

SOME REMARKS ON THE HEAT FLOW FOR FUNCTIONS AND FORMS

ANTON THALMAIER

Institut für Angewandte Mathematik

Universität Bonn, Wegelerstr. 6

D-53115 Bonn (Germany)

e-mail: `anton@wiener.iam.uni-bonn.de`*submitted* March 10, 1998; *revised* July 18, 1998

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Abstract

This note is concerned with the differentiation of heat semigroups on Riemannian manifolds. In particular, the relation $dP_t f = P_t df$ is investigated for the semigroup generated by the Laplacian with Dirichlet boundary conditions. By means of elementary martingale arguments it is shown that well-known properties which hold on complete Riemannian manifolds fail if the manifold is only BM-complete. In general, even if M is flat and f smooth of compact support, $\|dP_t f\|_\infty$ cannot be estimated on compact time intervals in terms of f or df .

1 Introduction

Let (M, g) be a Riemannian manifold and Δ its Laplacian. Consider the minimal heat semigroup associated to $\frac{1}{2}\Delta$ on functions given by

$$(P_t f)(x) = \mathbb{E}[(f \circ X_t(x)) 1_{\{t < \zeta(x)\}}] \quad (1.1)$$

where $X_\cdot(x)$ is Brownian motion on M starting at x , with (maximal) lifetime $\zeta(x)$. Denote by $W_{0,\cdot}: T_x M \rightarrow T_{X_\cdot(x)} M$ the linear transport on M along $X_\cdot(x)$ determined by the following pathwise covariant equation:

$$\begin{cases} \frac{D}{dr} W_{0,r} v = \frac{1}{2} \text{Ric}(W_{0,r} v, \cdot)^\# \\ W_{0,0} v = v. \end{cases} \quad (1.2)$$

By definition, $\frac{D}{dr} = //_{0,r} \frac{d}{dr} //_{0,r}^{-1}$ where $//_{0,\cdot}$ denotes parallel transport along $X_\cdot(x)$. For 1-forms $\alpha \in \Gamma(T^*M)$ let

$$(P_t^{(1)} \alpha)v = \mathbb{E}[\alpha_{X_t(x)} W_{0,t} v 1_{\{t < \zeta(x)\}}], \quad v \in T_x M. \quad (1.3)$$

It is a well-known consequence of the spectral theorem that on a complete Riemannian manifold M

$$dP_t f = P_t^{(1)} df \quad (1.4)$$

holds for all $f \in C_c^\infty(M)$ (compactly supported C^∞ functions on M) if, for instance,

$$\mathbb{E}[\|W_{0,t}\| \mathbf{1}_{\{X_t(x) \in K\}} \mathbf{1}_{\{t < \zeta(x)\}}] < \infty \quad (1.5)$$

for any $x \in M$ and any compact subset $K \subset M$. Indeed, (1.4) holds true for the semigroups associated to the self-adjoint extensions of the Laplacian on functions, resp. 1-forms. These semigroups defined by the spectral theorem can be identified with the stochastic versions (1.1) and (1.3) as soon as (1.3) is well-defined. The identification can be done, for instance, with straightforward martingale arguments by exhausting the manifold through a sequence of regular domains.

Note that from the defining equation (1.2) one gets

$$\|W_{0,t}\| \leq \exp \left\{ -\frac{1}{2} \int_0^t \underline{\text{Ric}}(X_s(x)) ds \right\}$$

where $\underline{\text{Ric}}(x)$ is the smallest eigenvalue of the Ricci tensor Ric_x at x . Thus (1.5) reads as a condition imposing lower bounds on the Ricci curvature of M .

