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STATE CLASSIFICATION FOR A CLASS OF INTERACTING SUPERPROCESSES WITH LOCATION DEPENDENT BRANCHING

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Abstract
The spatial structure of a class of superprocesses which arise as limits in distribution of a class of interacting particle systems with location dependent branching is investigated. The criterion of their state classification is obtained. Their effective state space is contained in the set of purely-atomic measures or the set of absolutely continuous measures according as one diffusive coefficient $c(x) \equiv 0$ *or* $|c(x)| \geq \varepsilon > 0$ *while another diffusive coefficient* $h \in C_b^2(\mathbb{R})$.

1 Introduction and main result

In Dawson-Li-Wang [3], a class of interacting branching particle systems with location dependent branching, which generalizes the model introduced in Wang [14], is introduced and the limiting superprocesses, which will be called superprocesses with dependent spatial motion and branching (SDSBs), are constructed and characterized. In Theorem 6.1 of [3], it is proved that when the motion coefficient satisfies uniformly elliptic condition (which means $|c(x)| \geq \varepsilon > 0$ in the following model), the effective state space of the SDSBs is contained in the space of all measures which are absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. It leaves an open problem whether the effective state space of the SDSBs is contained in the space of purely-atomic measures when the motion coefficient is degenerate (which means $c(x) \equiv 0$ in the following model). In our model, the motions of the particles are not independent. This can be seen from their non-zero quadratic variation processes. This is one essential difference from the Super-Brownian motion. Another essential difference is that the branching coefficient in our model depends on the spatial location. Therefore, motion of the particles affects the branching. This is a new class of interaction. To compare with other existing models, reader is referred to [1], [2], [4], [7], [8], [9], [10], to name only a few.
In the present paper, the spatial structure of the SDSBs is investigated. We will give solution to above mentioned open problem left in [3]. Combining with the result proved in [3], we will give a criterion of state classification for SDSBs. Before introducing our model, let us

give some notations. Let $\mathbb{R} = (-\infty, \infty)$, $\bar{\mathbb{R}} = \mathbb{R} \cup \{\partial\}$, the one-point compactification of $\mathbb{R}$, $\mathbb{N} = \{1, 2, 3, \cdots\}$, $\bar{\mathbb{N}} = \{0\} \cup \mathbb{N}$, $C(\mathbb{R})$ be the space of all continuous functions on $\mathbb{R}$, $C_b(\mathbb{R})$ be the space of all bounded continuous functions on $\mathbb{R}$, $C_0(\mathbb{R})$ be the space of all continuous functions vanishing at infinity, $C_L(\mathbb{R})$ be the space of all Lipschitz continuous functions on $\mathbb{R}$, and $C_b^n(\mathbb{R})$ be the space of all the functions which has bounded, continuous derivatives up until and including order n. Now, let us introduce our model.

Suppose that $\{W(x,t) : x \in \mathbb{R}, t \geq 0\}$ is a Brownian sheet (see [12]) and $\{B^i(t) : t \geq 0\}$,$i \in \mathbb{N}$, is a family of independent standard Brownian motions which are independent of $\{W(x,t) : x \in \mathbb{R}, t \geq 0\}$. For each natural number n which serves as a control parameter for our finite branching particle systems, we consider a system of particles (initially, there are m_0^n particles) which move, die and produce offspring in a random medium on $\mathbb{R}$.

The diffusive part of such a branching particle system has the form

$$dx_i^n(t) = c(x_i^n(t))\, dB^i(t) + \int_{\mathbb{R}} h(y - x_i^n(t))\, W(dy, dt),\ t \geq 0, \tag{1.1}$$

where $c \in C_L(\mathbb{R})$ and $h \in C_b^2(\mathbb{R})$ is a square-integrable function. By Lemma 3.1 of [3], for any initial conditions $x_i^n(0) = x_i$, the stochastic equations (1.1) have unique strong solution $\{x_i^n(t) : t \geq 0\}$ and, for each integer $m \geq 1$, $\{(x_1^n(t), \cdots, x_m^n(t)) : t \geq 0\}$ is an m-dimensional diffusion process which is generated by the differential operator

$$G^m := \frac{1}{2}\sum_{i=1}^{m} a(x_i)\frac{\partial^2}{\partial x_i^2} + \frac{1}{2}\sum_{i,j=1, i\neq j}^{m} \rho(x_i - x_j)\frac{\partial^2}{\partial x_i \partial x_j}. \tag{1.2}$$

In particular, $\{x_i^n(t) : t \geq 0\}$ is a one-dimensional diffusion process with generator $G := (a(x)/2)\Delta$, where Δ is the Laplacian operator,

$$\rho(x) := \int_{\mathbb{R}} h(y - x)h(y)\, dy, \tag{1.3}$$

and $a(x) := c^2(x) + \rho(0)$ for $x \in \mathbb{R}$. The function ρ is twice continuously differentiable with ρ' and ρ'' bounded since h is integrable and twice continuously differentiable with h' and h'' bounded. The quadratic variational process for the system given by (1.1) is

$$\langle x_i^n(t), x_j^n(t)\rangle = \int_0^t \rho(x_i^n(s) - x_j^n(s))\, ds + \delta_{\{i=j\}} \int_0^t c^2(x^i(s))\, ds, \tag{1.4}$$

