Electron. Commun. Probab. **20** (2015), no. 93, 1–11.
DOI: 10.1214/ECP.v20-4461
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

Contact process on one-dimensional long range percolation

Van Hao Can*

Abstract

Recently, by introducing the notion of cumulatively merged partition, Ménard and Singh provide in [6] a sufficient condition on graphs ensuring that the critical value of the contact process is positive. In this note, we show that the one-dimensional long range percolation with high exponent satisfies their condition and thus the contact process exhibits a non-trivial phase transition.

Keywords: Contact process; Cumulative merging; Long range percolation.
AMS MSC 2010: 82C22; 60K35; 05C80.
Submitted to ECP on August 3, 2015, final version accepted on December 17, 2015.

1 Introduction

In this paper, we study the contact process on G_s, the one-dimensional long range percolation graph with exponent $s > 1$, defined as follows: independently for any i and j in $\mathbb{Z}$ there is an edge connecting them with probability $|i-j|^{-s}$. In particular, G contains $\mathbb{Z}$ so it is connected.

On the other hand, the contact process was introduced in an article of T. E. Harris [4] and is defined as follows: given a locally finite graph $G = (V, E)$ and $\lambda > 0$, the contact process on G with infection rate λ is a Markov process $(\xi_t)_{t\geq 0}$ on $\{0,1\}^V$. Vertices of V (also called sites) are regarded as individuals which are either infected (state 1) or healthy (state 0). By considering ξ_t as a subset of V via $\xi_t \equiv \{v : \xi_t(v) = 1\}$, the transition rates are given by

$$\xi_t \to \xi_t \setminus \{v\} \text{ for } v \in \xi_t \text{ at rate } 1, \text{ and}$$
$$\xi_t \to \xi_t \cup \{v\} \text{ for } v \notin \xi_t \text{ at rate } \lambda \deg_{\xi_t}(v),$$

where $\deg_{\xi_t}(v)$ denotes the number of infected neighbors of v at time t. Given $A \subset V$, we denote by $(\xi_t^A)_{t\geq 0}$ the contact process with initial configuration A and if $A = \{v\}$ we simply write (ξ_t^v).

Since the contact process is monotone in λ, we can define the critical value

$$\lambda_c(G) = \inf\{\lambda : \mathbb{P}(\xi_t^v \neq \varnothing \, \forall t) > 0\}.$$

This definition does not depend on the choice of v if G is connected. If G has bounded degree, then there exists a non-trivial sub-critical phase, i.e. $\lambda_c > 0$, as the contact process is stochastically dominated by a continuous time branching random walk with

*Aix Marseille Université, CNRS, Centrale Marseille, I2M, UMR 7373, 13453 Marseille, France.
E-mail: cvhao89@gmail.com

reproduction rate λ. Thus for integer lattices and regular trees, the critical value is positive. The behavior of the contact process on these graphs was extensively investigated, see for instance [5, 7, 9].

In contrast, there is a little knowledge about the sub-critical phase on unbounded degree graphs. For Galton-Watson trees, Pemantle proved in [8] that if the reproduction law B asymptotically satisfies that $\mathbb{P}(B \geq x) \geq \exp(-x^{1-\varepsilon})$, for some $\varepsilon > 0$, then $\lambda_c = 0$. Recently, in [6], by introducing the notion of cumulatively merged partition (abbr. CMP) (see Section 2.2), the authors provided a sufficient condition on graphs ensuring that $\lambda_c > 0$. As an application, they show that the contact process on random geometric graphs and Delaunay triangulations exhibits a non-trivial phase transition.

The long range percolation graph was first introduced in [10, 11]. Then it gained interest in some contexts such as the graph distance, diameter, random walk, see [3] for a list of reference. The long range percolation is locally finite if and only if $s > 1$, so we only consider the contact process on such graphs. Moreover, it follows from the ergodicity of G_s that there is a non negative constant $\lambda_c(s)$, such that

$$\lambda_c(G_s) = \lambda_c(s) \text{ for almost all graphs } G_s. \tag{1.1}$$

It is clear that the sequence of graphs (G_s) is stochastically decreasing in s in the sense that G_{s_1} can be coupled as a subgraph of G_{s_2} if $s_1 \geq s_2$. Therefore $\lambda_c(s_1) \geq \lambda_c(s_2)$. Hence, we can define

$$s_c = \inf\{s : \lambda_c(s) > 0\}. \tag{1.2}$$

We will apply the method in [6] to show that $s_c < +\infty$. Here is our main result.