The Brownian motions $X_\cdot(x)$ may be constructed as solutions of a globally defined (non-intrinsic) Stratonovich SDE on M of the form

$$dX = A(X) \circ dZ + A_0(X) dt \quad (1.6)$$

with $A \in \Gamma(\mathbb{R}^r \otimes TM)$, $A_0 \in \Gamma(TM)$ and Z an \mathbb{R}^r -valued Brownian motion on some filtered probability space satisfying the usual completeness conditions. For $x \in M$, let

$$\mathcal{F}_t(x) := \mathcal{F}_t^{X(x)} \equiv \sigma\{X_s(x) : 0 \leq s \leq t\} \quad (1.7)$$

be the filtration generated by $X(x)$ starting at x . Then, by [4], A and A_0 in the SDE (1.6) can be chosen in such a way that

$$W_{0,t} v \mathbf{1}_{\{X_t(x) \in K\}} = //_{0,t} \mathbb{E}^{\mathcal{F}_t(x)} [//_{0,t}^{-1} (T_x X_t) v \mathbf{1}_{\{X_t(x) \in K\}}]. \quad (1.8)$$

Suppose that, instead of (1.5), we have

$$\mathbb{E}[\|T_x X_t\| \mathbf{1}_{\{X_t(x) \in K\}} \mathbf{1}_{\{t < \zeta(x)\}}] < \infty \quad (1.9)$$

for any $x \in M$ and any compact subset $K \subset M$. Then

$$(P_t^{(1)} df) v = \mathbb{E}[(df)_{X_t(x)} T_x X_t v \mathbf{1}_{\{t < \zeta(x)\}}], \quad v \in T_x M. \quad (1.10)$$

Thus, supposing for simplicity that (M, g) is BM-complete, i.e., $\zeta(x) \equiv \infty$ a.s. for all $x \in M$, relation (1.4) comes down to a matter of differentiation under the integral.

This point of view rises the question whether completeness of M is an essential ingredient for (1.4) to hold. However, we show that (1.4) may fail on metrically incomplete manifolds, even if the manifold is flat and BM-complete. Even then, $\limsup_{t \rightarrow 0^+} \|dP_t f\|_\infty$ may be infinite for compactly supported $f \in C^\infty(M)$.

2 Differentiation of semigroups

We follow the methods of [7]. In the sequel we write occasionally $T_x f$ instead of df_x for the differential of a function f to avoid mix-up with stochastic differentials. Finally, we denote by $B(M)$ the bounded measurable functions on M and by $bC^1(M)$ the bounded C^1 -functions on M with bounded derivative.

Lemma 2.1 *Let (M, g) be a Riemannian manifold and $f \in B(M)$. Fix $t > 0$, $x \in M$, and $v \in T_x M$. Then*

$$\begin{aligned} N_s &:= T_{X_s(x)}(P_{t-s}f) T_x X_s v, & 0 \leq s < t \wedge \zeta(x), \\ \tilde{N}_s &:= T_{X_s(x)}(P_{t-s}f) W_{0,s} v, & 0 \leq s < t \wedge \zeta(x), \end{aligned}$$

are local martingales (with respect to the underlying filtration).

Proof To see the first claim, note that $(P_{t-\cdot}f)(X_\cdot(x))$ is a local martingale depending on x in a differentiable way. Thus, the derivative with respect to x is again a local martingale, see [1]. The second claim is reduced to the first one by conditioning with respect to $\mathcal{F}_\cdot(x)$ to filter out redundant noise. The second part may also be checked directly using the Weitzenböck formula

$$d\Delta f \equiv \Delta^{(1)}df = \Delta^{\text{hor}}df - \text{Ric}(df^\#, \cdot) \quad (2.1)$$

where $\Delta^{(1)}$ is the Laplacian on 1-forms and $\Delta^{\text{hor}}df$ the horizontal Laplacian on $O(M)$ acting on df when considered as equivariant function on $O(M)$. Indeed, by lifting things up to the orthonormal frame bundle $O(M)$ over M , we can write

$$\tilde{N}_s = F(s, U_s) \cdot U_s^{-1} W_{0,s} v$$

where U is a horizontal lift of the BM $X_\cdot(x)$ to $O(M)$ (i.e., a horizontal BM on $O(M)$ with generator $\frac{1}{2}\Delta^{\text{hor}}$) and

$$F: [0, t] \times O(M) \rightarrow \mathbb{R}^d, \quad F_i(s, u) := (dP_{t-s}f)_{\pi(u)}(ue_i), \quad i = 1, \dots, d = \dim M.$$

Then $d\tilde{N}_s \stackrel{\text{m}}{=} 0$ (equality modulo differentials of local martingales) follows by means of Itô's formula. \square