where we set $\delta_{\{i=j\}} = 1$ or 0 according as $i = j$ or $i \neq j$, where $i, j \in \mathbb{N}$. Here $x_i^n(t)$ is the location of the i^{th} particle. We assume that each particle has mass $1/\theta^n$ and branches at rate $\gamma\theta^n$, where $\gamma \geq 0$ and $\theta \geq 2$ are fixed constants. We assume that when a particle $\frac{1}{\theta^n}\delta_x$, which has location at x, dies, it produces k particles with probability $p_k(x); x \in \mathbb{R}, k \in \bar{\mathbb{N}}$. This means that the branching mechanism depends on the spatial location. The offspring distribution is assumed to satisfy:

$$p_1(x) = 0, \quad \sum_{k=0}^{\infty} kp_k(x) = 1, \quad \text{and} \quad m_2(x) := \sum_{k=0}^{\infty} k^2 p_k(x) < \infty \quad \text{for all } x \in \mathbb{R}. \tag{1.5}$$

The second condition indicates that we are solely interested in the critical case. After branching, the resulting set of particles evolve in the same way as their parent and they start off from the parent particle's branching site. Let m_t^n denote the total number of particles at time t . Denote the empirical measure process by

$$\mu_t^n(\cdot) := \frac{1}{\theta^n} \sum_{i=1}^{m_t^n} \delta_{x_i^n(t)}(\cdot). \tag{1.6}$$

In order to obtain measure-valued processes by use of an appropriate rescaling, we assume that there is a positive constant $\xi > 0$ such that $m_0^n/\theta^n \leq \xi$ for all $n \geq 0$ and that weak convergence of the initial laws $\mu_0^n \Rightarrow \tilde{\mu}$ holds, for some finite measure $\tilde{\mu}$. As for the convergence from branching particle systems to a SDSB, reader is referred to [14] and [3].
Let $E := M(\mathbb{R})$ be the Polish space of all bounded Radon measures on $\mathbb{R}$ with the weak topology defined by

$$\mu^n \Rightarrow \mu \quad \text{if and only if} \quad \langle f, \mu^n \rangle \to \langle f, \mu \rangle \ \text{ for all } f \in C_b(\mathbb{R})\,.$$

By Ito's formula and the conditional independence of motions and branching, we can obtain the following formal generators (usually called pregenerators) for the limiting measure-valued processes:

$$\mathcal{L}_{c,\sigma} F(\mu) := \mathcal{A}_c F(\mu) + \mathcal{B}_\sigma F(\mu), \tag{1.7}$$

where

$$\mathcal{B}_\sigma F(\mu) \ := \tfrac{1}{2} \int_{\mathbb{R}} \sigma(x) \frac{\delta^2 F(\mu)}{\delta \mu(x)^2}\, \mu(dx), \tag{1.8}$$

and

$$\begin{aligned} \mathcal{A}_c F(\mu) := {} & \frac{1}{2} \int_{\mathbb{R}} a(x) \left(\frac{d^2}{dx^2} \right) \frac{\delta F(\mu)}{\delta \mu(x)}\, \mu(dx) \\ & + \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \rho(x-y) \left(\frac{d}{dx} \right) \left(\frac{d}{dy} \right) \frac{\delta^2 F(\mu)}{\delta \mu(x) \delta \mu(y)}\, \mu(dx)\, \mu(dy) \end{aligned} \tag{1.9}$$

for $F(\mu) \in \mathcal{D}(\mathcal{L}_{c,\sigma}) \subset C(E)$, where $\sigma(x) := \gamma(m_2(x) - 1)$ for any $x \in \mathbb{R}$, the variational derivative is defined by

$$\frac{\delta F(\mu)}{\delta \mu(x)} := \lim_{h \downarrow 0} \frac{F(\mu + h\delta_x) - F(\mu)}{h}, \tag{1.10}$$

$\mathcal{D}(\mathcal{L}_{c,\sigma})$ is the domain of the pregenerator $\mathcal{L}_{c,\sigma}$. Especially, we denote $\mathcal{L}_{0,\sigma} = \mathcal{A}_0 + \mathcal{B}_\sigma$ for $\mathcal{L}_{c,\sigma} = \mathcal{A}_c + \mathcal{B}_\sigma$ with $c(x) \equiv 0$. Let $B(\mathbb{R})^+$ be the space of all non-negative, bounded, measurable functions on $\mathbb{R}$. We cite two theorems proved in [3].

Theorem 1.1 *Let $c \in C_L(\mathbb{R})$, $h \in C_b^2(\mathbb{R})$ be a square-integrable function on $\mathbb{R}$, and $\sigma(x) \in B(\mathbb{R})^+$. Then, for any $\mu \in E$, $(\mathcal{L}_{c,\sigma}, \delta_\mu)$-martingale problem (MP) has a unique solution which is a diffusion process.*

Proof: For the proof of this theorem, reader is referred to the section 5 of [3]. □

Theorem 1.2 *Let $c \in C_L(\mathbb{R})$, $h \in C_b^2(\mathbb{R})$ be a square-integrable function on $\mathbb{R}$, and $\sigma(x) \in B(\mathbb{R})^+$. Suppose that there exists a constant $\varepsilon > 0$ such that $|c(x)| \geq \varepsilon > 0$. For any $\mu \in E$, let $\{\mu_t : t \geq 0\}$ be the unique solution to the $(\mathcal{L}_{c,\sigma}, \delta_\mu)$-MP with sample paths in $C([0,\infty), E)$. Then,*

$$\mathbb{P}\{\ \mu_t \ll \mathbb{L} \text{ on } \mathbb{R} \text{ for } t > 0\ |\mu_0 = \mu\} = 1,$$

where $\mathbb{L}$ is the Lebesgue measure on $\mathbb{R}$ and $\mu_t \ll \mathbb{L}$ means that μ_t is absolutely continuous with respect to Lebesgue measure $\mathbb{L}$ on $\mathbb{R}$.

