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A note on general Tauberian-type results for controlled stochastic dynamics*

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Abstract

We show that, in the context of stochastic control systems, the uniform existence of a limit of Cesàro averages implies the existence of uniform limits for averages with respect to a wide class of measures dominated by the Lebesgue measure and satisfying some asymptotic condition. It gives a partial answer to the problem mentioned in [18] and it provides an alternative method for the approach in [13] (in the deterministic control setting). Finally, we mention that the arguments rely essentially on integration-by-parts and is applicable to general deterministic or stochastic control problems.

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1 Introduction

In this paper, we consider a regular jump-diffusion stochastic control system. Nevertheless, the results of the main Section 3 are independent of the actual system considered, as soon as the dynamic programming-issued monotone result (in Proposition 3.2) holds-true. To fix the notations, we let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space supporting an $\mathbb{R}^d-$valued Brownian motion and an independent compound Poisson measure N with intensity $\widehat{N}(dedt) = \lambda(de)\,dt$ for some finite measure λ on a metric space $(E, \mathcal{E})$ endowed with its Borel σ-algebra. We consider a compact metric control space U. The coefficients $b : \mathbb{R}^N \times U \longrightarrow \mathbb{R}^N$, $\sigma : \mathbb{R}^N \times U \longrightarrow \mathbb{R}^{N\times d}$, $f : \mathbb{R}^N \times E \times U \longrightarrow \mathbb{R}^N$ are assumed to be uniformly continuous, bounded and Lipschitz-continuous in space, uniformly with respect to the control parameter. We consider the controlled system

$$dX_t^{x,u} = b\left(X_t^{x,u}, u_t\right)dt + \sigma\left(X_t^{x,u}, u_t\right)dW_t + \int_E f\left(X_{t-}^{x,u}, e, u_t\right)N(dedt),\ t \geq 0,\ X_0^{x,u} = x,$$

where $x \in \mathbb{R}^N$. The process u is $U-$valued and predictable (with respect to the natural filtration generated by W and N and completed by the $\mathbb{P}$-null sets) and the family of such controls is denoted by $\mathcal{U}_{ad}$.

We consider a cost criterion $g : \mathbb{R}^N \times U \longrightarrow [0, 1]$ assumed to be uniformly continuous. Whenever $(\mu_\delta)_{\delta>0}$ is a family of probability measures on $\mathbb{R}_+$, one considers the

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$\mu_\delta-$averaged value functions

$$v^\delta(x) := \inf_{u\in\mathcal{U}_{ad}} \mathbb{E}\left[\int_{\mathbb{R}_+} g\left(X_t^{x,u}, u_t\right)\mu_\delta(dt)\right],\ x\in\mathbb{R}^N,\ \text{for all } \delta>0. \tag{1.1}$$

Two particular classes are widely studied. The case when $\mu_\delta(dt) = \delta 1_{\left[0,\frac{1}{\delta}\right]}(t)\,dt$ leads to the Cesàro averages denoted, for convenience (and by setting $T=\delta^{-1}$),

$$V_T(x) := \inf_{u\in\mathcal{U}_{ad}} \frac{1}{T}\mathbb{E}\left[\int_0^T g\left(X_t^{x,u}, u_t\right)dt\right],\ x\in\mathbb{R}^N,\ \text{for all } T>0. \tag{1.2}$$

The case when μ_δ are exponentially distributed with parameter $\delta>0$ leads to the Abel means

$$v^\delta_{Abel}(x) := \inf_{u\in\mathcal{U}_{ad}} \mathbb{E}\left[\int_0^\infty \delta e^{-\delta t} g\left(X_t^{x,u}, u_t\right)dt\right],\ x\in\mathbb{R}^N,\ \text{for all } \delta>0.$$

In a discrete setting, for sequences of bounded real numbers $(x_n)_{n\geq 1}$, Hardy and Littlewood have proven in [11] that the convergence of the Cesàro means $\left(\frac{1}{n}\sum_{i=1}^n x_i\right)_{n\geq 1}$ is equivalent to the convergence of their Abel means $\left(\delta\sum_{i=1}^\infty (1-\delta)^i x_i\right)_{1>\delta>0}$. This result has been generalized by Feller (cf. [8], XIII.5) to the case of uncontrolled deterministic dynamics in continuous time. A further generalization to deterministic controlled dynamics with continuous time is available in [1]. However, the framework of the cited paper guarantees that the limit value function does not depend of the initial data. The general case for deterministic dynamics in which the limit value function may depend on the initial data has been considered in [15]. The main result in [15] states that, for deterministic control systems, V_t converges uniformly as $t\to\infty$ if and only if v^δ_{Abel} converges uniformly as $\delta\to 0$. Moreover, the two limits coincide. The authors of [15] also give an example proving that the limit value functions may not coincide if the convergence is not uniform. In the Brownian diffusion setting, similar results have been obtained in [5]. Finally, similar partial (Abelian) results for piecewise deterministic Markov processes make the object of [9].

The recent paper [18] considers a discrete control problem with arbitrary state space and bounded rewards and gives an affirmative answer to the existence of the limit for problems in which the averaging concerns general discrete measures, when the “patience” of the decision-maker tends to infinity. For a sequence of measures, a notion of "impatience" is translated in [18] by a total-variation decreasing to 0 condition. The method is adapted to a deterministic continuous control framework in the recent preprint [13] via what the authors call the "long-term condition". In both cases, the approach relies on the dynamic programming, reachability properties and an explicit candidate for limit (given in a sup/inf formulation and inspired by repeated games).

In this short note we show that, in the context of stochastic control systems, the uniform existence of a limit of Cesàro limits V_T implies the existence of uniform limits for averages v^δ with respect a wide class of measures dominated by Lebesgue measure and satisfying some asymptotic condition. It provides an alternative to the approach in [13]. Our approach requires some regularity of the density functions of the averaging measures and relies essentially on integration-by-parts formulae. Furthermore, it generalizes the method in [15] (in a deterministic setting) and [5, Section 4] (in a Brownian diffusion setting) and is applicable to general deterministic or stochastic control problems.

The specific assumptions on the measures μ_δ are given in Section 2. We give some examples of measures (Weibull, normal folded, uniform) satisfying these assumptions. In Section 3 we give the statement and the proof of the main Tauberian result and an example of piecewise diffusive switch inspired by Cook's genetic model introduced in [7].

