

A spectral decomposition for the Bolthausen-Sznitman coalescent and the Kingman coalescent

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Abstract

We consider both the Bolthausen-Sznitman and the Kingman coalescent restricted to the partitions of $\{1, \dots, n\}$. Spectral decompositions of the corresponding generators are derived. As an application we obtain a formula for the Green's functions and a short derivation of the well-known formula for the transition probabilities of the Bolthausen-Sznitman coalescent.

Keywords: Bolthausen-Sznitman coalescent; Kingman coalescent; spectral decomposition; Green's matrix; random recursive tree.

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1 Introduction

An exchangeable coalescent process is a discrete or continuous-time Markov chain that encodes the dynamics of particles grouped into so-called blocks. The jumps of the process consist of mergers of two or more blocks, and the rate at which a merger happens only depends on the current number of blocks, but not, for instance, on their sizes or the specific particles they contain. The theory of exchangeable coalescent processes has its origins in the study of genealogies in population genetics, culminating in the seminal work of Kingman [4].

Among exchangeable coalescent processes the so-called Λ -coalescents have received increasing attention in recent years. The latter were introduced independently by Donnelly and Kurtz [2], Pitman [7] and Sagitov [9]. A Λ -coalescent $\{\Pi(t), t \geq 0\}$ is a time-homogeneous exchangeable coalescent process in continuous time with state space $\mathcal{P}_{\mathbb{N}}$, the set of partitions of the non-negative integers $\mathbb{N} := \{1, 2, \dots\}$, that only allows for one merger of blocks at any jump. It can be characterized via its restrictions $\{\Pi^n(t), t \geq 0\}$ to $[n] := \{1, \dots, n\}$ as follows. If at any given time $\Pi^n(t)$ contains $b \geq 2$ blocks, then any $2 \leq k \leq b$ of these blocks merge at rate $\lambda_{b,k} := \int_0^1 x^{k-2}(1-x)^{b-k} \Lambda(dx)$, where Λ denotes a finite measure on the unit interval. This measure Λ together with the initial state $\Pi(0)$ uniquely determines Π , hence the name Λ -coalescent.

In this note we consider both the Kingman coalescent $\Pi^K = \{\Pi^K(t), t \geq 0\}$ and the Bolthausen-Sznitman coalescent $\Pi^{BS} = \{\Pi^{BS}(t), t \geq 0\}$, which are both Λ -coalescents. For convenience we drop the superscripts K and BS when there is no risk of ambiguity.

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The article is organized as follows. Our main results are spectral decompositions of the generator of the Bolthausen-Sznitman n -coalescent, Theorem 2.1, respectively of the generator of Kingman's n -coalescent, Theorem 2.7. As Corollaries we obtain for the Bolthausen-Sznitman coalescent a derivation of the formula for the transition probabilities that goes back to [1], and a formula for its Green's matrix. As a further application we obtain a spectral decomposition of the generator of the block counting process of the Bolthausen-Sznitman, respectively Kingman's coalescent.

2 Results

Let us introduce some notation. A partition of a set A is a set, π say, of nonempty pairwise disjoint subsets of A whose union is A . The members of π are called the blocks of π . Let $\#A$ denote the cardinality of A and let \mathcal{P}_A denote the set of partitions of A .

2.1 Bolthausen-Sznitman n -coalescent

The Bolthausen-Sznitman n -coalescent $\Pi^{n,BS} = \{\Pi^{n,BS}(t), t \geq 0\}$ is obtained by choosing Λ to be the uniform measure on $[0, 1]$. From the definition of the rates in the introduction it follows that the corresponding Q -matrix $Q := Q^{n,BS} = (q_{\pi\rho})_{\pi, \rho \in \mathcal{P}_{[n]}}$ is given by

$$q_{\pi\rho} = \begin{cases} \frac{(\#\rho-1)!(\#\pi-\#\rho-1)!}{(\#\pi-1)!} & \text{if } \pi \prec \rho, \\ -(\#\pi-1) & \text{if } \pi = \rho, \\ 0 & \text{otherwise,} \end{cases} \quad (2.1)$$

where $\pi \prec \rho$ if and only if ρ is obtained by exactly one merger of blocks of π . In this section we only consider the Bolthausen-Sznitman n -coalescent and therefore write Π instead of $\Pi^{n,BS}$.

For any two sets A and $B \subseteq A$ and a partition $\pi \in \mathcal{P}_A$ we call $\pi|_B := \{C \cap B : C \in \pi, C \cap B \neq \emptyset\} \in \mathcal{P}_B$ the restriction of π to B .

Theorem 2.1 (Spectral decomposition of the Bolthausen-Sznitman coalescent). *Let $L = (l_{\pi\rho})_{\pi, \rho \in \mathcal{P}_{[n]}}$ and $R = (r_{\pi\rho})_{\pi, \rho \in \mathcal{P}_{[n]}}$ be matrices defined by*

$$l_{\pi\rho} := \begin{cases} (-1)^{\#\pi-\#\rho} \frac{(\#\rho-1)!}{(\#\pi-1)!} & \text{if } \pi \leq \rho, \\ 0 & \text{otherwise,} \end{cases} \quad (2.2)$$

and

$$r_{\pi\rho} := \begin{cases} \frac{(\#\rho-1)!}{(\#\pi-1)!} \prod_{B \in \rho} (\#\pi|_B - 1)! & \text{if } \pi \leq \rho, \\ 0 & \text{otherwise,} \end{cases} \quad (2.3)$$

where $\pi \leq \rho$ if and only if each block of π is contained in a block of ρ . Then a spectral decomposition of Q is given by $Q = RDL$, where $D = (d_{\pi\rho})_{\pi, \rho \in \mathcal{P}_{[n]}}$ is defined by $d_{\pi\pi} = -(\#\pi - 1)$ and $d_{\pi\rho} = 0$ if $\pi \neq \rho$. In particular, $l_{\pi\pi} = r_{\pi\pi} = 1$ for any $\pi \in \mathcal{P}_{[n]}$.

The right eigenvectors $r_{\pi\rho}$ may be interpreted as the probability that a random recursive tree on the label set π can be cut down to a tree on the label set ρ , see the paragraph "A connection with random recursive trees" in Section 3.1 below. As an application of this spectral decomposition we derive the well-known formula for the transition probabilities of Π given by Bolthausen and Sznitman in [1], Proposition 1.4.

Corollary 2.2 (Transition probabilities of the Bolthausen-Sznitman coalescent). *For any two partitions $\pi, \rho \in \mathcal{P}_{[n]}$ and any time $t \geq 0$ the transition probabilities $p_{\pi\rho}(t) :=$*

$\mathbb{P}\{II(t) = \rho | II(0) = \pi\}$ of the Bolthausen-Sznitman n -coalescent are given by

$$p_{\pi\rho}(t) = (-1)^{\#\rho} e^{t \frac{(\#\rho - 1)!}{(\#\pi - 1)!}} \prod_{B \in \rho} (-e^{-t})^{\overline{\#\pi|_B}} = (e^{-t})^{\#\rho - 1} \frac{(\#\rho - 1)!}{(\#\pi - 1)!} \prod_{B \in \rho} (1 - e^{-t})^{\overline{\#\pi|_B - 1}},$$

where for $x \in \mathbb{R}$, $k \in \mathbb{N}$ we denote by $x^{\overline{k}} := x(x + 1) \cdots (x + k - 1)$ the ascending factorial power with the convention $x^{\overline{0}} := 1$.