Proof: For the proof of this theorem, reader is referred to the Theorem 6.1 of [3]. □

We have following main result:

Theorem 1.3 *Let $h \in C_b^2(\mathbb{R})$ be a square-integrable function on $\mathbb{R}$, $c(x) \equiv 0$, and $\sigma(x) \in B(\mathbb{R})^+$. Suppose that there exist constants $0 < \varepsilon < B$ such that $0 < \varepsilon \leq \inf_x \sigma(x) \leq \sup_x \sigma(x) \leq B < \infty$. For any $\mu_0 \in E$, let $\{\mu_t^\sigma : t \geq 0\}$ be the unique solution to the $(\mathcal{L}_{0,\sigma}, \delta_{\mu_0})$-MP with sample paths in $C([0,\infty), E)$. Then, for any $t > 0$, μ_t^σ is a purely-atomic measure. Furthermore, for any given $t_0 > 0$ and conditioned on $\mu_{t_0}^\sigma = \sum_{i \in I(t_0)} a_i(t_0)\delta_{x_i(t_0)}$ with $x_i(t_0) \neq x_j(t_0)$ if $i \neq j$ and $i,j \in I(t_0)$, for any $t \in [t_0, \infty)$, μ_t^σ has following representation:*

$$\mu_t^\sigma = \sum_{i \in I(t)} a_i(t)\delta_{x_i(t)},$$

where $x_i(t) \neq x_j(t)$ if $i \neq j$ and $i,j \in I(t)$ and $a_i(t)$ satisfies

$$a_i(t) - a_i(t_0) = \int_{t_0}^t \sqrt{\sigma(x_i(u))a_i(u)}\, dB^i(u), \tag{1.11}$$

and $x_i(t)$ satisfies

$$x_i(t) - x_i(t_0) = \int_{t_0}^t \int_{\mathbb{R}} h(y - x_i(u))\, W(dy, du), \tag{1.12}$$

where $W(y,u)$ is a Brownian sheet and $\{B^i(t) : i \geq 1\}$ are a sequence of independent one-dimensional Brownian motions which are independent of $W(y,u)$, $\{I(t) \subset \mathbb{N} : t > t_0\}$ is no-increasing random subsets in t in terms of set inclusion order.

2 Proof of the main result

The strategies to prove our main result can be described as follows:

(1) Generator Decomposition Technique : We decompose the branching generator as follows:

$$\mathcal{B}_\sigma F(\mu) = \mathcal{B}_d F(\mu) + \mathcal{B}_{\varepsilon/2} F(\mu),$$

where the operators are define by (1.8), (2.14), and (2.21), respectively. By virtue of $\mathcal{B}_{\varepsilon/2}$ and existing results of [13], this decomposition technique helps us to prove and explain that our concerned interacting superprocesses with a variable coefficient branching generator immediately enters into the purely-atomic measure valued state even if the initial state is an absolutely continuous measure.

(2) Branching Mechanism Reconstruction: By reconstruction of the branching mechanism, the variable coefficient $\sigma(x)$ of the branching generator is transformed as a location dependent branching rate with location independent, equal probability binary branching in the branching particle model. If $\inf_{x\in\mathbb{R}}\sigma(x)\ge\varepsilon>0$, then the mean life time of the particles with variable branching coefficient $\sigma(x)$ is shorter than that of the particles with constant branching coefficient ε.

(3) Trotter's Product Formula: Based on the branching mechanism reconstruction, we will use Trotter's product formula and a dominating method, which shows that the number of particles of a purely-atomic measure-valued superprocess with a variable branching coefficient $\sigma(x)$ is dominated by the number of particles of a purely-atomic measure-valued superprocess with a constant branching coefficient ε if $\sigma(x)\ge\varepsilon>0$, to reach our conclusion.

To prove our main result, we need two lemmas.

Lemma 2.1 *Let $h\in C_b^2(\mathbb{R})$ be a square-integrable function on $\mathbb{R}$, $c(x)\equiv 0$, and $\sigma(x)\in B(\mathbb{R})^+$. Suppose that there exist constants $0<\varepsilon<B$ such that $0<\varepsilon\le\inf_x\sigma(x)\le\sup_x\sigma(x)\le B<\infty$. Let*

$$\mathcal{L}_{0,d}F(\mu):=\mathcal{A}_0F(\mu)+\mathcal{B}_dF(\mu), \tag{2.13}$$

where $F(\mu)\in\mathcal{D}(\mathcal{L}_{0,d})$, $\mathcal{A}_0F(\mu)$ is defined by (1.9) with $c(x)\equiv 0$, and

$$\mathcal{B}_dF(\mu):=\frac{1}{2}\int_{\mathbb{R}}(\sigma(x)-\frac{\varepsilon}{2})\frac{\delta^2F(\mu)}{\delta\mu(x)^2}\,\mu(dx). \tag{2.14}$$

