DOI: 10.1214/ECP.v4-1011

Elect. Comm. in Probab. **4** (1999) 103–109

ELECTRONIC COMMUNICATIONS in PROBABILITY

CAPACITY ESTIMATES, BOUNDARY CROSSINGS AND THE ORNSTEIN–UHLENBECK PROCESS IN WIENER SPACE

Endre CSÁKI[1]
Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences
P. O. Box 127, H–1364 Budapest, Hungary
csaki@math-inst.hu

Davar KHOSHNEVISAN[2]
University of Utah, Department of Mathematics, Salt Lake City, UT 84112, U. S. A.
davar@math.utah.edu

Zhan SHI[3]
Laboratoire de Probabilités, Université Paris VI
4 Place Jussieu, F–75252 Paris Cedex 05, France
zhan@proba.jussieu.fr

submitted September 1, 1999; final version *accepted* in final form November 20, 1999

AMS subject classification: Primary: 60G60; Secondary: 60J60.
Keywords and phrases: Capacity on Wiener space, quasi-sure analysis, Ornstein–Uhlenbeck process, Brownian sheet

Abstract:
Let T_1 denote the first passage time to 1 of a standard Brownian motion. It is well known that as $\lambda \to \infty$, $\mathbb{P}\{T_1 > \lambda\} \sim c\lambda^{-1/2}$, where $c = (2/\pi)^{1/2}$. The goal of this note is to establish a capacitarian version of this result. Namely, we will prove the existence of positive and finite constants K_1 and K_2 such that for all $\lambda > e^e$,

$$K_1\lambda^{-1/2} \le \mathsf{Cap}\{T_1 > \lambda\} \le K_2\lambda^{-1/2}\log^3(\lambda)\cdot\log\log(\lambda),$$

where 'log' denotes the natural logarithm, and Cap is capacity on Wiener space.

[1]Supported, in part, by the Hungarian National Foundation for Scientific Research, Grant No. T 019346 and T 029621 and by the joint French-Hungarian Intergovernmental Grant "Balaton" No. F25/97.
[2]Supported, in part, by a grant from NSF and by NATO grant No. CRG 972075.
[3]Supported, in part, by the joint French-Hungarian Intergovernmental Grant "Balaton" No. F25/97 and by NATO grant No. CRG 972075.

1 Introduction

The goal of this note is to present a capacitarian extension of the classical fact that

$$\lim_{\lambda\to\infty} \lambda^{1/2}\ \mathbb{P}\{T_1 > \lambda\} = (2/\pi)^{1/2}, \tag{1.1}$$

where T_1 is the first passage time to 1 of a standard linear Brownian motion $B = \{B(t);\ t \geq 0\}$. Let $\Omega = C([0,\infty))$ denote the collection of all continuous real functions on $[0,\infty)$. As usual, Ω is made into a Banach space, once it is endowed with the supremum norm. Let $\mathcal{F}$ denote the collection of all of its Borel sets and let $\mathbb{W}$ denote Wiener's measure on $(\Omega, \mathcal{F})$. The probability triple $(\Omega, \mathcal{F}, \mathbb{W})$ is the classical *Wiener space*, and let $O = \big(O_s;\ s \geq 0\big)$ denote an *Ornstein–Uhlenbeck* process on $(\Omega, \mathcal{F}, \mathbb{W})$, which is an Ω-valued diffusion with stationary measure equal to $\mathbb{W}$ and whose increments are independent one-dimensional Ornstein–Uhlenbeck processes. Williams [5] has observed that O can be described path-by-path, using a two-parameter Brownian sheet $W = \{W(s,t);\ s,t \geq 0\}$. Namely, we can define O_s for each s as the random function

$$O_s(t) = e^{-s/2} W(e^s, t), \quad t \geq 0.$$

By *Fukushima–Malliavin capacity*, we mean the following: for all Borel sets $A \subset \Omega$,

$$\mathsf{Cap}(A) = \int_0^\infty e^{-\tau}\mathbb{P}\{O_s \in A \text{ for some } s \in [0,\tau]\}\ d\tau.$$

This is also called the 1-capacity of A, as it is related to the 1-potential measure of O. The following is the main result of this paper.