2 An Asymptotic Behavior Assumption

2.1 A Class of Lebesgue-Dominated Averaging Measures

The probability measures μ_δ are assumed to be dominated by Lebesgue measure on $\mathbb{R}_+$ and their densities $\xi(\delta,t)=\frac{d\mu_\delta(t)}{dt}$ to be locally absolutely continuous on the support of μ_δ for all $\delta>0$. Moreover, we assume the following asymptotic condition to hold true

$$\begin{array}{ll} i. & \lim\limits_{\delta\to0+}\int_0^t\left|\xi(\delta,t)-\xi(\delta,s)\right|ds=0,\text{ for all }t>0.\\ & \text{There exists } t^\delta\geq 0 \text{ s.t. } \xi(\delta,\cdot) \text{ is non-increasing on } \left[t^\delta,\infty\right),\text{ for }\delta>0,\\ ii. & \xi\left(\delta,t^\delta+\right)\neq0 \text{ and } \lim\limits_{\delta\to0+}\int_0^{t^\delta}\left|\xi\left(\delta,t^\delta\right)-\xi(\delta,s)\right|ds=0,\\ iii. & \limsup\limits_{\beta\to\infty,\ s\to\infty}\ \sup\limits_{\delta\leq\frac{1}{\beta}}\left[\mu_\delta\left(\left[\max\left(t^\delta,s\right),\beta s\right]\right)-s\xi(\delta,s)\right]=1.\end{array}\tag{A}$$

Remark 2.1. (i) Whenever

$$\limsup_{\delta\to0+}\mu_\delta([0,t])=\limsup_{\delta\to0+}\xi(\delta,t)=0,\text{ for all }t>0,\tag{2.1}$$

the condition $(A\ i)$ is satisfied. Moreover, let us assume that $\sup\limits_{\delta>0}\xi(\delta,\cdot)\in\mathbb{L}^1_{loc}(dt)$ and the (almost sure) existence of $\xi(t):=\lim\limits_{\delta\to0+}\xi(\delta,t)$. Then $(A\ i)$ is satisfied if and only if (2.1) holds true.

(ii) The condition $\xi\left(\delta,t^\delta+\right)\neq0$ guarantees that, for some $\varepsilon>0$, the interval $\left[t^\delta,t^\delta+\varepsilon\right]$ belongs to the support of μ^δ. Otherwise, t^δ can trivially be chosen as an upper-bound of this support set.

If, moreover, $\limsup\limits_{\delta\to0+}\mu_\delta\left(\left[0,t^\delta\right]\right)=\limsup\limits_{\delta\to0+}t^\delta\xi\left(\delta,t^\delta\right)=0$, then the limit condition in $(A\ ii)$ is also satisfied. In particular, this is the case if t^δ can be chosen independent of $\delta>0$ (i.e. if $\sup\limits_{\delta>0}t^\delta<\infty$) and the conditions (i) hold true. It is also satisfied when t^δ is a maximum point of $\xi(\delta,\cdot)$ (specific unimodal distributions) and $\limsup\limits_{\delta\to0+}t^\delta\xi\left(\delta,t^\delta\right)=0$.

(iii) Let us fix $t>0$. For $\delta>0$, we let $\overline{a}(\delta):=\sup\{r>0:\xi(\delta,r)>0\}\in\left(t^\delta,\infty\right]$. Then, for δ small enough, $\overline{a}(\delta)\geq t$. Otherwise, let us consider some sequence $\delta\to0$ for which $\overline{a}(\delta)<t$. It follows that $\int_0^t(\xi(\delta,t)-\xi(\delta,s))\,ds=-1$ which contradicts $(A\ i)$.

(iv) The $\limsup\limits_{\beta\to\infty,\ s\to\infty}$ should be understood as $\limsup\limits_{\beta\to\infty}\limsup\limits_{s\to\infty}$.

Let us assume that $\limsup\limits_{\delta\to0+}\mu_\delta([0,t])=\limsup\limits_{\delta\to0+}\xi(\delta,t)=0$, for all $t>0$. Then, the condition $(A\ iii)$ roughly states that, as the expansion factor β increases, any interval $\left[t^\delta,\beta s\right]$ has almost full μ_δ-measure, for some small δ. This is a tightness condition. Of course, our assumption is slightly stronger since $\delta=\delta(\beta,s)$ in $(A\ iii)$ and some uniform (tightness) property is required.

(v) If $\xi(\delta,\cdot)$ are non-increasing on $\mathbb{R}_+$, the condition (A) is implied by

$$\begin{array}{ll} i. & \lim\limits_{\delta\to0+}\mu_\delta([0,t])=0,\text{ for all }t>0.\\ & \text{there exists an increasing sequence } \beta_n\nearrow\infty \text{ and, for every } n\geq1,\\ & \text{an increasing, unbounded sequence } (s_{n,p})_{p\geq1} \text{ and}\\ ii. & \text{a sequence } (\delta_{n,p})_{p\geq1}\subset\left[0,\frac{1}{\beta_n}\right] \text{ s.t.}\\ & \lim_{n\to\infty}\lim_{p\to\infty}\mu_{\delta_{n,p}}([0,s_{n,p}])=0 \text{ and}\\ & \lim_{n\to\infty}\lim_{p\to\infty}\mu_{\delta_{n,p}}([0,\beta_n s_{n,p}])=1.\end{array}\tag{B}$$

2.2 Examples of Classical Laws Satisfying Our Assumption

Let us now mention some classes of distributions which satisfy these asymptotic conditions.

Example 2.2. The Weibull laws with scale parameter $\delta > 0$ and form $k(\delta) > 0$ and such that $\lim_{\delta\to 0} k(\delta) = 1$ given by the densities $\xi(\delta, r) = k(\delta)\,\delta^{k(\delta)} r^{k(\delta)-1} e^{-(\delta r)^{k(\delta)}} 1_{r>0}$, for $\delta > 0$ satisfy the previous assumptions. (Note that for $k(\delta) = 1$ one gets the exponential distribution). One can choose $t^\delta = \frac{1}{\delta}\left(\frac{k(\delta)-1}{k(\delta)}\right)^{\frac{1}{k(\delta)}}$. Note that t^δ may be unbounded (e.g. $k(\delta) = 1 + \sqrt{\delta}$). The conditions $(A\ i$ and $ii)$ follow from Remark 2.1 and

$$\lim_{\delta\to 0+} \mu_\delta([0,t]) = \lim_{\delta\to 0+} \left(1 - e^{-(\delta t)^{k(\delta)}}\right) = 0,\ \lim_{\delta\to 0+} t\xi(\delta, t) = \lim_{\delta\to 0+} k(\delta)(\delta t)^{k(\delta)} e^{-(\delta t)^{k(\delta)}} = 0.$$
$$\lim_{\delta\to 0+} \mu_\delta\left([0, t^\delta]\right) = \lim_{\delta\to 0+} \left(1 - e^{-\frac{k(\delta)-1}{k(\delta)}}\right) = 0,\ \lim_{\delta\to 0+} t^\delta \xi\left(\delta, t^\delta\right) = \lim_{\delta\to 0+} (k(\delta)-1)\, e^{-\frac{k(\delta)-1}{k(\delta)}} = 0.$$