Then, for any $t_0\ge 0$ and for any $\mu_{t_0}\in E$, $(\mathcal{L}_{0,d},\delta_{\mu_{t_0}})$-MP has a unique solution $\{\mu_t : t\ge t_0\ge 0\}$ which has sample paths in $C([t_0,\infty),E)$. If the initial state is given by $\mu_{t_0}=\sum_{i\in I(t_0)}a_i(t_0)\delta_{x_i(t_0)}$ which is a purely-atomic measure with $x_i(t_0)\ne x_j(t_0)$ if $i\ne j$ and $i,j\in I(t_0)$, where $I(t_0)$ is at most a countable set, then for any $t\in[t_0,\infty)$, μ_t has following representation:

$$\mu_t=\sum_{i\in I(t)}a_i(t)\delta_{x_i(t)},$$

where $x_i(t)\ne x_j(t)$ if $i\ne j$ and $i,j\in I(t)$, $I(t)$ is the random subset of $I(t_0)$ such that $a_i(t)>0$ if $i\in I(t)$, and $a_i(t)$ satisfies

$$a_i(t)-a_i(t_0)=\int_{t_0}^t\sqrt{(\sigma(x_i(u))-\frac{\varepsilon}{2})a_i(u)}\,dB^i(u), \tag{2.15}$$

and $x_i(t)$ satisfies

$$x_i(t)-x_i(t_0)=\int_{t_0}^t\int_{\mathbb{R}}h(y-x_i(u))\,W(dy,du), \tag{2.16}$$

where $W(y,u)$ is a Brownian sheet and $\{B^i(t):i\in\mathbb{N}\}$ are a sequence of independent one-dimensional Brownian motions which are independent of $W(y,u)$.

Proof: In order to simplify the notation, without loss of generality in the following we simply assume that $t_0 = 0$. The existence, uniqueness of the $(\mathcal{L}_{0,d}, \delta_{\mu_0})$-MP, and its solution being a diffusion process follow from Theorem 1.1 with $c(x) \equiv 0$. We will use Itô's formula to prove the remaining parts of the lemma. Suppose that $W(y,u)$ is a Brownian sheet and $\{B^i(t) : i \in \mathbb{N}, t \geq 0\}$ are a sequence of independent one-dimensional Brownian motions which are independent of $W(y,u)$. Let $\{a_i(t)\}$ be the unique solution of (2.15) and $\{x_i(t)\}$ be the unique solution of (2.16) with $t_0 = 0$. Define $\mu_t = \sum_{i \in I(t)} a_i(t)\delta_{x_i(t)}$. Since $h \in C_b^2(\mathbb{R})$ is a square-integrable function on $\mathbb{R}$ and $c \equiv 0$, according to the behavior of the generator $\mathcal{A}_0$, the location processes $\{x_i(t) : t \geq 0, i \in I(0)\}$ have following coalescence property (See section 1.2 and the proof of Lemma 1.2 in [13]).

Coalescence Property: A branching particle system is said to have coalescence property if the particle location processes are diffusion processes and for any two particles either they never separate or they never meet according as they start off from same initial location or not.

According to this coalescence property, $\mu_t = \sum_{i \in I(t)} a_i(t)\delta_{x_i(t)}$ is a purely-atomic measure valued process and $x_i(t) \neq x_j(t)$ for all $t \geq 0$ if $i \neq j$ and $i, j \in I(t)$. Now consider the following function in the general form:

$$F(\mu) = f(\langle \phi_1, \mu \rangle, \cdots, \langle \phi_n, \mu \rangle), \tag{2.17}$$

where $f \in C^2(\mathbb{R}^n)$ and $\{\phi_i \in \mathcal{S}(\mathbb{R}), i = 1, \cdots, n\}$ are test functions. According to Itô's formula, we have

$$\begin{aligned}
&f(\langle \phi_1, \mu_t \rangle, \cdots, \langle \phi_n, \mu_t \rangle) - f(\langle \phi_1, \mu_0 \rangle, \cdots, \langle \phi_n, \mu_0 \rangle) \\
&= \sum_{l=1}^{n} \int_0^t f_l^{'}(\langle \phi_1, \mu_s \rangle, \cdots, \langle \phi_n, \mu_s \rangle) \times \sum_{i \in I(0)} \Big[\phi_l^{'}(x_i(s)) a_i(s) \int_{\mathbb{R}} h(y - x_i(s))\, W(dy, ds) \\
&\quad + \phi_l(x_i(s)) \sqrt{(\sigma(x_i(s)) - \frac{\varepsilon}{2}) a_i(s)}\, dB^i(s) \Big] \\
&\quad + \frac{1}{2} \sum_{l=1}^{n} \int_0^t f_l^{'}(\langle \phi_1, \mu_s \rangle, \cdots, \langle \phi_n, \mu_s \rangle) \left[\sum_{i \in I(0)} \phi_l^{''}(x_i(s)) a_i(s) \rho(0) \right] ds \\
&\quad + \frac{1}{2} \sum_{l,m=1}^{n} \int_0^t f_{l,m}^{''}(\langle \phi_1, \mu_s \rangle, \cdots, \langle \phi_n, \mu_s \rangle) \times \Big[\sum_{k \in I(0)} \phi_l(x_k(s)) \phi_m(x_k(s)) (\sigma(x_k(s)) - \frac{\varepsilon}{2}) a_k(s) \\
&\quad + \sum_{i,j \in I(0)} \phi_l'(x_i(s)) \phi_m'(x_j(s)) a_i(s) a_j(s) \rho(x_i(s) - x_j(s)) \Big] ds \\
&= \int_0^t \int_{\mathbb{R}} \left\langle \frac{d}{dx} \frac{\delta F(\mu_s)}{\delta \mu(x)} h(y - x), \mu_s(dx) \right\rangle W(dy, ds) + \int_0^t \mathcal{A}_0 F(\mu_s)\, ds + \int_0^t \mathcal{B}_d F(\mu_s)\, ds \\
&\quad + \sum_{l=1}^{n} \int_0^t f_l^{'}(\langle \phi_1, \mu_s \rangle, \cdots, \langle \phi_n, \mu_s \rangle) \times \sum_{i \in I(0)} \phi_l(x_i(s)) \sqrt{(\sigma(x_i(s)) - \frac{\varepsilon}{2}) a_i(s)}\, dB^i(s).
\end{aligned} \tag{2.18}$$