Thanks to the spectral decomposition, Theorem 2.1, we obtain a formula for the Green's matrix $G = (g_{\pi\rho})_{\pi, \rho \in \mathcal{P}_{[n]}}$ of the Bolthausen-Sznitman n -coalescent defined by $g_{\pi\rho} := \int_0^\infty p_{\pi\rho}(t) dt$. Recall that $g_{\pi\rho} = \mathbb{E}[\int_0^\infty 1_{\{II(t)=\rho\}} dt | II(0) = \pi]$ is the expected total time that II spends in ρ starting from π . We denote by $\left[\begin{smallmatrix} i \\ j \end{smallmatrix} \right]$ the Stirling permutation numbers, which count the number of permutations of a set of i elements with j cycles.

Corollary 2.3 (Green's matrix of the Bolthausen-Sznitman coalescent). *The Green's matrix $G = (g_{\pi\rho})_{\pi, \rho \in \mathcal{P}_{[n]}}$ is given by*

$$g_{\pi\rho} = \begin{cases} (-1)^{\#\rho} \frac{(\#\rho - 1)!}{(\#\pi - 1)!} \sum_{(k_B)_{B \in \rho} \in \mathbb{N}^{\#\rho}} \frac{(-1)^{|k|}}{|k| - 1} \prod_{B \in \rho} \left[\begin{smallmatrix} \#\pi|_B \\ k_B \end{smallmatrix} \right] & \text{if } \pi \leq \rho \neq \{[n]\}, \\ \infty & \text{if } \pi \leq \rho = \{[n]\}, \\ 0 & \text{otherwise,} \end{cases} \quad (2.4)$$

where $|k| := \sum_{B \in \rho} k_B$.

Remark 2.4. Notice that since the return probability to any state $\pi \in \mathcal{P}_{[n]}$, $\pi \neq \{[n]\}$, equals 0 for any Λ -coalescent II there is a close connection between the Green's matrix G of II and its hitting probabilities defined by $h(\pi, \rho) := \mathbb{P}\{II \text{ hits } \rho \text{ when started from } \pi\}$, namely via $g_{\pi\rho} = h(\pi, \rho)/q_\rho = h(\pi, \rho)/(1 - \#\rho)$, cf. [6], p. 146, where $q_\rho := \sum_{\sigma \in \mathcal{P}_{[n]}, \sigma \neq \rho} q_{\rho\sigma}$ is the total rate in ρ .

For a set A and $j \in \mathbb{N}$ let $\mathcal{P}_{A,j}$ denote the set of partitions of A into j blocks. Moreover, $\left\{ \begin{smallmatrix} i \\ j \end{smallmatrix} \right\}$ denotes the Stirling partition numbers, which count the number of partitions into j blocks of a set of i elements.

Remark 2.5. To the best of the authors' knowledge, Corollary 2.3 is the first result on the Green's matrix, respectively hitting probabilities, of the (partition-valued) Bolthausen-Sznitman coalescent. However, the hitting probabilities of the corresponding block counting process have been considered before in [5, Corollary 1.5] and the references given in Remark 1.6 therein. From Corollary 2.3 it follows by a technical but straightforward computation that for any $\pi \in \mathcal{P}_{[n],i}$ and $j \leq i$

$$\sum_{\rho \in \mathcal{P}_{[n],j}} g_{\pi\rho} = (-1)^j \frac{(j - 1)!}{(i - 1)!} \sum_{k=j}^i \frac{(-1)^k}{k - 1} \left[\begin{smallmatrix} i \\ k \end{smallmatrix} \right] \left\{ \begin{smallmatrix} k \\ j \end{smallmatrix} \right\}$$

in agreement with the entries of the Green's matrix of the block counting process of the Bolthausen-Sznitman coalescent provided in the proof of Corollary 1.5 in [5].

As a further application of the spectral decomposition of Q , Theorem 2.1, we derive a spectral decomposition of the generator $Q' = (q'_{ij})_{i,j \in [n]}$ of the block counting process $\{N(t), t \geq 0\}$ of the Bolthausen-Sznitman n -coalescent defined by $N(t) := \#II(t)$. It is well-known that the matrix Q' is given by $q'_{ij} = i/((i - j)(i - j + 1))$ if $i > j$, $q'_{ij} = 1 - i$ if $i = j$, and $q'_{ij} = 0$ otherwise.

In [5] this spectral decomposition of Q' was derived by means of generating functions without recourse to the partition-valued process II .

Corollary 2.6. Let $L' = (l'_{ij})_{i,j \in [n]}$, $R' = (r'_{ij})_{i,j \in [n]}$, and $D' = (d'_{ij})_{i,j \in [n]}$ be matrices given by

$$l'_{ij} := (-1)^{i-j} \frac{(j-1)!}{(i-1)!} \begin{Bmatrix} i \\ j \end{Bmatrix}, \quad r'_{ij} := \frac{(j-1)!}{(i-1)!} \begin{bmatrix} i \\ j \end{bmatrix}, \quad d'_{ij} := (1-i)1_{\{i=j\}}. \quad (2.5)$$

Then a spectral decomposition of Q' is given by $Q' = R'D'L'$.

2.2 Kingman's n -coalescent

Kingman's n -coalescent $\Pi^{n,K} = \{\Pi^{n,K}(t), t \geq 0\}$ is obtained by choosing Λ to be δ_0 , the Dirac measure in 0, that is the corresponding Q -matrix $Q := Q^{n,K} = (q_{\pi\rho})_{\pi,\rho \in \mathcal{P}_{[n]}}$ is given by

$$q_{\pi\rho} = \begin{cases} 1 & \text{if } \pi \leq \rho, \\ -\binom{\#\pi}{2} & \text{if } \pi = \rho, \\ 0 & \text{otherwise,} \end{cases} \quad (2.6)$$

where $\pi \leq \rho$ if and only if $\pi \leq \rho$ and $\#\pi - \#\rho = 1$. In other words, the jump chain of Kingman's n -coalescent is the directed simple random walk on the partition lattice $\mathcal{P}_{[n]}$, where at each step the chain jumps into a coarser partition. From now on we only consider Kingman's n -coalescent and therefore write Π instead of $\Pi^{n,K}$.

Theorem 2.7 (Spectral decomposition of Kingman's coalescent). Define the matrices $L = (l_{\pi\rho})_{\pi,\rho \in \mathcal{P}_{[n]}}$ and $R = (r_{\pi\rho})_{\pi,\rho \in \mathcal{P}_{[n]}}$ by

$$l_{\pi\rho} := \begin{cases} (-1)^{\#\pi - \#\rho} \frac{(\#\pi + \#\rho - 2)!}{(2\#\pi - 2)!} \prod_{B \in \rho} \#\pi|_B! & \text{if } \pi \leq \rho, \\ 0 & \text{otherwise,} \end{cases} \quad (2.7)$$

and

$$r_{\pi\rho} := \begin{cases} \frac{(2\#\rho - 1)!}{(\#\pi + \#\rho - 1)!} \prod_{B \in \rho} \#\pi|_B! & \text{if } \pi \leq \rho, \\ 0 & \text{otherwise.} \end{cases} \quad (2.8)$$

Then a spectral decomposition of Q is given by $Q = RDL$, where $D = (d_{\pi\rho})_{\pi,\rho \in \mathcal{P}_{[n]}}$ is defined by $d_{\pi\pi} := -\binom{\#\pi}{2}$ and $d_{\pi\rho} = 0$ if $\pi \neq \rho$. In particular, $l_{\pi\pi} = r_{\pi\pi} = 1$ for any $\pi \in \mathcal{P}_{[n]}$.