Moreover, by picking $s \geq \sqrt{\beta}$ and $\delta_{s,\beta} := \frac{1}{s\sqrt{\beta}}$, one gets

$$\sup_{\delta \leq \frac{1}{\beta}} \left(\mu_\delta\left(\left[\max\left(t^\delta, s\right), \beta s\right]\right) - s\xi(\delta, s)\right)$$
$$\geq \min\left(e^{-\beta^{-\frac{k\left(\frac{1}{s\sqrt{\beta}}\right)}{2}}}, e^{\frac{1}{k\left(\frac{1}{s\sqrt{\beta}}\right)}-1}\right) - e^{-\beta^{\frac{k\left(\frac{1}{s\sqrt{\beta}}\right)}{2}}} - k\left(\frac{1}{s\sqrt{\beta}}\right)\beta^{-\frac{k\left(\frac{1}{s\sqrt{\beta}}\right)}{2}} e^{-\beta^{-\frac{k\left(\frac{1}{s\sqrt{\beta}}\right)}{2}}},$$

which implies $(A\ iii)$ by recalling that $\lim_{\delta\to 0} k(\delta) = 1$.

Example 2.3. The folded normal distributions $\xi(\delta, r) := \frac{\delta}{\sqrt{2\pi}}\left(e^{-\frac{\delta^2(r-a(\delta))^2}{2}} + e^{-\frac{\delta^2(r+a(\delta))^2}{2}}\right)$ such that $\lim_{\delta\to 0+} \delta a(\delta) = 0$. One picks $t^\delta = a(\delta)$. We have $\xi(\delta, r) \leq \sqrt{\frac{2}{\pi}}\delta$ and, thus,

$$\limsup_{\delta\to 0+} \left(\mu_\delta([0,t]) + t\xi(\delta, t)\right) \leq \limsup_{\delta\to 0+} \sqrt{\frac{8}{\pi}}\delta t = 0, \text{ for all } t > 0 \text{ and}$$
$$\limsup_{\delta\to 0+} \left(\mu_\delta\left([0, t^\delta]\right) + t^\delta \xi\left(\delta, t^\delta\right)\right) \leq \limsup_{\delta\to 0+} \sqrt{\frac{8}{\pi}}\delta a(\delta) = 0.$$

Finally, for $\beta > 0$, we pick $\delta_{\beta,s} = \frac{1}{s\sqrt{\beta}}$ and, for s_β great enough (s.t. $\frac{1}{s\sqrt{\beta}} a\left(\frac{1}{s\sqrt{\beta}}\right) \leq \frac{1}{\sqrt{\beta}}$, for $s \geq s_\beta$), we get

$$\mu_{\delta_{\beta,s}}\left(\left[\max\left(t^\delta, s\right), \beta s\right]\right) - s\xi\left(\delta_{\beta,s}, s\right)$$
$$\geq \int_s^{\beta s} \frac{\delta_{\beta,s}}{\sqrt{2\pi}}\left(e^{-\frac{\delta_{\beta,s}^2\left(r - a\left(\delta_{\beta,s}\right)\right)^2}{2}} + e^{-\frac{\delta_{\beta,s}^2\left(r + a\left(\delta_{\beta,s}\right)\right)^2}{2}}\right) dr - \sqrt{\frac{2}{\pi\beta}}$$
$$\geq \int_{\frac{1}{\sqrt{\beta}}}^{\sqrt{\beta} - \frac{1}{\sqrt{\beta}}} \frac{1}{\sqrt{2\pi}}\left(e^{-\frac{r^2}{2}} + e^{\frac{-\left(r + \frac{2}{\sqrt{\beta}}\right)^2}{2}}\right) dr - \sqrt{\frac{2}{\pi\beta}} \geq \int_{\frac{3}{\sqrt{\beta}}}^{\sqrt{\beta} - \frac{1}{\sqrt{\beta}}} \frac{2}{\sqrt{2\pi}} e^{-\frac{r^2}{2}} dr - \sqrt{\frac{2}{\pi\beta}}.$$

The latter expression increases to 1 as $\beta \to \infty$.

Example 2.4. The uniform laws $\xi(\delta, r) = \frac{1}{\overline{a}(\delta) - \underline{a}(\delta)} 1_{[\underline{a}(\delta), \overline{a}(\delta)]}(r)$ such that $\overline{a}$ and $\underline{a}$ are continuous and $\lim_{\delta\to 0} \frac{1+\underline{a}(\delta)}{\overline{a}(\delta) - \underline{a}(\delta)} = 0$. One picks $t^\delta = \underline{a}(\delta)$. Again, $\left(t^\delta\right)_{\delta>0}$ may be unbounded. For every $t > 0$, one gets

$$\limsup_{\delta\to 0+} \left(\mu_\delta([0,t]) + t\xi(\delta, t)\right) \leq \limsup_{\delta\to 0+} \frac{2t}{\overline{a}(\delta) - \underline{a}(\delta)} = 0, \text{ and}$$
$$\int_0^{t^\delta} \left|\xi\left(\delta, t^\delta\right) - \xi(\delta, s)\right| ds = \frac{\underline{a}(\delta)}{\overline{a}(\delta) - \underline{a}(\delta)},$$

for all $\delta > 0$. Also, for $\beta > 0$ and every (great enough) $s > 0$, we pick $\delta_{\beta,s} := \inf\{\delta > 0 : \overline{a}(\delta) = \beta s\}$ and have

$$\mu_{\delta_{\beta,s}}\left(\left[\max\left(t^{\delta_{\beta,s}}, s\right), \beta s\right]\right) - s\xi\left(\delta_{\beta,s}, s\right) \geq \frac{\overline{a}\left(\delta_{\beta,s}\right) - \max\left(\underline{a}\left(\delta_{\beta,s}\right), s\right) - s}{\overline{a}\left(\delta_{\beta,s}\right) - \underline{a}\left(\delta_{\beta,s}\right)}$$
$$\geq 1 - \frac{2}{\beta}\frac{\overline{a}\left(\delta_{\beta,s}\right)}{\overline{a}\left(\delta_{\beta,s}\right) - \underline{a}\left(\delta_{\beta,s}\right)}.$$

Remark 2.5. In fairness to the authors of [13], we point out that in the uniform example, our assumption is slightly stronger that the so-called LTC (long term condition) given in the deterministic framework. Indeed, the authors of [13] prove, for uniform laws (cf. [13, Example 3.3]), that the LTC condition holds true if and only if $\overline{a}(\delta) - \underline{a}(\delta)$ grows to infinity, while, in our case, we need to equally impose that this growth dominates $\underline{a}(\delta)$. This is essentially a consequence of the method we employ, based uniquely on integration by parts and implicitly requiring integrability conditions. More involved IPP formulae might allow this condition to be weakened.

