^{(n)} \mid n \in \mathbb{N}\}$ be a sequence of stochastic processes with sample paths in $\mathbb{D}([0,T];X)$, such that $x^{(n)} \in L^0(\Omega^{(n)},\mathcal{F}^{(n)},\mathbb{P}^{(n)})$, $n \in \mathbb{N}$. Let $(\mathcal{F}_t^{(n)})$ be the natural filtration induced by $x^{(n)}$, $n \in \mathbb{N}$. Then $\{x^{(n)} \mid n \in \mathbb{N}\}$ satisfies the Aldou's condition, if there exists an $p > 0$ and a family $\left\{f_{(n)}(\delta) : 0 < \delta < 1, n \in \mathbb{N}\right\}$ of nonnegative random variables, such that for all $t \in [0,T]$ and $h \in (0,\delta]$*

- $\mathbb{E}\left[\left|x^{(n)}(t+h) - x^{(n)}(t)\right|^p \mid \mathcal{F}_t^{(n)}\right] \le \mathbb{E}\left[f_{(n)}(\delta) \mid \mathcal{F}_t^{(n)}\right]$,
- $\sup_{n\in\mathbb{N}} \mathbb{E}\left[f_{(n)}(\delta) \mid \mathcal{F}_t^{(n)}\right] \to 0$ *as* $\delta \to 0$.

Similarly, the following lemma gives an integrability condition which implies A.3.

Lemma A.1. *Let $\{x^{(n)} \mid n \in \mathbb{N}\}$ be a sequence of stochastic processes with sample path in $\mathbb{D}([0,T];\mathbb{R})$, such that $x^{(n)} \in L^0(\Omega^{(n)},\mathcal{F}^{(n)},\mathbb{P}^{(n)})$, $n \in \mathbb{N}$. Let Γ be a subspace of E with norm $|\cdot|_\Gamma$, such that the embedding $\Gamma \hookrightarrow E$ is compact (see Remark B.2). Then the compact containment condition A.3 is satisfied, if some $p > 0$ exists, such that*

$$\sup_{n\in\mathbb{N}} \mathbb{E}^{(n)} \sup_{0\le t\le T} \left|x^{(n)}(t)\right|_\Gamma^p \le C < \infty.$$

Appendix B. Analytic semigroups and fractional powers of operators

For an introduction to analytic semigroups we refer to Engel and Nagel [22] or Pazy [37], for interpolation theory e.g. the lecture notes of Lunardi [33].

B.1. **Analytic Semigroups.** An important subclass of semigroups are so–called analytic semigroups. For any $\omega \in \mathbb{R}$ and $\theta \in (0,\pi)$ let

$$\Sigma_{\theta,\omega} = \{\lambda \in \mathbb{C} \setminus \{\omega\} \mid |\arg(\lambda - \omega)| \leq \theta\} .$$

Definition B.1. *(Engel and Nagel* [22, Definition II.4.2]*) A closed linear operator* $(A, D(A))$ *with dense domain* $D(A)$ *in a Banach space* E *is called sectorial if there exists a* $0 < \delta \leq \frac{\pi}{2}$ *and some* $\omega \in \mathbb{R}$ *such that the sector* $\Sigma_{\frac{\pi}{2}+\delta,\omega}$ *is contained in the resolvent set* $\rho(A)$ *and if for each* $\epsilon \in (0,\delta)$ *there exists* $M_\epsilon \geq 1$ *such that*

$$|R(\lambda : A)| \leq \frac{M_\epsilon}{|\lambda - \omega|} \quad \forall 0 \neq \lambda \in \bar{\Sigma}_{\frac{\pi}{2}+\delta-\epsilon,\omega},$$

where $R(\lambda : A) := (\lambda - A)^{-1}$ *denotes the resolvent of the operator* A*. If* $(A, D(A))$ *is sectorial with angle* δ *and* $\omega \in \mathbb{R}$*, we say* $(A, D(A)) \in \mathcal{H}(\omega, \delta)$*.*

For sectorial operators and appropriate paths γ in the complex plane, the associated semigroup can defined via the Cauchy integral formula.