Remark 2.8. i) Although the spectral decomposition of Q , Theorem 2.7, directly yields a spectral decomposition of the matrix of transition probabilities $p_{\pi\rho}(t) := \mathbb{P}\{\Pi(t) = \rho | \Pi(0) = \pi\}$, analogously to the one given in (3.10) for the Bolthausen-Sznitman n -coalescent, this expression is not particularly handy. In fact, a better approach to calculating $p_{\pi\rho}(t)$ is the one given by Kingman in [4] Equation (2.5).

ii) A combinatorial interpretation of the right eigenvectors $r_{\pi\rho}$ is given in the proofs. Unlike in the Bolthausen-Sznitman case, the authors are not aware of a probabilistic interpretation of the $r_{\pi\rho}$.

As in the case of the Bolthausen-Sznitman coalescent, consider the block counting process $\{N(t), t \geq 0\}$ of Kingman's n -coalescent defined by $N(t) := \#\Pi(t)$ and let $Q' = (q'_{ij})_{i,j \in [n]}$ denote the generator of $N(t)$. From Theorem 2.7 we obtain a spectral decomposition of Q' .

Corollary 2.9. Let $L' = (l'_{ij})_{i,j \in [n]}$, $R' = (r'_{ij})_{i,j \in [n]}$ be matrices defined by

$$l'_{ij} := (-1)^{i+j} \frac{(i+j-2)!}{(2i-2)!} \begin{bmatrix} i \\ j \end{bmatrix}, \quad r'_{ij} := \frac{(2j-1)!}{(i+j-1)!} \begin{bmatrix} i \\ j \end{bmatrix}, \quad (2.9)$$

where $\begin{bmatrix} i \\ j \end{bmatrix} := \binom{i-1}{j-1} \frac{i!}{j!}$ denotes the unsigned Lah numbers which count the number of partitions into j blocks of a set of i elements, where the elements in each block are ordered. Then $Q' = R'D'L'$ is a spectral decomposition of Q' , where $D' = (d'_{ij})_{i,j \in [n]}$ is defined by $d'_{ii} = -\binom{i}{2}$ and $d'_{ij} = 0$ if $i \neq j$.

3 Proofs

In order to prove our results, we need some notions and facts from the theory of lattices that we collect from [10]. Recall that a partially ordered set (poset for short) (P, \leq) is a set P together with a binary relation \leq satisfying:

1. For all $p \in P$, $p \leq p$ (reflexivity).
2. If $p \leq q$ and $q \leq p$, then $p = q$ (antisymmetry).
3. If $p \leq q$ and $q \leq r$, then $p \leq r$ (transitivity).

The binary relation \leq is called the order and the elements of P are said to be ordered with respect to \leq . Recall that two posets P, Q are called isomorphic, in which case we write $P \cong Q$, if there exists an order-preserving bijection $\phi: P \rightarrow Q$ whose inverse is order-preserving. The cartesian product $P \times Q$ of two posets P, Q is defined on the set $\{(p, q) : p \in P, q \in Q\}$ by letting $(p, q) \leq (p', q')$ in $P \times Q$ if and only if $p \leq p'$ in P and $q \leq q'$ in Q . For $p, q \in P$ an upper bound of p and q is an element $r \in P$ such that $p, q \leq r$. A least upper bound of p and q is an upper bound $s \in P$ of p and q such that for any upper bound r of p and q one has $s \leq r$. Clearly, if a least upper bound of two elements p and q exists, it is unique. A greatest lower bound is defined in complete analogy. A lattice is a poset with the property that any two of its elements have a least upper bound and a greatest lower bound. It is well-known, that $\mathcal{P}_{[n]}$ together with the relation \leq as defined in Theorem 2.1 is a lattice, the so-called partition lattice. For $\pi, \rho \in \mathcal{P}_{[n]}$ with $\pi \leq \rho$ we call the set $[\pi, \rho] := \{\sigma \in \mathcal{P}_{[n]} : \pi \leq \sigma \leq \rho\}$ an interval. We will make repeated use of the isomorphism

$$[\pi, \rho] \cong \prod_{B \in \rho} \mathcal{P}_{\pi|_B}, \quad (3.1)$$

cf. Example 3.10.4 in [10]. For more information on posets in general and the partition lattice in particular the reader is referred to [10]. Evidently, we have $\mathcal{P}_{[n]} = [\Delta_{[n]}, \{[n]\}]$, where for any set A we let $\Delta_A := \{\{a\} : a \in A\}$ be the partition of A into singletons. Occasionally, we write Δ_n instead of $\Delta_{[n]}$.

Using the notation we just introduced, the n - A -coalescent Π^n is a Markov chain with state space $\mathcal{P}_{[n]}$, initial state $\Delta_{[n]}$ and Q -matrix $Q^n = (q_{\pi\rho}^n)_{\pi, \rho \in \mathcal{P}_{[n]}}$ given by

$$q_{\pi\rho}^n := \begin{cases} \lambda_{\#\pi, \#\pi - \#\rho + 1} & \text{if } \pi \prec \rho, \\ -\lambda_{\#\pi} & \text{if } \pi = \rho, \\ 0 & \text{otherwise,} \end{cases} \quad (3.2)$$

where $\lambda_b := \sum_{k=2}^b \binom{b}{k} \lambda_{b,k}$, $b \geq 2$, is the infinitesimal rate.

There are several extensions of \leq to a linear order on $\mathcal{P}_{[n]}$. Let us fix such an extension \leq_{ex} and notice that the following quantities of Q^n that we are interested in do not depend on the specific extension chosen. The linear order \leq_{ex} induces a natural bijection ψ from $\mathcal{P}_{[n]}$ to $[B_n]$, where B_n denotes the n th Bell number, which is the number $\#\mathcal{P}_{[n]}$ of partitions of $[n]$, defined inductively by letting $\psi(\Delta_{[n]}) = 1$ and $\psi(\pi) \leq \psi(\rho)$ iff $\pi \leq_{\text{ex}} \rho$. Then Q^n can be seen as an upper right triangular matrix with entries ordered according to \leq_{ex} , if we define row/column π to be lower than row/column ρ iff $\pi \leq_{\text{ex}} \rho$. The determinant of Q^n is therefore given by the product of its diagonal entries. Hence, the characteristic polynomial of Q^n is given by $\chi_{Q^n}(x) = \det(Q^n - x\mathbb{I}_n) = (-1)^{B_n} \prod_{\pi \in \mathcal{P}_{[n]}} (\lambda_{\#\pi} + x) = (-1)^{B_n} \prod_{i=1}^n (\lambda_i + x)^{\binom{n}{i}}$, where \mathbb{I}_n is the identity matrix on $\mathcal{P}_{[n]}$. Hence, for each $i \in [n]$, $-\lambda_i$ is an eigenvalue of Q^n with algebraic multiplicity $\#\mathcal{P}_{[n], i} = \binom{n}{i}$.