Nevertheless, it is worth pointing out that our proof makes no use of the explicit type of problem and applies to both stochastic and deterministic frameworks. Similar assertions hold true for the folded normal distribution.

Inspired by the previous examples, let us assume that μ_δ admit finite first-order moments $\overline{\mu}_\delta := \int_{\mathbb{R}_+} r\xi(\delta, r)\, dr < \infty$ for all $\delta > 0$. A sufficient condition guaranteeing (A) is given by :

$$\begin{array}{ll} i) & \limsup_{\delta\to 0+} \xi(\delta, t) = \limsup_{\delta\to 0+} \mu_\delta([0,t]) = 0, \text{ for all } t > 0, \\ ii) & \limsup_{\delta\to 0+} \xi\left(\delta, t^\delta\right) = \limsup_{\delta\to 0+} \mu_\delta\left(\left[0, t^\delta\right]\right) = 0 \text{ (with } t^\delta \text{ as in (A))}, \\ iii) & \begin{array}{c}\text{There exist two sequences } a_n \nearrow \infty \text{ and } \delta_n \searrow 0 \text{ s.t.} \\ \mu_{\delta_n}([0,a_n]) = o(1),\ \xi(\delta_n, a_n) = o\left(\frac{1}{a_n}\right) \text{ and } \overline{\mu}_{\delta_n} = O(a_n) \text{ as } n\to\infty.\end{array} \end{array}, \quad \text{(A')}$$

Remark 2.6. (i) The condition (A'iii) reads $\lim_{n\to\infty} \mu_{\delta_n}([0,a_n]) = \lim_{n\to\infty} a_n\xi(\delta_n, a_n) = 0$ and $\limsup_{n\to\infty} \frac{\overline{\mu}_{\delta_n}}{a_n} < \infty$.

(ii) In particular, (A'iii) holds true if $\sup_{\delta>0} \overline{\mu}_\delta < \infty$ and if (A'i) is satisfied.

3 The Main Tauberian Result

3.1 Theoretical Result

The main result of our note is the following.

Theorem 3.1. *(i) If the sequence $(V_t)_{t>0}$ converges to some function v uniformly on compact sets as $t\to\infty$, then, for all $\varepsilon > 0$ and all $k > 0$, there exists $\delta_{\varepsilon,k} > 0$ such that*

$$v^\delta(x) \geq v(x) - \varepsilon,$$

for all $x \in \mathbb{R}^N$ such that $|x| \leq k$ and all $\delta < \delta_{\varepsilon,k}$.

(ii) If the sequence $(V_t)_{t>0}$ converges to some function v uniformly on $\mathbb{R}^N$ as $t\to\infty$, then there exists a sequence $(\delta_n)_{n\geq 1}$ such that $\left(v^{\delta_n}\right)_{n\geq 1}$ converges uniformly to v.

Proof. (i) To prove the first assertion, let us fix $\varepsilon > 0$ and $k > 0$. Then, there exists some $t_{\varepsilon,k} > 0$ such that

$$\sup_{x\in\mathbb{R}^N,\ |x|\leq k} |V_t(x) - v(x)| \leq \frac{\varepsilon}{3},$$

for all $t \geq t_{\varepsilon,k}$. Due to $(A\ i)$ and $(A\ ii)$, we can set $\delta_{\varepsilon,k}$ such that

$$\int_0^{\max\left(t_{\varepsilon,k},t^\delta\right)} \left|\xi\left(\delta,\max\left(t_{\varepsilon,k},t^\delta\right)\right)-\xi\left(\delta,s\right)\right| ds \leq \frac{\varepsilon}{6},$$

for all $\delta \leq \delta_{\varepsilon,k}$. For $\delta > 0$, we let $\overline{a}\left(\delta\right) := \sup\left\{r>0 : \xi\left(\delta,r\right)>0\right\} \in \left(t^\delta,\infty\right]$. We can assume, without loss of generality, that $\overline{a}\left(\delta\right) > t_{\varepsilon,k}$ (see Remark 2.1 (iii)). Then, for some $\max\left(t_{\varepsilon,k},t^\delta\right) \leq \overline{a}\left(\delta,\varepsilon\right) < \overline{a}\left(\delta\right)$,

$$\begin{aligned}1-\frac{\varepsilon}{3} &\leq 1+\int_0^{\max\left(t_{\varepsilon,k},t^\delta\right)}\left(\xi\left(\delta,\max\left(t_{\varepsilon,k},t^\delta\right)\right)-\xi\left(\delta,s\right)\right)ds-\frac{\varepsilon}{6}\\ &\leq \max\left(t_{\varepsilon,k},t^\delta\right)\xi\left(\delta,\max\left(t_{\varepsilon,k},t^\delta\right)\right)+\mu_\delta\left(\left[\max\left(t_{\varepsilon,k},t^\delta\right),\overline{a}\left(\delta,\varepsilon\right)\right]\right).\end{aligned}$$

An integration-by-parts argument implies that, for every $x \in \mathbb{R}^N$ such that $|x| \leq k$, every $\delta \leq \delta_{\varepsilon,k}$ and every admissible control process $u \in \mathcal{U}_{ad}$, we have

$$\begin{aligned}v\left(x\right)-\frac{2\varepsilon}{3} &\leq \left(1-\frac{\varepsilon}{3}\right)\left(v\left(x\right)-\frac{\varepsilon}{3}\right)\\ &\leq \left(\max\left(t_{\varepsilon,k},t^\delta\right)\xi\left(\delta,\max\left(t_{\varepsilon,k},t^\delta\right)\right)+\int_{\max\left(t_{\varepsilon,k},t^\delta\right)}^{\overline{a}\left(\delta,\varepsilon\right)}\xi\left(\delta,s\right)ds\right)\left(v\left(x\right)-\frac{\varepsilon}{3}\right)\\ &\leq \overline{a}\left(\delta,\varepsilon\right)\xi\left(\delta,\overline{a}\left(\delta,\varepsilon\right)\right)\left(v\left(x\right)-\frac{\varepsilon}{3}\right)+\int_{\max\left(t_{\varepsilon,k},t^\delta\right)}^{\overline{a}\left(\delta,\varepsilon\right)}-s\partial_s\xi\left(\delta,s\right)\left(v\left(x\right)-\frac{\varepsilon}{3}\right)dt\\ &\leq \overline{a}\left(\delta,\varepsilon\right)\xi\left(\delta,\overline{a}\left(\delta,\varepsilon\right)\right)\left(v\left(x\right)-\frac{\varepsilon}{3}\right)+\int_{\max\left(t_{\varepsilon,k},t^\delta\right)}^{\overline{a}\left(\delta,\varepsilon\right)}-\partial_s\xi\left(\delta,s\right)sV_s\left(x\right)ds\\ &\leq \overline{a}\left(\delta,\varepsilon\right)\xi\left(\delta,\overline{a}\left(\delta,\varepsilon\right)\right)\left(v\left(x\right)-\frac{\varepsilon}{3}\right)+\int_{\max\left(t_{\varepsilon,k},t^\delta\right)}^{\overline{a}\left(\delta,\varepsilon\right)}-\partial_s\xi\left(\delta,s\right)\int_0^s\mathbb{E}\left[g\left(X_l^{x,u},u_l\right)\right]dlds\end{aligned}$$