Theorem 1.1 *There exist positive and finite constants K_1 and K_2 such that for all $\lambda > e^e$,*

$$\frac{K_1}{\lambda^{1/2}} \leq \mathsf{Cap}\{T_1 > \lambda\} \leq \frac{K_2\ (\log\lambda)^3 \cdot \log\log\lambda}{\lambda^{1/2}}.$$

Remark 1.2 For all $\omega \in \Omega$, $T_1(\omega)$ denotes the first passage time of ω to the level 1: $T_1(\omega) = \inf\{t \geq 0 : \omega(t) \geq 1\}$. In this notation, Eq. (1.1) states that $\mathbb{W}\big\{T_1 > \lambda\big\} \sim \sqrt{2/\pi}\ \lambda^{-1/2}$ $(\lambda \to \infty)$. □

There is a relation to the recent results of Csáki, Khoshnevisan and Shi [1]. Namely, by Lemma 2.2 below, and stated in terms of the observation of D. Williams, Theorem 1.1 asserts the existence of finite and positive constants K_3 and K_4, such that for all $\lambda > e^e$,

$$\frac{K_3}{\lambda^{1/2}} \leq \mathbb{P}\Big\{\inf_{1\leq s\leq e}\sup_{0\leq t\leq\lambda} W(s,t) \leq 1\Big\} \leq \frac{K_4\ (\log\lambda)^3 \cdot \log\log\lambda}{\lambda^{1/2}}, \tag{1.2}$$

while [1, Theorem 1.5] states that

$$\exp\Big(-K_3\ (\log\lambda)^2\Big) \leq \mathbb{P}\Big\{\sup_{0\leq s\leq 1}\sup_{0\leq t\leq\lambda} W(s,t) \leq 1\Big\} \leq \exp\Big(-K_4\frac{(\log\lambda)^2}{\log\log\lambda}\Big).$$

Above and hereafter, we designate uninteresting constants by K, K_5, K_6, …. These may change from line to line as well as within the lines.

2 Background Estimates

In this section, we present two basic estimates. For this first estimate, let $U = \{U(x);\ x \in \mathbb{R}\}$ denote an Ornstein–Uhlenbeck process that is indexed by $\mathbb{R}$ and is speeded up so that U is a centered Gaussian process with covariance

$$\mathbb{E}\{U(x)U(y)\} = e^{-|x-y|}, \qquad x, y \in \mathbb{R}. \tag{2.1}$$

Lemma 2.1 *There exist two finite constants $x_0 \in (0,1)$ and $t_0 > 0$, such that for all $x \in (0, x_0)$ and all $t > t_0$,*

$$\mathbb{P}\{\sup_{0\le s\le t} U(s) \le x\} \le 2e^{-(1-x)t}.$$

Proof The process $\{U(x);\ x \ge 0\}$ is a diffusion with generator $\mathcal{A}f(x) = f''(x) - xf'(x)$ whose symmetrizing measure is the standard Gaussian. Thus, a routine application of the spectral theorem shows that the probability in the statement of the lemma has an eigenfunction expansion in terms of the (countable) eigenvalues of the (compact operator) $\mathcal{A}$. Ref. [4] contains all of the delicate information that we will need about these eigenvalues to which the reader is referred for further details. Let $\lambda_1(x) \le \lambda_2(x) \le \cdots$ and $h_1^x, h_2^x, \ldots$ denote the ordered eigenvalues and the orthonormalized (in $L^2(e^{-t^2/2}dt)$) eigenfunctions of $\mathcal{A}$ on $(-\infty, x)$ with zero boundary conditions. Then,

$$\mathbb{P}\{\sup_{0\le s\le t} U(s) \le x\} = (2\pi)^{-1/2}\sum_{j=1}^{\infty} e^{-t\lambda_j(x)}\Big(\int_{-\infty}^{x} h_j^x(t)e^{-t^2/2}\, dt\Big)^2.$$