From now on we fix an $n \in \mathbb{N}$, $n \geq 2$, and drop this index in the notation, if there is no risk of confusion.

3.1 Bolthausen-Sznitman n -coalescent

In order to prepare the proof of the spectral decomposition of $Q = Q^{n,BS}$, Theorem 2.1, we calculate the left and right eigenvectors of Q . In the sequel we give two proofs for the right eigenvectors of Q that are of rather different flavours. The first proof is completely self-contained and only makes use of the partition lattice $\mathcal{P}_{[n]}$. Together with the proof of Lemma 3.5 it might serve as a starting point to find a spectral decomposition for more general coalescents, e.g. beta coalescents. There is a probabilistic interpretation of the right eigenvector of Q in terms of random recursive trees which then motivates our second proof that heavily draws on random recursive trees and their connection to the Bolthausen-Sznitman coalescent as explored in Goldschmidt and Martin [3].

Lemma 3.1. For $\rho \in \mathcal{P}_{[n]}$ the vector $(r_{\pi\rho})_{\pi \in \mathcal{P}_{[n]}}$ defined by

$$r_{\pi\rho} := \begin{cases} \frac{(\#\rho-1)!}{(\#\pi-1)!} \prod_{B \in \rho} (\#\pi|_B - 1)! & \text{if } \pi \leq \rho, \\ 0 & \text{otherwise,} \end{cases} \quad (3.3)$$

is a right eigenvector of Q with corresponding eigenvalue $1 - \#\rho$.

Proof. (first proof of Lemma 3.1) In order to carry out the following calculations, for any two partitions $\pi, \rho \in \mathcal{P}_{[n]}$ we need to parameterize the set $\{\sigma: \pi \prec \sigma \leq \rho\}$. To construct an arbitrary partition σ such that $\pi \prec \sigma$, we could choose a subset $C \subseteq \pi$ of at least two blocks of π and merge them in order to obtain σ . In this case $\#\sigma = \#\pi - \#C + 1$. If, additionally, we require $\sigma \leq \rho$, we certainly cannot choose any collection C of blocks in π . Instead, all blocks chosen have to be in $\pi|_B$ for some block $B \in \rho$, in which case $\#\sigma|_B = \#\pi|_B - \#C + 1$. To summarize, we have $\{\sigma: \pi \prec \sigma \leq \rho\} = \{\{\bigcup_{D \in C} D\} \cup (\pi \setminus C) : B \in \rho, C \subseteq \pi|_B, \#C \geq 2\}$. Using this parametrization, we obtain

$$\begin{aligned} \sum_{\sigma \in \mathcal{P}_{[n]}} q_{\pi\sigma} r_{\sigma\rho} &= \sum_{\sigma: \pi \prec \sigma \leq \rho} q_{\pi\sigma} r_{\sigma\rho} - (\#\pi - 1) r_{\pi\rho} \\ &= \sum_{B \in \rho} \sum_{\substack{C \subseteq \pi|_B \\ \#C \geq 2}} \frac{(\#\pi - \#C)! (\#C - 2)! (\#\rho - 1)!}{(\#\pi - 1)! (\#\pi - \#C)!} \\ &\quad \times (\#\pi|_B - \#C)! \prod_{D \in \rho \setminus \{B\}} (\#\pi|_D - 1)! - (\#\pi - 1) r_{\pi\rho} \\ &= \sum_{B \in \rho} \sum_{c=2}^{\#\pi|_B} \binom{\#\pi|_B}{c} \frac{(c-2)!}{(\#\pi - 1)!} (\#\rho - 1)! \frac{(\#\pi|_B - c)!}{(\#\pi|_B - 1)!} \prod_{D \in \rho} (\#\pi|_D - 1)! - (\#\pi - 1) r_{\pi\rho} \\ &= \left(\sum_{B \in \rho} \sum_{c=2}^{\#\pi|_B} \binom{\#\pi|_B}{c} (c-2)! \frac{(\#\pi|_B - c)!}{(\#\pi|_B - 1)!} - (\#\pi - 1) \right) r_{\pi\rho} \\ &= \left(\sum_{B \in \rho} \#\pi|_B \sum_{c=2}^{\#\pi|_B} \frac{1}{c(c-1)} - (\#\pi - 1) \right) r_{\pi\rho} \\ &= \left(\sum_{B \in \rho} (\#\pi|_B - 1) - (\#\pi - 1) \right) r_{\pi\rho} = (1 - \#\rho) r_{\pi\rho}, \end{aligned}$$

and the claim follows. \square

A connection with random recursive trees. Let us now recall the notion of a random recursive tree in order to prepare our second proof of Lemma 3.1, where we closely follow Goldschmidt and Martin [3]. We call a tree on n nodes labelled by $1, 2, \dots, n$

an increasing tree if the root has label 1 and the labels in any path from the root to another node are increasing.

If we have an increasing tree on $n - 1$ nodes, we obtain an increasing tree on n nodes by adding a node labelled n and attaching it by an edge to one of the nodes in the given tree. Starting from the tree that only consists of the root node 1 this gives an explicit construction of all increasing trees on n nodes. Consequently, there are $(n - 1)!$ increasing trees on n nodes. A random recursive tree is a tree chosen uniformly at random from all increasing trees on n nodes. An explicit construction of a random recursive tree on n nodes is the following. Start with the root node labelled 1. If the tree has k nodes, choose one of these nodes uniformly at random and attach to it node $k + 1$ by an edge. Stop after attaching node n .

For any partition $\pi \in \mathcal{P}_{[n]}$ an increasing tree on π is a tree with $\#\pi$ nodes that are labelled by the blocks in π such that the labels in any path from the root to another node are increasing with respect to their least element. We denote a random recursive tree on π by \mathfrak{T}_π .

Crucial to the construction of Π^n via random recursive trees is the following cutting procedure. When given a tree \mathcal{T} on π , a cut is performed by picking an edge, removing the subtree above this edge (here we picture trees as they grow in nature: from the root at the bottom to the leaves at the top) and adding the labels of this subtree to the labels of the node below the edge. For a simple tree the cutting procedure is depicted in Figure 1. We denote by $c\mathcal{T}$ the tree obtained by cutting \mathcal{T} at an edge chosen uniformly at

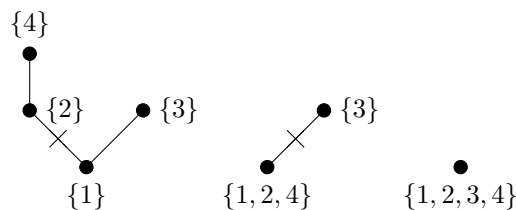


Figure 1: An increasing tree on $\{\{1\}, \{2\}, \{3\}, \{4\}\}$ cut down successively to a tree on $\{\{1, 2, 3, 4\}\}$.

random. For any tree \mathcal{T} on the label set π let $p(\mathcal{T}) = \pi$. A striking result is [3, Proposition 2.1], which states that after cutting a random recursive tree \mathfrak{T}_π at an edge chosen uniformly at random, the new tree $c\mathfrak{T}_\pi$ is a random recursive tree on the new label set (which is again a partition of $[n]$), more formally: $c\mathfrak{T}_\pi \stackrel{d}{=} \mathfrak{T}_{p(c\mathfrak{T}_\pi)}$.