Again, by an integration-by-parts argument, we have

$$\begin{aligned}v\left(x\right)-\frac{2\varepsilon}{3} \leq{}& \overline{a}\left(\delta,\varepsilon\right)\xi\left(\delta,\overline{a}\left(\delta,\varepsilon\right)\right)\left[v\left(x\right)-\frac{\varepsilon}{3}-\frac{1}{\overline{a}\left(\delta,\varepsilon\right)}\int_0^{\overline{a}\left(\delta,\varepsilon\right)}\mathbb{E}\left[g\left(X_l^{x,u},u_l\right)\right]dl\right]\\ &+\int_0^{\max\left(t_{\varepsilon,k},t^\delta\right)}\left[\xi\left(\delta,\max\left(t_{\varepsilon,k},t^\delta\right)\right)-\xi\left(\delta,t\right)\right]g\left(X_t^{x,u},u_t\right)dt\\ &+\mathbb{E}\left[\int_0^{\overline{a}\left(\delta,\varepsilon\right)}\xi\left(\delta,t\right)g\left(X_t^{x,u},u_t\right)dt\right].\end{aligned} \tag{3.1}$$

Recalling that $\overline{a}\left(\delta,\varepsilon\right) \geq t_{\varepsilon,k}$, one gets

$$\frac{1}{\overline{a}\left(\delta,\varepsilon\right)}\int_0^{\overline{a}\left(\delta,\varepsilon\right)}\mathbb{E}\left[g\left(X_l^{x,u},u_l\right)\right]dl \geq V_{\overline{a}\left(\delta,\varepsilon\right)}\left(x\right) \geq v\left(x\right)-\frac{\varepsilon}{3}. \tag{3.2}$$

Also,

$$\begin{aligned}&\int_0^{\max\left(t_{\varepsilon,k},t^\delta\right)}\left[\xi\left(\delta,\max\left(t_{\varepsilon,k},t^\delta\right)\right)-\xi\left(\delta,t\right)\right]g\left(X_t^{x,u},u_t\right)dt\\ &\leq \int_0^{\max\left(t_{\varepsilon,k},t^\delta\right)}\left|\xi\left(\delta,\max\left(t_{\varepsilon,k},t^\delta\right)\right)-\xi\left(\delta,t\right)\right|dt \leq \frac{\varepsilon}{6}.\end{aligned}$$

Substituting this inequality and (3.2) in (3.1), one gets

$$v\left(x\right)-\frac{2\varepsilon}{3} \leq \mathbb{E}\left[\int_0^\infty \xi\left(\delta,t\right)g\left(X_t^{x,u},u_t\right)dt\right]+\frac{\varepsilon}{6}.$$

The conclusion follows by picking some admissible control process $u \in \mathcal{U}_{ad}$ which is $\frac{\varepsilon}{6}$-optimal for $v^{\delta}(x)$.

(ii) Before proving the second assertion, we state the following monotonicity result

Proposition 3.2. *For every $T_0 > s \geq 0$, $x \in \mathbb{R}^N$ and admissible control process $u \in \mathcal{U}_{ad}$, one has*

$$\liminf_{t\to\infty} V_t(x) \leq \liminf_{t\to\infty} \mathbb{E}\left[V_t\left(X_{T_0}^{x,u}\right)\right] \text{ and } (T_0 - s)\,\mathbb{E}\left[V_{T_0-s}\left(X_s^{x,u}\right)\right] \leq \mathbb{E}\left[\int_s^{T_0} g\left(X_r^{x,u}, u_r\right) dr\right]. \tag{3.3}$$

We postpone the proof of this proposition to the end of the subsection and complete our theorem. We fix $\varepsilon > 0$ and $\alpha(\varepsilon) > 0$ to be specified later on. Our assumption yields the existence of some $t_{\varepsilon,\alpha(\varepsilon)} > 0$ such that

$$\sup_{x\in\mathbb{R}^N} |V_s(x) - v(x)| \leq \alpha^2(\varepsilon),$$

for all $s \geq \alpha(\varepsilon)\, t_{\varepsilon,\alpha(\varepsilon)}$. We fix, for the time being, the time horizon $t \geq t_{\varepsilon,\alpha(\varepsilon)}$ and an admissible control $u^{t,\alpha(\varepsilon)} \in \mathcal{U}_{ad}$ for which

$$\frac{1}{t}\mathbb{E}\left[\int_0^t g\left(X_r^{x,u^{t,\alpha(\varepsilon)}}, u_r^{t,\alpha(\varepsilon)}\right) dr\right] \leq V_t(x) + \alpha^2(\varepsilon).$$

Using the first inequality in Proposition 3.2, we get

$$v(x) \leq \liminf_{T\to\infty} \mathbb{E}\left[V_T\left(X_s^{x,u^{t,\alpha(\varepsilon)}}\right)\right] = \mathbb{E}\left[v\left(X_s^{x,u^{t,\alpha(\varepsilon)}}\right)\right] \leq \mathbb{E}\left[V_{t-s}\left(X_s^{x,u^{t,\alpha(\varepsilon)}}\right)\right] + \alpha^2(\varepsilon),$$

for all $s \leq (1-\alpha(\varepsilon))\,t$ (the last inequality is a consequence of the fact that $t - s \geq \alpha(\varepsilon)\, t_{\varepsilon,\alpha(\varepsilon)}$). Combining this estimate with the second inequality in Proposition 3.2 and recalling the choice of $u^{t,\alpha(\varepsilon)}$, one has

$$\begin{aligned}
tv(x) &\geq tV_t(x) - t\alpha^2(\varepsilon) \geq \mathbb{E}\left[\int_0^t g\left(X_r^{x,u^{t,\alpha(\varepsilon)}}, u_r^{t,\alpha(\varepsilon)}\right) dr\right] - 2t\alpha^2(\varepsilon)\\
&\geq \mathbb{E}\left[\int_0^s g\left(X_r^{x,u^{t,\alpha(\varepsilon)}}, u_r^{t,\alpha(\varepsilon)}\right) dr\right] + (t-s)\,\mathbb{E}\left[V_{t-s}\left(X_s^{x,u^{t,\alpha(\varepsilon)}}\right)\right] - 2t\alpha^2(\varepsilon)\\
&\geq \mathbb{E}\left[\int_0^s g\left(X_r^{x,u^{t,\alpha(\varepsilon)}}, u_r^{t,\alpha(\varepsilon)}\right) dr\right] + (t-s)\,v(x) - (3t-s)\,\alpha^2(\varepsilon),
\end{aligned}$$

for all $\alpha(\varepsilon)\,t \leq s \leq (1-\alpha(\varepsilon))\,t$. This implies that whenever $s \in [\alpha(\varepsilon)\,t, (1-\alpha(\varepsilon))\,t]$,

$$v(x) \geq \frac{1}{s}\mathbb{E}\left[\int_0^s g\left(X_r^{x,u^{t,\alpha(\varepsilon)}}, u_r^{t,\alpha(\varepsilon)}\right) dr\right] - 3\alpha(\varepsilon). \tag{3.4}$$