We will need the following three facts about these eigenvalues: (i) for all $j \ge 1$, $\lambda_j(x) \ge \lambda_1(x) + j - 1$; (ii) $\lambda_1(0) = 1$; and (iii) $\lambda_1'(0) = -(2/\pi)^{1/2}$. See Uchiyama [4, Prop. 1.1], all the time noting that our speed measure is twice that of Uchiyama. This accounts for our doubling of the eigenvalues. Applying these facts, in conjunction with the Cauchy–Schwarz inequality, yields

$$\begin{aligned}\mathbb{P}\{\sup_{0\le s\le t} U(s) \le x\} &\le (2\pi)^{-1/2}\sum_{j=1}^{\infty} e^{-t\left\{\lambda_1(x)+j-1\right\}} \times \\ &\qquad \times \int_{-\infty}^{x} \left|h_j^x(t)\right|^2 e^{-t^2/2}\, dt \times \int_{-\infty}^{x} e^{-t^2/2}\, dt \\ &\le (1-e^{-t})^{-1}e^{-t\lambda_1(x)}.\end{aligned}$$

The result follows from facts (iii) and (ii). □

For all $r \ge 0$ and for all Borel sets $A \subset \Omega$, we define the *incomplete r-capacity* $\mathsf{Cap}_r(A)$ as

$$\mathsf{Cap}_r(A) = \mathbb{P}\big\{O_s \in A \text{ for some } s \in [0, r]\big\}.$$

Our second background estimate relates capacities to incomplete capacities and is an exercise in Laplace transforms. We point out that this result has already been used in the Introduction to establish Eq. (1.2).

Lemma 2.2 *There exists a finite constant $K > 1$, such that for all Borel sets $A \subset \Omega$,*

$$K^{-1}\mathsf{Cap}_1(A) \le \mathsf{Cap}(A) \le K\mathsf{Cap}_1(A).$$

Proof Clearly,

$$\mathsf{Cap}(A) \geq \int_0^1 e^{-\tau}\mathbb{P}\big\{O_s \in A \text{ for some } s \in [0,1]\big\}\, d\tau.$$

This implies the lower bound. For the upper bound, note that

$$\begin{aligned}\mathsf{Cap}(A) &\leq \sum_{j=0}^{\infty}\int_j^{j+1} e^{-\tau}\mathbb{P}\big\{O_s \in A \text{ for some } s \in [0,j+1]\big\}\, d\tau \\ &\leq \sum_{j=0}^{\infty} e^{-j} \sum_{\ell=0}^{j}\mathbb{P}\big\{O_s \in A \text{ for some } s \in [\ell,\ell+1]\big\}.\end{aligned}$$

By stationarity, $\mathsf{Cap}(A) \leq \mathsf{Cap}_1(A) \sum_{j=0}^{\infty}(j+1)e^{-j}$, and the lemma follows. □

3 The Proof of Theorem 1.1

Throughout this proof, $B = \{B(t);\ t \geq 0\}$ denotes a standard linear Brownian motion and ε stands for a small positive number. We will also need three variables all of which are functions of ε as follows:

$$\delta = \varepsilon^2 \log^2(1/\varepsilon), \tag{3.1}$$

$$a = 1 + \frac{1}{c_0^2 \log^2(1/\varepsilon)\log\log(1/\varepsilon)}, \tag{3.2}$$

where $c_0 \in (0,\infty)$ is chosen to satisfy

$$\mathbb{P}\Big\{\sup_{\delta\leq t\leq 1}\frac{B(t)}{t^{1/2}} \leq c_0\sqrt{\log\log(1/\delta)}\Big\} \geq \frac{1}{2}. \tag{3.3}$$