Notation For the Brownian motion $X_\cdot(x)$ on M , let

$$B = \int_0^\cdot //_{0,r}^{-1} \circ dX_r(x)$$

denote the anti-development of $X_\cdot(x)$ taking values in $T_x M$. By definition, B is a BM on $T_x M$ satisfying

$$A(X(x)) dZ = //_{0,\cdot} dB.$$

Lemma 2.2 *Let (M, g) be a Riemannian manifold, $f \in B(M)$, $x \in M$ and $t > 0$. Let $\Theta_{0,\cdot}: T_x M \rightarrow T_{X_\cdot(x)} M$ be linear maps such that*

$$T_{X_s(x)}(P_{t-s}f) \Theta_{0,s} v, \quad 0 \leq s < t \wedge \zeta(x),$$

is a continuous local martingale. Then

$$T_{X_s(x)}(P_{t-s}f) \Theta_{0,s} h_s - \int_0^s (T_{X_r(x)} P_{t-r}f) \Theta_{0,r} dh_r, \quad 0 \leq s < t \wedge \zeta(x), \quad (2.2)$$

is again a continuous local martingale for any adapted $T_x M$ -valued process h of locally bounded variation. In particular,

$$T_{X_s(x)}(P_{t-s}f) \Theta_{0,s} h_s - (P_{t-s}f)(X_s(x)) \int_0^s \langle \Theta_{0,r} \dot{h}_r, //_{0,r} dB_r \rangle, \quad 0 \leq s < t \wedge \zeta(x),$$

is a local martingale for any adapted process h with paths in the Cameron-Martin space $\mathbb{H}([0, t], T_x M)$, i.e., $h_\cdot(\omega) \in \mathbb{H}([0, t], T_x M)$ for almost all ω .

Proof Indeed, by Itô's lemma,

$$\begin{aligned} d(T_{X_s(x)}(P_{t-s}f) \Theta_{0,s} h_s) &= (T_{X_s(x)}(P_{t-s}f) \Theta_{0,s}) dh_s + d(T_{X_s(x)}(P_{t-s}f) \Theta_{0,s}) \cdot h_s \\ &\stackrel{\text{m}}{=} (T_{X_s(x)}(P_{t-s}f) \Theta_{0,s}) dh_s \end{aligned}$$

where $\stackrel{\text{m}}{=}$ stands for equality modulo local martingales. The second part can be seen using the formula

$$(P_{t-s}f)(X_s(x)) = \int_0^s T_{X_r(x)}(P_{t-r}f) //_{0,r} dB_r.$$

This proves the Lemma. □

Lemma 2.2 leads to explicit formulae for $dP_t f$ by means of appropriate choices for h which make the local martingales in Lemma 2.2 to uniformly integrable martingales. This can be done as in [7].

Theorem 2.3 [7] *Let $f: M \rightarrow \mathbb{R}$ be bounded measurable, $x \in M$ and $v \in T_x M$. Then, for any bounded adapted process h with paths in $\mathbb{H}(\mathbb{R}_+, T_x M)$ such that $(\int_0^{\tau_D \wedge t} |\dot{h}_s|^2 ds)^{1/2} \in L^1$, and the property that $h_0 = v$, $h_s = 0$ for all $s \geq \tau_D \wedge t$, the following formula holds:*

$$d(P_t f)_x v = -\mathbb{E} \left[f(X_t(x)) 1_{\{t < \zeta(x)\}} \int_0^{\tau_D \wedge t} \langle W_{0,s}(\dot{h}_s), //_{0,s} dB_s \rangle \right] \quad (2.3)$$

where τ_D is the first exit time of $X(x)$ from some relatively compact open neighbourhood D of x .

Theorem 2.4 *Let (M, g) be a BM-complete Riemannian manifold such that $\text{Ric} \geq \alpha$ for some constant α .*

(i) *For $f \in bC^1(M)$ the relation $dP_s f = P_s^{(1)} df$ holds for $0 \leq s \leq t$ if and only if*

$$\sup_{0 \leq s \leq t} \|dP_s f\|_\infty < \infty. \quad (2.4)$$

(ii) *Let $f \in C^1(M)$ be bounded such that (2.4) is satisfied. Then, for $t > 0$,*

$$\|dP_t f\|_\infty \leq \left(\left(\frac{1 - e^{-\alpha t}}{\alpha} \right)^{1/2} \frac{1}{t} \|f\|_\infty \right) \wedge \left(e^{-\alpha t/2} \|df\|_\infty \right) \quad (2.5)$$

with the convention $(1 - e^{-\alpha t})/\alpha = t$ for $\alpha = 0$.