Theorem 1.1. *We have*

$$s_c \leq 102.$$

There is a phase transition in the structure of the long range percolation. If $s < 2$, the graph G_s exhibits the small-world phenomenon. More precisely, the distance between x and y is of order $(\log|x-y|)^{\varkappa+o(1)}$ with $\varkappa = \varkappa(s) > 1$, with probability tending to 1 as $|x-y| \to \infty$, see for instance [2]. In contrast, if $s > 2$, the graph somehow looks like $\mathbb{Z}$ (see Section 2.1) and the distance now is of order $|x-y|$, see [1]. On the other hand, as mentioned above, we know that $\lambda_c(\mathbb{Z}) > 0$. Hence, we conjecture that

$$s_c \leq 2.$$

The results in [6] can be slightly improved and thus we could get a better bound on s_c, but it would still be far from the critical value 2.

The paper is organized as follows. In Section 2, we first describe the structure of the graph and show that G_s can be seen as the gluing of i.i.d. finite subgraphs. Then we recall the definitions and results of [6] on the CMP. By studying the moment of the total weight of a subgraph, we are able to apply the results from [6] and prove our main theorem.

2 Proof of Theorem 1.1

2.1 Structure of the graph

We fix $s > 2$. For any $k \in \mathbb{Z}$, we say that k is a *cut-point* if there is no edge (i, j) with $i < k$ and $j > k$.

Lemma 2.1. *The following statements hold.*

(i) For all $k \in \mathbb{Z}$

$$\mathbb{P}(k \text{ is a cut-point}) = \mathbb{P}(0 \text{ is a cut-point}) > 0.$$

As a consequence, almost surely there exist infinitely many cut-points.

(ii) The subgraphs induced in the intervals between consecutive cut-points are i.i.d. In particular, the distances between consecutive cut-points form a sequence of i.i.d. random variables.

Proof. We first prove (i). Observe that

$$\begin{aligned}
\mathbb{P}(k \text{ is a cut-point}) &= \mathbb{P}(0 \text{ is a cut-point}) \\
&= \prod_{i<0<j} \left(1 - |i-j|^{-s}\right) \\
&\geq \exp\left(-2 \sum_{i<0<j} |i-j|^{-s}\right) \\
&\geq e^{2/(2-s)},
\end{aligned}$$

where we used that $1 - x \geq \exp(-2x)$ for $0 \leq x \leq 1/2$ and

$$\begin{aligned}
\sum_{i<0<j} |i-j|^{-s} = \sum_{i,j\geq 1} (i+j)^{-s} &\leq \frac{1}{s-1} \sum_{i \geq 1} i^{1-s} \\
&\leq \frac{1}{s-1}\left(1 + \frac{1}{s-2}\right) = \frac{1}{s-2},
\end{aligned}$$

using series integral comparison.

Then the ergodic theorem implies that there are infinitely many cut-points a.s.

Part (ii) is immediate, since there are no edges between different intervals between consecutive cut-points. □

We now study some properties of the distance between two consecutive cut-points.

Proposition 2.2. *Let D be the distance between two consecutive cut-points. Then there exists a sequence of integer-valued random variables $(\varepsilon_i)_{i\geq 0}$ with $\varepsilon_0 = 1$, such that*

(i) $D = \sum_{i=0}^{T} \varepsilon_i$ *with* $T = \inf\{i \geq 1 : \varepsilon_i = 0\}$,

(ii) T *is stochastically dominated by a geometric random variable with mean* $e^{2/(2-s)}$,

(iii) for all $i, \ell \geq 1$

$$\mathbb{P}(\varepsilon_i > \ell \mid T \geq i) \leq \ell^{2-s}/(s-2).$$

Proof. To simplify notation, we assume that 0 is a cut-point. Set $X_{-1} = 0$ and $X_0 = 1$, then we define for $i \geq 1$

$$\begin{aligned}
&X_i = \max\{k : \exists\, X_{i-2} \leq j \leq X_{i-1} - 1,\ j \sim k\}, \\
&\varepsilon_i = X_i - X_{i-1}.
\end{aligned}$$

Then $\varepsilon_i \geq 0$ and we define

$$T = \inf\{i \geq 1 : X_i = X_{i-1}\} = \inf\{i \geq 1 : \varepsilon_i = 0\}.$$

We have $X_i = X_{i-1}$ for all $i \geq T$, or equivalently $\varepsilon_i = 0$ for all $i \geq T$.