By the law of the iterated logarithm, such a c_0 must exist and can be chosen independently of the values of δ and ε. Consider the following random time that is finite (a.s., but this is taken care of in the usual way by adding in appropriate null sets):

$$\sigma = \inf\Big\{s \geq 1:\ \sup_{\delta\leq t\leq 1}\frac{W(s,t)}{t^{1/2}} \leq \frac{\varepsilon}{\delta^{1/2}}\Big\},$$

where $W = \{W(s,t);\ s,t \geq 0\}$ is a two-parameter Brownian sheet. Let $\mathcal{F}^1$ denote the (complete, right continuous) filtration of the infinite-dimensional process $\{W(s,\bullet);\ s \geq 0\}$. It is easy to see that σ is a stopping time with respect to the one-parameter filtration $\mathcal{F}^1$.
Next, we define two events E and F:

$$\begin{aligned}\mathsf{E} &= \Big\{\sup_{\delta\leq t\leq 1}\frac{W(a,t)-W(\sigma,t)}{t^{1/2}} \leq c_0\sqrt{(a-1)\log\log(1/\delta)}\Big\}, \\ \mathsf{F} &= \Big\{\sup_{\delta\leq t\leq 1}\frac{W(a,t)}{t^{1/2}} \leq \frac{\varepsilon}{\delta^{1/2}} + c_0\sqrt{(a-1)\log\log(1/\delta)}\Big\}.\end{aligned}$$

Since $\{W(s,\bullet);\ s \geq 0\}$ is a Lévy process on Ω, the following lemma can be easily verified:

Lemma 3.1 *σ is a finite stopping time with respect to $\mathcal{F}^1$. Moreover, for any fixed $a > 0$,*

(i) conditional on $\{\sigma \le a\}$, $W(a,\bullet) - W(\sigma,\bullet)$ is independent of $\mathcal{F}^1$, and has the same distribution as $(a-\sigma)^{1/2}B(\bullet)$;

(ii) by the triangle inequality, $\mathsf{E} \cap \{\sigma \le a\} \subset \mathsf{F}$.

The following lemma partly shows our interest in the event F. The event E is used in our derivation that is to come.

Lemma 3.2 $\mathbb{P}\big\{\inf_{1\le s\le a}\sup_{0\le t\le 1} W(s,t) \le \varepsilon\big\} \le 2\mathbb{P}\{\mathsf{F}\}$.

Proof By Lemma 3.1, on $\{\sigma \le a\}$,

$$\begin{aligned}\mathbb{P}\{\mathsf{E} \mid \sigma\} &= \mathbb{P}\Big\{\sup_{\delta\le t\le 1}\frac{B(t)}{t^{1/2}} \le c_0\sqrt{\frac{a-1}{a-\sigma}\ \log\log(1/\delta)}\ \Big|\ \sigma\Big\} \\ &\ge \mathbb{P}\Big\{\sup_{\delta\le t\le 1}\frac{B(t)}{t^{1/2}} \le c_0\sqrt{\log\log(1/\delta)}\Big\} \\ &\ge \frac{1}{2}.\end{aligned}$$

The last line follows from (3.3). Using Lemma 3.1 (ii), we can deduce

$$\mathbb{P}\{\sigma \le a\} \le 2\mathbb{P}\{\mathsf{F}\}.$$

On the other hand,

$$\Big\{\inf_{1\le s\le a}\sup_{0\le t\le 1} W(s,t) \le \varepsilon\Big\} \subset \{\sigma \le a\}.$$

The lemma follows. □

To estimate $\mathbb{P}\{\mathsf{F}\}$, we observe that when ε is small,

$$\frac{\varepsilon/\delta^{1/2} + c_0\sqrt{(a-1)\log\log(1/\delta)}}{a^{1/2}} \le \frac{2}{a^{1/2}\log(1/\varepsilon)} \le \frac{2}{\log(1/\varepsilon)},$$

so that by scaling,

$$\mathbb{P}\{\mathsf{F}\} \le \mathbb{P}\Big\{\sup_{\delta\le t\le 1}\frac{B(t)}{t^{1/2}} \le \frac{2}{\log(1/\varepsilon)}\Big\}. \tag{3.4}$$

Define the process U by $U(x) = B(e^{-2x})/e^{-x}$, $x \in \mathbb{R}$. It follows from direct covariance computations that U is the same (in law) as the Ornstein–Uhlenbeck process in (2.1). Moreover,

$$\sup_{\delta\le t\le 1}\frac{B(t)}{t^{1/2}} = \sup_{0\le x\le \frac{1}{2}\log(1/\delta)} U(x).$$

Combining this with (3.4) and Lemma 2.1, we readily obtain the following:

$$\mathbb{P}\{\mathsf{F}\} \le K\varepsilon\log(1/\varepsilon).$$

Lemma 3.2 and the stationarity of the Ornstein–Uhlenbeck process, imply the following result that is interesting in its own right.