Proof (i) Of course, $dP_s f = P_s^{(1)} df$ implies (2.4) in case df is bounded. On the other hand, let $f \in C^1(M)$ such that (2.4) holds. Condition (2.4) ensures the local martingale

$$\bar{N}_s = (dP_{t-s} f)_{X_s(x)} W_{0,s} v, \quad v \in T_x M,$$

of Lemma 2.1 to be a martingale for $0 \leq s \leq t$, which gives by taking expectations

$$(dP_t f)_x v = \mathbb{E}[(df)_{X_t(x)} W_{0,t} v] = P_t^{(1)} df(v).$$

(ii) As in (i), condition (2.4) for $f \in C^1(M)$ implies $(dP_t f)_x v = \mathbb{E}[(df)_{X_t(x)} W_{0,t} v]$ which shows $|d(P_t f)_x| \leq e^{-\alpha t/2} \|df\|_\infty$. On the other hand, by Lemma 2.2,

$$T_{X_\cdot(x)}(P_{t-\cdot} f) W_{0,\cdot} h_\cdot - (P_{t-\cdot} f)(X_\cdot(x)) \int_0^\cdot \langle W_{0,r} \dot{h}_r, //_{0,r} dB_r \rangle \quad (2.6)$$

is a local martingale for any adapted process h with $h_\cdot(\omega) \in \mathbb{H}([0, t], T_x M)$. If we take $h_s := (1 - s/t)v$ where $v \in T_x M$, then by means of assumption (2.4) and the bound on the Ricci curvature, (2.6) is seen to be a uniformly integrable martingale, hence

$$d(P_t f)_x v = -\frac{1}{t} \mathbb{E} \left[f \circ X_t(x) \int_0^t \langle W_{0,r} v, //_{0,r} dB_r \rangle \right].$$

Thus

$$\begin{aligned} |d(P_t f)_x| &\leq \frac{1}{t} \|f\|_\infty \left(\mathbb{E} \int_0^t \|W_{0,r}\|^2 dr \right)^{1/2} \\ &\leq \frac{1}{t} \|f\|_\infty \left(\int_0^t e^{-\alpha r} dr \right)^{1/2} \leq \frac{1}{t} \left(\frac{1 - e^{-\alpha t}}{\alpha} \right)^{1/2} \|f\|_\infty \end{aligned}$$

which shows part (ii). \square

Remark 2.5 [8] Let M be an arbitrary Riemannian manifold and $D \subset M$ an open set with compact closure and nonempty smooth boundary. Let $f \in B(M)$. Then, for $x \in D$ and $t > 0$,

$$|d(P_t f)_x| \leq c \|f\|_\infty$$

with a constant c depending on t , $\dim M$, $\text{dist}(x, \partial D)$ and the lower bound of Ric on D . This follows from Theorem 2.3 with an explicit choice for h . See [8] for the details.

Remark 2.6 In the abstract framework of the Γ_2 -theory of Bakry and Emery (e.g. [2]) lower bounds on the Ricci curvature $\text{Ric} \geq \alpha$ (i.e. $\Gamma_2 \geq \alpha\Gamma$) may be expressed equivalently in terms of the semigroup as

$$|dP_t f|^2 \leq e^{-\alpha t} P_t |df|^2, \quad t \geq 0,$$

for f in a sufficiently large algebra of bounded functions on M . However, in general, the setting does not include the Laplacian on metrically incomplete manifolds. On such spaces, we may have $\limsup_{t \rightarrow 0+} \|dP_t f\|_\infty = \infty$ for $f \in C_c^\infty(M)$, as can be seen from the examples below.