Note that X_T is the closest cut-point on the right of 0, so it has the same law as D, by definition. Moreover

$$X_T = \sum_{i=0}^{T} \varepsilon_i, \tag{2.1}$$

which implies (i). Observe that for $i \geq 1$ we have $\{T \geq i\} = \{X_{i-2} < X_{i-1}\}$ and

$$\begin{aligned}
\mathbb{P}(T = i \mid T \geq i) &= \mathbb{P}(X_i = X_{i-1} \mid X_{i-2} < X_{i-1}) \\
&= \mathbb{P}(\nexists\, X_{i-2} \leq j < X_{i-1} < k : j \sim k \mid X_{i-2} < X_{i-1}) \\
&\geq \prod_{j<0<k} \left(1 - |j-k|^{-s}\right) \\
&\geq e^{2/(2-s)}.
\end{aligned}$$

This implies (ii). For (iii), we note that for $i, \ell \geq 1$,

$$\begin{aligned}
\mathbb{P}(X_i \leq X_{i-1} + \ell \mid X_{i-2} < X_{i-1}) &\geq \prod_{\substack{j<0 \\ k>\ell}} \left(1 - |j-k|^{-s}\right) \\
&\geq 1 - \sum_{\substack{j<0 \\ k>\ell}} |j-k|^{-s}.
\end{aligned}$$

We have

$$\begin{aligned}
\sum_{\substack{j<0 \\ k>\ell}} |j-k|^{-s} &= \sum_{j=1}^{\infty} \sum_{k=\ell+1}^{\infty} (k+j)^{-s} \\
&\leq \frac{1}{s-1} \sum_{j=1}^{\infty} (j+\ell)^{1-s} \\
&\leq \ell^{2-s}/(s-2).
\end{aligned}$$

Therefore,

$$\mathbb{P}(\varepsilon_i > \ell \mid T \geq i) \leq \ell^{2-s}/(s-2),$$

which proves (iii). □

Since the definition of λ_c is independent of the starting vertex, we can assume that the initially infected vertex is a cut-point.

It will be convenient to assume that 0 is a cut-point. Suppose that conditioned on 0 being a cut-point and infected at the beginning, we can prove that $\lambda_c > 0$. Since the distribution is invariant under translations, we have $\lambda_c > 0$ for the contact process starting from any cut point.

Hence, from now on we condition on the event 0 is a cut-point. Set $K_0 = 0$, for $i \geq 1$, we call K_i (resp. K_{-i}) the i^{th} cut point from the right (resp. left) of 0. By Lemma 2.1 (ii), the graphs induced in the intervals $[K_i, K_{i+1})$ are i.i.d. Therefore, G_s is isomorphic to the graph $\tilde{G}_s$ obtained by gluing an i.i.d. sequence of graphs with distribution of the graph $[0, K_1)$. We have to prove that the contact process on $\tilde{G}_s$ exhibits a non-trivial phase transition.

2.2 Cumulatively merged partition

We recall here the definitions introduced in [6]. Given a locally finite graph $G = (V, E)$, an expansion exponent $\alpha \geq 1$, and a sequence of non-negative weights defined on the vertices

$$(r(x), x \in V) \in [0, \infty)^V,$$

a partition $\mathcal{C}$ of the vertex set V is said to be (r, α)-admissible if it satisfies

$$\forall\, C, C' \in \mathcal{C}, \quad C \neq C' \qquad \Longrightarrow \qquad d(C, C') > \min\{r(C), r(C')\}^\alpha,$$

with

$$r(C) = \sum_{x \in C} r(x).$$

We call *cumulatively merged partition* (CMP) of the graph G with respect to r and α the finest (r, α)- admissible partition and denote it by $\mathscr{C}(G, r, \alpha)$. It is the intersection of all (r, α)-admissible partitions of the graph, where the intersection is defined as follows: for any sequence of partitions $(\mathcal{C}_i)_{i \in I}$,

$$x \sim y \text{ in } \cap_{i \in I} \mathcal{C}_i \quad \text{ if } \quad x \sim y \text{ in } \mathcal{C}_i \text{ for all } i \in I.$$

As for Bernoulli percolation on $\mathbb{Z}^d$, the question we are interested in is the existence of an infinite cluster (here an infinite partition). For the CMP on $\mathbb{Z}^d$ with i.i.d. weights, we have the following result.