We then use the splitting

$$[0,\infty) = (\alpha(\varepsilon)\,t, (1-\alpha(\varepsilon))\,t] \cup ([0,\infty) \setminus (\alpha(\varepsilon)\,t, (1-\alpha(\varepsilon))\,t]),$$

and recall that $0 \leq g \leq 1$ to get

$$\begin{aligned}
v^{\delta}(x) &\leq \int_0^{\alpha(\varepsilon)t} \xi(\delta,r)\,dr + 1 - \int_0^{(1-\alpha(\varepsilon))t} \xi(\delta,r)\,dr\\
&\quad + \mathbb{E}\left[\int_{\alpha(\varepsilon)t}^{(1-\alpha(\varepsilon))t} \xi(\delta,r)\,g\left(X_r^{x,u^{t,\alpha(\varepsilon)}}, u_r^{t,\alpha(\varepsilon)}\right) dr\right]\\
&\leq \int_0^{\alpha(\varepsilon)t} \xi(\delta,r)\,dr + 1 - \int_0^{(1-\alpha(\varepsilon))t} \xi(\delta,r)\,dr\\
&\quad + \int_{\alpha(\varepsilon)t}^{\max\left(\alpha(\varepsilon)t,t^{\delta}\right)} \xi(\delta,r)\,dr + \mathbb{E}\left[\int_{\max(\alpha(\varepsilon)t,t^{\delta})}^{(1-\alpha(\varepsilon))t} \xi(\delta,r)\,g\left(X_r^{x,u^{t,\alpha(\varepsilon)}}, u_r^{t,\alpha(\varepsilon)}\right) dr\right],
\end{aligned}$$

for all $\delta > 0$. Using, as we have already done in the first part, an integration-by-parts formula and the inequality (3.4), it follows that

$$\begin{aligned}
& v^{\delta}(x) \\
& \leq 1 - \int_{\alpha(\varepsilon)t}^{(1-\alpha(\varepsilon))t} \xi(\delta, r)\, dr + \int_{\alpha(\varepsilon)t}^{\max\left(\alpha(\varepsilon)t, t^{\delta}\right)} \xi(\delta, r)\, dr \\
& + (1-\alpha(\varepsilon))\, t\xi(\delta, (1-\alpha(\varepsilon))\, t)\, \frac{1}{(1-\alpha(\varepsilon))\, t} \mathbb{E}\left[\int_0^{(1-\alpha(\varepsilon))t} g\left(X_r^{x, u^{t,\alpha(\varepsilon)}}, u_r^{t,\alpha(\varepsilon)}\right) dr\right] \\
& + \int_{\max(\alpha(\varepsilon)t, t^{\delta})}^{(1-\alpha(\varepsilon))t} -s\partial_s \xi(\delta, s)\, \mathbb{E}\left[\frac{1}{s}\int_0^s g\left(X_r^{x, u^{t,\alpha(\varepsilon)}}, u_r^{t,\alpha(\varepsilon)}\right) dr\right] ds \\
& \leq 1 - \mu_{\delta}\left(\left[\max\left(\alpha(\varepsilon)\, t, t^{\delta}\right), (1-\alpha(\varepsilon))t\right]\right) \\
& + (v(x) + 3\alpha(\varepsilon))\left[(1-\alpha(\varepsilon))\, t\xi(\delta, (1-\alpha(\varepsilon))\, t) + \int_{\max(\alpha(\varepsilon)t, t^{\delta})}^{(1-\alpha(\varepsilon))t} -s\partial_s \xi(\delta, s)\, ds\right] \\
& = 1 - \mu_{\delta}\left(\left[\max\left(\alpha(\varepsilon)\, t, t^{\delta}\right), (1-\alpha(\varepsilon))t\right]\right) + (v(x) + 3\alpha(\varepsilon)) \times \\
& \left(\max\left(\alpha(\varepsilon)\, t, t^{\delta}\right) \xi\left(\delta, \max\left(\alpha(\varepsilon)\, t, t^{\delta}\right)\right) + \mu_{\delta}\left(\left[\max\left(\alpha(\varepsilon)\, t, t^{\delta}\right), (1-\alpha(\varepsilon))t\right]\right)\right) \\
& \leq 1 - \mu_{\delta}\left(\left[\max\left(\alpha(\varepsilon)\, t, t^{\delta}\right), (1-\alpha(\varepsilon))t\right]\right) \\
& + (v(x) + 3\alpha(\varepsilon))\left(\max\left(\alpha(\varepsilon)\, t, t^{\delta}\right) \xi\left(\delta, \max\left(\alpha(\varepsilon)\, t, t^{\delta}\right)\right) - \mu_{\delta}\left(\left[0, \max\left(\alpha(\varepsilon)\, t, t^{\delta}\right)\right]\right) + 1\right).
\end{aligned} \tag{3.5}$$

The reader is invited to note that, by our assumptions $(A\ ii)$ and $(A\ iii)$, there exists $\beta_{\varepsilon} > \frac{1}{\varepsilon}$ and some $s_{\varepsilon} > t_{\varepsilon,\alpha(\varepsilon)}$ such that

$$\begin{cases} \sup_{\delta \leq \frac{1}{\beta_{\varepsilon}}} \left[\mu_{\delta}\left(\left[\max\left(t^{\delta}, s_{\varepsilon}\right), \beta_{\varepsilon}s_{\varepsilon}\right]\right) - s_{\varepsilon}\xi(\delta, s_{\varepsilon})\right] \geq 1 - \varepsilon, \text{ for all } \varepsilon > 0 \text{ and} \\ \int_0^{t^{\delta}} \left(\xi\left(\delta, t^{\delta}\right) - \xi(\delta, s)\right) ds \in [-\varepsilon, \varepsilon], \text{ for all } \delta < \frac{1}{\beta_{\varepsilon}}. \end{cases} \tag{3.6}$$