Since $W(y,s)$ is a Brownian sheet and $\{B^i(t)\}$ are independent one-dimensional Brownian motions, μ_t is the unique solution to the $(\mathcal{L}_{0,d}, \delta_{\mu_0})$-MP and the lemma is proved. □

Lemma 2.2 *Let $h \in C_b^2(\mathbb{R})$ be a square-integrable function on $\mathbb{R}$, $c(x) \equiv 0$, and $\sigma(x) \in B(\mathbb{R})^+$. Suppose that there exist constants $0 < \varepsilon < B$ such that $0 < \varepsilon \leq \inf_x \sigma(x) \leq \sup_x \sigma(x) \leq B < \infty$. Let*

$$\mathcal{L}_{0,d}F(\mu) := \mathcal{A}_0 F(\mu) + \mathcal{B}_d F(\mu), \tag{2.19}$$

$$\mathcal{L}_{0,\varepsilon/2}F(\mu) := \mathcal{A}_0 F(\mu) + \mathcal{B}_{\varepsilon/2} F(\mu), \tag{2.20}$$

where $F(\mu) \in \mathcal{D}(\mathcal{L}_{0,d})$, $\mathcal{A}_0 F(\mu)$ is defined by (1.9) with $c(x) \equiv 0$, and $\mathcal{B}_d F(\mu)$ is defined by (2.14) and

$$\mathcal{B}_{\varepsilon/2}F(\mu) := \frac{\varepsilon}{4}\int_{\mathbb{R}} \frac{\delta^2 F(\mu)}{\delta\mu(x)^2}\,\mu(dx). \tag{2.21}$$

Then, for any $\mu_0, \nu_0 \in E$, $(\mathcal{L}_{0,d}, \delta_{\mu_0})$-MP ($(\mathcal{L}_{0,\varepsilon/2}, \delta_{\nu_0})$-MP) has a unique solution $\{\mu_t : t \geq 0\}$ ($\{\nu_t : t \geq 0\}$) which has sample paths in $C([0,\infty), E)$. If the initial state is given by $\mu_0 = a(0)\delta_{x(0)} = \nu_0$ which is a single atom measure with $a(0) > 0$ and $x(0) \in \mathbb{R}$, then for any $t \in [0,\infty)$, μ_t and ν_t have following representations:

$$\mu_t = a(t)\delta_{x(t)}, \qquad \nu_t = b(t)\delta_{x(t)},$$

where $a(t)$ satisfies

$$a(t) - a(0) = \int_0^t \sqrt{(\sigma(x(u)) - \frac{\varepsilon}{2})a(u)}\,dB(u), \tag{2.22}$$

$b(t)$ satisfies

$$b(t) - a(0) = \int_0^t \sqrt{\frac{\varepsilon}{2} b(u)}\,dB(u), \tag{2.23}$$

and $x(t)$ satisfies

$$x(t) - x(0) = \int_0^t \int_{\mathbb{R}} h(y - x(u))\,W(dy, du), \tag{2.24}$$

where $W(y,u)$ is a Brownian sheet and $\{B(t) : t \geq 0\}$ is an one-dimensional Brownian motion which is independent of $W(y,u)$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define $\tau_a := \inf\{t : a(t) = 0\}$ and $\tau_b := \inf\{t : b(t) = 0\}$. Then $\mathbb{P}(\tau_a \leq \tau_b) = 1$.

Proof: The conclusion that for any $\mu_0, \nu_0 \in E$, $(\mathcal{L}_{0,d}, \delta_{\mu_0})$-MP ($(\mathcal{L}_{0,\varepsilon/2}, \delta_{\nu_0})$-MP) has a unique solution $\{\mu_t : t \geq 0\}$ ($\{\nu_t : t \geq 0\}$) which has sample paths in $C([0,\infty), E)$ is proved in [3]. The purely-atomic representation is proved by above lemma 2.1. To complete the proof, it only needs to prove that $\mathbb{P}(\tau_a \leq \tau_b) = 1$. In order to prove this result, we will compare two operators $\mathcal{L}_{0,d}$ and $\mathcal{L}_{0,\varepsilon/2}$. We will use different point of view to explain the behaviors of $\mathcal{B}_d$ and $\mathcal{B}_{\varepsilon/2}$. At the beginning of this paper, we have introduced our model of interacting branching particle systems, where $\sigma(x) = \gamma(m_2(x) - 1)$, γ is a constant branching rate and the offspring distribution depends on spatial location. Now we remodel the interacting branching particle systems. For both the operator $\mathcal{L}_{0,d}$ and the operator $\mathcal{L}_{0,\varepsilon/2}$, their corresponding interacting branching systems can be alternatively described as follows:

For each n which serves as a control parameter for a finite branching particle system, we consider a system of particles (initially, there are m_0^n particles) which move, die and produce offspring in a random medium on $\mathbb{R}$. The diffusive part of such a branching particle system has the form

$$dx_i^n(t) = \int_{\mathbb{R}} h(y - x_i^n(t))\, W(dy, dt), \quad t \geq 0, \tag{2.25}$$

where $W(y,t)$ is a Brownian sheet and $h \in C_b^2(\mathbb{R})$ is a square-integrable function. Here $x_i^n(t)$ is the location of the i^{th} particle. The branching mechanisms for the operator $\mathcal{L}_{0,d}$ and the operator $\mathcal{L}_{0,\varepsilon/2}$ are different.
(1) For the operator $\mathcal{L}_{0,d}$, we assume that each particle has mass $1/\theta^n$ and branches at rate $\tilde{\gamma}(x)\theta^n$ if the particle's current location is x, where $\theta \geq 2$ is a fixed constant and $\tilde{\gamma}(x) := \sigma(x) - \varepsilon/2 \geq \varepsilon/2 > 0$. We assume that all particles undergo binary branching with equal probability $\frac{1}{2}$ or more precisely after a particle dies, it is replaced by 0 or 2 particles of same kind with equal probability $\frac{1}{2}$. Thus, the offspring distribution is independent of spatial location. Therefore, in other words, this says that particles undergo binary branching with equal probability $\frac{1}{2}$ and each particle's lifetime is measured by a clock whose speed changes as this particle's location changes. By Itô's formula, it is not difficult to find that the pregenerator of the limiting superprocess of the interacting branching particle systems is $\mathcal{L}_{0,d}$. In [3], it is proved that the martingale problem for $\mathcal{L}_{0,d}$ is well-posed.
(2) For the operator $\mathcal{L}_{0,\varepsilon/2}$, we assume that each particle has mass $1/\theta^n$ and branches at rate $(\varepsilon/2)\theta^n$ which is independent of the particle's current location x, where $\theta \geq 2$ and ε are two fixed constants. We assume that all particles undergo binary branching with equal probability $\frac{1}{2}$. Thus, the offspring distribution is independent of spatial location. Therefore, this means that a particle's lifetime is measured by a clock whose speed is fixed. After this particle dies, it is replaced by 0 or 2 particles with equal probability $\frac{1}{2}$. By Itô's formula, it is not difficult to find that the pregenerator of the limiting superprocess of the interacting branching particle systems is $\mathcal{L}_{0,\varepsilon/2}$. In [14] or [3] it is already proved that the martingale problem for $\mathcal{L}_{0,\varepsilon/2}$ is well-posed.
Based on above reconstruction of the models of the interacting branching particle systems, we have following comparison for $a(t)\delta_{x(t)}$ and $b(t)\delta_{x(t)}$, the unique solutions of $(\mathcal{L}_{0,d}, \delta_{(a(0)\delta_{x(0)})})$-MP and $(\mathcal{L}_{0,\varepsilon/2}, \delta_{(a(0)\delta_{x(0)})})$-MP, respectively. First, they have same location trajectory $x(t)$. Second, they have same binary branching mechanism. Third, the only difference is their mass processes. This difference is produced by their different branching rates. Since $\inf_{x\in\mathbb{R}}(\sigma(x) - \varepsilon/2) \geq \varepsilon/2$, the continuous branching process $a(t)$'s clock speed is uniformly quicker than or equal to that of the continuous branching process $b(t)$. Thus, $\mathbb{P}(\tau_a \leq \tau_b) = 1$ holds. $\square$

Proof of Theorem 1.3: For any measure $\mu_0 \in E$, let $\{\gamma(t) : t \geq 0\}$ be the unique solution to the $(\mathcal{B}_{\varepsilon/2}, \delta_{\mu_0})$-MP with sample paths in $C([0,\infty), E)$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then, by Theorem 1.1 of [13], for any $t > 0$, $\gamma(t)$ is a purely-atomic-measure (This is just the mutation-free Fleming-Viot process, see [6] [5]). For any natural integer k, we assume that at t/k, $\gamma(t/k)$ can be represented as follows:

$$\gamma(t/k) = \sum_{i\in I(t/k)} c_i(t/k)\delta_{x_i(0)},$$

where $I(t/k)$ is at most a countable set such that $x_i(0) \neq x_j(0)$ if $i \neq j$ and $i,j \in I(t/k)$, and $c_i(t/k) > 0$ for all $i \in I(t/k)$. Let $\{T_t^{\varepsilon/2}\}$ be the Feller semigroup generated by $\mathcal{B}_{\varepsilon/2}$, $\{U_t^d\}$ be the Feller semigroup generated by $\mathcal{L}_{0,d}$, and $\{U_t^\sigma\}$ be the Feller semigroup generated by $\mathcal{L}_{0,\sigma}$. By Trotter's product formula (See [11]), for any $F \in C_0(E)$, we have

$$\lim_{k\to\infty}[U^d_{t/k}\cdot T^{\varepsilon/2}_{t/k}]^k F = U_t^\sigma F \tag{2.26}$$

for all $t \geq 0$, uniformly on bounded intervals. Define

$$\mathcal{L}_{0,\varepsilon}F(\mu) := \mathcal{A}_0 F(\mu) + \mathcal{B}_\varepsilon F(\mu), \tag{2.27}$$