For $\pi \in \mathcal{P}_{[n]}$ define a time-homogeneous continuous-time Markov chain $\mathcal{R}_\pi := \{\mathcal{R}_\pi(t), t \geq 0\}$ with values in the set of random recursive trees as follows. The initial state $\mathcal{R}_\pi(0)$ is the random recursive tree \mathfrak{T}_π on π . The process \mathcal{R}_π evolves according to the following dynamics. Suppose \mathcal{R}_π is in state \mathcal{T} . If \mathcal{T} has only one vertex, do nothing. Otherwise, attach to each edge in \mathcal{T} an exponential 1 clock, all clocks being independent. When the first clock rings, cut \mathcal{T} at the associated edge to obtain the next state of \mathcal{R}_π . Proposition 2.2 in [3] then establishes that Π^n is equal in distribution to $\{p(\mathcal{R}_{\Delta_n}(t)), t \geq 0\}$.

Let $\{R_\pi(k), k \geq 0\}$ denote the jump chain of \mathcal{R}_π . From the definition of \mathcal{R} it follows that a process that is equal in distribution to the jump chain R can be constructed recursively by letting

$$R_\pi(0) := \mathfrak{T}_\pi, \quad R_\pi(k + 1) := cR_\pi(k), \quad k \geq 0.$$

Fix two partitions $\pi, \rho \in \mathcal{P}_{[n]}$ such that $\pi \leq \rho$. We say that an increasing tree \mathcal{T} on π

contains ρ (in symbols: $\rho \sqsubset \mathcal{T}$) iff one can obtain an increasing tree on ρ by successively cutting \mathcal{T} . A simple question then is: what is the probability $\mathbb{P}\{\rho \sqsubset \mathfrak{T}_\pi\}$ that a random recursive tree on π contains ρ ? Notice first, that an increasing tree \mathcal{T} on π contains ρ if for each block $B \in \rho$ we can find a node $v = v(B)$ in \mathcal{T} such that the labels in the subtree above v coincide with the elements in $\pi|_B$. In other words, we can construct all increasing trees on π that contain ρ by first constructing an increasing tree \mathcal{T} on ρ . Then each node $v \in \mathcal{T}$ is labelled by some block $B \in \rho$. Now if $\#\pi|_B > 1$, build an increasing tree on $\pi|_B$ and replace v by this tree. This procedure is done for all nodes $v \in \mathcal{T}$ to obtain an increasing tree on π that contains ρ . Therefore, the number of increasing trees on π containing ρ is $\#\{\mathcal{T} : \mathcal{T} \text{ increasing tree on } \pi, \rho \sqsubset \mathcal{T}\} = (\#\rho - 1)! \prod_{B \in \rho} (\#\pi|_B - 1)!$. On the other hand, the total number of increasing trees on π is $(\#\pi - 1)!$. Since a random recursive tree on π is a tree chosen uniformly at random from all increasing trees on π , we obtain

$$\mathbb{P}\{\rho \sqsubset \mathfrak{T}_\pi\} = \begin{cases} (\#\rho - 1)! \prod_{B \in \rho} (\#\pi|_B - 1)! / (\#\pi - 1)! & \text{if } \pi \leq \rho, \\ 0 & \text{otherwise,} \end{cases} \tag{3.4}$$

which is just $r_{\pi\rho}$. We can now turn to our second proof of Lemma 3.1 in terms of random recursive trees.

Proof. (second proof of Lemma 1) We rewrite the statement as $\sum_{\sigma: \pi \prec \sigma} q_{\pi\sigma} r_{\sigma\rho} = (\#\pi - \#\rho)r_{\pi\rho}$. Let $J = \{J(k), k \geq 0\}$ denote the jump chain of $\Pi = \Pi^{BS,n}$. Since $\mathbb{P}\{J(1) = \sigma | J(0) = \pi\} = q_{\pi\sigma}/q_\pi$ is the probability that J jumps from π to σ , after dividing the rewritten statement by $q_\pi = \#\pi - 1$ we obtain for the left hand side

$$\begin{aligned} \sum_{\sigma: \pi \prec \sigma} \frac{q_{\pi\sigma}}{q_\pi} r_{\sigma\rho} &= \sum_{\sigma: \pi \prec \sigma} \mathbb{P}\{J(1) = \sigma | J(0) = \pi\} \mathbb{P}\{\rho \sqsubset \mathfrak{T}_\sigma\} \\ &= \sum_{\sigma: \pi \prec \sigma} \mathbb{P}\{p(c\mathfrak{T}_\pi) = \sigma\} \mathbb{P}\{\rho \sqsubset \mathfrak{T}_\sigma\} = \mathbb{P}\{\rho \sqsubset \mathfrak{T}_{p(c\mathfrak{T}_\pi)}\} \\ &= \mathbb{P}\{\rho \sqsubset c\mathfrak{T}_\pi\} = \mathbb{P}\{\rho \sqsubset c\mathfrak{T}_\pi | \rho \sqsubset \mathfrak{T}_\pi\} \mathbb{P}\{\rho \sqsubset \mathfrak{T}_\pi\}, \end{aligned}$$

where we used Propositions 2.2 and 2.1 of [3]. Conditional on $\rho \sqsubset \mathfrak{T}_\pi$, there are precisely $\#\pi - \#\rho$ among the $\#\pi - 1$ edges in \mathfrak{T}_π that we may cut in order to obtain a tree $c\mathfrak{T}_\pi$ that contains ρ . Since, by definition, $c\mathfrak{T}_\pi$ is obtained by cutting \mathfrak{T}_π at an edge chosen uniformly at random, we have $\mathbb{P}\{\rho \sqsubset c\mathfrak{T}_\pi | \rho \sqsubset \mathfrak{T}_\pi\} = (\#\pi - \#\rho) / (\#\pi - 1)$. The claim follows. \square

Lemma 3.2. For $x \in \mathbb{R} \setminus \{0\}$ we have

$$\sum_{\sigma \in \mathcal{P}_{[n]}} r_{\pi\sigma} x^{\#\sigma - 1} l_{\sigma\rho} = (-1)^{\#\rho} x^{-1} \frac{(\#\rho - 1)!}{(\#\pi - 1)!} \prod_{B \in \rho} (-x)^{\overline{\#\pi|_B}}. \tag{3.5}$$

Proof. Notice that

$$\begin{aligned} \sum_{\sigma \in \mathcal{P}_{[n]}} r_{\pi\sigma} x^{\#\sigma - 1} l_{\sigma\rho} &= \frac{(\#\rho - 1)!}{(\#\pi - 1)!} \sum_{\sigma \in [\pi, \rho]} (-1)^{\#\sigma - \#\rho} x^{\#\sigma - 1} \prod_{B \in \sigma} (\#\pi|_B - 1)! \\ &= (-1)^{\#\rho} x^{-1} \frac{(\#\rho - 1)!}{(\#\pi - 1)!} \sum_{\sigma \in [\pi, \rho]} \prod_{B \in \sigma} -x(\#\pi|_B - 1)!. \end{aligned} \tag{3.6}$$