We set $\alpha(\varepsilon) := \frac{1}{\beta_{\varepsilon}+1} < \varepsilon$ (the reader will note that $(1-\alpha(\varepsilon)) = \beta_{\varepsilon}\alpha(\varepsilon)$). Then, by setting $t := \frac{s_{\varepsilon}}{\alpha(\varepsilon)}$, the first inequality in (3.6) yields the existence of some $\delta_{\varepsilon} < \frac{1}{\beta_{\varepsilon}}$ such that

$$\mu_{\delta_{\varepsilon}}\left(\left[\max\left(t^{\delta_{\varepsilon}}, \alpha(\varepsilon)\, t\right), (1-\alpha(\varepsilon))\, t\right]\right) \geq 1 - 2\varepsilon \text{ and } \alpha(\varepsilon)\, t\xi(\delta_{\varepsilon}, \alpha(\varepsilon)\, t) \leq 2\varepsilon.$$

Moreover, using the second inequality in (3.6), we get

$$t^{\delta_{\varepsilon}}\xi\left(\delta_{\varepsilon}, t^{\delta_{\varepsilon}}\right) - \mu_{t^{\delta_{\varepsilon}}}\left(\left[0, t^{\delta_{\varepsilon}}\right]\right) ds \in [-\varepsilon, \varepsilon].$$

Then, for $\varepsilon < \frac{1}{6}$, the inequality (3.5) implies

$$v^{\delta_{\varepsilon}}(x) \leq 2\varepsilon + (v(x) + 3\varepsilon)(1 + 2\varepsilon) \leq v(x) + 8\varepsilon.$$

Our result is now complete by recalling that $\delta_{\varepsilon} \leq \frac{1}{\beta_{\varepsilon}}$ and using the first assertion of our theorem. □

Remark 3.3. (i) In the last part of our proof, the choice of δ_{ε} explicitly relies on s_{ε}. Since the condition $(A\ iii)$ can only produce a sequence of such s_{ε}, we can only infer that some subsequence v^{δ} converges to v. However, in our explicit examples, $\delta_{\varepsilon} = \delta(\beta_{\varepsilon}, s)$ for all s large enough and this dependence is continuous in s. It follows that, at least for our examples, the second part can be given with respect to any sequence $(\delta_n)_{n\geq 1}$. Hence, in this case, we have the existence of a unique limit for $\left(v^{\delta}\right)_{\delta>0}$ as $\delta \to 0$. To achieve this, in general, one could strengthen the condition by asking the existence of a continuous function $\delta : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$ such that

$$\begin{aligned}
&\delta(\beta, s) \leq \frac{1}{\beta}, \\
&\delta(\beta, \cdot) \text{ is non-increasing, } \lim_{s\to\infty} \delta(\beta, s) = 0, \\
&\limsup_{\beta\to\infty} \liminf_{s\to\infty} \left[\mu_{\delta(\beta,s)}\left(\left[\max\left(t^{\delta(\beta,s)}, s\right), \beta s\right]\right) - s\xi\left(\delta(\beta, s), s\right)\right] = 1.
\end{aligned}$$

In this way, when a convenient bound is got for $\delta_n := \delta(n, s_n)$, it holds true for all $\delta \leq \delta_n$ (due to Darboux property and the $\liminf$ formulation.)

(ii) The essential assumption in the main result is the uniform convergence of the sequence $(V_t)_{t>0}$. Minimal non-expansive conditions guaranteeing this convergence can be found in [5, Theorem 8] with no jumps ($f = 0$). Adapting this approach (see also [10] in a framework where the jump mechanism is more complicated), a non-expansive condition in this setting would be

$$\sup_{u\in U} \inf_{v\in U} \max \begin{pmatrix} \left(\langle b(x, u) - b(y, v), x - y\rangle + \frac{1}{2}\left|\sigma(x, u) - \sigma(y, v)\right|^2\right), \\ \sup_{e\in Supp(\lambda)} \left|\left|x + f(x, e, u) - y - f(y, e, v)\right| - |x - y|\right| \\ |g(x, u) - g(y, v)| - Lip(g)|x - y| \end{pmatrix} \leq 0, \tag{3.7}$$

where

$$\left|\sigma(x, u) - \sigma(y, v)\right|^2 = Tr\left[\left(\sigma(x, u) - \sigma(y, v)\right)\left(\sigma^*(x, u) - \sigma^*(y, v)\right)\right],$$

for all $(x, y, u, v) \in \mathbb{R}^{2N} \times U^2$, $Supp(\lambda) \subset E$ denotes the support of the measure λ and $Lip(g)$ denotes the Lipschitz constant of g with respect to the state parameter. Let us also assume that there exists some compact set $\mathbb{K}$ which is invariant with respect to the dynamics (see [16] for explicit conditions). Then it can be shown (in the same way as [5, Proposition 4]) that the functions V_t are equicontinuous on $\mathbb{K}$ and they converge uniformly on $\mathbb{K}$. The reader will note that, in this invariant case, the condition (3.7) needs only be checked for $(x, y) \in \mathbb{K}$.

To complete the subsection, we sketch the proof of the monotonicity result. It is a mere consequence of the dynamic programming principle.

Proof of Proposition 3.2. Using the dynamic programming principle (cf. [17], [14], [4], etc.), one gets, for every $t > T_0$,

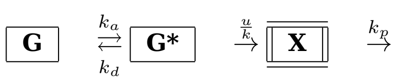
$$tV_t(x) = \inf_{u\in\mathcal{U}_{ad}} \left(\mathbb{E}\left[\int_0^{T_0} g\left(X_s^{x,u}, u_s\right) ds\right] + \mathbb{E}\left[(t - T_0) V_{t-T_0}\left(X_{T_0}^{x,u}\right)\right]\right)$$

and the conclusion follows by dividing the equality by $t > 0$ and letting $t \to \infty$.

The second assertion follows similar patterns. (For a proof based only on Itô's formula and Krylov's shaking the coefficients method, the reader may want to take a look at [5, Proposition 19]. Finally, we mention that an adaptation of Krylov's method [12] to Lévy processes can be found in [3].) □

3.2 A Gene-inspired Piecewise Diffusive Switch Example

We recall the diagram of Cook's model of gene expression, product accumulation and product degradation and its implications on haploinsufficiency (cf. [7]).



This model considers a gene (x_0) to switch randomly between inactive state (G) and active state (G*). The activation (respectively deactivation) rate is denoted by k_a (respectively k_d) and, to simplify the framework, we assume $k_a = k_d = 1$. When active, a single burst of $\frac{u}{k}$ (u is an exogenous control and k a volume normalization coefficient) units of the (concentration) vector X occurs. We consider a simple model in which two products X are of interest : a monomer (x_1) and its dimer (x_2). There is a continuous transition from monomer to dimer and conversely and the monomer is subject to degradation with a stochastic perturbation. We deal with a three-dimensional state space ($N = 3$, $x = (x_0, x_1, x_2)$). We have a unidimensional Brownian motion ($d = 1$). The jump mechanism is driven by activation and deactivation. The Poisson measure only counts the jumps and, as a new jump occurs, x_0 switches from 0 (inactive) to 1 (active) or vice versa. For the dimerization and degradation (which is a high speed reaction with $k_p > 2$), we have

$$2\boxed{\mathbf{X}_1} \underset{u}{\overset{u}{\rightleftarrows}} \boxed{\mathbf{X}_2} \text{ and } \boxed{\mathbf{X}_1} \overset{k_p u}{\rightarrow}$$

with a random fluctuation occurring only in the degradation. The control space is set to be $U = [0, 1]$.