Let $\{V_t^{\varepsilon/2}\}$ be the Feller semigroup generated by $\mathcal{L}_{0,\varepsilon/2}$, and $\{V_t^\varepsilon\}$ be the Feller semigroup generated by $\mathcal{L}_{0,\varepsilon}$. By Trotter's product formula (See [11]), for any $F \in C_0(E)$, we have

$$\lim_{k\to\infty}[V^{\varepsilon/2}_{t/k}\cdot T^{\varepsilon/2}_{t/k}]^k F = V_t^\varepsilon F \tag{2.28}$$

for all $t \geq 0$, uniformly on bounded intervals. Let $\{\mu_t^\sigma : t \geq 0\}$ be the unique solution to the $(\mathcal{L}_{0,\sigma}, \delta_{\mu_0})$-MP with $\mu_0 \in E$ and Let $\{\nu_t^\varepsilon : t \geq 0\}$ be the unique solution to the $(\mathcal{L}_{0,\varepsilon}, \delta_{\mu_0})$-MP with same $\mu_0 \in E$. We already proved in [13] that $\{\nu_t^\varepsilon : t > 0\}$ is a purely-atomic measure valued process. Now we want to prove that for any $t > 0$, μ_t^σ is also a purely-atomic measure and the number of atoms of μ_t^σ is at most equal to that of ν_t^ε. To this end, we will construct the stochastic representations for both $[U^d_{t/k}\cdot T^{\varepsilon/2}_{t/k}]^k$ and $[V^{\varepsilon/2}_{t/k}\cdot T^{\varepsilon/2}_{t/k}]^k$ conditionally on $\gamma(t/k)$. On the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $W(x,t)$ be a Brownian sheet and $\{B^i(t) : i \in \mathbb{N}\}$ be a sequence of independent one-dimensional Brownian motions which are independent of $W(x,t)$. Conditioned on $\gamma(t/k)$, for each $i \in I(t/k)$ we construct following sequences: For the location processes, define

$$x_i(t) := x_i(0) + \int_0^t h(y - x_i(s))\, W(dy, ds) \qquad t \geq 0, \tag{2.29}$$

and for $1 \leq m \leq 2k$ define

$$\tilde{x}_i(mt/k) := \begin{cases} x_i(jt/k) & \text{if} \quad m = 2j, \\ x_i(jt/k) & \text{if} \quad m = 2j+1. \end{cases} \tag{2.30}$$

For the mass processes, define $\tilde{c}_i(t/k) := c_i(t/k)$ and

$$\begin{aligned}\tilde{c}_i(2jt/k) \;&:= \tilde{c}_i((2j-1)t/k) \\ &+ \int_{(2j-1)t/k}^{2jt/k} \sqrt{(\sigma(x_i(s - jt/k)) - \frac{\varepsilon}{2})\tilde{c}_i(s)}\, dB^i(s) \qquad 1 \leq j \leq k;\end{aligned} \tag{2.31}$$

$$\tilde{c}_i((2j+1)t/k) \;:= \tilde{c}_i(2jt/k) + \int_{2jt/k}^{(2j+1)t/k} \sqrt{\frac{\varepsilon}{2}\tilde{c}_i(s)}\, dB^i(s) \qquad 1 \leq j < k. \tag{2.32}$$

Based on above definitions, for any $1 \leq m \leq 2k$ we can define

$$\mu^{\sigma,k}_{mt/k} := \sum_{i\in I(mt/k)} \tilde{c}_i(mt/k)\delta_{\tilde{x}_i(mt/k)}, \tag{2.33}$$

where $I(mt/k)$ is a random subset of $I(t/k)$ such that $\tilde{c}_i(mt/k) > 0$ if $i \in I(mt/k)$. For any function $F \in C(E)$ and any natural integer l satisfying $1 \leq l \leq k$, by (2.29),(2.31), and Lemma 2.1, we can get

$$U^d_{t/k} F(\mu^{\sigma,k}_{(2l-1)t/k}) = \mathbb{E}_{\mu^{\sigma,k}_{(2l-1)t/k}} F(\mu^{\sigma,k}_{(2l)t/k}) \tag{2.34}$$

and by (2.30)and (2.32), we can get

$$T^{\varepsilon/2}_{t/k} F(\mu^{\sigma,k}_{(2l-2)t/k}) = \mathbb{E}_{\mu^{\sigma,k}_{(2l-2)t/k}} F(\mu^{\sigma,k}_{(2l-1)t/k}). \tag{2.35}$$

Thus, we have

$$\begin{aligned} &\mathbb{E}_{\mu_0} F(\mu^{\sigma,k}_{(2k)t/k}) \\ &= \mathbb{E}_{\mu_0}\{\mathbb{E}_{\mu^{\sigma,k}_{t/k}}\{\mathbb{E}_{\mu^{\sigma,k}_{2t/k}} \cdots \{\mathbb{E}_{\mu^{\sigma,k}_{(2k-1)t/k}} F(\mu^{\sigma,k}_{(2k)t/k})\} \cdots \} \\ &= [U^d_{t/k} \cdot T^{\varepsilon/2}_{t/k}]^k F(\mu_0). \end{aligned} \tag{2.36}$$