Proposition 3.3 *For all positive and finite K_5, there exists a finite $K_6 > 1$, such that whenever I is an interval in $[1, K_5+1]$ whose length is bounded above by $\{c_0^2 \log^2(1/\varepsilon)\log\log(1/\varepsilon)\}^{-1}$,*

$$\mathbb{P}\Big\{\inf_{s\in I}\sup_{0\le t\le 1} W(s,t) \le \varepsilon\Big\} \le K_6\ \varepsilon\log(1/\varepsilon), \quad \forall \varepsilon \in (0, K_6^{-1}).$$

Proof of Theorem 1.1 Since $[1, e]$ can be covered by $2c_0^2\log^2(1/\varepsilon)\log\log(1/\varepsilon)$ many intervals I of the above type, we deduce the following estimate: for all $\varepsilon > 0$ small,

$$\mathbb{P}\Big\{\inf_{1\le s\le e}\sup_{0\le t\le 1} W(s,t) \le \varepsilon\Big\} \le K\varepsilon\log^3(1/\varepsilon)\log\log(1/\varepsilon). \tag{3.5}$$

By scaling, we obtain the upper bound of Theorem 1.1 from Eq. (3.5). The lower bound of Theorem 1.1 follows immediately from Eq. (1.1). This completes our proof. □

It is possible to refine the rate given by Proposition 3.3, if the intervals are kept to small sizes. We conclude this article with a precise statement of this claim and its proof.

Proposition 3.4 *For all positive and finite C_1, there exists a finite $C_2 > 1$, such that whenever I is an interval in $[1, 1 + C_1]$ whose length is at most $C_1\varepsilon^2$,*

$$\mathbb{P}\Big\{\inf_{s\in I}\sup_{0\le t\le 1} W(s,t) \le \varepsilon\Big\} \le C_2\ \varepsilon, \quad \forall \varepsilon \in (0, C_2^{-1}).$$

By (1.1), this is sharp, up to a constant.

Proof Without loss of generality, $I = [p, p + C_1\varepsilon^2]$, where $p \in [1, C_2]$. Define

$$J = \int_I \mathbf{1}\Big\{\sup_{0\le t\le 1} W(s,t) \le \varepsilon\Big\}\ ds = \int_p^{p+C_1\varepsilon^2} \mathbf{1}\Big\{\sup_{0\le t\le 1} W(s,t) \le \varepsilon\Big\}\ ds,$$

where $\mathbf{1}\{\cdots\}$ denotes the indicator function of the events in the parentheses. Since the elements of I are greater than 1, Eq. (1.1) implies that for all small $\varepsilon > 0$,

$$K_7^{-1}\ \varepsilon|I| \le \mathbb{E}\{J\} \le K_7\ \varepsilon|I|, \tag{3.6}$$

where $|I| = C_1\varepsilon^2$ denotes the length of I. We now compute a conditional version of this calculation. Recalling the 1-parameter filtration $\mathcal{F}^1$, we define the martingale M as a continuous modification of the following

$$M_r = \mathbb{E}\{J \mid \mathcal{F}_r^1\}, \qquad r \ge 0.$$

Observe that for all $r \ge 0$,

$$M_r \ge \int_r^{p+C_1\varepsilon^2} \mathbb{P}\Big\{\sup_{0\le t\le 1} W(s,t) \le \varepsilon \;\Big|\; \mathcal{F}_r^1\Big\}\ ds \cdot \mathbf{1}\Big\{\sup_{0\le t\le 1} W(r,t) \le \frac{\varepsilon}{2}\Big\}.$$