Definition B.2. *(Engel and Nagel* [22, Definition II.4.2]*) Let* $(A, D(A))$ *be a sectorial operator of angle* δ*. Define* $T(0) := I$ *and operators* $T(z)$*, for* $z \in \sigma_\delta$*, by*

$$T(z) := \frac{1}{2\pi i} \int_\gamma R(\lambda; A)\, d\lambda,$$

where γ *is any piecewise smooth curve in* $\Sigma_{\frac{\pi}{2}+\delta}$ *going from* $\infty e^{-i(\frac{\pi}{2}+\delta')}$ *to* $\infty e^{i(\frac{\pi}{2}+\delta')}$ *for some* $\delta' \in (|\arg(z)|, \delta))$*.*

It can be shown (see e.g. [22, Proposition II.4.3] or [37, Theorem 2.5.2]), that the in Definition B.1 defined semigroup of an sectorial operator is analytic.

Definition B.3. *(Engel and Nagel* [22, Definition II.4.5]*) A family of operators* $(T(z))_{z\in\Sigma_\delta\cup\{0\}} \subset L(E)$ *is called an analytic semigroup if:*

(i) $T(0) = I$ *and* $T(z_1 + z_2) = T(z_1)T(z_2)$ *for all* $z_1, z_2 \in \Sigma_\delta$*.*
(ii) *The map* $z \to T(z)$ *is analytic in* Σ_δ*.*
(iii) $\lim_{z\to 0, z\in\Sigma_{\delta'}} T(z)x = x$ *for all* $x \in E$ *and* $0 < \delta' < \delta$*.*

B.2. **Fractional Power of Operators.** For a sectorial operator A, where $(A, D(A)) \in \mathcal{H}(0, \delta)$, $0 < \delta < \frac{\pi}{2}$ arbitrary and $\alpha > 0$ one can define fractional powers of the operator.

Definition B.4. *Let* $\sigma \subset \mathbb{C}$ *be an open sector, such that* $\mathbb{R}^+ \subset \Sigma \subset \rho(A)$*. For* $\alpha > 0$ *the bounded linear operator* $(-A)^\alpha$ *is defined by*

$$(-A)^\alpha := \frac{1}{2\pi i} \int_\gamma \lambda^{-\alpha} R(\lambda, A) d\lambda,$$

where the path γ is a piecewise smooth part $\Sigma \setminus \mathbb{R}^+$ going from $\infty e^{-i\delta}$ to $\infty e^{i\delta}$ for some $\delta > 0$.

Definition B.5. *Let A be a sectorial operator. For every $\alpha > 0$ we define*

$$A^\alpha = ((-A)^{-\alpha})^{-1}.$$

For $\alpha = 0$, $A^\alpha = I$.

In order to study regularity property of solution of some Cauchy problem it is convenient to introduce several scales of subspaces of E, e.g. domain of fractional powers of operators. In particular, if $\alpha \geq 0$, then $V_\alpha := D((-A)^\alpha)$ equipped with norm $|\cdot|_\alpha := |(I-A)^\alpha \cdot|$. Let V_{-n}, $n \in \mathbb{N}$ be the completion of E with respect to the norm $|\cdot|_{-n} := |(-A)^{-n} \cdot|$. If $\alpha < 0$ such that $\alpha = -n + \beta$ for some $n \in \mathbb{N}$ and $\beta \in (0, 1]$, then $V_\alpha := \{x \in V_{-n} \mid |(I-A)^\beta x|_{-n} < \infty\}$ equipped with norm $|\cdot| := |(I-A)^\beta \cdot|_{-n}$.

Remark B.1. *In the following we will use certain interpolation inequalities. In particular let E be a separable Banach space, A be a operator generating an analytic semigroup* $(\exp(-\omega t)S_t)_{t\geq 0}$ *of contraction on E. Then there exists a constant $C < \infty$, such that we have (see e.g. (see e.g.* [22, Theorem II.4.6] *or Pazy* [37, Theorem 2.6.3]*)*

$$|(-A)^\alpha S(t)x|_\alpha \leq C \, \exp(\omega t) \, t^{-\alpha}, \quad \alpha \geq 0.$$

Remark B.2. *(See Bergh and Löfström* [5, Corollary 3.8.2]*) Let E be a separable Banach space and $A : E \to E$ a sectorial operator. If the embedding $V_1 = D(A) \hookrightarrow E$ is compact, then for any $\delta \in (0, 1)$ the embedding $V_\delta \hookrightarrow E$ is compact as well.*

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