Proposition 2.3. *[6, Proposition 3.7] For any $\alpha \geq 1$, there exists a positive constant $\beta_c = \beta_c(\alpha)$, such that for any positive random variable Z satisfying $\mathbb{E}(Z^\gamma) \leq 1$ with $\gamma = (4\alpha d)^2$ and any $\beta < \beta_c$, almost surely $\mathscr{C}(\mathbb{Z}^d, \beta Z, \alpha)$-the CMP on $\mathbb{Z}^d$ with expansion exponent α and i.i.d. weights distributed as βZ-has no infinite cluster.*

We note that in [6, Proposition 3.7], the authors only assume that $\mathbb{E}(Z^\gamma) < \infty$ and they do not precise the dependence of β_c with $\mathbb{E}(Z^\gamma)$. However, we can deduce from their proof a lower bound on β_c depending only on $\mathbb{E}(Z^\gamma)$ (and only on α, γ, d if we suppose $\mathbb{E}(Z^\gamma) \leq 1$), see Appendix for more details. Finally, our $\beta_c(\alpha)$ is a lower bound of the critical parameter $\lambda_c(\alpha)$ introduced by Ménard and Singh.

Using the notion of CMP, they give a sufficient condition on a graph G ensuring that the critical value of the contact process is positive.

Theorem 2.4. *[6, Theorem 4.1] Let $G = (V, E)$ be a locally finite connected graph. Consider $\mathscr{C}(G, r_\triangle, \alpha)$ the CMP on G with expansion exponent α and degree weights*

$$r_\triangle(x) = \deg(x) 1(\deg(x) \geq \triangle).$$

Suppose that for some $\alpha \geq 5/2$ and $\triangle \geq 0$, the partition $\mathscr{C}(G, r_\triangle, \alpha)$ has no infinite cluster. Then

$$\lambda_c(G) > 0.$$

Thanks to this result, Theorem 1.1 will follow from the following proposition.

Proposition 2.5. *Fix $s > 102$. There exists a positive constant $\triangle$, such that the partition $\mathscr{C}(\tilde{G}_s, r_\triangle, 5/2)$ has no infinite cluster a.s.*

2.3 Proof of Proposition 2.5

Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be two CMPs. We write $\mathscr{C}_1 \preceq \mathscr{C}_2$, if there is a coupling such that $\mathscr{C}_1$ has an infinite cluster only if $\mathscr{C}_2$ has an infinite cluster.

Lemma 2.6. *We have*

$$\mathscr{C}(\tilde{G}_s, r_\triangle, 5/2) \preceq \mathscr{C}(\mathbb{Z}, Z_\triangle, 5/2), \tag{2.2}$$

with

$$Z_\triangle = \sum_{0 \le x < K_1} \deg(x) 1(\deg(x) \ge \triangle).$$

Proof. For $i \in \mathbb{Z}$, we define

$$Z_i = \sum_{K_i \le x < K_{i+1}} \deg(x) 1(\deg(x) \ge \triangle).$$

Then $(Z_i)_{i \in \mathbb{Z}}$ is a sequence of i.i.d. random variables with the same distribution as $Z_\triangle$, since the graph $\tilde{G}_s$ is composed of i.i.d. subgraphs $[K_i, K_{i+1})$. Therefore, $\mathscr{C}(\mathbb{Z}, (Z_i), 5/2)$ has the same law as $\mathscr{C}(\mathbb{Z}, Z_\triangle, 5/2)$. Thus to prove Lemma 2.6, it remains to show that

$$\mathscr{C}(\tilde{G}_s, r_\triangle, 5/2) \preceq \mathscr{C}(\mathbb{Z}, (Z_i), 5/2). \tag{2.3}$$

For any subset A of the vertices of $\tilde{G}_s$, we define its projection

$$p(A) = \{i \in \mathbb{Z} : A \cap [K_i, K_{i+1}) \neq \varnothing\}.$$

Since all intervals $[K_i, K_{i+1})$ have finite mean, if $|A| = \infty$ then $|p(A)| = \infty$. Therefore, to prove (2.3), it suffices to show that

$$x \sim y \text{ in } \mathscr{C}(\tilde{G}_s, r_\triangle, 5/2) \quad \text{implies} \quad p(x) \sim p(y) \text{ in } \mathscr{C}(\mathbb{Z}, (Z_i), 5/2). \tag{2.4}$$

We prove (2.4) by contradiction. Suppose that there exist x_0 and y_0 such that $x_0 \sim y_0$ in $\mathscr{C}(\tilde{G}_s, r_\triangle, 5/2)$ and $p(x_0) \not\sim p(y_0)$ in $\mathscr{C}(\mathbb{Z}, (Z_i), 5/2)$. Then by definition there exists $\mathcal{C}$, a $((Z_i), 5/2)$-admissible partition of $\mathbb{Z}$, such that $p(x_0) \not\sim p(y_0)$ in $\mathcal{C}$.