3 An example

Let $\mathbb{R}^2 \setminus \{0\}$ be the plane with origin removed. For $n \geq 2$, let M_n be an n -fold covering of $\mathbb{R}^2 \setminus \{0\}$ equipped with the flat Riemannian metric. See [6] for a detailed analysis of the heat kernel on such BM-complete spaces. In terms of polar coordinates $x = (r, \vartheta)$ on M_n with $0 < r < \infty$, $0 \leq \vartheta < 2n\pi$,

$$h(x) = \cos(\vartheta/n) J_{1/n}(r) \tag{3.1}$$

is a bounded eigenfunction of Δ on M_n (with eigenvalue -1); here $J_{1/n}$ denotes the Bessel function of order $1/n$. Note that $J_{1/n}(r) = O(r^{1/n})$ as $r \searrow 0$, consequently dh is unbounded on M_n . The martingale property of

$$m_t = e^{t/2} (h \circ X_t(x)), \quad t \geq 0,$$

implies $P_t h = e^{-t/2} h$ which means that $dP_t h$ is unbounded on M_n as well.

Example 3.1 On M_n the relation $dP_t f = P_t^{(1)} df$ fails in general for compactly supported $f \in C^\infty(M_n)$. If this happens, then by Theorem 2.4 (i),

$$\sup_{0 \leq s \leq t} \|dP_s f\|_\infty = \infty \tag{3.2}$$

for $f \in C^\infty(M_n)$ of compact support.

Proof Otherwise (3.2) holds true for all compactly supported $f \in C^\infty(M_n)$. Fix $t > 0$. Then by Theorem 2.4 (ii)

$$\|dP_t f\|_\infty \leq \frac{1}{\sqrt{t}} \|f\|_\infty \tag{3.3}$$

for any compactly supported $f \in C^\infty(M_n)$. On the other hand, we may choose a sequence (f_ℓ) of nonnegative, compactly supported elements in $C^\infty(M_n)$ such that $f_\ell \nearrow h^c := h + c$ with h given by (3.1) and c a constant such that $h + c \geq 0$. But then (see Chavel [3] p.187 Lemma 3; note that this is a local argument which can be applied on any open relatively compact subset of M)

$$P_t f_\ell \nearrow P_t h^c \quad \text{and} \quad dP_t f_\ell \rightarrow dP_t h^c \quad \text{as } \ell \rightarrow \infty.$$

By (3.3) we would have

$$\|dP_t h^c\|_\infty \leq \frac{1}{\sqrt{t}} \|h^c\|_\infty,$$

in contradiction to $\|dP_t h^c\|_\infty = e^{-t/2} \|dh\|_\infty = \infty$. □

Remark 3.2 In [5] it is shown that if a stochastic dynamical system of the type (1.6) is strongly 1-complete, and if for each compact set K there is a $\delta > 0$ such that

$$\sup_{x \in K} \mathbb{E} \|T_x X_s\|^{1+\delta} < \infty,$$

then $dP_t f = P_t^{(1)} df$ holds true for functions $f \in bC^1(M)$. Example 3.1 shows that the strong 1-completeness is necessary and can not be replaced by completeness.

On M_n consider the heat equation for 1-forms

$$\begin{cases} \frac{\partial}{\partial t} \alpha = \frac{1}{2} \Delta^{(1)} \alpha \\ \alpha|_{t=0} = df \end{cases} \quad (3.4)$$

where $f \in C^\infty(M_n)$. Take $f \in C^\infty(M_n)$ of compact support with $dP_t f \neq P_t^{(1)} df$. Then

$$\alpha_t^1 := P_t^{(1)} df \quad \text{and} \quad \alpha_t^2 := dP_t f$$

define two different smooth solutions to (3.4). Note that $\|\alpha_t^i\| \in L^2$, $i = 1, 2$.

Corollary 3.3 *On the n -fold cover M_n of the punctured plane ($n \geq 2$) there are infinitely many nontrivial classical solutions to*

$$\begin{cases} \frac{\partial}{\partial t} \alpha = \frac{1}{2} \Delta^{(1)} \alpha \\ \alpha|_{t=0} = 0 \end{cases}$$

of the form $\alpha_t = P_t^{(1)} df - dP_t f$ with $f \in C^\infty(M_n)$ of compact support.

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