Remark 3.4. Transcription can occur either in averaged form (units per second) or, as we meant here, as a single burst of $\frac{u}{k}$ units at the activation of the gene. With this in mind, a proper way of writing

$$\boxed{\mathbf{G}} \underset{k_d}{\overset{k_a}{\rightleftarrows}} \boxed{\mathbf{G^*}} \overset{\frac{u}{k}}{\rightarrow} \boxed{\mathbf{X}}$$

would be

$$\boxed{\mathbf{G}} \overset{k_a}{\rightarrow} \boxed{\mathbf{G}^*} + \frac{u}{k}\boxed{\mathbf{X}_1} \text{ and } \boxed{\mathbf{G}^*} \overset{k_d}{\rightarrow} \boxed{\mathbf{G}}.$$

We get the following coefficients

$$b\left(\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}, u\right) = \begin{pmatrix} 0 \\ -2ux_1^2 + 2ux_2 - k_p u x_1 \\ ux_1^2 - ux_2 \end{pmatrix},\ \sigma\left(\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}, u\right) = \begin{pmatrix} 0 \\ u\,(1 - x_1)\,x_1 \\ 0 \end{pmatrix}$$

$$f\left(\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}, 1, u\right) = \begin{pmatrix} 1 - 2x_0 \\ \min\left(\frac{u}{k}, 1 - x_1\right)(1 - x_0) \\ 0 \end{pmatrix},\ x_0 \in \{0, 1\},\ \ (x_1, x_2) \in \mathbb{R}^2.$$

One easily notes that $\mathbb{K} := \{0, 1\} \times [0, 1]^2$ is invariant with respect to the system. Indeed, x_0 does not change between jumps and, when jumps occur, it switches between 0 and 1 (according to f, it changes from x_0 to $1 - x_0$). The x_1 component increases with $\frac{u}{k}$ but cannot exceed 1 (at gene activation, i.e. when, previously, $x_0 = 0$ and a jump occurs). Jumps do not change x_2.

Step 1. Invariance. To check invariance between jumps, one can use the results in [6] or [2]. Alternatively, one may note that, for $(x_0, x_1, x_2) \in \mathbb{K}$,

$$\sigma\left(\begin{pmatrix} x_0 \\ 1 \\ x_2 \end{pmatrix}, u\right) = \sigma\left(\begin{pmatrix} x_0 \\ 0 \\ x_2 \end{pmatrix}, u\right) = 0,$$

$$b\left(\begin{pmatrix} x_0 \\ 1 \\ x_2 \end{pmatrix}, u\right) = \begin{pmatrix} 0 \\ (2x_2 - 2 - k_p)\,u \le 0 \\ u - ux_2 \end{pmatrix},\ b\left(\begin{pmatrix} x_0 \\ 0 \\ x_2 \end{pmatrix}, u\right) = \begin{pmatrix} 0 \\ 2ux_2 \ge 0 \\ -ux_2 \end{pmatrix},$$

$$b\left(\begin{pmatrix} x_0 \\ x_1 \\ 0 \end{pmatrix}, u\right) = \begin{pmatrix} 0 \\ -2ux_1^2 - k_p u x_1 \\ ux_1^2 \ge 0 \end{pmatrix},\ b\left(\begin{pmatrix} x_0 \\ x_1 \\ 1 \end{pmatrix}, u\right) = \begin{pmatrix} 0 \\ -2ux_1^2 + 2u - k_p u x_1 \\ \left(x_1^2 - 1\right) u \le 0 \end{pmatrix}$$

to conclude that $\mathbb{K}$ is invariant.

Step 2. Non-expansivity. For every $u \in [0,1]$,

$$\begin{aligned}
&\inf_v \left(\langle b(x,u) - b(y,v), x-y \rangle + \frac{1}{2} |\sigma(x,u) - \sigma(y,v)|^2 \right) \\
&\leq \left(\langle b(x,u) - b(y,u), x-y \rangle + \frac{1}{2} |\sigma(x,u) - \sigma(y,u)|^2 \right) \\
&= -u \begin{bmatrix} \left(2(x_1+y_1) - \frac{1}{2}u(x_1+y_1-1)^2 + k_p\right)(x_1-y_1)^2 \\ -(x_1+y_1+2)(x_1-y_1)(x_2-y_2) + (x_2-y_2)^2 \end{bmatrix} \leq 0.
\end{aligned}$$

The last inequality is a consequence of the fact that $x_1, y_1 \in [0,1]$, $k_p \geq 2$ and

$$\begin{aligned}
\Delta &= (x_1+y_1+2)^2 - 4\left(2(x_1+y_1) - \frac{1}{2}u(x_1+y_1-1)^2 + k_p\right) \\
&= (x_1+y_1-2)^2 - 4\left(-\frac{1}{2}u(x_1+y_1-1)^2 + k_p\right) < 4 - 4\left(-\frac{1}{2} + k_p\right) < 0.
\end{aligned}$$

For the jumps, since the first component of f does not depend on u, the inequality can be written for vectors sharing the same $x_0 \in \{0,1\}$. We note that the function $x_1 \mapsto x_1 + \min\left(\frac{u}{k}, 1-x_1\right)(1-x_0)$ is $1-$Lipschitz continuous. Then our system is non-expansive.

Using the Remark 3.3 (ii), in this setting, the Cesàro means converge uniformly on $\mathbb{K}$ and Theorem 3.1 holds true.

Step 3. Non-dissipativity. We also note the fact that our system is not dissipative and classical results do not apply. Indeed, for $u = 0$ and $y_1 = y_2 = 0$,

$$\inf_v \left(\langle b(x,u) - b(y,v), x-y \rangle + \frac{1}{2} |\sigma(x,u) - \sigma(y,v)|^2 \right) = 0,$$

for all $x \in \mathbb{K}$. Hence, we are unable to find a $C > 0$ such that

$$\inf_v \left(\langle b(x,u) - b(y,v), x-y \rangle + \frac{1}{2} |\sigma(x,u) - \sigma(y,v)|^2 \right) \leq -C|x-y|^2,$$

for all $u \in [0,1]$ and all $(x,y) \in \mathbb{K}^2$.

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