We define a partition $\tilde{\mathcal{C}}$ of $\tilde{G}_s$ as follows:

$$x \sim y \text{ in } \tilde{\mathcal{C}} \quad \text{if and only if} \quad p(x) \sim p(y) \text{ in } \mathcal{C}.$$

In other words, an element in $\tilde{\mathcal{C}}$ is $\cup_{i \in C}[K_i, K_{i+1})$ with C a set in $\mathcal{C}$. We now claim that $\tilde{\mathcal{C}}$ is $(r_\triangle, 5/2)$-admissible. Indeed, let $\tilde{C}$ and $\tilde{C}'$ be two different sets in $\tilde{\mathcal{C}}$. Then by the definition of $\tilde{\mathcal{C}}$, we have $p(\tilde{C})$ and $p(\tilde{C}')$ are two different sets in $\mathcal{C}$ and

$$Z(p(\tilde{C})) := \sum_{i \in p(\tilde{C})} Z_i = \sum_{x \in \tilde{C}} \deg(x) 1(\deg(x) \ge \triangle) = r_\triangle(\tilde{C}).$$

Moreover, since these intervals $[K_i, K_{i+1})$ are disjoint,

$$d(\tilde{C}, \tilde{C}') \ge d(p(\tilde{C}), p(\tilde{C}')).$$

On the other hand, as $\mathcal{C}$ is $((Z_i), 5/2)$-admissible,

$$d(p(\tilde{C}), p(\tilde{C}')) > \min\{Z(p(\tilde{C})), Z(p(\tilde{C}'))\}^{5/2}.$$

It follows from the last three inequalities that

$$d(\tilde{C}, \tilde{C}') > \min\{r_\triangle(\tilde{C}), r_\triangle(\tilde{C}')\}^{5/2},$$

which implies that $\tilde{\mathcal{C}}$ is $(r_\triangle, 5/2)$-admissible.

Let C_0 and C_0' be the two sets in the partition $\mathcal{C}$ containing $p(x_0)$ and $p(y_0)$ respectively. Then by assumption $C_0 \neq C_0'$. We define

$$\tilde{C}_0 = \bigcup_{i \in C_0} [K_i, K_{i+1}) \qquad \text{and} \qquad \tilde{C}_0' = \bigcup_{i \in C_0'} [K_i, K_{i+1}).$$

Then both $\tilde{C}_0$ and $\tilde{C}_0'$ are in $\tilde{\mathcal{C}}$, and $\tilde{C}_0 \neq \tilde{C}_0'$. Moreover $\tilde{C}_0$ contains x_0 and $\tilde{C}_0'$ contains y_0. Hence $x_0 \not\sim y_0$ in $\tilde{\mathcal{C}}$ which is a $(r_\triangle, 5/2)$-admissible partition. Therefore, $x_0 \not\sim y_0$ in $\mathscr{C}(\tilde{G}_s, r_\triangle, 5/2)$, which leads to a contradiction. Thus (2.4) has been proved. □

We now apply Proposition 2.3 and Lemma 2.6 to prove Proposition 2.5. To do that, we fix a positive constant $\beta < \beta_c(5/2)$ with $\beta_c(5/2)$ as in Proposition 2.3 with $d = 1$ and rewrite

$$Z_\triangle = \beta \frac{Z_\triangle}{\beta}.$$

If we can show that there is $\triangle$=$\triangle$ (β, s), such that

$$\mathbb{E}\left(\left(\frac{Z_\triangle}{\beta}\right)^{100}\right) \leq 1, \tag{2.5}$$

then Proposition 2.3 implies that a.s. $\mathscr{C}(\mathbb{Z}, Z_\triangle, 5/2)$ has no infinite cluster. Therefore, by Lemma 2.6, there is no infinite cluster in $\mathscr{C}(\tilde{G}_s, r_\triangle, 5/2)$ and thus Proposition 2.5 follows. Now it remains to prove (2.5).

It follows from Proposition 2.2 (i) that

$$\mathbb{E}(K_1^{100}) = \mathbb{E}(D^{100}) = \mathbb{E}\left(\left(\sum_{i=0}^{T} \varepsilon_i\right)^{100}\right), \tag{2.6}$$

where T and (ε_i) are as in Proposition 2.2.