Similarly if we define $\tilde{b}_i(t/k) := c_i(t/k)$ and

$$\tilde{b}_i(jt/k) := \tilde{b}_i((j-1)t/k) + \int_{(j-1)t/k}^{jt/k} \sqrt{\frac{\varepsilon}{2}\tilde{b}_i(s)}\, dB^i(s) \qquad 2 \leq j \leq 2k. \tag{2.37}$$

and for any $1 \leq m \leq 2k$ we define

$$\nu^{\varepsilon,k}_{mt/k} := \sum_{i \in I'(mt/k)} \tilde{b}_i(mt/k)\delta_{\tilde{x}_i(mt/k)}, \tag{2.38}$$

where $I'(mt/k)$ is a random subset of $I(t/k)$ such that $\tilde{b}_i(mt/k) > 0$ if $i \in I'(mt/k)$. By (2.29),(2.37), and Lemma 2.1 with $\sigma(x) \equiv \varepsilon$, for any $1 \leq l \leq k$ we can get

$$V^{\varepsilon,k}_{t/k} F(\nu^{\varepsilon,k}_{(2l-1)t/k}) = \mathbb{E}_{\nu^{\varepsilon,k}_{(2l-1)t/k}} F(\nu^{\varepsilon,k}_{(2l)t/k}) \tag{2.39}$$

Thus, we have

$$\begin{aligned} &\mathbb{E}_{\mu_0} F(\nu^{\varepsilon,k}_{(2k)t/k}) \\ &= \mathbb{E}_{\mu_0}\{\mathbb{E}_{\nu^{\varepsilon,k}_{t/k}}\{\mathbb{E}_{\nu^{\varepsilon,k}_{2t/k}} \cdots \{\mathbb{E}_{\nu^{\varepsilon,k}_{(2k-1)t/k}} F(\nu^{\varepsilon,k}_{(2k)t/k})\} \cdots \} \\ &= [V^{\varepsilon/2,k}_{t/k} \cdot T^{\varepsilon/2}_{t/k}]^k F(\mu_0). \end{aligned} \tag{2.40}$$

By Lemma 2.2, we know that for any natural integer k and for any $1 \leq m \leq 2k$, $I(mt/k) \subseteq I'(mt/k)$ holds almost surely with respect to $\mathbb{P}$. Thus, this is true in distribution for the limiting processes $\{\mu^\sigma_t : t \geq 0\}$ and $\{\nu^\varepsilon_t : t \geq 0\}$ and we conclude that for any $t > 0$, μ^σ_t is a purely-atomic-measure. The remaining conclusion follows from Itô's formula. □

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References

[1] Dawson, D. A. (1992). Infinitely divisible random measures and superprocesses. in *Proc. 1990 Workshop on Stochastic Analysis and Related Topics*, Silivri, Turkey, 1992.

[2] Dawson, D. A. (1993). Measure-valued Markov processes. *Lecture Notes in Math., Springer, Berlin*, 1541:1–260, 1993.

[3] Dawson, D. A.; Li, Z. and Wang, H. (2001). Superprocesses with dependent spatial motion and general branching densities. *Electronic Journal of Probability*, V6, 25:1–33, 2001.

[4] Dynkin, E. B. (1994). *An Introduction to Branching Measure-Valued Processes*. CRM Monograph Series, Vol. 6, Amer. Math. Soc., Providence, 1994.

[5] Ethier, S. N. and Kurtz, T. G. (1986). *Markov Processes*: *Characterization and Convergence*. John Wiley and Sons, New York, 1986.

[6] Ethier, S. N. and Kurtz, T. G. (1987). The infinitely-many-alleles model with selection as a measure-valued diffusion. *Lecture Notes in Biomath.*, 70:72–86, 1987.

[7] Ethier, S. N. and Kurtz, T. G. (1993). Fleming-Viot processes in population genetics. *SIAM J. Control Optim.*, 31:345–386, 1993.

[8] Kurtz, T. G. (1981). *Approximation of Population Processes*. CBMS-NSF Regional Conf. Series in Appl. Math. 36, SIAM, Philadelpha, 1981.

[9] Le Gall, J. F. (1999). *Spatial Branching Processes, Random Snakes and Partial Differential Equations*. Birkhäuser Verlag, 1999.

[10] Perkins, E.(2000). *Dawson-Watanabe Superprocesses and Measure-valued Diffusions*. St. Flour Notes, Springer-Verlag, 2000.

[11] Trotter, H. F. (1959). On the product of semigroups of operators. *Proc. Amer. Math. Soc.*, 10:545–551, 1959.

[12] Walsh, J. B. (1986). An introduction to stochastic partial differential equations. *Lecture Notes in Math.*, 1180:265–439, 1986.

[13] Wang, H. (1997).

State classification for a class of measure-valued branching diffusions in a Brownian medium. *Probab. Th. Rel. Fields*, 109:39–55, 1997.

[14] Wang, H. (1998). A class of measure-valued branching diffusions in a random medium. *Stochastic Anal. Appl.*, 16(4):753–786, 1998.