Since $\begin{bmatrix} n \\ i \end{bmatrix}$ counts the number of permutations of $[n]$ with i cycles it is clear that $\begin{bmatrix} n \\ i \end{bmatrix} = \sum_{\pi \in \mathcal{P}_{[n], i}} \prod_{B \in \pi} (\#B - 1)!$, cf. equation (1.15) in [8]. Using the isomorphism $[\pi, \rho] \cong$

$\times_{B \in \rho} \mathcal{P}_{\pi|_B}$, we calculate

$$\begin{aligned} \sum_{\sigma \in [\pi, \rho]} \prod_{B \in \sigma} -x(\#\pi|_B - 1)! &= \sum_{\tau' \in \times_{B \in \rho} \mathcal{P}_{\pi|_B}} \prod_{B \in \rho} \prod_{C \in \tau'_B} -x(\#C - 1)! \\ &= \prod_{B \in \rho} \sum_{\tau \in \mathcal{P}_{\pi|_B}} \prod_{C \in \tau} -x(\#C - 1)! = \prod_{B \in \rho} \left(\sum_{k=1}^{\#\pi|_B} (-x)^k \sum_{\tau \in \mathcal{P}_{\pi|_B, k}} \prod_{C \in \tau} (\#C - 1)! \right) \\ &= \prod_{B \in \rho} \sum_{k=1}^{\#\pi|_B} (-x)^k \begin{bmatrix} \#\pi|_B \\ k \end{bmatrix} \end{aligned} \tag{3.7}$$

$$= \prod_{B \in \rho} (-x)^{\overline{\#\pi|_B}}, \tag{3.8}$$

where in the last step we used $x^{\overline{n}} = \sum_{k=1}^n \binom{n}{k} x^k$, cf. equation (1.16) in [8]. \square

Lemma 3.3. The matrix $L = (l_{\pi\rho})_{\pi, \rho \in \mathcal{P}_{[n]}}$ defined by

$$l_{\pi\rho} := \begin{cases} (-1)^{\#\pi - \#\rho} \frac{(\#\rho - 1)!}{(\#\pi - 1)!} & \text{if } \pi \leq \rho, \\ 0 & \text{otherwise,} \end{cases} \tag{3.9}$$

is the inverse matrix of R , i.e. $\sum_{\sigma \in \mathcal{P}_{[n]}} r_{\pi\sigma} l_{\sigma\rho} = \delta_{\pi\rho}$.

Proof. Choosing $x = 1$ in Lemma 3.2 we have that $\sum_{\sigma \in \mathcal{P}_{[n]}} r_{\pi\sigma} l_{\sigma\rho} = \delta_{\pi\rho}$. \square

Proof. (of Theorem 2.1) The claim follows by Lemmata 3.1 and 3.3. \square

Proof. (of Corollary 2.2) Since Π is a continuous-time Markov chain with finite state space, we have for $P(t) = (p_{\pi\rho}(t))_{\pi, \rho \in \mathcal{P}_{[n]}}$ the identity $P(t) = \exp(tRDL) = R \exp(tD)L$. In the last step we made use of the spectral decomposition, Theorem 2.1. In particular, this yields

$$p_{\pi\rho}(t) = \sum_{\sigma \in \mathcal{P}_{[n]}} r_{\pi\sigma} e^{-t(\#\sigma - 1)} l_{\sigma\rho} = \sum_{\sigma \in [\pi, \rho]} r_{\pi\sigma} e^{-t(\#\sigma - 1)} l_{\sigma\rho}. \tag{3.10}$$

Letting $x = e^{-t}$ in Lemma 3.2 proves the claim. \square

Proof. (of Corollary 2.3) From (3.7) we have

$$\begin{aligned} x^{-1} \sum_{\sigma \in [\pi, \rho]} \prod_{B \in \sigma} -x(\#\pi|_B - 1)! &= x^{-1} \prod_{B \in \rho} \sum_{k=1}^{\#\pi|_B} (-x)^k \begin{bmatrix} \#\pi|_B \\ k \end{bmatrix} \\ &= \sum_{(k_B)_{B \in \rho} \in \mathbb{N}^{\#\rho}} (-1)^{|k|} x^{|k| - 1} \prod_{B \in \rho} \begin{bmatrix} \#\pi|_B \\ k_B \end{bmatrix}, \end{aligned}$$

where $|k| := \sum_{B \in \rho} k_B$. Letting $x = e^{-t}$ and integrating out with respect to t we see for $\rho \neq \{[n]\}$

$$\int_0^\infty e^t \sum_{\sigma \in [\pi, \rho]} \prod_{B \in \sigma} -e^{-t(\#\pi|_B - 1)} dt = \sum_{(k_B)_{B \in \rho} \in \mathbb{N}^{\#\rho}} \frac{(-1)^{|k|}}{|k| - 1} \prod_{B \in \rho} \begin{bmatrix} \#\pi|_B \\ k_B \end{bmatrix},$$

where for $\rho = \{[n]\}$

$$\int_0^\infty e^t \sum_{\sigma \in [\pi, \rho]} \prod_{B \in \sigma} -e^{-t(\#\pi|_B - 1)} dt = \sum_{k=1}^{\#\pi} (-1)^k \int_0^\infty e^{-t(k-1)} dt \begin{bmatrix} \#\pi \\ k \end{bmatrix} = -\infty.$$

The claim follows from (3.6) and the definition of $g_{\pi\rho}$. \square

Proof. (of Corollary 2.6) Evidently, the rate at which a jump from i to j blocks occurs equals the sum of all rates at which a jump from a partition $\pi \in \mathcal{P}_{[n],i}$ to a partition $\rho \in \mathcal{P}_{[n],j}$ occurs, i.e. $q'_{ij} = \sum_{\rho \in \mathcal{P}_{[n],j}} q_{\pi\rho}$, where $\pi \in \mathcal{P}_{[n],i}$ is fixed arbitrarily. The quantity $\sum_{\rho \in \mathcal{P}_{[n],j}} q_{\pi\rho}$ does not depend on the choice of $\pi \in \mathcal{P}_{[n],i}$, as the following calculation shows. By the spectral decomposition of Q , Theorem 2.1, we obtain