Applying the inequality $(x_1 + \ldots + x_n)^{100} \leq n^{99}(x_1^{100} + \ldots + x_n^{100})$ for any $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \mathbb{R}$, we get

$$\begin{aligned} \mathbb{E}\left(\left(\sum_{i=0}^{T} \varepsilon_i\right)^{100}\right) &\leq \mathbb{E}\left[(T+1)^{99} \sum_{i=0}^{T} \varepsilon_i^{100}\right] \\ &= \sum_{i=0}^{\infty} \mathbb{E}\left[(T+1)^{99} \varepsilon_i^{100} 1(T \geq i)\right]. \end{aligned} \tag{2.7}$$

Let $p = 1 + (s - 102)/200 > 1$ and q be its conjugate, i.e. $p^{-1} + q^{-1} = 1$. Then applying Hölder's inequality, we obtain

$$\mathbb{E}\left[(T+1)^{99} \varepsilon_i^{100} 1(T \geq i)\right] \leq \mathbb{E}\left((T+1)^{99q}\right)^{1/q} \mathbb{E}\left(\varepsilon_i^{100p} 1(T \geq i)\right)^{1/p}. \tag{2.8}$$

On the other hand,

$$\mathbb{E}\left(\varepsilon_i^{100p} 1(T \geq i)\right) = \mathbb{E}\left(\varepsilon_i^{100p} \mid T \geq i\right) \mathbb{P}(T \geq i). \tag{2.9}$$

Using Proposition 2.2 (iii) we have for $i \geq 1$

$$\begin{aligned} \mathbb{E}\left(\varepsilon_i^{100p} \mid T \geq i\right) &\leq 100p \sum_{\ell=0}^{\infty} \mathbb{P}(\varepsilon_i > \ell \mid T \geq i)(\ell+1)^{100p-1} \\ &\leq 100p \left[1 + \sum_{\ell \geq 1} \ell^{2-s}(1+\ell)^{100p-1}/(s-2)\right] \\ &\leq C_1 = C_1(s) < \infty, \end{aligned}$$

since by definition

$$2 - s + 100p - 1 = -1 - (s - 102)/2 < -1.$$

Hence for all $i \geq 1$

$$\mathbb{E}\left(\varepsilon_i^{100p} 1(T \geq i)\right) \leq C_1 \mathbb{P}(T \geq i). \tag{2.10}$$

It follows from (2.6), (2.7), (2.8) and (2.10) that

$$\begin{aligned} \mathbb{E}(K_1^{100}) &\leq \mathbb{E}\left[(T+1)^{99q}\right]^{1/q} \left[1 + \sum_{i=1}^{\infty} \left(C_1 \mathbb{P}(T \geq i)\right)^{1/p}\right] \\ &= M < \infty, \end{aligned} \tag{2.11}$$

since T is stochastically dominated by a geometric random variable.

For any $j \in \mathbb{Z}$ and any interval I, we denote by $\deg_I(j)$ the number of neighbors of j in I when we consider the original graph (without conditioning on 0 being a cut-point).

Now for any non decreasing sequence $(x_k)_{k\geq 1}$ with $x_1 \geq 1$, conditionally on $\varepsilon_1 = x_1 - 1, \varepsilon_2 = x_2 - x_1, \ldots$, we have for all $j \in (x_{k-1}, x_k)$,

$$\deg(j) \prec 1 + \deg_{[x_{k-2}, x_{k+1})}(j),$$

where $\prec$ means stochastic domination.

Indeed, the conditioning implies that j is only connected to vertices in $[x_{k-2}, x_{k+1}]$ and that there is a vertex in $[x_{k-1}, x_k)$ connected to x_{k+1}.

Similarly, if $j = x_k$, it is only connected to vertices in $[x_{k-2}, x_{k+2}]$. Moreover, j is connected to at least one vertex in $[x_{k-2}, x_{k-1})$ and there is a vertex in $[x_k, x_{k+1})$ connected to x_{k+2}. Therefore,

$$\deg(x_k) \prec 2 + \deg_{[x_{k-2}, x_{k+2})}(x_k).$$

In conclusion, conditionally on $j \in [0, K_1)$,

$$\deg(j) \prec 2 + Y,$$

where

$$Y = \deg_{(-\infty, +\infty)}(j).$$

Hence,

$$\mathbb{E}\left(\deg(j)^{100} 1(\deg(j) \geq \triangle) \mid j \in [0, K_1)\right) \leq \mathbb{E}\left((2+Y)^{100} 1(Y \geq \triangle - 2)\right). \tag{2.12}$$