Since $\{W(s,t) - W(r,t);\ s \ge r,\ t \ge 0\}$ is independent of $\mathcal{F}_r^1$, it follows that for all $p \le r \le p + C_1\varepsilon^2/2$,

$$\begin{aligned} M_r &\ge \int_r^{p+C_1\varepsilon^2} \mathbb{P}\Big\{\sup_{0\le t\le 1}(W(s,t) - W(r,t)) \le \frac{\varepsilon}{2}\Big\}\ ds \cdot \mathbf{1}\Big\{\sup_{0\le t\le 1} W(r,t) \le \frac{\varepsilon}{2}\Big\} \\ &= \int_r^{p+C_1\varepsilon^2} \mathbb{P}\Big\{\sup_{0\le t\le 1} W(s-r,t) \le \frac{\varepsilon}{2}\Big\}\ ds \cdot \mathbf{1}\Big\{\sup_{0\le t\le 1} W(r,t) \le \frac{\varepsilon}{2}\Big\} \\ &\ge \int_0^{C_1\varepsilon^2/2} \mathbb{P}\Big\{\sup_{0\le t\le 1} W(s,t) \le \frac{\varepsilon}{2}\Big\}\ ds \cdot \mathbf{1}\Big\{\sup_{0\le t\le 1} W(r,t) \le \frac{\varepsilon}{2}\Big\}, \end{aligned}$$

almost surely. Moreover, continuity considerations imply that the above holds a.s., simultaneously for all $r \in [p, p + C_1\varepsilon^2/2]$. By scaling, this leads to:

$$M_r \geq K_8\ \varepsilon^2 \cdot \mathbf{1}\Big\{ \sup_{0\leq t\leq 1} W(r,t) \leq \frac{\varepsilon}{2} \Big\}.$$

Consider the $\mathcal{F}^1$ stopping time $T = \inf\{s \geq p : \ \sup_{0\leq t\leq 1} W(s,t) \leq \varepsilon/2\}$, where $\inf \varnothing = +\infty$. Applying $r \equiv T$ and taking expectations in the above to see that

$$\mathbb{E}\big[M_T \mathbf{1}\{T < \infty\}\big] \geq K_8\ \varepsilon^2\ \mathbb{P}\Big\{ \inf_{p\leq s\leq p+C_1\varepsilon^2/2} \sup_{0\leq t\leq 1} W(s,t) \leq \frac{\varepsilon}{2} \Big\}.$$

Since M is a bounded martingale, by the optional stopping theorem and by Eq. (3.6), $\mathbb{E}\big[M_T \mathbf{1}\{T < \infty\}\big] = \mathbb{E}\{M_0\} = \mathbb{E}\{J\} \leq K_9\ \varepsilon^3$. The proposition follows upon relabeling the parameters C_1 and C_2. □

Acknowledgement

We are grateful to the editor as well as an anonymous referee for providing us with helpful suggestions and insightful comments.

References

[1] Csáki, E., Khoshnevisan, D. and Shi, Z.: Boundary crossings and the distribution function of the maximum of Brownian sheet. (1999) Preprint.

[2] Fukushima, M.: Basic properties of Brownian motion and a capacity on the Wiener space. *J. Math. Soc. Japan* **36**, (1984) 161–176.

[3] Malliavin, P.: Stochastic calculus of variation and hypoelliptic operators. In: *Proc. Intern. Symp. Stoch. Diff. Eq.* (Kyoto 1976), pp. 195–263. Wiley, New York, 1978.

[4] Uchiyama, K.: Brownian first exit from and sojourn over one sided moving boundary and application. *Z. Wahrsch. Verw. Gebiete*, **54**, pp. 75–116, 1980.

[5] Williams, D.: Appendix to P.-A. Meyer: Note sur les processus d'Ornstein–Uhlenbeck. *Sém. de Probab. XVI.* Lecture Notes in Mathematics, **920**, p. 133. Springer, Berlin, 1982.