$$\begin{aligned} q'_{ij} &= \sum_{\rho \in \mathcal{P}_{[n],j}} \sum_{\sigma \in [\pi, \rho]} r_{\pi\sigma} d_{\sigma\sigma} l_{\sigma\rho} \\ &= \sum_{\substack{\sigma: \pi \leq \sigma \\ \#\sigma \geq j}} \sum_{\substack{\rho: \sigma \leq \rho \\ \#\rho = j}} \frac{(\#\sigma - 1)!}{(\#\pi - 1)!} \prod_{B \in \sigma} (\#B - 1)! (1 - \#\sigma) (-1)^{\#\sigma - j} \frac{(j - 1)!}{(\#\sigma - 1)!} \\ &= \frac{(j - 1)!}{(i - 1)!} (-1)^j \sum_{\substack{\sigma: \pi \leq \sigma \\ \#\sigma \geq j}} (\#\sigma - 1)! \prod_{B \in \sigma} (\#B - 1)! (1 - \#\sigma) \frac{(-1)^{\#\sigma}}{(\#\sigma - 1)!} \left\{ \begin{matrix} \#\sigma \\ j \end{matrix} \right\} \\ &= \frac{(j - 1)!}{(i - 1)!} (-1)^j \sum_{k=j}^i (k - 1)! \begin{bmatrix} i \\ k \end{bmatrix} (1 - k) \frac{(-1)^k}{(k - 1)!} \left\{ \begin{matrix} k \\ j \end{matrix} \right\} = \sum_{k=j}^i r'_{ik} d'_{kk} l'_{kj}, \end{aligned}$$

hence, $Q' = R'D'L'$. □

3.2 Kingman's n -coalescent

For any two partitions $\pi, \rho \in \mathcal{P}_{[n]}$ such that $\pi \leq \rho$ one may ask in how many different ways the jump chain of Kingman's coalescent may reach ρ when started in π . Since at each step only one merger of a pair of blocks occurs, there are $\#\pi - \#\rho + 1$ steps to be taken, and so the set of different ways is $C(\pi, \rho) := \{(\pi_1, \dots, \pi_m) : \pi = \pi_1 < \dots < \pi_m = \rho, m = \#\pi - \#\rho + 1\}$, where we defined $<$ in the paragraph preceding Theorem 2.7. We call each element in $C(\pi, \rho)$ a maximal chain in $[\pi, \rho]$ and denote by $m(\pi, \rho) := \#C(\pi, \rho)$ the number of maximal chains in $[\pi, \rho]$. Before we turn to the proof of the spectral decomposition of Q , Theorem 2.7, we count the number of maximal chains $m(\pi, \rho)$ in $[\pi, \rho]$ in the next Lemma.

Lemma 3.4 (Number of maximal chains). *For $\pi, \rho \in \mathcal{P}_{[n]}$ with $\pi \leq \rho$ we have that*

$$m(\pi, \rho) = 2^{\#\rho - \#\pi} (\#\pi - \#\rho)! \prod_{B \in \rho} \#\pi|_B!. \tag{3.11}$$

Proof. Notice that any maximal chain (π_1, \dots, π_n) in $[\Delta_{[n]}, \{[n]\}]$ can be constructed as follows. Let $\pi_1 := \Delta_{[n]}$, and if π_i with $i < n$ is constructed, π_{i+1} is obtained by merging two blocks in π_i , which can be done in $\binom{\#\pi_i}{2}$ ways. When $\pi_n = \{[n]\}$ is reached, the construction is finished. This construction shows that there are $m(\Delta_{[n]}, \{[n]\}) = \binom{n}{2} \binom{n-1}{2} \dots \binom{2}{2} = 2^{1-n} n!(n-1)!$ maximal chains in $[\Delta_{[n]}, \{[n]\}]$. Hence (3.11) holds in the case $(\pi, \rho) = (\Delta_{[n]}, \{[n]\})$.

For the general case, recall the isomorphism $[\pi, \rho] \cong \times_{B \in \rho} \mathcal{P}_{\pi|_B}$. As a consequence, any maximal chain in $[\pi, \rho]$ can be built by choosing a maximal chain in each factor $\mathcal{P}_{\pi|_B}$ and then 'intertwining' these chains, i.e. ordering their elements (excluding the first element — the partition into singletons — in each chain) in any order subject to the restriction that the order of elements of the same chain is preserved. Consequently, we have

$$\begin{aligned} m(\pi, \rho) &= (\#\pi - \#\rho)! \prod_{B \in \rho} [(\#\pi|_B - 1)!]^{-1} \prod_{B \in \rho} m(\Delta_{\pi|_B}, \{\pi|_B\}) \\ &= (\#\pi - \#\rho)! \prod_{B \in \rho} 2^{1 - \#\pi|_B} \#\pi|_B! = 2^{\#\rho - \#\pi} (\#\pi - \#\rho)! \prod_{B \in \rho} \#\pi|_B!. \end{aligned} \quad \square$$

Lemma 3.5. For any $\rho \in \mathcal{P}_{[n]}$ the vector $(r_{\pi\rho})_{\pi \in \mathcal{P}_{[n]}}$ defined by

$$r_{\pi\rho} := \begin{cases} \frac{2^{\#\pi-\#\rho}(2\#\rho-1)!}{(\#\pi-\#\rho)!(\#\pi+\#\rho-1)!} m(\pi, \rho) & \text{if } \pi \leq \rho, \\ 0 & \text{otherwise,} \end{cases} \quad (3.12)$$

$$= \begin{cases} \frac{(2\#\rho-1)!}{(\#\pi+\#\rho-1)!} \prod_{B \in \rho} \#\pi|_B! & \text{if } \pi \leq \rho, \\ 0 & \text{otherwise,} \end{cases} \quad (3.13)$$

is a right eigenvector of Q with corresponding eigenvalue $-\binom{\#\rho}{2}$.

Proof. Fix $\pi, \rho \in \mathcal{P}_{[n]}$. If $\pi < \rho$, we have

$$\begin{aligned} \sum_{\sigma \in \mathcal{P}_{[n]}} q_{\pi\sigma} r_{\sigma\rho} &= \sum_{\sigma: \pi < \sigma} r_{\sigma\rho} - \binom{\#\pi}{2} r_{\pi\rho} \\ &= \frac{2^{\#\pi-1-\#\rho}(2\#\rho-1)!}{(\#\pi-1-\#\rho)!(\#\pi+\#\rho-2)!} \sum_{\sigma: \pi < \sigma} m(\sigma, \rho) - \binom{\#\pi}{2} r_{\pi\rho} \\ &= \left(\frac{(\#\pi-\#\rho)(\#\pi+\#\rho-1)}{2} - \binom{\#\pi}{2} \right) r_{\pi\rho} = -\binom{\#\rho}{2} r_{\pi\rho}, \end{aligned}$$

where we used $\sum_{\sigma: \pi < \sigma} m(\sigma, \rho) = m(\pi, \rho)$. If $\pi = \rho$, we have $\sum_{\sigma \in \mathcal{P}_{[n]}} q_{\pi\sigma} r_{\sigma\rho} = q_{\pi\pi} = -\binom{\#\rho}{2} r_{\pi\rho}$, since $m(\pi, \pi) = 1$, hence $r_{\pi\pi} = 1$. Finally, if $\pi \leq \rho$ does not hold, thus $r_{\pi\rho} = 0$, we cannot have $\pi \leq \sigma \leq \rho$ for any $\sigma \in \mathcal{P}_{[n]}$ and therefore $\sum_{\sigma \in \mathcal{P}_{[n]}} q_{\pi\sigma} r_{\sigma\rho} = 0$. This shows (3.12). Now (3.13) follows from Lemma 3.4 on the number of maximal chains. \square

Evidently, the B_n eigenvectors of Q defined by (3.12) are linearly independent. We are now interested in the inverse matrix of $R = (r_{\pi\rho})_{\pi, \rho \in \mathcal{P}_{[n]}}$, that is the matrix $L = (l_{\pi\rho})_{\pi, \rho \in \mathcal{P}_{[n]}}$ such that $\delta_{\pi\rho} = \sum_{\sigma \in \mathcal{P}_{[n]}} r_{\pi\sigma} l_{\sigma\rho}$ for all $\pi, \rho \in \mathcal{P}_{[n]}$.