On the other hand,

$$\begin{aligned} \mathbb{P}(Y = k) &= \mathbb{P}(\deg_{(-\infty,+\infty)}(0) = k) \\ &\leq \mathbb{P}(\deg_{(-\infty,+\infty)}(0) \geq k) \\ &\leq \sum_{i_1 < i_2 < \ldots < i_k} |i_1|^{-s} |i_2|^{-s} \ldots |i_k|^{-s} \\ &\leq \frac{1}{k!} \sum_{i_1, i_2, \ldots, i_k} |i_1|^{-s} |i_2|^{-s} \ldots |i_k|^{-s} \\ &= \frac{1}{k!} \left(2 \sum_{i \geq 1} i^{-s}\right)^k = \frac{C^k}{k!}, \end{aligned}$$

with $C = 2\sum_{i\geq 1} i^{-s}$. Therefore,

$$\begin{aligned}\mathbb{E}\left((2+Y)^{100}1(Y \geq \triangle -2)\right) &\leq \sum_{k\geq \triangle -2} \frac{C^k(k+2)^{100}}{k!} \\ &:= f(\triangle). \end{aligned} \tag{2.13}$$

It follows from (2.11), (2.12) and (2.13) that

$$\begin{aligned}\mathbb{E}(Z_\triangle^{100}) &= \mathbb{E}\left[\left(\sum_{0\leq j<K_1} \deg(j)1(\deg(j)\geq\triangle)\right)^{100}\right] \\ &\leq \mathbb{E}\left[K_1^{99}\sum_{0\leq j<K_1} \deg(j)^{100}1(\deg(j)\geq\triangle)\right] \\ &\leq \mathbb{E}(K_1^{100})f(\triangle) \\ &\leq Mf(\triangle).\end{aligned}$$

Since $f(\triangle)\to 0$ as $\triangle\to\infty$, there exists $\triangle_0\in(0,\infty)$, such that $Mf(\triangle_0)\leq\beta^{100}$ and thus (2.5) is satisfied. □

Appendix: a lower bound on β_c

In [6], Proposition 3.7 (our Proposition 2.3) follows from Lemmas 3.9, 3.10, 3.11 and a conclusion argument. Let us find in their proof a lower bound on β_c.

At first, they define a constant $c = 2\alpha d + 1$ and some sequences

$$L_n = 2^{c^n} \quad \text{and} \quad R_n = L_1 \dots L_n \quad \text{and} \quad \varepsilon_n = 2^{-2dc^{n+1}}.$$

In Lemma 3.9, the authors do not use any information on Z and β. They set a constant $k_0 = [2^{d+1}(c+1)]$.

In Lemma 3.10, they suppose that $\beta\leq 1$ and the information concerning Z is as follows. There exists n_0, such that for all $n\geq n_0$, we have

$$2^d\mathbb{E}(Z^\gamma)L_{n+1}^{-\mu}\leq 1/2,$$

with

$$\mu = \frac{\gamma-1}{2\alpha} - 3d - 4\alpha d^2 > 0.$$

In fact, under the assumption $\mathbb{E}(Z^\gamma)\leq 1$, we can take

$$n_0 = \left[\frac{\log\left(\frac{d+1}{\mu}\right)}{\log c}\right]. \tag{2.14}$$

In Lemma 3.10, they also assume that $\beta\leq 1$ and define a constant n_1, such that $n_1\geq n_0$ and for all $n\geq n_1$

$$3k_0^{\alpha+1}L_{n+1}\leq\frac{R_{n+1}}{20},$$

or equivalently,

$$60k_0^{\alpha+1}\leq R_n. \tag{2.15}$$

In the conclusion leading to the proof of [6, Proposition 3.7], a lower bound on β_c is implicit. Indeed, with Lemmas 3.9, 3.10, 3.11 in hand, the authors only require that

$$\mathbb{P}(\mathcal{E}(R_{n_1})) \geq 1 - \varepsilon_{n_1}, \tag{2.16}$$

where for any $N \geq 1$

$$\mathcal{E}(N) = \{\text{there exits a stable set } S \text{ such that } [\![N/5, 4N/5]\!]^d \subset S \subset [\![1, N]\!]^d\}.$$

We do not recall the definition of stable sets here. However, we notice that by the first part of Proposition 2.5 and Corollary 2.13 in [6], the event $\mathcal{E}(N)$ occurs when the weights of all vertices in $[\![1, N]\!]^d$ are less than $1/2$. Therefore

$$\begin{aligned}
\mathbb{P}(\mathcal{E}(N)) &\geq \mathbb{P}\left(r(x) \leq 1/2 \text{ for all } x \in [\![1, N]\!]^d\right) \\
&= \mathbb{P}(\beta Z \leq 1/2)^{N^d} \\
&= (1 - \mathbb{P}(\beta Z > 1/2))^{N^d} \\
&= \left(1 - \mathbb{P}(Z^\gamma > (2\beta)^{-\gamma})\right)^{N^d} \\
&\geq (1 - (2\beta)^\gamma \mathbb{E}(Z^\gamma))^{N^d}.
\end{aligned}$$

Hence (2.16) is satisfied if

$$(1 - (2\beta)^\gamma \mathbb{E}(Z^\gamma))^{R_{n_1}^d} \geq (1 - \varepsilon_{n_1}),$$

or equivalently

$$(2\beta)^\gamma \mathbb{E}(Z^\gamma) \leq 1 - (1 - \varepsilon_{n_1})^{R_{n_1}^{-d}}.$$

Hence, under the assumption $\mathbb{E}(Z^\gamma) \leq 1$, we can take

$$\beta_c = \frac{1}{2}\left(1 - (1 - \varepsilon_{n_1})^{R_{n_1}^{-d}}\right)^{1/\gamma},$$

with n_1 as in (2.15).

References

[1] I. Benjamini, N. Berger. *The diameter for long range percolation cluster on finite cycles*, Random Struct. Algorithms, **19** (2001), no. 2, 102–111. MR-1848786

[2] M. Biskup. *On the scaling of the chemical distance in long range percolation models*, Ann. Probab. **32** (2004), no. 4, 2938–2977. MR-2094435

[3] N. Crawford, A. Sly. *Simple random walk on long range percolation cluster II: Scaling limits*, Ann. probab. **41** (2013), 445–502. MR-3077517

[4] T. E. Harris. *Contact interactions on a lattice*, Ann. probab. **2** (1974), 969–988. MR-0356292

[5] T.M. Liggett. *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes.* Grundlehren de Mathematischen Wissenschaften **324**, Springer (1999). MR-1717346

[6] L. Ménard, A. Singh. *Percolation by cumulative merging and phase transition for the contact process on random graphs*, arXiv:1502.06982v1.

[7] R. Pemantle. *The contact process on trees*, Ann. Probab. **20** (1992), no. 4, 2089–2116. MR-1188054

[8] R. Pemantle, A. M. Stacey. *The branching random walk and contact process on Galton-Watson and nonhomogeneous trees*, Ann. Probab. **29** (2001), no. 4, 1563–1590. MR-1880232

[9] A. M. Stacey. *The existence of an intermediate phase for the contact process on trees*, Ann. Probab. **24** (1996), no. 4, 1711–1726. MR-1415226

[10] L. S. Schulman. *Long range percolation in one dimension*, J. Phys. A. Lett **16** (1983). MR-0701466

[11] Z. Q. Zhang, F. C. Pu, B. Z. Li. *Long range percolation in one dimension*, J. Phys. A: Math. Gen. **16** (1983). MR-0701466

Acknowledgments. I am deeply grateful to my advisor Bruno Schapira for his help and many suggestions during the preparation of this work. I would like also to thank the referee for a careful reading this paper as well as many valuable comments.

Electronic Journal of Probability
Electronic Communications in Probability

Advantages of publishing in EJP-ECP

- Very high standards
- Free for authors, free for readers
- Quick publication (no backlog)

Economical model of EJP-ECP

- Low cost, based on free software (OJS[1])
- Non profit, sponsored by IMS[2], BS[3], PKP[4]
- Purely electronic and secure (LOCKSS[5])

Help keep the journal free and vigorous

- Donate to the IMS open access fund[6] (click here to donate!)
- Submit your best articles to EJP-ECP
- Choose EJP-ECP over for-profit journals

[1] OJS: Open Journal Systems http://pkp.sfu.ca/ojs/
[2] IMS: Institute of Mathematical Statistics http://www.imstat.org/
[3] BS: Bernoulli Society http://www.bernoulli-society.org/
[4] PK: Public Knowledge Project http://pkp.sfu.ca/
[5] LOCKSS: Lots of Copies Keep Stuff Safe http://www.lockss.org/
[6] IMS Open Access Fund: http://www.imstat.org/publications/open.htm