Lemma 3.6. For any $\pi \in \mathcal{P}_{[n]}$ the vector $(l_{\pi\rho})_{\rho \in \mathcal{P}_{[n]}}$ given by

$$l_{\pi\rho} := \begin{cases} (-1)^{\#\pi-\#\rho} \frac{2^{\#\pi-\#\rho}(\#\pi+\#\rho-2)!}{(2\#\pi-2)!(\#\pi-\#\rho)!} m(\pi, \rho) & \text{if } \pi \leq \rho, \\ 0 & \text{otherwise,} \end{cases} \quad (3.14)$$

$$= \begin{cases} (-1)^{\#\pi-\#\rho} \frac{(\#\pi+\#\rho-2)!}{(2\#\pi-2)!} \prod_{B \in \rho} \#\pi|_B! & \text{if } \pi \leq \rho, \\ 0 & \text{otherwise,} \end{cases} \quad (3.15)$$

is a left eigenvector of Q with corresponding eigenvalue $-\binom{\#\pi}{2}$.

Proof. Use $\sum_{\sigma: \sigma < \rho} m(\pi, \sigma) = m(\pi, \rho)$ to obtain

$$\begin{aligned} \sum_{\sigma \in \mathcal{P}_{[n]}} l_{\pi\sigma} q_{\sigma\rho} &= \sum_{\sigma: \sigma < \rho} l_{\pi\sigma} - \binom{\#\rho}{2} l_{\pi\rho} \\ &= (-1)^{\#\pi-\#\rho-1} \frac{2^{\#\pi-\#\rho-1}(\#\pi+\#\rho-1)!}{(2\#\pi-2)!(\#\pi-\#\rho-1)!} \sum_{\sigma: \sigma < \rho} m(\pi, \sigma) - \binom{\#\rho}{2} l_{\pi\rho} \\ &= \left(-\frac{(\#\pi+\#\rho-1)(\#\pi-\#\rho)}{2} - \binom{\#\rho}{2} \right) l_{\pi\rho} = -\binom{\#\pi}{2} l_{\pi\rho}, \end{aligned}$$

thus (3.14) holds. Equation (3.15) follows from the Lemma on the number of maximal chains, Lemma 3.4. \square

Proof. (of Theorem 2.7) The inverse matrix of R , let us call it $U = (u_{\pi\rho})_{\pi,\rho \in \mathcal{P}_{[n]}}$, is uniquely determined, and is a matrix of left eigenvectors of Q , i.e. $UQ = DU$. Moreover, for any $\pi \in \mathcal{P}_{[n]}$ we have by assumption $u_{\pi\pi} = \sum_{\sigma: \pi \leq \sigma \leq \rho} u_{\pi\sigma} r_{\sigma\rho} = \delta_{\pi\pi} = 1$. This uniquely determines a matrix of left eigenvectors of Q , since for any $\pi, \rho \in \mathcal{P}_{[n]}$ with $\pi < \rho$ we have $-\binom{\#\rho}{2} u_{\pi\rho} = \sum_{\sigma \in [\pi, \rho]} u_{\pi\sigma} q_{\sigma\rho} = 1_{\{\pi < \rho\}} + \sum_{\sigma: \pi < \sigma \leq \rho} u_{\pi\sigma} q_{\sigma\rho}$, and $u_{\pi\rho} = 0$ for $\pi \not\leq \rho$. Since $QR = RD$ by Lemma 3.6, and evidently $l_{\pi\pi} = 1$ for any $\pi \in \mathcal{P}_{[n]}$, we have $U = L$ and the claim follows. \square

Remark 3.7. Instead of calculating the hitting probabilities $h(\pi, \rho)$ of Kingman's n -coalescent via the spectral decomposition, Theorem 2.7, we use the observation from section 2 that the jump chain of Π can be interpreted as the directed simple random walk on $\mathcal{P}_{[n]}$. This implies that the jump chain of Π (when started from π) traces out any maximal chain in $[\pi, \{[n]\}]$ with equal probability $m(\pi, \{[n]\})^{-1}$. Clearly, the total number of maximal chains in $[\pi, \{[n]\}]$ that contain ρ is $m(\pi, \rho)m(\rho, \{[n]\})$, and thus

$$h(\pi, \rho) = \frac{m(\pi, \rho)m(\rho, \{[n]\})}{m(\pi, \{[n]\})} = \binom{\#\pi - 1}{\#\rho - 1}^{-1} \frac{\#\rho!}{\#\pi!} \prod_{B \in \rho} \#\pi|_B! = \left[\frac{\#\pi}{\#\rho} \right]^{-1} \prod_{B \in \rho} \#\pi|_B!,$$

where we used the Lemma on the number of maximal chains, Lemma 3.4, in the second step. In the special case $\pi = \Delta_{[n]}$ this formula was given by Kingman in [4], equation (2.3).

Proof. (of Corollary 2.9) In complete analogy to the argument in Corollary 2.6, we have $q'_{ij} = \sum_{\rho \in \mathcal{P}_{[n],j}} q_{\pi\rho}$ independent of the particular partition $\pi \in \mathcal{P}_{[n],i}$, as the following calculation shows. Using the spectral decomposition of Q , Theorem 2.7, we obtain

$$\begin{aligned} q'_{ij} &= \sum_{\rho \in \mathcal{P}_{[n],j}} \sum_{\sigma \in [\pi, \rho]} r_{\pi\sigma} d_{\sigma\sigma} l_{\sigma\rho} \\ &= - \sum_{\substack{\sigma: \pi \leq \sigma \\ \#\sigma \geq j}} \frac{(2\#\sigma - 1)!}{(i + \#\sigma - 1)!} \left(\prod_{B \in \sigma} \#\pi|_B! \right) \binom{\#\sigma}{2} \sum_{\substack{\rho: \sigma \leq \rho \\ \#\rho = j}} (-1)^{\#\sigma + j} \frac{(\#\sigma + j - 2)!}{(2\#\sigma - 2)!} \prod_{B \in \rho} \#\sigma|_B! \\ &= - \sum_{k=j}^i \frac{(2k - 1)!}{(i + k - 1)!} \begin{bmatrix} i \\ k \end{bmatrix} \binom{k}{2} (-1)^{k+j} \frac{(k + j - 2)!}{(2k - 2)!} \begin{bmatrix} k \\ j \end{bmatrix} = \sum_{k=j}^i r'_{ik} d'_{kk} l'_{kj}, \end{aligned}$$

where in the third step we used the identity $\begin{bmatrix} i \\ k \end{bmatrix} = \sum_{\sigma \in \mathcal{P}_{[i],k}} \prod_{B \in \sigma} \#B!$ twice. This identity is obvious from the interpretation of $\begin{bmatrix} i \\ k \end{bmatrix}$ as the number of partitions into k ordered blocks of a set of i elements, where the elements in each block are ordered. \square

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